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MAHAVIER LIMITS; THEORY AND APPLICATIONS

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ABSTRACT. Generalised inverse limits were introduced in 2006 by Ingram and Mahavier as a generalisation of the classical notion of inverse limit of an inverse system of topological spaces. There followed a rather intensive period of research on the subject with many results established, some of which are direct generalisations of classical results, and others that attest more to the differences between the classical and the generalised notions. It is well-known that inverse limits of spaces are precisely categorical limits in the category of spaces and continuous functions. It is also known that generalised inverse limits are not limits in the category of spaces and upper semicontinuous multivalued functions. In this work we present a categorical extension of the notion of limit in a category to what we call Mahavier limit. We show that the new concept is a generalisation of categorical limit, and that generalised inverse limits of spaces are precisely Mahavier limits in the category of spaces and upper semicontinuous multivalued functions. Foundational categorical tools are extended to the new setting, which are then applied to topological spaces to obtain results regarding a subsequence theorem and mapping theorems.

1. INTRODUCTION

Motivated by an ongoing stream of results directly influenced by work of Mahavier and Ingram that started a decade ago, we develop a generalisation of the notion of categorical limit which extends the link between

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category theory and inverse limits in topology. We present some fundamental category theoretic results for these new limits, and we employ categorical techniques to obtain results about generalised inverse limits of topological spaces. Why we find the link with category theory important is briefly outlined below, followed by more details on the main results.

Generalised inverse limits arise when one considers diagrams of spaces with multivalued functions between them rather than single-valued functions. The definition is not given in terms of a universal property, but rather as an ad-hoc adaptation of the classical construction of inverse limits in the category **Top** of topological spaces. The fundamentals are reviewed in detail in Section 2 below. We mention here just one difference between the classical theory of inverse limits and this generalisation of it, namely the subsequence theorem. Intuitively, the subsequence theorem states that the inverse limit of an inverse sequence of spaces is homeomorphic to the inverse limit computed out of any subsequence of the given inverse system. The subsequence theorem is a property of all classical inverse limits, and in fact is a consequence of a well-known categorical result regarding final functors. In contrast, the subsequence theorem fails dramatically for the notion of generalised inverse limits for nearly all inverse systems of spaces with multivalued functions. In particular, it is thus implied that any categorical framework in which the subsequence theorem holds cannot fully capture the notion of generalised inverse limits. This explains, for instance, the difficulties encountered in the early attempt [?] for a categorical approach.

We mention here the independently developed formalism presented in [?], a formalism subsumed by the formalism we develop below. The results presented in [?] are, just like our results, geared toward generalised inverse limits in topology, and yet there is little overlap between the two articles, attesting to the richness of the categorical framework. We further mention [?], where the notion of Mahavier completeness is studied. In particular, in such complete categories the precise difference between the ordinary notion of limit and the generalised notion is quantified in terms of classifying diagrams. As far as content is concerned, the present work and [?] are perpendicular, each offering a different perspective on the relationship between classical category theory and the extension of it that we propose. Finally, we mention [?], where the authors study a particular kind of 2-categorical limit. Interestingly, even though the motivations are disjoint, some aspects are shared among the two approaches. However, either by inspection or by noting that the Lack-Shulman formalism supports the subsequence theorem, the notion of Mahavier limit we study is not subsumed under the results of [?], nor is it the other way around. In slight more detail, utilising the language used in [?], the

notion of \mathcal{F} -enriched category corresponds well with our notion of order extension, giving rise to notions of tight and loose morphisms. The idea of [?] is to consider diagrams with a portion required to be tight. The corresponding notion of limit is then required to have corresponding tightness of a certain portion of its projections so that the tightness in the diagram is correlated with the tightness of the projections. If our approach is phrased using similar terms, then the diagrams we consider do not have any tight portions, but all of the projections in our notion of limit are required to be tight. A more detailed comparison with [?] is beyond the scope of this work.

1.1. Motivation for a categorical approach. It is, of course, very common to study a space (or other structures) by exhibiting the space as the (essentially) unique solution of a problem stated in terms of simpler spaces, and then study the simpler spaces together with the dynamics of the problem whose solution defines the space. In topology, and particularly in compactum theory, exhibiting a space as the inverse limit of a sequence of spaces, or of spaces indexed by a directed set, is a well-established technique with more successes than can effectively be recounted here. General information on inverse limits in topology, which appeared first in the work of Alexandrov in 1926 and further developed by Lefschetz in 1952, can be found in [?]. An informative short discussion of inverse limits in the context of continua theory can be found in the introduction of [?].

Inverse limits are also ubiquitous outside of topology, for instance the p -adic integers are the inverse limit of the cyclic groups $\mathbb{Z}/p^n \cdot \mathbb{Z}$. The common generalisation which embodies all examples of inverse limits under a single unifying formalism is the notion of limit in a category. The formalism of category theory, developed in 1945 by Eilenberg and Mac Lane, addressed pressing issues in topology, primarily homology, where category theory became an arguably invaluable language. Within just a few decades the language of category theory was used to unify and to simplify ideas and results in many areas, and here we only mention [?] for a very elegant exposition of inverse systems and entropy from the category theoretic point-of-view applied to ergodic theory.

The notion of limit in a category can be seen as a way of constructing new objects from given objects and relationships among them by means of the universal property. In more detail, one considers a diagram of objects and morphisms in the given category and asks for a single object that represents the entire diagram. The precise meaning of what it means for an object to represent the diagram is the content of the definition of the universal property. As already stated, this notion of categorical

limit captures precisely the various notions of inverse limits of inverse systems in, among others, topology. From a point-of-view that values both topology and category theory, such a formulation of inverse limits allows one to distinguish between properties of inverse limits of, say, compacta, that are generic as opposed to those that are topological.

In 2006 Ingram and Mahavier introduced a generalisation of the common explicit definition of the inverse limit of an inverse system of compacta. The new definition, the result of considering multivalued functions instead of single-valued functions, is very natural and, quite fascinatingly, gives rise to highly complicated spaces even from very simple inverse systems. The elegance and utility of these generalised inverse limits resulted in a stream of results over the decade of their existence (e.g., the recent publications [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?]). Many similarities, as well as stark differences, between inverse limits and generalised inverse limits were noted, and the situation is somewhat a reflection of the state of affairs in topology of the pre category theory days, namely numerous results are present, some of which rely heavily on topology, but some are of a more general flavour, and the language to pinpoint the difference is missing. The aim of this article is to fully restore the categorical perspective by showing that while generalised inverse limits are not limits in the categorical sense, they do correspond, precisely, to a generalisation of the concept of categorical limit.

1.2. Main results and plan of the paper. Section 2 serves to further elucidate our motivation and to establish the universal property of generalised inverse limits (Theorem 2.3), all within the context of topology, i.e., the classical Ingram-Mahaiver setting of generalised inverse limits. This section can harmlessly be skipped by an uninterested reader. With the motivation in place, Section 3 is concerned with developing category-theoretic machinery. Using ordered categories, the basic setting is that of what we call an order extension, given in Definition 3.2, within which the notion of Mahavier limit (Definition 3.6) can be interpreted. A main conceptual result is that Mahavier limits, when interpreted in the category of topological spaces, are classical inverse limits, while when interpreted in a suitable order extension Mahavier limits are precisely generalised inverse limits. An important result is Theorem 3.15, stating that an order extension admits all small Mahavier limits provided certain conditions are met. These conditions are verified for topological spaces in Theorem 3.16, thus proving that generalised inverse limits of any shape exist.

The full generality of Mahavier limits in arbitrary order extensions requires more examples than just one, and thus Section 4 introduces monads as a convenient manner to obtain order extensions of interest. The

classical setting for topology corresponds then to that obtained by the hyperspace monad. Other monads are presented as well, though the main focus of the article remains grounded in topology. Section 5 is concerned with extending the usual category theoretic results about functoriality of limits and canonical morphisms resulting from a change of diagram functor to the context of Mahavier limits. When applied to topological spaces this approach unifies numerous instances in the literature of the ad-hoc constructions of such mappings. Section 6 addresses the subsequence theorem, from a categorical perspective. In particular, we obtain conditions under which the subsequence theorem holds in an arbitrary order extension of a category, a result whose technical heart is Lemma 6.3. These conditions are effectively applied to topological spaces to obtain a solution to the subsequence theorem problem. Section 7 is concerned with another extension of category theoretic results, regarding limits and adjunctions, to Mahavier limits. The results are applied to relate the categories of compacta, topological spaces, and sets in a way useful for the study of generalised inverse limits of compacta. Finally, Section 8 relays some concluding remarks.

2. THE UNIVERSAL PROPERTY OF THE INGRAM-MAHAVIER NOTION OF GENERALISED INVERSE LIMIT OF TOPOLOGICAL SPACES

The aim of this section is to motivate the general category theoretic machinery developed in later sections as well as to acquaint the reader with generalised inverse limits of spaces, and with the classical notion of limits in category theory. Other than presenting the universal property of the generalised inverse limit, for the reasons just given, the presentation is somewhat expository (yet short) and is meant to bring readers of different backgrounds to a common base line.

Remark 2.1. The same characterisation of generalised inverse limits of compacta was arrived at, independently, in [?].

The (now classical) notion of generalised inverse limit of an inverse system of spaces with set-valued bonding functions is a generalisation of the classical notion of limit of an inverse system of spaces with single-valued bonding functions, as follows. For simplicity we consider systems indexed by the natural numbers, postponing full generality to Section 3. Let **Top** be the category of topological spaces and continuous functions. A *diagram* in **Top** is a functor $\mathcal{D} \rightarrow \mathbf{Top}$, where \mathcal{D} is a small category which is called the *shape* of the diagram. When \mathcal{D} is the category

$$\cdots \longrightarrow n + 1 \longrightarrow n \longrightarrow \cdots \longrightarrow 2 \longrightarrow 1$$

whose objects are the natural numbers and with a single morphism $m \rightarrow n$ for all $m, n \in \mathbb{N}$ with $m \geq n$, a diagram

$$\cdots \longrightarrow X_{n+1} \xrightarrow{f_n} X_n \xrightarrow{f_{n-1}} \cdots \longrightarrow X_2 \xrightarrow{f_1} X_1$$

in **Top** of shape \mathcal{D} is called an *inverse sequence* and it amounts to spaces $\{X_n\}_{n \in \mathbb{N}}$ and continuous functions $\{f_n: X_{n+1} \rightarrow X_n\}_{n \in \mathbb{N}}$. We denote an inverse sequence by $\{f_n: X_{n+1} \rightarrow X_n\}_{n \in \mathbb{N}}$.

The *inverse limit*, or simply the *limit*, of an inverse sequence $\{f_n: X_{n+1} \rightarrow X_n\}_{n \in \mathbb{N}}$ is an object of **Top** which is defined only up to a homeomorphism, and it may be described in one of two ways, namely by an explicit construction or by stating the universal property it satisfies, and we present both approaches. Explicitly, the inverse limit is represented by the space $X = \{(x_n)_{n \in \mathbb{N}} \mid x_n = f_n(x_{n+1}), \forall n \in \mathbb{N}\}$ endowed with the subspace topology of the product space $\prod_{n \in \mathbb{N}} X_n$. There are then projection functions $\pi_n: X \rightarrow X_n$, given by $(x_k)_{k \in \mathbb{N}} \mapsto x_n$, and it is immediate that $\pi_n = f_n \circ \pi_{n+1}$ for all $n \in \mathbb{N}$. More generally, any topological space Y together with continuous functions $\psi_n: Y \rightarrow X_n$, $n \in \mathbb{N}$, such that

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{f_n} & X_n \\ & \swarrow \psi_{n+1} & \searrow \psi_n \\ & Y & \end{array}$$

commutes for all $n \in \mathbb{N}$ is called a *cone* to the diagram. In particular, X above together with the projections $\{\pi_n\}_{n \in \mathbb{N}}$ is a cone. The *universal property* of the limit states that every cone to the diagram factors uniquely through the limit cone, as expressed in the diagram

$$\begin{array}{ccccc} \cdots & & X_{n+1} & \xrightarrow{f_n} & X_n & & \cdots \\ & & \swarrow \pi_{n+1} & & \searrow \pi_n & & \\ & & & X & & & \\ & & \swarrow \forall \psi_{n+1} & \uparrow \exists! \varphi & \searrow \forall \psi_n & & \\ & & & Y & & & \end{array}$$

which in full detail reads as: given any cone $\{\psi_n: Y \rightarrow X_n\}_{n \in \mathbb{N}}$ to the diagram, there exists a unique continuous function $\varphi: Y \rightarrow X$ such that $\psi_n = \pi_n \circ \varphi$.

A limit of a diagram in **Top** is thus a space X together with a cone $\{\pi_n: X \rightarrow X_n\}_{n \in \mathbb{N}}$ which among all cones to the diagram has the universal property stated above. The object X is then called a *limiting object* and the cone $\{\pi_n: X \rightarrow X_n\}_{n \in \mathbb{N}}$ a *limiting cone*. It is easily established that if $\{\pi_n: X \rightarrow X_n\}_{n \in \mathbb{N}}$ is a limiting cone, X' is a space, and $s: X' \rightarrow X$

is a homeomorphism, then $\{\pi_n \circ s: X' \rightarrow X_n\}_{n \in \mathbb{N}}$ is also a limiting cone. Conversely, if $\{\pi_n: X \rightarrow X_n\}_{n \in \mathbb{N}}$ and $\{\pi'_n: X' \rightarrow X_n\}_{n \in \mathbb{N}}$ are limiting cones to the same diagram, then there exists a unique homeomorphism $s: X' \rightarrow X$ such that

$$\begin{array}{ccc} & X_n & \\ \pi'_n \nearrow & & \nwarrow \pi_n \\ X' & \xrightarrow{s} & X \end{array}$$

commutes for all $n \in \mathbb{N}$. It is in this precise sense that the limiting object of a diagram, if it exists, is unique up to a homeomorphism, which itself is unique if the cones are specified.

The passage to generalised inverse limits is arrived at by allowing the so called *bonding functions* $f_n: X_{n+1} \rightarrow X_n$, given by the diagram $\mathcal{D} \rightarrow \mathbf{Top}$, to be upper semicontinuous set-valued functions. In more detail, consider the functor $T: \mathbf{Top} \rightarrow \mathbf{Top}$ where $T(X)$ is the hyperspace of all non-empty compact subsets of X , endowed with the upper Vietoris topology (a basis for which is given by the sets $U^+ = \{S \in T(X) \mid S \subseteq U\}$, as U ranges over the open sets in X), and on morphisms $f: X \rightarrow Y$ the function $T(f): T(X) \rightarrow T(Y)$ is the direct image function f_{\rightarrow} . By an *upper semicontinuous set-valued function* $X \rightsquigarrow Y$ is meant a continuous function $X \rightarrow T(Y)$. It is easy to see that this notion of upper semicontinuity coincides with the definition commonly used in the literature (e.g., in [?]). A *generalised inverse sequence*

$$\dots \rightsquigarrow X_{n+1} \xrightarrow{f_n} X_n \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1$$

is then defined to be spaces $\{X_n\}_{n \in \mathbb{N}}$ and, for each $n \in \mathbb{N}$, an upper semicontinuous set-valued function $f_n: X_{n+1} \rightsquigarrow X_n$. Such sequences will be denoted by $\{f_n: X_{n+1} \rightsquigarrow X_n\}$. The ad-hoc definition of the *generalised inverse limit*, or simply the *generalised limit*, of this system is the space $X = \{(x_n)_{n \in \mathbb{N}} \mid x_n \in f_n(x_{n+1}), \forall n \in \mathbb{N}\}$ endowed, again, with the subspace topology of the product. Obviously, it is a generalisation of the explicit construction of the limit of an inverse sequence presented above, and the subject of intensive study since its introduction in 2006 in [?] building upon precursory results presented in [?] in 2004. We complete the picture now by expressing the universal property of the generalised inverse limit.

