

# Crown reductions for the Minimum Weighted Vertex Cover problem

Miroslav Chlebík<sup>1</sup> and Janka Chlebíková<sup>2</sup>

<sup>1</sup> Max Planck Institute for Mathematics in the Sciences  
Inselstraße 22-26, D-04103 Leipzig, Germany  
`chlebik@mis.mpg.de`

<sup>2</sup> Faculty of Mathematics, Physics and Informatics  
Comenius University, Mlynská dolina,  
824 28 Bratislava, Slovakia  
`chlebikova@fmpf.uniba.sk`

**Abstract.** The paper studies crown reductions for the Minimum Weighted Vertex Cover problem introduced recently in the unweighted case by Fellows et al. ([20], [1]). We describe in detail a close relation of crown reductions to Nemhauser and Trotter reductions that are based on the linear programming relaxation of the problem. We introduce and study the so called strong crown reductions, suitable for finding (or counting) all minimum vertex covers, or finding a minimum vertex cover under some additional constraints. It is described how crown decompositions and strong crown decompositions suitable for such problems can be computed in polynomial time. For weighted König-Egerváry graphs  $(G, w)$  we observe that the set of vertices belonging to all minimum vertex covers, and the set of vertices belonging to no minimum vertex covers, can be efficiently computed.

Further, for some specific classes of graphs, simple algorithms for the MIN-VC problem with a constant approximation factor  $r < 2$  are provided. On the other hand, we conclude that for the regular graphs, or for the Hamiltonian connected graphs, the problem is as hard to approximate as for general graphs.

It is demonstrated how the results about strong crown reductions can be used to achieve a linear size problem kernel for some related vertex cover problems.

## 1 Introduction

The Minimum (weighted) VERTEX COVER problem (shortly, MIN- $w$ -VC) is one of the fundamental NP-hard problems in the combinatorial optimization:

MINIMUM WEIGHTED VERTEX COVER (MIN- $w$ -VC)

*Instance:* A simple graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ .

*Feasible solution:* A vertex cover  $C$  for  $G$ , i.e., a subset  $C \subseteq V$  such that for each  $e \in E$ ,  $e \cap C \neq \emptyset$ .

*Goal:* To minimize the weight  $w(C) := \sum_{u \in C} w(u)$  of the vertex cover  $C$ .

The unweighted version of the MINIMUM VERTEX COVER problem (shortly, MIN-VC) is the special case of MIN- $w$ -VC with uniform weights  $w(u) = 1$  for each  $u \in V$ .

As the MINIMUM VERTEX COVER problem cannot be solved exactly in polynomial time, unless  $P = NP$ , approaches have concentrated on the design of polynomial time approximation algorithms. In spite of a great deal of efforts, the tight bound on its approximability by a polynomial time algorithm is left open. Recall that the problem has a simple 2-approximation algorithm and, for any constant  $r < 2$ , no  $r$ -approximation algorithm is known, even in the unweighted case. Currently the best lower bound on polynomial time approximability is  $10\sqrt{5} - 21 \approx 1.36067$ , due to Dinur and Safra [17]. They proved that achieving a smaller approximation factor is NP-hard.

Recently, there has been increasing interest and progress in lowering the exponential running time of algorithms that solve NP-hard optimization problems, like MIN-VC, exactly. The theory of parametrized computation and fixed parameter tractability is a newly developed approach dealing with exact algorithms for such intractable problems. Many hard problems can be associated with a parameter in such a way that the problems are tractable when the parameter is fixed or varies within a small range. Such parametrized problems are now known as fixed parameter tractable (FPT) [18]. The parametrized version of the VERTEX COVER problem is a well known FPT problem and has received considerable interest:

PARAMETRIZED WEIGHTED VERTEX COVER

*Parameter:*  $k > 0$  is a fixed constant

*Instance:* A graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ .

*Question (decision version):* Does  $G$  have a vertex cover of weight at most  $k$ ?

*Solution (search version):* A vertex cover  $C$  in  $G$  of weight at most  $k$ , or a report that no such vertex cover exists.

Very important methods employed in the development of algorithms (both, exact and approximation) are reductions to the problem kernel. These are efficient transformations (referred to as *kernelization*) that reduce input instances to instances of smaller size and special structure. For example, in parametrized version of the MIN-VC problem they reduce in conjunction or independently both, the graph size and the parameter size. After applying a polynomial time reduction to problem kernel as a preprocessing step, either the branch-and-search process based on bounded search trees can be applied to design an *exact* algorithm (running in polynomial time if the parameter is fixed) or, in nonparametrized version, the special structure of the problem kernel allows to design a simple polynomial time approximation algorithm.

The following reduction is based on a simple local sufficient condition of optimality for the MIN- $w$ -VC problem, that was first mentioned by Nemhauser and Trotter in [31].

### ***Commitment Reduction***

Consider a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ . For  $U \subseteq V$ , let  $N(U)$  denote the set of its neighbors in  $G$ ,  $N(U) := \