To express a generalised inverse sequence as a diagram, similarly to how an inverse sequence is a diagram, we construct a suitable category, the category \mathbf{Top}_T . In technical terms, the category \mathbf{Top}_T is the Kleisli category of the hyperspace monad $X \mapsto T(X)$ (see Section 4 for details). There are several hyperspace monads to choose from, and we consider now

the one with $T(X)$ the full power set of X , given the upper Vietoris topology. The objects of \mathbf{Top}_T are all topological spaces, and the morphisms from a space X to a space Y are the continuous functions $X \rightarrow T(Y)$, namely all upper semicontinuous set-valued functions $X \rightsquigarrow Y$. For morphisms $g: X \rightsquigarrow Y$ and $f: Y \rightsquigarrow Z$, the composition $f \circ g: X \rightsquigarrow Z$ is defined by $(f \circ g)(x) = \bigcup_{y \in g(x)} f(y)$. Identity morphisms $\text{id}_X: X \rightsquigarrow X$ are provided by $\eta_X: X \rightarrow T(X)$ given by $x \mapsto \{x\}$. When this category is restricted to the compact Hausdorff spaces and to closed valued functions, the result is precisely the category \mathcal{CHU} described in [?]. Now, just as an inverse sequence is a diagram $\mathcal{D} \rightarrow \mathbf{Top}$ for a suitable shape category \mathcal{D} , a generalised inverse sequence is precisely a diagram $\mathcal{D} \rightarrow \mathbf{Top}_T$ of the same shape. Note that \mathbf{Top}_T contains an isomorphic copy of \mathbf{Top} , namely the subcategory consisting of all spaces, but only of morphisms $X \rightsquigarrow Y$ with $|f(x)| = 1$, for all $x \in X$, and we consider \mathbf{Top} as a subcategory of \mathbf{Top}_T .

We thus have an inclusion of categories $\mathbf{Top} \subseteq \mathbf{Top}_T$. The category \mathbf{Top}_T has a natural ordering on each hom-set, as follows. Given morphisms $f, g: X \rightsquigarrow Y$ in \mathbf{Top}_T , declare that $f \leq g$ precisely when $f(x) \subseteq g(x)$, for all $x \in X$. Notice that in \mathbf{Top} , as a subcategory of \mathbf{Top}_T , the ordering relation \leq is that of equality.

In the presence of an ordering of morphisms in \mathbf{Top}_T it is natural to consider weakening the meaning of the commutativity of a triangle

$$\begin{array}{ccc} X & \overset{g}{\rightsquigarrow} & Y \\ & \swarrow f \quad \searrow h & \\ & Z & \end{array}$$

of morphisms in \mathbf{Top}_T from $h = g \circ f$ to $h \leq g \circ f$, a condition we call *order commutativity*.

Remark 2.2. The authors of [?] refer to this condition as weak commutativity. We note as well that categories in which each hom-set is enriched with an ordering as above are a particular case of enrichment of categories, namely **Ord**-enriched categories where **Ord** is the category of ordered sets (often also named **Pos**). Such enriched categories are used in aspects of computer science, for instance see [?, ?, ?]. The theory we develop in the study of \mathbf{Top}_T as an order enriched category is different and does not simply fall under the umbrella of enriched category theory.

Notice that in \mathbf{Top} , as a subcategory of \mathbf{Top}_T , the meaning of order commutativity coincides with commutativity in the usual sense.

By a *generalised cone* to a generalised sequence $\{f_n: X_{n+1} \rightsquigarrow X_n\}$ is meant a space Y , together with continuous functions $\psi_n: Y \rightarrow X_n$ for all

$n \in \mathbb{N}$, such that

$$\begin{array}{ccc}
 X_{n+1} & \overset{f_n}{\rightsquigarrow} & X_n \\
 \swarrow \psi_{n+1} & & \searrow \psi_n \\
 & Y &
 \end{array}$$

order commutes for all $n \in \mathbb{N}$.

Theorem 2.3. *Given a generalised inverse sequence $\{f_n: X_{n+1} \rightsquigarrow X_n\}_{n \in \mathbb{N}}$, its generalised limit $X = \{(x_n) \mid x_n \in f_n(x_{n+1}), \forall n \in \mathbb{N}\}$ together with the projections $\pi_n: X \rightarrow X_n$ given by $(x_k)_{k \in \mathbb{N}} \mapsto x_n$, have the universal property described by the diagram*

$$\begin{array}{ccccc}
 \cdots & & X_{n+1} & \overset{f_n}{\rightsquigarrow} & X_n & \cdots \\
 & & \swarrow \pi_{n+1} & & \searrow \pi_n & \\
 \cdots & & & X & & \cdots \\
 & \swarrow \forall \psi_{n+1} & & \uparrow \exists! \varphi & & \searrow \forall \psi_n \\
 & & & Y & &
 \end{array}$$

which reads as follows. Firstly, X with the projections $\{\pi_n: X \rightarrow X_n\}_{n \in \mathbb{N}}$ is a generalised cone to the diagram. Second, if Y together with continuous functions $\{\psi_n: Y \rightarrow X_n\}_{n \in \mathbb{N}}$ is a generalised cone to the diagram, then there exists a unique continuous function $\varphi: Y \rightarrow X$ such that $\psi_n = \pi_n \circ \varphi$, for all $n \in \mathbb{N}$.

Proof. The equality $\psi_n = \pi_n \circ \varphi$ dictates that $\varphi(y) = (\psi_n(y))_{n \in \mathbb{N}}$, for all $y \in Y$, which indeed lands in X , i.e., $\psi_n(y) \in f_n(\psi_{n+1}(y))$, precisely by the definition of generalised cone, and so one only needs to verify the continuity of φ , which is a standard exercise. \square

The ad-hoc definition of generalised inverse limit of a generalised sequence can now be replaced by a categorical one, as follows. Any generalised cone $\{Y \rightarrow X_n\}_{n \in \mathbb{N}}$ to the diagram which has the above universal property is called a *generalised limiting cone*, and the object Y is called a *generalised limiting object*. A precise generalisation of the notion of limit of an inverse sequence in **Top**. In particular, the generalised inverse limit too can easily be shown to be defined up to a homeomorphism, which is unique if the generalised cones are specified. All of these claims, and more, are proven below. Taking these results as motivation we develop the theory of Mahavier limits; a theory which specialises both to ordinary limits and to the above notion of generalised limits, as special cases.

Remark 2.4. For a function $f: X \rightarrow Y$ we write f_{\rightarrow} for the direct image of f . For a multivalued function $f: X \rightsquigarrow Y$ we write $f_{\rightarrow}(S) = \bigcup_{s \in S} f(s)$, for all $S \subseteq X$.

3. MAHAVIER LIMITS IN ORDERED CATEGORIES

In this section we introduce the main category theoretic machinery, a generalisation of the ordinary notion of limit in a category, which provides for a categorical approach to generalised inverse limits of spaces as presented in Section 2. Following the trend set in [?] we call these more general limits *Mahavier limits*. In more detail, the motivating scenario is the inclusion $\mathbf{Top} \subseteq \mathbf{Top}_T$, where \mathbf{Top}_T is enriched with an ordering on morphisms as described in Section 2. Mahavier limits are defined in any ordered category \mathcal{C} relative to a subcategory \mathcal{B} , and they reduce to ordinary categorical limits when $\mathcal{B} = \mathcal{C}$. The notion of Mahavier limit in \mathbf{Top}_T relative to \mathbf{Top} , for various diagram shapes, captures precisely the notions of generalised inverse limits of spaces one can find in the literature. Thus, from the categorical point-of-view, we place generalised inverse limits on an equal footing with ordinary inverse limits, allowing the development of category theoretic tools to handle formal aspects of generalised inverse limits, and permitting a seamless translation mechanism of ideas across different categories to consider generalised inverse limits not just of spaces but of any kind of structure at all.

The level of generality we employ, while far greater than is required to achieve the above topology oriented goal, is quite intuitive and seems to us to represent an adequate level of generality. Expectedly, we engage in importing elementary category theoretic results to our setting. In particular, it is well-known that a category is complete, i.e., it has all small limits, if it has all small products and equalisers. The main result of this section is the analogue of that theorem for Mahavier limits, which, when applied to topological spaces, yields the fact that \mathbf{Top}_T is Mahavier complete relative to \mathbf{Top} .

Let us fix some basic notational conventions. Recall that a category is small if its class of objects is a set. Throughout, we adopt the usual convention of imposing smallness conditions where needed to avoid set-theoretic difficulties. For more on such size issues the reader is referred to [?]. We also adopt the convention of using script upper case letters $\mathcal{B}, \mathcal{C}, \mathcal{D}$ for categories, $C \in \mathcal{C}$ to denote objects in a category, and $c \in \mathcal{C}$, or $c: C \rightarrow C'$ if specifying the domain and codomain is required, for morphisms in \mathcal{C} . $\mathcal{C}(C, C')$ denotes the *hom-set* of all morphisms $c: C \rightarrow C'$. The collection of objects of \mathcal{C} is denoted by $\text{ob}(\mathcal{C})$, and the collection of morphisms is denoted by $\text{mor}(\mathcal{C})$.

3.1. Ordered categories. An *ordered category* is a category \mathcal{C} together with an ordering \leq on each hom-set $\mathcal{C}(C, C')$, subject to the condition that composition is monotone, i.e., if $c_1 \leq c'_1$ and $c_2 \leq c'_2$, then, provided the compositions exist, $c_1 \circ c_2 \leq c'_1 \circ c'_2$. An *order functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ between two ordered categories consists of an object part assignment $F: \text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{D})$ together with monotone functions $F: \mathcal{C}(C, C') \rightarrow \mathcal{D}(FC, FC')$, subject to the conditions that $F(c \circ c') \leq F(c) \circ F(c')$, for all composable morphisms $c, c' \in \mathcal{C}$, and that $F(\text{id}_C) = \text{id}_{FC}$, for all objects $C \in \mathcal{C}$. We denote the resulting category of all small ordered categories and order functors by \mathbf{Cat}_{\leq} , where an ordered category is small if its underlying category is small. Note that an order functor, typically, is not a functor between the underlying categories.

The full subcategory of \mathbf{Cat}_{\leq} spanned by the discretely ordered categories, i.e., those where each hom-set is ordered by the equality relation, is obviously isomorphic to the category \mathbf{Cat} , and we thus identify the two. An *ordered subcategory* of an ordered category \mathcal{C} is a subcategory of its underlying category, endowed with an ordering on each hom-set so that it becomes an ordered category, and so that if $f \leq g$ in \mathcal{B} , then $f \leq g$ in \mathcal{C} .

Definition 3.1. A *pair of ordered categories* $(\mathcal{C}, \mathcal{B})$, or more simply a *pair* $\mathcal{B} \subseteq \mathcal{C}$, is an ordered category \mathcal{C} and an ordered subcategory \mathcal{B} . A morphism $F: (\mathcal{C}, \mathcal{B}) \rightarrow (\mathcal{C}', \mathcal{B}')$ between pairs is an order functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ whose restriction to \mathcal{B} yields an order functor $\mathcal{B} \rightarrow \mathcal{B}'$.

We denote the category of pairs of ordered categories and their morphisms by \mathbf{pCat}_{\leq} . The forgetful functor $\mathbf{pCat}_{\leq} \rightarrow \mathbf{Cat}_{\leq}$, mapping $(\mathcal{C}, \mathcal{B})$ to \mathcal{C} , has a left adjoint given by $\mathcal{C} \mapsto (\mathcal{C}, \emptyset)$, and a right adjoint given by $\mathcal{C} \mapsto (\mathcal{C}, \mathcal{C})$. We identify \mathbf{Cat}_{\leq} as a subcategory of \mathbf{pCat}_{\leq} by means of the right adjoint.

In particular, any inclusion of categories $\mathcal{B} \subseteq \mathcal{C}$ is a pair, albeit one where order does not really play any role. The motivating scenario described in Section 2 takes place in the pair $\mathbf{Top} \subseteq \mathbf{Top}_T$, where the hom-sets in \mathbf{Top}_T are ordered by point wise set inclusion.

The ambient structure required for the definition of Mahavier limits is given in the following definition.

Definition 3.2. A pair $\mathcal{B} \subseteq \mathcal{C}$ is an *order extension* if $\text{ob}(\mathcal{B}) = \text{ob}(\mathcal{C})$, \mathcal{B} is a category, i.e., the ordering on each hom-set $\mathcal{B}(B, B')$ is by equality, and if that ordering is precisely the one induced by $\mathcal{C}(B, B')$. We then say that \mathcal{C} is an *order extension* of \mathcal{B} .

Restating the definition, given a category \mathcal{B} , an order extension of it is an ordered category \mathcal{C} obtained from \mathcal{B} , without changing the objects,

by adding morphisms and an ordering on morphisms in such a way that no distinct morphisms in \mathcal{B} become comparable.

Remark 3.3. Within the context of a pair $\mathcal{B} \subseteq \mathcal{C}$, we shall distinguish notationally between morphisms \rightsquigarrow belonging to the ambient ordered category \mathcal{C} and morphisms \rightarrow emphasised to belong to \mathcal{B} .

3.2. Order commutativity and order cones. Utilising the presence of an ordering on each hom-set in an ordered category \mathcal{C} , we declare that a triangle

$$\begin{array}{ccc} C & \overset{c}{\rightsquigarrow} & C' \\ & \swarrow c' & \searrow c'' \\ & C'' & \end{array}$$

of morphisms in \mathcal{C} is *order commutative* if $c'' \leq c \circ c'$. Note that, quite trivially, if \mathcal{C} is a category, i.e., the ordering on each hom-set is the equality relation, then order commutativity and commutativity coincide.

Let \mathcal{D} be a category and \mathcal{C} an ordered category. An *order diagram* in \mathcal{C} of shape \mathcal{D} is an order functor $F: \mathcal{D} \rightarrow \mathcal{C}$. The order diagram is *small* if \mathcal{D} is small.

Example 3.4. Consider the ordered category \mathbf{Top}_T . As noted in Section 2, a generalised inverse sequence is a functor $\mathcal{D} \rightarrow \mathbf{Top}_T$ for the shape category

$$\cdots \longrightarrow n+1 \longrightarrow n \longrightarrow \cdots \longrightarrow 2 \longrightarrow 1$$

and since, more generally, if \mathcal{D} is a category and \mathcal{C} is an ordered category, then any functor $\mathcal{D} \rightarrow \mathcal{C}$ is also an order functor, generalised inverse sequences of spaces as considered in Section 2 are order diagrams of shape \mathcal{D} in the ordered category \mathbf{Top}_T . The converse does not hold. In more detail, any directed set Λ may be viewed as a category (just like any pre-ordered set can), and then a diagram $\Lambda \rightarrow \mathbf{Top}_T$ corresponds precisely to an inverse system of spaces, i.e., spaces $\{X_\lambda\}_{\lambda \in \Lambda}$ and for all $\lambda \geq \mu$ in Λ , an upper semicontinuous set-valued function $f_{\mu,\lambda}: X_\lambda \rightsquigarrow X_\mu$, such that $f_{\nu,\mu} \circ f_{\mu,\lambda} = f_{\nu,\lambda}$, for all $\lambda \geq \mu \geq \nu$, and with $f_{\lambda,\lambda} = \text{id}_{X_\lambda}$. An order diagram $\Lambda \rightarrow \mathbf{Top}_T$ consists again of such a family of spaces and set-valued functions, but the condition on compositions is replaced by $f_{\nu,\lambda} \subseteq f_{\nu,\mu} \circ f_{\mu,\lambda}$, for all $\lambda \geq \mu \geq \nu$, where the inclusion is meant point wise. This notion of order diagram in \mathbf{Top}_T of the shape of a directed set Λ is precisely the notion of generalised inverse system used in [?], which is the most general type of generalised inverse systems of compacta studied in the literature.

Of particular importance to classical generalised inverse limits is the case where all spaces are compact Hausdorff, and the bonding functions are

required to be upper semicontinuous and closed valued. Let \mathbf{Comp} be the full subcategory of \mathbf{Top} spanned by the compacta, and let \mathbf{Comp}_T be the order subcategory of \mathbf{Top}_T consisting of all compacta and only the closed valued morphisms, with hom-set orderings inherited from \mathbf{Top}_T . Then $\mathbf{Comp} \subseteq \mathbf{Comp}_T$ is an order extension and it accommodates diagrams of compacta along the same lines as in the preceding paragraph.

Definition 3.5. Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension, and $F: \mathcal{D} \rightarrow \mathcal{C}$ an order diagram. An *order cone* to F relative to \mathcal{B} consists of an object $B \in \mathcal{B}$ and morphisms $\{\psi_D: B \rightarrow FD\}_{D \in \mathcal{D}}$ in \mathcal{B} subject to the condition that the triangle

$$\begin{array}{ccc} FD & \overset{Fd}{\rightsquigarrow} & FD' \\ \psi_D \swarrow & & \searrow \psi_{D'} \\ & B & \end{array}$$

order commutes for all $d: D \rightarrow D'$ in \mathcal{D} . When \mathcal{B} is understood we simply say *order cone*. Obviously, when \mathcal{B} is a category, order cones in \mathcal{B} relative to \mathcal{B} coincide with cones in \mathcal{B} in the usual sense.

3.3. Mahavier limits. We now give the definition of Mahavier limit in an order extension by means of a universal property which, when compared to the usual one in the definition of categorical limits, requires no explanation. When applied to the order extension $\mathbf{Comp} \subseteq \mathbf{Comp}_T$ of compacta and closed valued upper semicontinuous functions we note some similarities between our approach and [?, ?], though the latter do not obtain a universal property characterisation, and consequently their results and techniques are quite different than what follows below. The concept of Mahavier limit in \mathbf{Comp}_T relative to \mathbf{Comp} is precisely the notion of limit presented in [?].

Fix an order extension $\mathcal{B} \subseteq \mathcal{C}$.

Definition 3.6. Given an order diagram $F: \mathcal{D} \rightarrow \mathcal{C}$, a *Mahavier limit* of the diagram is depicted as

$$\begin{array}{ccc} FD & \overset{Fd}{\rightsquigarrow} & FD' \\ \pi_D \swarrow & & \searrow \pi_{D'} \\ & B & \\ \forall \psi_D \swarrow & \exists! \varphi \uparrow & \searrow \forall \psi_{D'} \\ & \forall B' & \end{array}$$

where $\{\pi_D: B \rightarrow FD\}_{D \in \mathcal{D}}$ is an order cone to the diagram relative to \mathcal{B} and such that for any other order cone $\{\psi_D: B' \rightarrow FD\}_{D \in \mathcal{D}}$ relative to

\mathcal{B} to the same order diagram, there exists a unique morphism $\varphi: B' \rightarrow B$ in \mathcal{B} with $\psi_D = \pi_D \circ \varphi$, for all $D \in \mathcal{D}$.

We then call B a *Mahavier limiting object* of the diagram, and its corresponding order cone a *Mahavier limiting cone*. We denote any Mahavier limiting object by $\varprojlim_{\mathcal{B}}^M F$, or by $\varprojlim^M F$ if \mathcal{B} is understood.

The usual basic facts regarding the extent to which limiting objects in the usual sense are unique carry over seamlessly to Mahavier limits, as follows.

Theorem 3.7. *Let $F: \mathcal{D} \rightarrow \mathcal{C}$ be an order diagram. If $\{\pi_D: B \rightarrow FD\}_{D \in \mathcal{D}}$ is a Mahavier limiting cone of F , and $s: B' \rightarrow B$ is an isomorphism in \mathcal{B} , then $\{\pi_D \circ s: B' \rightarrow FD\}_{D \in \mathcal{D}}$ is also a Mahavier limiting cone of F . Moreover, if $\{\pi_D: B \rightarrow FD\}_{D \in \mathcal{D}}$ and $\{\pi'_D: B' \rightarrow FD\}_{D \in \mathcal{D}}$ are Mahavier limiting cones to the same order diagram F , then there exists a unique isomorphism $s: B \rightarrow B'$ in \mathcal{B} for which $\pi_D = \pi'_D \circ s$, for all $D \in \mathcal{D}$. In particular, any two Mahavier limiting objects of a given order diagram are isomorphic in \mathcal{B} , and canonically so if the Mahavier limiting cones are specified.*

Proof. The first claim is trivial. The claim about B and B' follows by noting that unique morphisms $\varphi_1: B \rightarrow B'$ and $\varphi_2: B' \rightarrow B$ in \mathcal{B} exist by the universal property, and that both the identity and $\varphi_1 \circ \varphi_2$ factorise the same order cone, and thus are equal. Similarly, $\varphi_2 \circ \varphi_1$ is the identity, and so $\varphi_1: B \rightarrow B'$ is an isomorphism. \square

Example 3.8. *It is easily verified that for a category \mathcal{B} ordinary categorical limits in \mathcal{B} coincide with Mahavier limits in \mathcal{B} relative to \mathcal{B} , and thus the theory of Mahavier limits extends that of ordinary categorical limits. Moreover, Mahavier limits in a pair $\mathcal{B} \subseteq \mathcal{C}$ of categories, viewed trivially as an order extension, is precisely the notion of relative limit used in [?].*

Example 3.9. *Consider again the order extension $\mathbf{Top} \subseteq \mathbf{Top}_T$. It is obvious that the material presented in Section 2 can now be summarised as the claim that generalised inverse limits of generalised inverse sequences in \mathbf{Top} are Mahavier limits in \mathbf{Top}_T relative to \mathbf{Top} . Continuing Example 3.4, Mahavier limits in \mathbf{Top}_T relative to \mathbf{Top} of shape Λ for a directed set Λ encompass the most general notions of generalised inverse limits of generalised inverse systems found in the literature, including both inverse systems indexed by various structures (e.g., [?] and [?] for inverse systems indexed by the integers and, respectively, by totally ordered sets) as well as inverse systems with weaker conditions on the bonding functions as in [?].*

3.4. Mahavier equalisers. Equalisers in category theory, namely limits of diagrams of the form $f, g: C \rightrightarrows C'$, play an important role in the general theory. We now briefly look at Mahavier equalisers, i.e., Mahavier limits of order diagrams of the form $f, g: C \rightrightarrows C'$ in an arbitrary order extension $\mathcal{B} \subseteq \mathcal{C}$. By definition, such a Mahavier equaliser is

$$E \xrightarrow{e_1} C \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{f} \end{array} C'$$

(Note: In the original image, there is a curved arrow labeled e_2 from E to C' above the diagram.)

where e_1, e_2 are universal with respect to the property that $e_2 \leq f \circ e_1$ and $e_2 \leq g \circ e_1$. Note that e_1 need not determine e_2 . The situation becomes simpler if one of the two parallel morphisms, say g , is in \mathcal{B} , since then the condition $e_2 \leq g \circ e_1$ reduces to $e_2 = g \circ e_1$, and so e_2 is redundant. In other words, for *mixed parallel morphisms* $f, g: C \rightrightarrows C'$ a Mahavier equaliser consists of

$$E \xrightarrow{e} C \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{f} \end{array} C'$$

with $g \circ e \leq f \circ e$, with the usual statement of universality with respect to that condition.

Recall again the order extensions $\mathbf{Top} \subseteq \mathbf{Top}_T$ of topological spaces and $\mathbf{Comp} \subseteq \mathbf{Comp}_T$ of compact Hausdorff spaces and closed valued functions.

Proposition 3.10. *All mixed parallel morphisms*

$$X \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{f} \end{array} Y$$

in \mathbf{Top}_T (respectively \mathbf{Comp}_T) have Mahavier equalisers relative to \mathbf{Top} (respectively \mathbf{Comp}).

Proof. The equaliser is given by $E = \{x \in X \mid g(x) \in f(x)\}$, as a subspace of X , as we now verify the universal property for the inclusion $e: E \rightarrow X$. By the identification of \mathbf{Top} as a subcategory of \mathbf{Top}_T , we have that $(g \circ e)(x) = \{g(x)\}$ and that $(f \circ e)(x) = f(x)$, and thus $g \circ e \leq f \circ e$. Now, if W is any space and $r: W \rightarrow X$ a continuous function satisfying $g \circ r \leq f \circ r$, namely $g(r(w)) \in f(r(w))$, for all $w \in W$, then r lands in E , and thus factorises uniquely through e .

The above establishes the claim for $\mathbf{Top} \subseteq \mathbf{Top}_T$. To establish the claim for $\mathbf{Comp} \subseteq \mathbf{Comp}_T$ it remains to verify that E is in fact an object in \mathbf{Comp} , assuming X is compact Hausdorff and f is closed valued. Since X is Hausdorff, so is E . To verify that E is also compact it suffices, since X is compact and Hausdorff, to show that E is closed in X . Let $x \in X \setminus E$, namely $g(x) \notin f(x)$. As $f(x)$ is closed, and X is compact Hausdorff, and

thus regular, there exist disjoint open sets $V_{g(x)}, V_{f(x)} \subseteq Y$ with $g(x) \in V_{g(x)}$ and $f(x) \in V_{f(x)}$. By continuity of g and by upper semicontinuity of f , there exist open sets $U_g, U_f \subseteq X$ with $x \in U_g$ and $g \rightarrow (U_g) \subseteq V_{g(x)}$, and $x \in U_f$ and $f \rightarrow (U_f) \subseteq V_{f(x)}$. But then $x \in U_f \cap U_g \subseteq X \setminus E$, and thus E is closed. \square

3.5. Max products. Recall that a *weak product* of objects $\{B_i\}_{i \in I}$ in a category \mathcal{B} is an object B together with morphisms $\{\pi_i: B \rightarrow B_i\}_{i \in I}$ such that for any choice of morphisms $\{f_i: B' \rightarrow B_i\}_{i \in I}$, there exists a (perhaps not unique) morphism $g: B' \rightarrow B$ with $f_i = \pi_i \circ g$, for all $i \in I$. Such weak products are generally not unique, not even up to isomorphism. Weak products appear, for instance, in categorical techniques in programming semantics and in logic (see, e.g., [?, ?, ?]).

Our motivating example \mathbf{Top}_T as well as its subcategory \mathbf{Comp}_T , admit weak products, as follows (for simplicity we only consider binary products). We only give the details for \mathbf{Comp}_T . Given compact Hausdorff spaces X and Y , the ordinary topological product $X \times Y$, i.e., the product in \mathbf{Comp} , fails to be a product in \mathbf{Comp}_T , for obvious reasons. However, given morphisms $f: Z \rightsquigarrow X$ and $g: Z \rightsquigarrow Y$ in \mathbf{Comp}_T , the morphism $f \times g: Z \rightsquigarrow X \times Y$, given by $z \mapsto f(z) \times g(z)$, the topological product of the closed spaces $f(z)$ and $g(z)$, shows that $X \times Y$ with the canonical projections as a product in \mathbf{Comp} , constitute a weak product in \mathbf{Comp}_T . The same argument holds for any collection of objects, and in fact the concept of product in an ordered category we develop below captures more of the close relationship between products in \mathbf{Top} and what we call max products in \mathbf{Top}_T .

For the rest of this discussion let us fix an order extension $\mathcal{B} \subseteq \mathcal{C}$, and a collection of objects $\{C_i\}_{i \in I}$ in \mathcal{C} .

Definition 3.11. A *max product relative to \mathcal{B}* of the collection is an object C and morphisms $\{\pi_i: C \rightsquigarrow C_i\}_{i \in I}$, as in the diagram

$$\begin{array}{ccccc}
 C_1 & \xleftarrow{\pi_1} & C & \xrightarrow{\pi_2} & C_2 \\
 \swarrow \forall f & & \exists h & & \searrow \forall g \\
 & & C' & & \\
 & & \uparrow u & & \downarrow v \\
 & & C'' & &
 \end{array}$$

depicting, for simplicity, the case of just two objects, with the property that for all objects C' and morphisms $\{f_i: C' \rightsquigarrow C_i\}_{i \in I}$, there exists a morphism $h: C' \rightsquigarrow C$ such that $f_i = \pi_i \circ h$, and so that the following holds: for all objects C'' and morphisms $u: C'' \rightarrow C'$ in \mathcal{B} and $v: C'' \rightsquigarrow C$ in \mathcal{C} , if $\pi_i \circ v \leq \pi_i \circ h \circ u$, for all $i \in I$, then $v \leq h \circ u$.

Obviously, a max limit is a weak limit, but more is true. Firstly, consider, for a given collection $\{f_i: C' \rightsquigarrow C_i\}_{i \in I}$ as above, the set S of all morphisms $h': C' \rightsquigarrow C$ which satisfy $f_i = \pi_i \circ h'$, for all $i \in I$. By taking $v = h'$ and $u = \text{id}_{C'}$, we see that h is maximal in S with respect to the ordering in $\mathcal{C}(C', C)$, and in particular, once C is chosen, h is unique. Hence the term max product for this concept. Second, the usual argument shows that if C and \hat{C} are both max products for the same collection of objects, then the respective canonical projections yield morphisms $\psi: C \rightsquigarrow \hat{C}$ and $\varphi: \hat{C} \rightsquigarrow C$ with $\text{id}_C \leq \psi \circ \varphi$, and $\text{id}_{\hat{C}} \leq \varphi \circ \psi$, a remnant of uniqueness up to isomorphism.

Definition 3.12. For an order extension $\mathcal{B} \subseteq \mathcal{C}$, we say that products in \mathcal{B} induce max products in \mathcal{C} if whenever $\{\pi_i: B \rightarrow B_i\}_{i \in I}$ is a product in the usual category theoretic sense in \mathcal{B} , the collection $\{\pi_i: B \rightarrow B_i\}_{i \in I}$ is a max product in \mathcal{C} .

Proposition 3.13. *Products in **Top** (respectively **Comp**) induce max products in **Top_T** (respectively **Comp_T**).*

Proof. We only give the details for the binary product $X \times Y$ in the compact Hausdorff case. Given morphisms $f: Z \rightsquigarrow X$ and $g: Z \rightsquigarrow Y$ in **Comp_T**, the morphism $h: Z \rightsquigarrow X \times Y$, given by $h(z) = f(z) \times g(z)$ is easily seen to be upper semicontinuous, and it follows that $X \times Y$, with the canonical projections, is a max product in **Comp_T**. Indeed, let W be an arbitrary compact Hausdorff space, and consider morphisms $u: W \rightarrow Z$, and $v: W \rightsquigarrow X \times Y$ satisfying $\pi_X \circ v \leq \pi_X \circ h \circ u$ and $\pi_Y \circ v \leq \pi_Y \circ h \circ u$. For all $w \in W$, clearly $(\pi_X \circ h \circ u)(w) = f(u(w))$ and $(\pi_Y \circ h \circ u)(w) = g(u(w))$, and thus the conditions become $\pi_X(v(w)) \subseteq f(u(w))$ and $\pi_Y(v(w)) \subseteq g(u(w))$. But then $v(w) \subseteq f(u(w)) \times g(u(w)) = h(u(w))$ follows at once, and thus $v \leq h \circ u$, as required. The same argument works for arbitrary set-indexed products. \square

3.6. Completeness. A very useful and foundational fact of category theory is that a category which admits all small products and all equalisers admits all small limits. We now prove the analogous result for Mahavier limits.

Definition 3.14. An order extension $\mathcal{B} \subseteq \mathcal{C}$ is *Mahavier complete* if every small order diagram $\mathcal{D} \rightarrow \mathcal{C}$ has a Mahavier limit. We also say then that \mathcal{C} is *Mahavier complete relative to \mathcal{B}* .

Theorem 3.15. *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension. If the conditions*

- \mathcal{B} has all small products;
- products in \mathcal{B} induce max products in \mathcal{C} ; and

constitute an order cone to the diagram since, chasing the first diagram, $\pi_{D'} \circ e = \pi_{D'} \circ g \circ e \leq \pi_{D'} \circ f \circ e = Fd \circ \pi_D \circ e$. The rest of the proof establishes the universal property by constructing φ which factorises the given order cone.

The morphisms $\{\psi_D\}_{D \in \mathcal{D}}$ give rise to the unique morphism $\psi: C \rightarrow \prod FD$ with $\psi_D = \pi_D \circ \varphi$ by virtue of the codomain being a product in \mathcal{B} . We now wish to establish that $g \circ \psi \leq f \circ \psi$. By the definition of max products in \mathcal{C} it suffices to show that $\pi_{D'} \circ g \circ \psi \leq \pi_{D'} \circ f \circ \psi$, for all $D' \in \mathcal{D}$. Chasing the diagrams reveals that $\pi_{D'} \circ g \circ \psi = \pi_{D'} \circ \psi = \psi_{D'} \leq Fd \circ \psi_D = Fd \circ \pi_D \circ \psi = \pi_{D'} \circ f \circ \psi$, as required. The universal property of e furnishes a unique $\varphi: C \rightarrow E$ with $\psi = e \circ \varphi$. It follows at once that φ yields a factorisation of the order cone in the second diagram. A very similar argument shows that any such factorisation in the second diagram fits in the first diagram as another morphism $C \rightarrow E$ which factorises ψ , and so the uniqueness of φ in the first diagram forces the uniqueness of φ in the second diagram too. \square

As a corollary we obtain the following result.

Theorem 3.16. *\mathbf{Top}_T is Mahavier complete relative to \mathbf{Top} and \mathbf{Comp}_T is Mahavier complete relative to \mathbf{Comp} .*

Proof. \mathbf{Comp} has, by Tychonoff's theorem, all small products, and by Proposition 3.13 products in \mathbf{Comp} induce max products in \mathbf{Comp}_T . By Proposition 3.10, Mahavier equalisers of mixed parallel morphisms relative to \mathbf{Comp} exist in \mathbf{Comp}_T , and thus Theorem 3.15 applies. A similar argument for \mathbf{Top}_T relative to \mathbf{Top} completes the proof. \square

The literature on generalised inverse limits of spaces is primarily concerned with generalised inverse limits of *thin* order diagrams, namely order diagrams of shape \mathcal{D} , where in \mathcal{D} each hom-set has at most one morphism (i.e., it is a pre-ordered set). The notion of Mahavier limit in the order extensions described above unifies all of the notions found in the literature and eliminates the ad-hoc nature of the definitions, as they are now fortified with a universal property. We close this section by examining Mahavier limits in $\mathbf{Top} \subseteq \mathbf{Top}_T$ of particularly simple diagrams, and consider pullbacks and monomorphisms in the Mahavier context.

Firstly, $\varprojlim_{\mathbf{Top}}^M (X \overset{f}{\rightsquigarrow} Y) \cong \text{Gr}(f) = \{(x, y) \in X \times Y \mid y \in f(x)\}$, the graph of f . This is interesting since the graphs of the bonding functions play an important role in the study of generalised inverse limits, and thus Mahavier limits capture more than just the immediate definition of the generalised inverse limit. Recall that along the proof of Proposition 3.10 it was established that the Mahavier equaliser of a mixed diagram in \mathbf{Comp}_T relative to \mathbf{Comp} , when constructed as a subspace

of the domain, is closed. Incidentally, we immediately obtain the ubiquitous result that the graph of an upper semicontinuous function f as above is closed in $X \times Y$, simply by following the recipe in the proof of Theorem 3.15. Another aspect of this simple computation is that Mahavier limits behave quite differently with respect to initial functors, as we discuss in detail in Section 6. For now, we just note that for ordinary categorical limits, the limit of a diagram $f: X \rightarrow Y$, where f is a continuous function, is simply X with the obvious cone given by $\text{id}_X: X \rightarrow X$ and $f: X \rightarrow Y$. Of course, it is also given by the graph of f with the usual projections; after all, X is homeomorphic to the graph of f . For Mahavier limits the two approaches yield different concepts, only one of which, i.e., the graph solution, is the limit.

Second, $\lim_{\mathbf{Top}}^M (X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y) \cong \{(x, y) \in X \times Y \mid y \in f(x) \cap g(x)\}$.

We thus see that the general form of Mahavier equalisers is a very sensible notion, implying that the study of Mahavier limits of arbitrary order diagrams, not just thin ones, is probably relevant to the general theory.

Finally, we consider pullbacks, and their relationship with monomorphisms. Recall that a monomorphism f in a category is a left-cancelable morphism, i.e., $f \circ u = f \circ v$ implies $u = v$, for all morphisms u, v to the domain of f . In \mathbf{Top} the monomorphisms are precisely the injective continuous functions. In \mathbf{Top}_T the monomorphisms are the upper semicontinuous functions $f: X \rightsquigarrow Y$ satisfying the condition that $f(x) = f(x')$ implies $x = x'$, for all $x, x' \in X$, in other words, precisely those morphisms $f: X \rightsquigarrow Y$ whose corresponding function $X \rightarrow T(Y)$ is injective. However, in the context of multivalued functions it is also natural to consider the stronger condition, namely when $f(x) \cap f(x') \neq \emptyset$ implies $x = x'$. The following definition of order monomorphism in an order extension precisely captures this condition.

Definition 3.17. Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension. A morphism $f: C \rightsquigarrow C'$ is an *order monomorphism relative to \mathcal{B}* if for all morphisms $b, b': B \rightarrow C$, the existence of a morphism $b'': B \rightarrow C'$ with $b'' \leq f \circ b$ and $b'' \leq f \circ b'$ implies that $b = b'$.

A fundamental result of category theory is that monomorphisms are stable under pullbacks, namely in a pullback square

$$\begin{array}{ccc} W & \xrightarrow{\pi_X} & X \\ \downarrow \pi_Y & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

if f is a monomorphism, then so is π_Y . By a *Mahavier pullback* we mean a Mahavier limit of the shape of a pullback. In more detail, given

morphisms $f: X \rightsquigarrow Z$ and $g: Y \rightsquigarrow Z$ in \mathcal{C} , a Mahavier pull back is an object W and projections in \mathcal{B} , π_X, π_Y, π_Z with domain W , such that $\pi_Z \leq f \circ \pi_X$ and $\pi_Z \leq g \circ \pi_Y$, subject to the universal property.

Proposition 3.18. *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension, $f: X \rightsquigarrow Z$ in \mathcal{C} , and $g: Y \rightarrow Z$ in \mathcal{B} . If the diagram*

$$\begin{array}{ccc} W & \xrightarrow{\pi_X} & X \\ \downarrow \pi_Y & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

is a Mahavier pullback, and f is an order monomorphism, then π_Y is a monomorphism in \mathcal{B} .

Proof. Consider morphisms $b, b': W' \rightarrow W$ with $\pi_Y \circ b = \pi_Y \circ b'$, and the aim is to show that $b = b'$. Forming the diagram

$$\begin{array}{ccc} W' & & X \\ & \searrow b & \downarrow \pi_X \\ & \searrow b' & W \\ & & \downarrow \pi_Y \\ & & Y \\ & & \downarrow g \\ & & Z \end{array} \quad \begin{array}{l} \xrightarrow{\pi_X \circ b = \pi_X \circ b'} \\ \xrightarrow{\pi_Y \circ b = \pi_Y \circ b'} \end{array}$$

where the top curved morphism is justified by noting that $g \circ \pi_Y \circ b \leq f \circ \pi_X \circ b$ and $g \circ \pi_Y \circ b' \leq f \circ \pi_X \circ b'$, and since f is an order monomorphism relative to \mathcal{B} , the morphisms are equal as claimed. It is now easily verified that we obtain an order cone to the diagram defining the Mahavier pullback, and that it is factorised through both b and b' , which must thus be equal, completing the proof. \square

Remark 3.19. Recall that if \mathcal{B} is a category, then its *opposite category* \mathcal{B}^{op} is the category with the same objects as in \mathcal{B} , and whose hom-sets are $\mathcal{B}^{\text{op}}(B, B') = \mathcal{B}(B', B)$. For every morphism $b: B \rightarrow B'$ in \mathcal{B} , we denote by $b^{\text{op}}: B' \rightarrow B$ the corresponding morphism in \mathcal{B}^{op} . The composition in \mathcal{B}^{op} is then given, for composable morphisms, by $g^{\text{op}} \circ f^{\text{op}} = (f \circ g)^{\text{op}}$. Any concept or property in a category \mathcal{B} immediately gives rise to a *dual* concept, simply by interpreting it in \mathcal{B}^{op} . Thus, for instance, categorical colimits in a category \mathcal{B} are precisely categorical limits in \mathcal{B}^{op} .

Duality in the context of pairs $\mathcal{B} \subseteq \mathcal{C}$ is straightforward. Firstly, the *opposite* of an ordered category \mathcal{C} is the ordered category \mathcal{C}^{op} whose underlying category is the opposite of the underlying category of \mathcal{C} , and where the ordering is given by $c^{\text{op}} \leq c'^{\text{op}}$ if $c \leq c'$ (so the order is not reversed). For pairs we define $(\mathcal{C}, \mathcal{B})^{\text{op}} = (\mathcal{C}^{\text{op}}, \mathcal{B}^{\text{op}})$, noting that if $\mathcal{B} \subseteq \mathcal{C}$ is an order extension, then so is its opposite. Mahavier colimits

in \mathcal{C} relative to \mathcal{B} are then defined to be Mahavier limits in \mathcal{C}^{op} relative to \mathcal{B}^{op} . Every categorical result in this work has a dual, which we shall not bother to make explicit.

4. ORDER EXTENSIONS FROM MONADS

The theory of monads is well-developed and we only tap it at its surface, using monads as a convenient formalism for constructing order extensions of interest, including the main examples $\mathbf{Top} \subseteq \mathbf{Top}_T$ and $\mathbf{Comp} \subseteq \mathbf{Comp}_T$, through the construction of the Kleisli category of a monad. For the convenience of the reader this section attempts to remain self-contained by providing standard definitions and results, without proofs. For more details the reader may consult [?] for an introductory text, [?] for a detailed account of the fundamental theory of monads, and [?] for more recent work. The reader primarily interested in compacta may safely only skim the contents of this section.

4.1. Monads. A monad is a monoid object in the category of endofunctors of a category. When spelled out, the definition is as follows.

Definition 4.1. A *monad* (T, μ, η) on a category \mathcal{B} is a functor $T: \mathcal{B} \rightarrow \mathcal{B}$ and natural transformations $\mu: T^2 \rightarrow T$ and $\eta: \text{Id}_{\mathcal{B}} \rightarrow T$ such that the diagrams

$$\begin{array}{ccc} T^3 & \xrightarrow{T \circ \mu} & T^2 \\ \mu \circ T \downarrow & & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{T \circ \eta} & T^2 & \xleftarrow{\eta \circ T} & T \\ & \searrow \text{id} & \downarrow \mu & \swarrow \text{id} & \\ & & T & & \end{array}$$

commute. The natural transformation μ is called the *multiplication* of the monad and η is called the *unit*. Typically, we refer to a monad T without explicit mention of μ or η . The left diagram above expresses an associativity property for the multiplication, while the diagram on the right expresses left and right identity laws for the unit.

In more detail, all of the arrows are natural transformations between iterates of the functor T , whose components at an object C are as follows:

- $(T \circ \mu)_B = T(\mu_B: T^2 B \rightarrow TB)$.
- $(\mu \circ T)_B = \mu_{TB}: T^2(TB) \rightarrow T(TB)$.
- $(T \circ \eta)_B = T(\eta_B: B \rightarrow TB)$.
- $(\eta \circ T)_B = \eta_{TB}: TB \rightarrow T(TB)$.

The following examples serve to clarify the definition as well as to provide us with relevant testing grounds for the notion of Mahavier limit. The verification of the monad axioms is quite straightforward, typically requiring little more than correctly deciphering the diagrams.

4.1.1. *The identity monad.* For any category \mathcal{B} , the *identity monad* on it is the monad (T, μ, η) where $T = \text{Id}_{\mathcal{B}}$, and $\mu = \eta = \text{Id}_T$.

4.1.2. *The Manes monad.* The Manes monad is a monad on **Set** where T is the covariant power set functor, namely $T(X) = \mathcal{P}(X)$ on objects, and for morphisms f , $T(f) = f_{\rightarrow}$ is the direct image function. The multiplication components $\mu_X: \mathcal{P}(\mathcal{P}(X)) \rightarrow \mathcal{P}(X)$ are given by $\mu_X(\mathcal{S}) = \bigcup_{S \in \mathcal{S}} S$, and the unit components $\eta_X: X \rightarrow \mathcal{P}(X)$ are given by $\eta_X(x) = \{x\}$, for all $x \in X$.

4.1.3. *Hyperspace monads.* For a compact Hausdorff space X let $T(X)$ be the set of all compact subsets of X , endowed with the upper Vietoris topology. Given a continuous function $f: X \rightarrow Y$, the function $T(f): T(X) \rightarrow T(Y)$ is given by the direct image function f_{\rightarrow} , which is, of course, well-defined, yielding a functor $T: \mathbf{Comp} \rightarrow \mathbf{Comp}$. Defining $\eta_X: X \rightarrow T(X)$ by $x \mapsto \{x\}$, which is again trivially well-defined, is evidently a natural transformation. Defining $\mu_X: T^2(X) \rightarrow T(X)$ by $\mathcal{S} \mapsto \bigcup_{S \in \mathcal{S}} S$ is well-defined but less trivially so, relying on the fact that a compact union of compact subsets is compact (see [?] or, for a treatment more geared toward upper semicontinuous functions, see [?]). The verification of the monad axioms is straightforward, being formally the same as the verification for the Manes monad. See [?] for more details on the hyperspace monad, albeit with the Vietoris topology rather than the upper Vietoris topology (the details are similar). Further variations include considering $T: \mathbf{Top} \rightarrow \mathbf{Top}$ where $T(X)$ is the full power set $\mathcal{P}(X)$, again with the upper Vietoris topology - our central scenario - as well as other topologies.

4.1.4. *A monad on monoids.* Recall that a monoid (M, \cdot, e) is a set M together with an associative binary operation \cdot on M for which e is a unit. Let **Mon** be the category of monoids and their homomorphisms. Given a monoid M let $T(M) = \mathcal{P}(M)$ be the set of all subsets of M , together with the element-wise extension $S \cdot S' = \{s \cdot s' \mid s \in S, s' \in S'\}$ of the operation \cdot to subsets of M . $T(M)$ is obviously a monoid and if $f: M \rightarrow M'$ is a homomorphism, then so is the direct image function, and thus, setting $T(f) = f_{\rightarrow}$ yields a functor $T: \mathbf{Mon} \rightarrow \mathbf{Mon}$. Defining $\eta_M: M \rightarrow T(M)$ by $m \mapsto \{m\}$ and $\mu_X: T^2(M) \rightarrow T(M)$ by $\mathcal{S} \mapsto \bigcup_{S \in \mathcal{S}} S$ again, clearly endows T with the structure of a monad.

Remark 4.2. There are two types of modifications that can be performed on the monads above to obtain more monads. We shall exemplify using the Manes monad. Firstly, one can replace the power set functor \mathcal{P} by \mathcal{P}_* , where $\mathcal{P}_*(S)$ is the set of all *non-empty* subsets of S . Another possibility is to replace \mathcal{P} by \mathcal{P}_κ where κ is an infinite cardinal, and

$\mathcal{P}_\kappa(S)$ is the set of all subsets $S' \subseteq S$ with $|S'| \leq \kappa$. The effects of such modifications, in particular with respect to Mahavier limits, can be substantial, but we shall not delve into the details in this work.

4.2. The Kleisli category. Let (T, μ, η) be a monad on a category \mathcal{B} . The *Kleisli category* of the monad is denoted by \mathcal{B}_T , and it consists of all the objects of \mathcal{B} , with hom-sets $\mathcal{B}_T(B, B') = \mathcal{B}(B, T(B'))$. We denote morphisms in \mathcal{B}_T by $f: B \rightsquigarrow B'$, and write $f^\sharp: B \rightarrow T(B')$ for the corresponding morphism in \mathcal{B} . The composition in \mathcal{B}_T is given by the *Kleisli composition rule*:

$$\begin{array}{ccc}
 & & (g \circ f)^\sharp \\
 & \curvearrowright & \\
 B & \xrightarrow{f} & B' \xrightarrow{g} B'' \\
 & \curvearrowleft & \\
 & & (g \circ f)^\sharp \\
 & & B \xrightarrow{f^\sharp} T(B') \xrightarrow{T(g^\sharp)} T^2(B'') \xrightarrow{\mu_{B''}} T(B'')
 \end{array}$$

i.e., $(g \circ f)^\sharp = \mu_{B''} \circ T(g^\sharp) \circ f^\sharp$. Identity morphisms are given, at each object B , by $\text{id}_B^\sharp = \eta_B: B \rightarrow T(B)$.

The verification that the Kleisli category \mathcal{B}_T is indeed a category is very well-established and will not be repeated here. It is easily seen that the Kleisli category of the identity monad on a category \mathcal{B} is isomorphic to \mathcal{B} . The Kleisli category of the Manes monad is the category **Rel** of sets and relations, and, what is from the perspective of Section 2 most important, the Kleisli category of hyperspace monads on topological spaces or on compact Hausdorff spaces yield various categories, including the central instances **Top_T**, where $T(X)$ is the full power set endowed with the upper Vietoris topology, and **Comp_T**, where $T(X)$ is the collection of closed subsets of X , again with the upper Vietoris topology, which featured above as the ambient categories where the diagrams of generalised inverse limits reside. The Kleisli category corresponding to the monad on monoids consists of all monoids and all *multihomomorphisms* $f: G \rightarrow H$ namely, a multivalued function between the underlying sets such that $f(e) = \{e\}$ and $f(g \cdot g') = f(g) \cdot f(g')$, obviously a generalisation of ordinary homomorphisms. Such multihomomorphisms are considered when G, H are groups in [?, ?].

Remark 4.3. Note that a naive attempt to interpret generalised inverse limits simply as limits in the Kleisli category **Comp_T** is bound to fail. The behaviour of limits (and colimits) in the Kleisli category is sufficiently understood (see [?]), and in particular conditions that guarantee the existence of limits in \mathcal{B}_T in terms of limits in \mathcal{B} are known. However, these conditions are rarely met. In fact, very often limits in the Kleisli category differ wildly from limits in \mathcal{B} . For instance, considering the Manes monad again, products in **Rel** coincide with coproducts in **Set**, namely are given

by disjoint union, and equalisers in general do not exist. Change the power set monad to the non-empty power set monad, and now arbitrary products no longer exist. For \mathbf{Comp}_T this is already noted in [?] where the category \mathcal{CHU} , which is precisely \mathbf{Comp}_T , is constructed and it is shown that generalised inverse limits need not be (and in fact very rarely are) limits in the classical sense. A more precise treatment is given in [?].

Note that there is a functor $\mathcal{B} \rightarrow \mathcal{B}_T$ which is the identity on objects and which sends $b: B \rightarrow B'$ to $B \xrightarrow{b} B' \xrightarrow{\eta_{B'}} TB'$. We identify \mathcal{B} , along this functor, as a subcategory of \mathcal{B}_T . Note that when applied to the suitable hyperspace monads this statement is precisely the content of the identification made in Section 2 regarding \mathbf{Top} as a subcategory of \mathbf{Top}_T , and similarly for \mathbf{Comp} .

Remark 4.4. Pairs of the form $\mathcal{B} \subseteq \mathcal{B}_T$ are often considered in applications of monads, see e.g., [?, ?].

4.3. Order on the Kleisli category. We are of course primarily interested in obtaining an ordered category structure on \mathcal{B}_T , so that the Kleisli category becomes an order extension of \mathcal{B} . An ordering on each hom-set $\mathcal{B}_T(B, B')$ is precisely an ordering on the hom-sets $\mathcal{B}(B, T(B'))$. Obviously, one can figure out conditions on the latter ordering, in terms of compatibility with T , which are equivalent to the Kleisli category being an ordered category. However, in practice it is quite straightforward to check directly that a proposed ordering on the Kleisli category is compatible with composition, and thus we consider order enriched monads following [?], with only a slight variation required for our notion of order extension.

Definition 4.5. An *order enriched monad* is a monad T on \mathcal{B} together with an ordering on $\mathcal{B}(B, T(B'))$ for all $B, B' \in \mathcal{B}$ such that the Kleisli category becomes an ordered category, and if for all morphisms $b_1, b_2: B \rightarrow B'$ in \mathcal{B} , if $\eta_{B'} \circ b_1 \leq \eta_{B'} \circ b_2$ then $\eta_{B'} \circ b_1 = \eta_{B'} \circ b_2$.

If T is an order enriched monad, then we always assume the Kleisli category is endowed with its corresponding ordered category structure, and it then follows immediately that $\mathcal{B} \subseteq \mathcal{B}_T$ is an order extension.

For the Manes monad, the hyperspace monads, and the monad on monoids, defining the ordering \leq on the hom-sets by means of point-wise set inclusion, i.e., $f \leq g$ when $f(x) \subseteq g(x)$ for all x in the domain, endows each monad with the structure of an order enriched monad, thus yielding order extensions. The motivating order extensions from Section 2 are, expectedly, obtained in this way for the right choice of hyperspace monads.

4.4. Productive monads. Now that any order enriched monad T on \mathcal{B} furnishes an order extension $\mathcal{B} \subseteq \mathcal{B}_T$, we turn to consider conditions on the monad that will guarantee a reasonable transfer of products from \mathcal{B} to \mathcal{B}_T , ultimately hoping for all products in \mathcal{B} to induce max products in \mathcal{B}_T . Such a notion is obviously related to the theory of monoidal monads (see [?] for an explicit construction of a tensor product on the Kleisli category for a monoidal monad, and [?] for general theory), however what we require is particular attention on products not merely as a monoidal structure. We thus proceed attending to products and ignoring the potentially greater generality. Since the term "cartesian monad" is already in use, we call monads which behave well with respect to products *productive monads*.

Definition 4.6. Let T be a monad on a category \mathcal{B} and assume \mathcal{B} has all small products. Suppose further that for all set-indexed collections $\{B_i\}_{i \in I}$ of objects in \mathcal{B} , a morphism $\alpha: \prod_{i \in I} T(B_i) \rightarrow T(\prod_{i \in I} B_i)$ is given. With the respective canonical projections for each product denoted by π with a suitable subscript, if

$$\begin{array}{ccc} \prod_{i \in I} T(B_i) & \xrightarrow{\alpha} & T(\prod_{i \in I} B_i) \\ \swarrow \pi_{T(B_i)} & & \nwarrow T(\pi_{B_i}) \\ & T(B_i) & \end{array}$$

commutes for all $i \in I$, then T , together with all the structure morphisms α , is said to be a *productive monad*.

Of course, the morphisms α must be chosen for each choice of product, and a more pedantic approach would reflect that by a subscript, and some further compatibility requirements are natural. However, we do not require any of that and thus opt to reduce the notational clutter altogether. We shall also abuse the notation slightly and refer to a productive monad to assume suitable morphisms α are fixed.

Proposition 4.7. Let T be a productive monad on a category \mathcal{B} . Then for all set-indexed objects $\{B_i\}_{i \in I}$ and a product B with canonical projections $\{\pi_i: B \rightarrow B_i\}_{i \in I}$, the same projections form a weak product in \mathcal{B}_T .

Proof. For simplicity only, the accompanying diagrams are for binary weak products. So, we seek to solve the weak product problem

$$\begin{array}{ccccc} B_1 & \xleftarrow{\pi_1} & B_1 \prod B_2 & \xrightarrow{\pi_2} & B_2 \\ & \swarrow \forall f_1 & \uparrow \exists h & \searrow \forall f_2 & \\ & & B & & \end{array}$$

which, when converted to morphisms in \mathcal{B} according to the definition of the Kleisli category and the canonical embedding of \mathcal{B} in it, amounts to the outer part of the diagram

$$\begin{array}{ccccc}
 T(B_1) & \xleftarrow{T(\pi_1)} & T(B_1 \amalg B_2) & \xrightarrow{T(\pi_2)} & T(B_2) \\
 & \swarrow \pi_1 & \uparrow \alpha & \searrow \pi_2 & \\
 & & T(B_1) \amalg T(B_2) & & \\
 & \swarrow f_1^\# & \uparrow f_1^\# \amalg f_2^\# & \searrow f_2^\# & \\
 & & B & &
 \end{array}$$

and we seek a suitable morphism $h^\# : B \rightarrow T(B_1 \amalg B_2)$. The morphisms inside the diagram are the respective canonical projections for the depicted products in \mathcal{B} , the morphism to the product induced by the given morphisms $f_i^\#$, and the structure morphisms α . We then define $h^\# = \alpha \circ (f_1^\# \amalg f_2^\#)$, and it is now easy to verify that $\pi_i \circ h = f_i$ holds in the Kleisli category, for all i , by chasing around the diagram. \square

We conclude this section by summarising the results above into a list of conditions on a monad that imply properties of the Kleisli category relevant to Mahavier limits.

Proposition 4.8. *Let T be a productive order enriched monad on a category \mathcal{B} . Then, the pair $\mathcal{B} \subseteq \mathcal{B}_T$ is an order extension and products in \mathcal{B} induce weak products in \mathcal{B}_T .*

Verifying, in particular cases, that the induced weak products are in fact max products is, typically, straightforward, as done in Proposition 3.13 for spaces.

This result furnishes plenty of order extensions, though, obviously not all of them. When applied to the by now obvious hyperspace monads on topological spaces or on compacta, one obtains the order extensions $\mathbf{Top} \subseteq \mathbf{Top}_T$ and $\mathbf{Comp} \subseteq \mathbf{Comp}_T$, with all of the relevant structure so that Mahavier limits recover generalised inverse limits. When applied to the identity monad on a category \mathcal{B} , one obtains the trivial order extension $\mathcal{B} \subseteq \mathcal{B}$, where Mahavier limits coincide with ordinary categorical limits. One can now describe the notion of generalised inverse limit of spaces as an extension of the ordinary notion of categorical limit on \mathbf{Top} by means of passing from the identity monad on \mathbf{Top} to a hyperspace monad.

5. MAPPINGS BETWEEN MAHAVIER LIMITS

For ordinary categorical limits it is well-known that, when all relevant limits exist, given functors $\mathcal{D}_0 \rightarrow \mathcal{D} \rightarrow \mathcal{C}$ one obtains a morphism $\varprojlim(\mathcal{D} \rightarrow \mathcal{C}) \rightarrow \varprojlim(\mathcal{D}_0 \rightarrow \mathcal{D} \rightarrow \mathcal{C})$, and given a natural transformation between two diagrams $F, F': \mathcal{D} \rightarrow \mathcal{C}$, say from F to F' , one obtains a morphism $\varprojlim F \rightarrow \varprojlim F'$. The aim of this section is to obtain the analogous results for Mahavier limits. We mention [?] where issues of obtaining functions between generalised inverse limits of compacta are considered, but differently to our approach. In particular, as the authors of [?] state clearly, in the setting they present, taking limits is not quite functorial. Taking Mahavier limits is functorial.

5.1. Shape change transformations. Let $F: \mathcal{D} \rightarrow \mathcal{C}$ be an order diagram. If \mathcal{D}_0 is a category and $S: \mathcal{D}_0 \rightarrow \mathcal{D}$ is a functor, then $\mathcal{D}_0 \xrightarrow{S} \mathcal{D} \xrightarrow{F} \mathcal{C}$ is an order diagram which we say is obtained by a *shape change* of F along S . Suppose that the Mahavier limits of F and of $F \circ S$ exist relative to a subcategory $\mathcal{B} \subseteq \mathcal{C}$, with Mahavier limiting cones $\{\pi_D: \varprojlim_{\mathcal{B}}^M F \rightarrow FD\}_{D \in \mathcal{D}}$ and $\{\rho_D: \varprojlim_{\mathcal{B}}^M (F \circ S) \rightarrow FSD\}_{D \in \mathcal{D}_0}$. Just as for ordinary categorical limits, it then follows that the Mahavier limiting objects are related by a morphism $b \in \mathcal{B}$ simply because $\{\pi_{SD}\}_{D \in \mathcal{D}_0}$ forms an order cone to $F \circ S$, and then b is the unique morphism with $\pi_{SD} = \rho_D \circ b$, for all $D \in \mathcal{D}_0$.

Let \mathcal{C} be a small ordered category, and consider the category \mathbf{Cat}/\mathcal{C} whose objects are order functors $\mathcal{D} \rightarrow \mathcal{C}$ and whose morphisms S

$$\begin{array}{ccc} \mathcal{D}_0 & \xrightarrow{S} & \mathcal{D} \\ & \searrow F & \swarrow F' \\ & & \mathcal{C} \end{array}$$

are commuting triangles. When \mathcal{C} is a category, this construction coincides with the comma category of \mathbf{Cat} over \mathcal{C} . The preceding discussion shows that if $\mathcal{B} \subseteq \mathcal{C}$ is Mahavier complete, so that $\varprojlim_{\mathcal{B}}^M$ is defined on the objects of \mathbf{Cat}/\mathcal{C} , one obtains a *shape change* morphism $\varprojlim_{\mathcal{B}}^M S: \varprojlim_{\mathcal{B}}^M F \rightarrow \varprojlim_{\mathcal{B}}^M (F \circ S)$.

Theorem 5.1. *Let $\mathcal{B} \subseteq \mathcal{C}$ be a Mahavier complete order extension. Then $\varprojlim_{\mathcal{B}}^M: \mathbf{Cat}/\mathcal{C} \rightarrow \mathcal{B}$ is a contravariant functor.*

Proof. The proof is immediate, resorting to Theorem 3.7 (several times). \square

This theorem, specialised to $\mathbf{Top} \subseteq \mathbf{Top}_T$ and $\mathbf{Comp} \subseteq \mathbf{Comp}_T$, is used repeatedly in the literature, e.g., [?, Theorem 5.1].

5.2. Transformations between diagrams of the same shape. For categories \mathcal{C} and \mathcal{D} we write $[\mathcal{D}, \mathcal{C}]$ for the functor category whose objects are all functors $\mathcal{D} \rightarrow \mathcal{C}$ and whose morphisms are natural transformations. When \mathcal{D} is thought of as a shape for diagrams we regard $[\mathcal{D}, \mathcal{C}]$ as the category of diagrams in \mathcal{C} of shape \mathcal{D} . Assume now that \mathcal{C} has limits of shape \mathcal{D} . It is well-known that an arbitrary choice for a limiting object for each diagram gives rise to a functor $\varprojlim: [\mathcal{D}, \mathcal{C}] \rightarrow \mathcal{C}$. We now present the analogous result for Mahavier limits.

Of course, the result hinges on defining the order version of $[\mathcal{D}, \mathcal{C}]$, and we do so in full generality, namely in the category \mathbf{pCat}_{\leq} of pairs of ordered enriched categories. Thus, given morphisms $F, G: (\mathcal{C}, \mathcal{B}) \rightarrow (\mathcal{C}', \mathcal{B}')$ of pairs, an *order natural transformation* $\alpha: F \rightarrow G$ relative to \mathcal{B}' is a family $\{\alpha_C: FC \rightarrow GC\}_{C \in \mathcal{C}}$ of morphisms in \mathcal{B}' such that the diagram

$$\begin{array}{ccc} FC & \xrightarrow{\sim}^{Fc} & FC' \\ \downarrow \alpha_C & & \downarrow \alpha_{C'} \\ GC & \xrightarrow{\sim}^{Gc} & GC' \end{array}$$

order commutes in the sense that $\alpha_{C'} \circ Fc \leq Gc \circ \alpha_C$, for all $c: C \rightsquigarrow C'$ in \mathcal{C} . When \mathcal{B}' is clear, we simply say *order natural transformation*. If $H: (\mathcal{C}, \mathcal{B}) \rightarrow (\mathcal{C}', \mathcal{B}')$ is another morphism, and $\beta: G \rightarrow H$ another order natural transformation, then it is immediate to verify that $\beta \circ \alpha$, given component-wise by $\beta_C \circ \alpha_C$, is an order natural transformation, and that the collection $[(\mathcal{C}, \mathcal{B}), (\mathcal{C}', \mathcal{B}')]$ of all order functors $(\mathcal{C}, \mathcal{B}) \rightarrow (\mathcal{C}', \mathcal{B}')$ and all order natural transformations between them is a category.

For a category \mathcal{D} and an order extension $\mathcal{B} \subseteq \mathcal{C}$ we denote $[(\mathcal{D}, \emptyset), (\mathcal{C}, \mathcal{B})]$ by $[\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$. When \mathcal{D} is thought of as the shape of diagrams, this defines the category of order diagrams in \mathcal{C} of shape \mathcal{D} relative to \mathcal{B} .

Theorem 5.2. *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension, and \mathcal{D} a category, and assume that \mathcal{C} has all Mahavier limits of shape \mathcal{D} relative to \mathcal{B} . Then any arbitrary choice of limiting object for each order diagram $F: \mathcal{D} \rightarrow \mathcal{C}$ extends canonically to a functor $\varprojlim_{\mathcal{B}}^M: [\mathcal{D}, \mathcal{C}]_{\mathcal{B}} \rightarrow \mathcal{B}$.*

Proof. Given an order diagram $F: \mathcal{D} \rightarrow \mathcal{C}$, let $\varprojlim_{\mathcal{B}}^M F$ be a Mahavier limiting object of the diagram, which comes together with a Mahavier limiting cone, denoted, somewhat ambiguously, by $\{\pi_D\}_{D \in \mathcal{D}}$. For an order natural transformation $\alpha: F \rightarrow G$ between two such order diagrams, we must construct a canonical morphism $\varprojlim_{\mathcal{B}}^M(\alpha): \varprojlim_{\mathcal{B}}^M(F) \rightarrow \varprojlim_{\mathcal{B}}^M(G)$ in \mathcal{B} , and, given a second order natural transformation $\beta: G \rightarrow H$, show the functoriality of the construction. Consider, for a morphism $d \in \mathcal{D}$,

the diagram

$$\begin{array}{ccccc}
 FD & \overset{Fd}{\rightsquigarrow} & & FD' & \\
 \downarrow \alpha_D & \swarrow & \lim_{\leftarrow \mathcal{B}}^M F & \searrow & \downarrow \alpha_{D'} \\
 GD & \overset{Gd}{\rightsquigarrow} & & GD' & \\
 \downarrow \beta_D & \swarrow & \lim_{\leftarrow \mathcal{B}}^M G & \searrow & \downarrow \beta_{D'} \\
 HD & \overset{Hd}{\rightsquigarrow} & & HD' & \\
 & \swarrow & \lim_{\leftarrow \mathcal{B}}^M H & \searrow &
 \end{array}$$

where the existence and commutativity of the dotted arrows is the aim. Emanating from $\lim_{\leftarrow \mathcal{B}}^M F$, the morphisms $\{\alpha_D \circ \pi_D\}_{D \in \mathcal{D}}$ form an order cone to G , since $\alpha_{D'} \circ \pi_{D'} \leq \alpha_{D'} \circ Fd \circ \pi_D \leq Gd \circ \alpha_D \circ \pi_D$. The universal property of the chosen Mahavier limiting cone for G yields one of the dotted arrows, and a similar argument gives the other two. Their commutativity follows again from the universal property, namely by means of Theorem 3.7. \square

Remark 5.3. It is a triviality that a functor preserves isomorphisms, and thus now equally trivial that any two Mahavier limits of, respectively, order diagrams $F, F': \mathcal{D} \rightarrow \mathcal{C}$, in some order extension $\mathcal{B} \subseteq \mathcal{C}$, are isomorphic provided the two diagrams are isomorphic in the category $[\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$.

As an application we consider [?, Theorem 5.3], according to which if X is a compactum and $f, g: X \rightsquigarrow X$ are upper semicontinuous set-valued functions, which are topologically conjugate, namely there exists a homeomorphism $h: X \rightarrow X$ with $h \circ f = g \circ h$, then $\lim_{\leftarrow \mathcal{B}}^M F \cong \lim_{\leftarrow \mathcal{B}}^M G$, where we denote by F the constant sequence with f as bonding functions, and G is similarly defined. It is immediate that the condition of topological conjugacy implies that $\alpha: F \rightarrow G$, given by $\alpha_n = h$ for all $n \geq 1$, is a natural isomorphism, namely $F \cong G$ in the category of diagrams. That $\lim_{\leftarrow \mathcal{B}}^M F \cong \lim_{\leftarrow \mathcal{B}}^M G$ now follows from the previous remark.

Recall that for ordinary categorical limits, if \mathcal{B} is a category admitting all limits of shape \mathcal{D} , for some fixed category \mathcal{D} , then the functor

$\varprojlim: [\mathcal{D}, \mathcal{B}] \rightarrow \mathcal{B}$ is right adjoint to the diagonal functor $\Delta: \mathcal{B} \rightarrow [\mathcal{D}, \mathcal{B}]$, where $\Delta(B)$ is the functor which is constantly B on objects and constantly id_B on morphisms. Noting that for an order extension $\mathcal{B} \subseteq \mathcal{C}$, the diagonal functor naturally extends to $\Delta: \mathcal{B} \rightarrow [\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$, simply via the inclusion of \mathcal{B} in \mathcal{C} , this result extended immediately to the setting of Mahavier limits, as follows.

Theorem 5.4. *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension admitting all Mahavier limits of shape \mathcal{D} , where \mathcal{D} is a fixed category. Then the functor $\varprojlim_{\mathcal{B}}^M: [\mathcal{D}, \mathcal{C}]_{\mathcal{B}} \rightarrow \mathcal{B}$ is right adjoint to the diagonal functor $\Delta: \mathcal{B} \rightarrow [\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$.*

Proof. Noting, for a given order functor $F: \mathcal{D} \rightarrow \mathcal{C}$, the trivial fact that an order natural transformation $\Delta(B) \rightarrow F$ relative to \mathcal{B} is precisely an order cone from B to the order diagram F relative to \mathcal{B} , the proof is formally identical to the classical proof. \square

Let us now look at [?, Theorem 5.2] through the lens of adjunctions. The theorem states that if $\{f_n: X_{n+1} \rightsquigarrow X_n\}_{n \geq 1}$ and $\{g_n: Y_{n+1} \rightsquigarrow Y_n\}_{n \geq 1}$ are generalised inverse systems of compacta, which we will denote by F and G , and $\alpha_n: X_n \rightarrow Y_n$, $n \geq 1$, are given such that $\alpha_n \circ f_n = g_n \circ \alpha_{n+1}$, then the induced function $\varprojlim_{\mathcal{B}}^M F \rightarrow \varprojlim_{\mathcal{B}}^M G$ is continuous, and if each α_n is injective (and surjective), then so is the induced function. Now, the condition on α implies that it is a natural transformation $\alpha: F \rightarrow G$, and the induced function is precisely $\varprojlim_{\mathcal{B}}^M(\alpha)$, so it is continuous. An elementary result of category theory is that right adjoints preserve monomorphisms, and it is easily seen that if each α_n is injective, then the natural transformation α is a monomorphism in the relevant category of order diagrams. Since taking Mahavier limits is a right adjoint functor, it follows that $\varprojlim_{\mathcal{B}}^M(\alpha)$ is a monomorphism in **Comp**, namely an injective function. The surjectivity part of the claim cannot be addressed in the same way (since right adjoint need not preserve epimorphisms). The true nature of that part of the theorem is revealed in Section 7.

6. THE SUBSEQUENCE THEOREM

When the category \mathcal{D} is

$$\cdots \longrightarrow n+1 \longrightarrow n \longrightarrow \cdots \longrightarrow 2 \longrightarrow 1$$

order diagrams $F: \mathcal{D} \rightarrow \mathcal{C}$ are sequences, and if \mathcal{D}_0 is an infinite full subcategory of \mathcal{D} , then $F \circ S: \mathcal{D}_0 \rightarrow \mathcal{C}$, where $S: \mathcal{D}_0 \rightarrow \mathcal{D}$ is the inclusion functor, is said to yield a subsequence of F . For ordinary categorical limits it is well-known that the canonical morphism $\varprojlim F \rightarrow \varprojlim(F \circ S)$ is then an

isomorphism. That the same does not generally hold for Mahavier limits was noted early on, and the task of identifying conditions under which Mahavier limits of infinite subsequences agree with the Mahavier limit of the original sequence became known as the *subsequence theorem problem* (see [?] for details). Without any conditions on the bonding functions the subsequence theorem may fail dramatically, even for diagrams with a single bonding function. For instance, in [?] an upper semicontinuous function $f: X \rightsquigarrow X$, with $X = [0, 1]$, is constructed with the property that for the diagram $F: \mathcal{D} \rightarrow \mathbf{Comp}_T$ which is constantly f , the Mahavier limits of any two different subsequences are non-homeomorphic.

Before addressing the subsequence theorem in the context of Mahavier limits in full generality, we wish to emphasise an important detail.

Remark 6.1. In [?, Theorem 5.3] it is noted that the subsequence theorem holds for very special diagrams of compacta, namely those with a single bonding function $f: X \rightsquigarrow X$ satisfying $f^2 = f$. Even this case is worth looking into in more detail, keeping the category theoretic perspective in mind. Recall again that without any conditions at all, ordinary categorical limits do satisfy the subsequence theorem, and in a strong sense, namely the canonical shape change morphism $\varprojlim F \rightarrow \varprojlim (F \circ S)$ is an isomorphism. Let us now fix a very concrete scenario, for which we choose an arbitrary compact Hausdorff space X and the morphism $f: X \rightsquigarrow X$ given by $f(x) = X$, for all $x \in X$. Construct now the sequence $F: \mathcal{D} \rightarrow \mathbf{Comp}_T$ with $F(n) = X$, and with $F(m \rightarrow n) = f$, for all $n < m$, and let \mathcal{D}_0 be the subcategory of \mathcal{D} spanned by, say, the even numbers, and $S: \mathcal{D}_0 \rightarrow \mathcal{D}$ the inclusion. Now, it is easily seen that both $\varprojlim_{\mathbf{Comp}}^M F$ and $\varprojlim_{\mathbf{Comp}}^M (F \circ S)$ are given by $X \times X \times X \times \dots$, with the evident Mahavier cones given by the usual projections. The shape change morphism $\varprojlim_{\mathbf{Comp}}^M F \rightarrow \varprojlim_{\mathbf{Comp}}^M (F \circ S)$ is then given by $(x_n)_{n \in \mathbb{N}} \mapsto (x_{2n})_{n \in \mathbb{N}}$, and so it is not an isomorphism (unless X is very trivial).

Put simply, the reason that the subsequence theorem holds for a sequence with a single bonding function f satisfying $f^2 = f$, is that the given sequence $\mathcal{D} \rightarrow \mathbf{Comp}_T$ and any subsequence $\mathcal{D}_0 \rightarrow \mathbf{Comp}_T$ are isomorphic objects in $\mathbf{Cat}/\mathbf{Comp}_T$, and the result follows by Theorem 5.1.

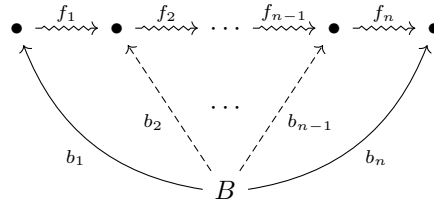
There are thus (at least) two reasons a sequence may have the subsequence property. One, observed in the case $f^2 = f$, is when even before the Mahavier limit is computed, the relevant diagrams are already isomorphic. This is a combinatorial condition on the bonding functions which has little to do with the actual construction of the Mahavier limits. The other reason for having the subsequence property is because some conditions are satisfied which guarantee the shape change morphism is

an isomorphism. We view the latter as the statement of the subsequence theorem property in the context of this work.

Let us fix some terminology, for the sake of precision. Throughout, by *sequence* we mean an order diagram $F: \mathcal{D} \rightarrow \mathcal{C}$ where \mathcal{D} may be either \mathbb{N} or \mathbb{Z} , viewed as posets, and thus as categories with a single morphism $k \rightarrow m$ whenever $k \geq m$. By a *subsequence* we mean an inclusion $S: \mathcal{D}_0 \rightarrow \mathcal{D}$ of an infinite full subcategory \mathcal{D}_0 into \mathcal{D} , inducing the order diagram $F \circ S$, all within an ambient order extension $\mathcal{B} \subseteq \mathcal{C}$. We say that a sequence F has the *subsequence property with respect to a subsequence S* , if whenever the Mahavier limit $\varprojlim_{\mathcal{B}}^M(F \circ S)$ exists, so does the Mahavier limit $\varprojlim_{\mathcal{B}}^M(F)$, and the canonical shape change morphism $\varprojlim_{\mathcal{B}}^M(F) \rightarrow \varprojlim_{\mathcal{B}}^M(F \circ S)$ is an isomorphism. We say that F has the *subsequence property* if F has the subsequence property with respect to every subsequence S . Finally, we say that the order extension $\mathcal{B} \subseteq \mathcal{C}$ is *sequentially conservative* if every sequence F has the subsequence property.

We now prove a subsequence result which is valid in any order extension, using the following concepts.

Definition 6.2. Consider the diagram

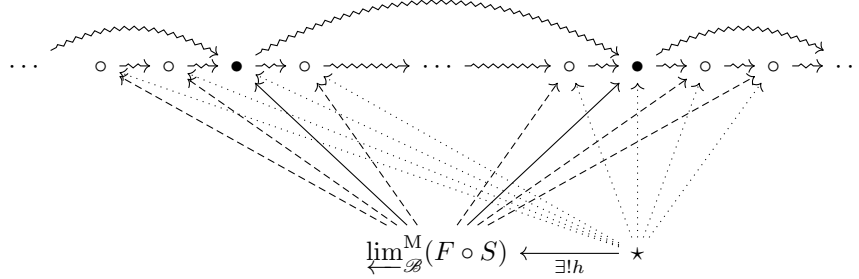


where the morphisms f_1, \dots, f_n are any morphisms in \mathcal{C} (whose domains and codomains can be arbitrary, as long as the morphisms compose as shown in the diagram), and b_1, b_n are morphisms in \mathcal{B} . We refer to the entire diagram as a *system*. We say that b_1 and b_n form *boundary conditions* for the system if $b_n \leq f_n \circ f_{n-1} \circ \dots \circ f_2 \circ f_1 \circ b_1$, namely if the diagram order commutes. We then also say that the system is a *system with boundary conditions*. For a fixed system with boundary conditions, a *solution* of the system consists of morphisms b_2, \dots, b_{n-1} such that $\{b_1, \dots, b_n\}$ is an order cone to the finite sequence, namely $b_{k+1} \leq f_k \circ b_k$, for all $1 \leq k < n$. A sequence (f_1, \dots, f_n) of morphisms in \mathcal{C} is *solvable* if every system with boundary conditions as above admits a unique solution.

Lemma 6.3. Consider an order extension $\mathcal{B} \subseteq \mathcal{C}$, a sequence $F: \mathcal{D} \rightarrow \mathcal{C}$, and an inclusion $S: \mathcal{D}_0 \rightarrow \mathcal{D}$, yielding a subsequence $F \circ S$ of F . By a *gap in the sequence F determined by S* we mean the image of F on a full subcategory of \mathcal{D} spanned by $\{k, k + 1, \dots, k + t\}$ where of these the only objects that belong to \mathcal{D}_0 are k and $k + t$, and $t \geq 2$. If all gaps

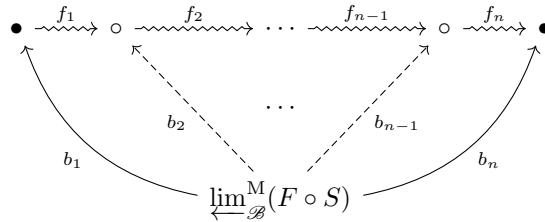
determined by the subsequence are solvable, then F has the subsequence property with respect to S .

Proof. Consider the diagram



where the top part depicts the given sequence F , with the subsequence $F \circ S$ indicated by the \bullet 's and the curved morphisms; a gap is then precisely a finite consecutive portion of morphisms in the top row between two consecutive \bullet 's enclosing at least one \circ , and the curved morphism is simply the composition of the morphisms in the gap. The assumed Mahavier limit of the subsequence is given with its canonical projections as solid arrows. The morphisms emanating from \star are an arbitrary order cone to the entire sequence. We show that $\varprojlim_{\mathcal{B}}^M(F \circ S)$ is a Mahavier limit of the entire sequence by first obtaining the dashed arrow so as to complete the solid arrows into an order cone to the entire sequence, and then obtain the morphism $h: \star \rightarrow \varprojlim_{\mathcal{B}}^M(F \circ S)$ which uniquely factorises the order cone from \star through the solid and dashed morphisms.

Consider the diagram



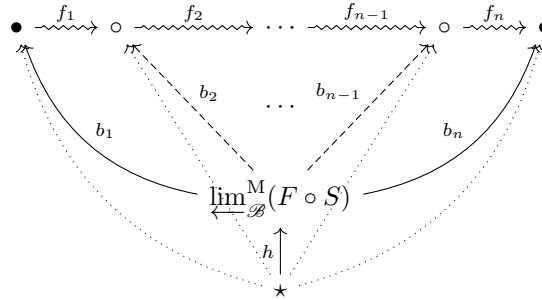
whose outer part is an arbitrary gap in the sequence F determined by S , i.e., b_1 and b_n are the canonical projections to the \bullet 's. It follows at once that this diagram is a system with boundary conditions, and thus a unique solution exists, giving the dashed arrows. Thus, each gap in the diagram admits a unique solution, and it is easy to see that all of these solutions together fit into the first diagram above to form an order cone from $\varprojlim_{\mathcal{B}}^M(F \circ S)$ to the entire sequence.

We now attend to the required morphism h . For the given order cone from \star to the sequence, i.e., the dotted arrows above, it is easily seen that the collection of just those dotted arrows with a \bullet for a codomain constitutes an order cone to the subsequence $F \circ S$. The universal property of the Mahavier limit yields a unique $h: \star \rightarrow \varprojlim_{\mathcal{B}}^M(F \circ S)$ with the property that

$$\begin{array}{ccc} & \bullet & \\ & \nearrow & \nwarrow \\ \varprojlim_{\mathcal{B}}^M(F \circ S) & \xleftarrow{h} & \star \end{array}$$

commutes. In particular, since any morphism $\star \rightarrow \varprojlim_{\mathcal{B}}^M(F \circ S)$ which factorises the entire cone through the dashed and solid arrows in particular factorises the cone to the subsequence through the solid arrows, the uniqueness of a factorisation h of the entire order cone through $\varprojlim_{\mathcal{B}}^M(F \circ S)$ is guaranteed.

We must now establish that h factorises all of the dotted morphisms, rather than just the ones with a \bullet for a codomain. Consider thus



which is an arbitrary gap in the sequence together with the dashed morphisms as constructed above and the dotted morphisms from the given order cone to the sequence. Consider now the system with boundary conditions given by the outer part of the diagram, namely the squiggly morphisms together with the curved dotted morphisms. Obviously, the remaining dotted morphisms are a solution. Note that a simple diagram chase reveals that the compositions $\{b_k \circ h\}_{1 \leq k \leq n}$ are also a solution to the same system with the same boundary conditions, and thus, by the assumption on the gaps in the sequence, the two solutions coincide, which is precisely the required commutativity of the dotted morphisms via h and the solid and dashed morphisms.

Pasting all of that information together at each gap of the sequence shows that the solid and dashed morphisms together form a Mahavier limiting cone from $\varprojlim_{\mathcal{B}}^M(F \circ S)$ to the entire sequence, and thus $\varprojlim_{\mathcal{B}}^M(F)$

exists and is given by $\varprojlim_{\mathcal{B}}^M(F \circ S)$. The claim that whenever $\varprojlim_{\mathcal{B}}^M(F)$ and $\varprojlim_{\mathcal{B}}^M(F \circ S)$ are Mahavier limits, then the canonical shape change morphism $\varprojlim_{\mathcal{B}}^M(F) \rightarrow \varprojlim_{\mathcal{B}}^M(F \circ S)$ is an isomorphism follows from Theorem 3.7. \square

Definition 6.4. An order extension $\mathcal{B} \subseteq \mathcal{C}$ is *sequentially solvable* if any finite sequential system in \mathcal{C} as above, with boundary conditions in \mathcal{B} has a unique solution in \mathcal{B} .

As an immediate corollary of Lemma 6.3 we obtain the following theorem.

Theorem 6.5. *A sequentially solvable order extension is sequentially conservative.*

As an application, let us consider the subsequence theorem presented in [?], which we briefly recall. Let X be a compactum, and $F: X \rightsquigarrow X$ an upper semicontinuous set-valued function, and denote by \mathbf{F} the sequence which is constantly F . Greenwood and Youl identify the following conditions on F to ensure that \mathbf{F} has the subsequence property. Consider a collection \mathcal{C} of continuous functions $X \rightarrow X$. We shall say that \mathcal{C} is *conservative* if $\text{Fix}(f) = \text{Fix}(f')$ holds for all $f, f' \in \mathcal{C} \setminus \{\text{id}_X\}$ (where $\text{Fix}(f)$ is the set of fixed points of the function), the common set D of all fixed points is repelling, i.e., for all $f \in \mathcal{C}$ and all $x \in X$, if $f(x) \in D$, then $x \in D$, and further for all $f, f' \in \mathcal{C}$, with $f \neq f'$, if $f(x) = f'(x)$, then $x \in D$. Now, a finite collection $C = \{f_i: X \rightarrow X\}_{i \in I}$ of continuous functions is an (X, D) -collection if its composition closure, namely the smallest set of continuous functions $X \rightarrow X$ containing C and closed under composition, is conservative. Given such an (X, D) -collection, which contains the identity as an element, consider the function $X \rightsquigarrow X$ determined by the property that its graph is the union of the graphs $\text{Gr}(f_i)$, $i \in I$. Under these conditions, \mathbf{F} satisfies the subsequence theorem. Using the above machinery the result follows by noting that the conditions of Lemma 6.3 are easily verified: given any boundary condition problem involving F , the strong conditions on the composition closure immediately imply the existence of at most one solution, and compactness allows one to produce a solution. In fact, with the same argument, we may formulate a slightly stronger result, as follows. For an (X, D) -collection and F as above, consider the smallest ordered subcategory \mathbf{Comp}_F of \mathbf{Comp}_T consisting of all compacta, all of the single-valued functions, and the set-valued function F . Then the order extension $\mathbf{Comp} \subseteq \mathbf{Comp}_F$ is sequentially conservative.

To treat the most general form of diagrams, rather than just linear ones giving rise to sequences, let us briefly recall the classical result of category

theory regarding final functors. We follow [?, Subsection 2.11]. Recall, that a functor $G: \mathcal{D}_0 \rightarrow \mathcal{D}$ is *final* if for every diagram $F: \mathcal{D} \rightarrow \mathcal{B}$ in any category \mathcal{B} , whenever $\{\pi_D: B \rightarrow FD\}_{D \in \mathcal{D}}$ is a limiting cone for F , the collection $\{\pi_{G(D_0)}: B \rightarrow F(G(D_0))\}_{D_0 \in \mathcal{D}_0}$ is a limiting cone for $F \circ G$, and if the limit of $F \circ G$ exists, then the limit of F also exists. Notice that it then follows automatically that the shape change morphism is an isomorphism.

Definition 6.6. A functor $F: \mathcal{D}_0 \rightarrow \mathcal{D}$ is *Mahavier final* if the same conditions as above hold with all references to 'limit' replaced by 'Mahavier limit'.

Now, sufficient conditions for a functor $F: \mathcal{D}_0 \rightarrow \mathcal{D}$ to be final are ([?, Proposition 2.11.1]) that for every $D \in \mathcal{D}$ there exists $D_0 \in \mathcal{D}_0$ and a morphism $d: G(D_0) \rightarrow D$, and further that for all $D_0, D'_0 \in \mathcal{D}_0$, $D \in \mathcal{D}$, and morphisms $d: G(D_0) \rightarrow D$ and $d': G(D'_0) \rightarrow D$, there exist $D''_0 \in \mathcal{D}_0$ and morphisms $d_0: D''_0 \rightarrow D_0$ and $d'_0: D''_0 \rightarrow D'_0$ with $d \circ G(d_0) = d' \circ G(d'_0)$. We shall call these the *sufficiency conditions (for finality)*. Notice that for a discrete linear diagram $\mathcal{D} = \mathbb{N}$ or $\mathcal{D} = \mathbb{Z}$ as above, and any infinite full subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$, the inclusion functor trivially satisfies these conditions, and thus is final. The classical subsequence theorems follow at once.

We can now point at what is perhaps the most profound difference between ordinary categorical limits and Mahavier limits: final functors are in relative abundance while Mahavier final functors are rare. For example, recall that in $\mathbf{Comp} \subseteq \mathbf{Comp}_T$ we have that $\lim_{\leftarrow \mathcal{B}}^M (X \overset{f}{\rightsquigarrow} Y) \cong \text{Gr}(f)$ which, unless f is single-valued, need not be homeomorphic to X . In particular, the inclusion functor $\bullet \rightarrow (\bullet \rightarrow \circ)$ is final but not Mahavier final.

Given an order extension $\mathcal{B} \subseteq \mathcal{C}$, let us say that \mathcal{B} is a *right ideal* in \mathcal{C} if $c \circ b$ is an arrow in \mathcal{B} for all morphisms $b \in \mathcal{B}$ and $c \in \mathcal{C}$. The proof of the following theorem is identical to the classical proof, and we thus keep the details at a minimum.

Theorem 6.7. *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension with \mathcal{B} a right ideal in \mathcal{C} . Then any functor $F: \mathcal{D}_0 \rightarrow \mathcal{D}$ satisfying the sufficiency conditions is Mahavier final.*

Proof. The important ingredient in the proof is the ability to extend an order cone $\{\pi_{D_0}: B \rightarrow F(G(D_0))\}_{D_0 \in \mathcal{D}_0}$ to $F \circ G$, to an order cone to F . Given an object $D \in \mathcal{D}$ use the first sufficiency condition to obtain an object $D_0 \in \mathcal{D}_0$ and a morphism $d: G(D_0) \rightarrow D$. Then consider the composition $p_D = Fd \circ \pi_{D_0}$, which is a morphism in \mathcal{B} . The second sufficiency condition guarantees (by repeatedly relying on the fact that the order in

\mathcal{B} is trivial) just as in the classical argument that p_D is independent of the choice of d . The rest of the proof, namely that $\{p_D: B \rightarrow FD\}_{D \in \mathcal{D}}$ is a Mahavier limiting cone, is essentially the same as the classical proof. \square

Clearly, if \mathcal{B} is a category, then for the trivial order extension $\mathcal{B} \subseteq \mathcal{B}$, \mathcal{B} is a right ideal in itself, and Mahavier limits coincide with ordinary limits; the entire theorem collapses to the classical one. Of course, the condition on \mathcal{B} being a right ideal in \mathcal{C} is quite drastic, and is certainly not satisfied in $\mathbf{Comp} \subseteq \mathbf{Comp}_T$, a fact which is at the heart of the failure of the subsequence theorem without conditions on the bonding functions.

To conclude, an unrestricted analogue of the classical behaviour of limits with respect to initial functors is obtained under very restricting assumptions on the order extension. In more general extensions, such as $\mathbf{Comp} \subseteq \mathbf{Comp}_T$, Lemma 6.3 identifies conditions for a subsequence theorem, but these are specific to the form of a discrete linear diagram \mathcal{D} . For more general diagrams \mathcal{D} and an arbitrary initial functor $\mathcal{D}_0 \rightarrow \mathcal{D}$, one can adapt the proof of Lemma 6.3, and obtain similar conditions for the shape change morphism to be an isomorphism. The extra ingredient required, compared to the classical proofs, is that of a *segmentation* of the shape category \mathcal{D} , which above was given by the gaps in the sequence determined by the subsequence. Roughly, let us say that \mathcal{D} admits a segmentation if cones to an order functor on \mathcal{D} can be pasted together from smaller segments of \mathcal{D} . Then, assuming similar solvability of systems determined by the shapes of the segments, one can adapt the proof of Lemma 6.3. We shall leave the argument at this level of lack of rigour. Needless to say, obtaining good segmentations in practice is highly dependent on the shape of \mathcal{D} .

7. MAHAVIER LIMITS AND ADJUNCTIONS

A fundamental elementary result of category theory is that right adjoints are continuous functors, namely they preserve all small limits. The goal of this section is to prove a similar result for Mahavier limits.

We already observed some of the interaction between the order extension $\mathbf{Top} \subseteq \mathbf{Top}_T$ and its sub-extension $\mathbf{Comp} \subseteq \mathbf{Comp}_T$. For instance, obviously, diagrams in \mathbf{Comp}_T can also be seen as diagrams in \mathbf{Top}_T , and a-priori there is no reason for Mahavier limits computed in the ambient category to agree with the Mahavier limits computed in the smaller category. The fact that they actually do agree is a particular detail of the relationship between the two scenarios. Now, every topological space has an underlying set and thus forgetful functors $\mathbf{Comp} \rightarrow \mathbf{Set}$ and $\mathbf{Top} \rightarrow \mathbf{Set}$ exist. The functor $\mathbf{Top} \rightarrow \mathbf{Set}$ has a left adjoint, associating

with a set X its discrete topology, and a right adjoint, associating to X its indiscrete topology. Obviously, these adjoints are not related to the forgetful functor $\mathbf{Comp} \rightarrow \mathbf{Set}$, since the Hausdorff and the compactness conditions complicate things considerably. However, since limits of diagrams in \mathbf{Comp} are the same whether computed in \mathbf{Comp} or in \mathbf{Top} , treating them as limits in \mathbf{Top} allows one to utilise properties of limits and adjunctions to study limits of inverse systems of compacta.

It is instructive to consider the persistence of these details when the categories \mathbf{Comp} , \mathbf{Top} , and \mathbf{Set} are order extended, each by means of its own monad. Recall the full power set hyperspace monad $T: \mathbf{Top} \rightarrow \mathbf{Top}$ with the upper Vietoris topology and the monad $T': \mathbf{Comp} \rightarrow \mathbf{Comp}$ with $T'(X)$ the collection of closed subsets of X , again with the upper Vietoris topology. Somewhat abusively, we shall use T for both monads. Finally, even more abusively, consider the Manes monad $T: \mathbf{Set} \rightarrow \mathbf{Set}$, and endow all three monads with orderings given by point-wise set-inclusion. For each of these monads T the obvious extra structure, the conditions of Proposition 4.8, hold, resulting in the expected order extensions $\mathbf{Set} \subseteq \mathbf{Set}_T$, $\mathbf{Top} \subseteq \mathbf{Top}_T$, and $\mathbf{Comp} \subseteq \mathbf{Comp}_T$, which we now investigate as a trio.

Proposition 7.1. *Each of the three order extensions, $\mathbf{Set} \subseteq \mathbf{Set}_T$, $\mathbf{Top} \subseteq \mathbf{Top}_T$, and $\mathbf{Comp} \subseteq \mathbf{Comp}_T$ is Mahavier complete. If $\mathcal{D} \rightarrow \mathbf{Comp}_T$ is an order diagram, then, via the inclusion $\iota: \mathbf{Comp}_T \rightarrow \mathbf{Top}_T$, it can be seen as an order diagram $\iota \circ F: \mathcal{D} \rightarrow \mathbf{Top}_T$. It then holds that $\varprojlim_{\mathbf{Comp}}^M (F) \cong \varprojlim_{\mathbf{Top}}^M (\iota \circ F)$, where the isomorphism is in \mathbf{Top} . Finally, if $F: \mathcal{D} \rightarrow \mathbf{Top}_T$ is an order diagram, then ignoring the topology yields an order diagram $G: \mathcal{D} \rightarrow \mathbf{Set}_T$, and it then holds that the underlying set of $\varprojlim_{\mathbf{Top}}^M (F)$ is in bijection with the set $\varprojlim_{\mathbf{Set}}^M (G)$.*

Proof. Firstly, obviously, small products in each of the three relevant categories certainly exist, and it is easily verified that the induced weak products in each case are max products. Next, the proof of Proposition 3.10, which yields mixed Mahavier equalisers in \mathbf{Top}_T and in \mathbf{Comp}_T can easily be adapted to yield mixed Mahavier equalisers in \mathbf{Set}_T . Theorem 3.15 can then be applied to conclude that each of these order extensions is Mahavier complete. The rest of the claims can be deduced by inspecting the construction of Mahavier limits in the relevant order extensions. \square

We now turn to a categorical approach to the above result, eliminating the need to inspect any explicit constructions of the limits.

Definition 7.2. Let $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{B}' \subseteq \mathcal{C}'$ be order extensions, $F: \mathcal{C} \rightarrow \mathcal{C}'$ a functor whose restriction $F_{\mathcal{B}}$ to \mathcal{B} is a functor to \mathcal{B}' , and $G: \mathcal{C}' \rightarrow \mathcal{C}$ a functor whose restriction $G_{\mathcal{B}'}$ to \mathcal{B}' is a functor to \mathcal{B} . We say

that an adjunction between $F_{\mathcal{B}}$ and $G_{\mathcal{B}'}$, with right adjoint $G_{\mathcal{B}'}$, namely a bijection $\mathcal{B}'(F_{\mathcal{B}}(B), B') \cong \mathcal{B}(B, G(B'))$, for all $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$, which is natural in each variable, *extends* to the order extensions if there is an adjunction between F and G , with right adjoint G , such that the natural bijection $\mathcal{C}'(F(C), C') \cong \mathcal{C}(C, G(C'))$ is an extension of the natural bijection on the subcategories.

Let $D, I: \mathbf{Set} \rightarrow \mathbf{Top}$ be the functors obtained by placing the discrete, respectively indiscrete, topology on a set. These functors extend to functors $D_T, I_T: \mathbf{Set}_T \rightarrow \mathbf{Top}_T$, since for a multivalued function $f: S_1 \rightsquigarrow S_2$, namely a function $f: S_1 \rightarrow T(S_2)$, we have that $f: D(S_1) \rightarrow D(T(S_2))$ is certainly continuous, so we obtain a multivalued function $f: D(S_1) \rightsquigarrow D(S_2)$, while the function $f: I(S_1) \rightarrow T(I(S_2))$ is continuous since the upper Vietoris topology resulting from an indiscrete space is itself indiscrete, namely $T(I(S_2)) = I(T(S_2))$, so we obtain the morphism $f: I(S_1) \rightsquigarrow I(S_2)$. Similar arguments show that the underlying set functor $U: \mathbf{Top} \rightarrow \mathbf{Set}$ extends to a functor $U_T: \mathbf{Top}_T \rightarrow \mathbf{Set}_T$, and that the adjunction exhibiting D as left adjoint to U , as well as I as right adjoint to U extend, in the above sense, to adjunctions exhibiting D_T as left adjoint to U_T , and I_T as right adjoint to U_T .

Theorem 7.3. *Let $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{B}' \subseteq \mathcal{C}'$ be order extensions, with functors F and G as in Definition 7.2, together with an extended adjunction exhibiting G as right adjoint to F . Then G is Mahavier continuous, i.e., it preserves all Mahavier limits that exist in $\mathcal{B}' \subseteq \mathcal{C}'$. In more detail, if $H: \mathcal{D} \rightarrow \mathcal{C}'$ is any order diagram for which $\varprojlim_{\mathcal{B}'}^M(H)$ exists, then $\varprojlim_{\mathcal{B}}^M(G \circ H)$ exists as well and $G(\varprojlim_{\mathcal{B}'}^M H) \cong \varprojlim_{\mathcal{B}}^M(G \circ H)$ as objects in \mathcal{B} .*

Proof. The proof is formally identical to the proof that right adjoints are continuous functors. \square

We may now conclude our treatment of [?, Theorem 5.2], namely the surjectivity claims, illuminating why they are purely set-theoretic arguments. Indeed, the computation of the Mahavier limit can be seen to take place in $\mathbf{Top} \subseteq \mathbf{Top}_T$, and applying the forgetful functor, which is an extended right adjoint, yields a computation in $\mathbf{Set} \subseteq \mathbf{Set}_T$. Moreover, if the original diagram had surjective bonding functions, then so does the set theoretic diagram, and so one only needs to conclude the surjectivity claims for Mahavier limits of sets - a simple exercise.

8. CONCLUDING REMARKS

Through the extension of the notion of limit in a category to Mahavier limit in an order extension of a category we have shown that the link

between inverse limits of inverse systems of compacta and limits in the category \mathbf{Comp} extends seamlessly to a link between generalised inverse limits of generalised inverse systems of compacta and Mahavier limits in $\mathbf{Comp} \subseteq \mathbf{Comp}_T$. The emphasis throughout was on the applicability of the general machinery to central issues of classical generalised inverse limits in topology, turning ad-hoc definitions and techniques into facts that follow from very general principles. It remains to be seen how effective the formalism above can be in the study of compacta in general, or of ergodic concepts (e.g., [?]) directly related to generalised inverse limits. We conclude this work by briefly considering further research that goes beyond this immediate query.

8.1. Mahavier limits of sets and of arbitrary topological spaces.

The categorical approach is highly suitable to tackle questions of mappings between limits and the subsequence theorem, where results are given in arbitrary order extensions, involving no particularities of specific categories. However, questions relating to whether a limiting object of a given diagram is empty or not, or questions about conditions that guarantee a limiting object is connected (issues that are treated extensively in the literature on generalised inverse limits) or results such as those given in [?], cannot be hoped to be fully answered purely categorically. It would be interesting to see if the categorical perspective can shed some light on these issues.

Section 7 shows that an understanding of the relationship between $\mathbf{Set} \subseteq \mathbf{Set}_T$ and $\mathbf{Top} \subseteq \mathbf{Top}_T$ is useful for the study of $\mathbf{Comp} \subseteq \mathbf{Comp}_T$, and so it becomes valuable to study Mahavier limits of sets, as well as of arbitrary topological spaces. From a more extensive perspective, how do Mahavier limits relate to the field of set-valued analysis?

8.2. Mahavier limits for the Giry monad. The Giry monad (appearing already in [?] and studied in [?]) is at the heart of a categorical approach to probability theory. Much theory is developed around the Giry (and related) monad, of which we only mention [?] where a Kleisli construction is used for a categorical approach to Bayesian machine learning. It is conceivable that some concepts of stochastic ordering can be placed on the Giry monad so that the resulting Kleisli category yields an interesting order extension. It should be interesting to understand the role of Mahavier limits in the categorical perspective to probability and machine learning.

8.3. Mahavier limits in programming semantics. Order enriched monads, and thus ordered Kleisli categories, and so order extensions, are

common place in programming semantics (e.g., [?]), where weak products also play a role. The concept of max limits above already offers an improvement over weak products, namely the uniqueness of the induced morphism. It would be interesting to see if the concept of max products can further simplify arguments, and if Mahavier limits enter the discussion naturally. Moreover, ordered enriched categories are useful in computer science (i.e., [?, ?, ?]). The work above offers a different approach to doing category theory in order enriched categories. Again, it is a natural to investigate the applicability of our methods to this problem domain.

8.4. Mahavier limits in algebra. Obviously, interesting cases where an object can be studied effectively by exhibiting it as a Mahavier limiting object of a relatively simple order diagram is a technique that should not be limited to compacta alone, or just to topology. The monad on monoids already gives a relevant order extension for the study of monoids and multihomomorphisms, and one may restrict to subcategories of it to study, say, groups and multihomomorphisms (for instance, the results of [?, ?] can naturally be framed inside such an order extension). It would be valuable if a group (or a ring, or a module, ...) of interest can be shown to arise as the Mahavier limiting object of an inverse system, much as the p -adic integers admit such a representation.

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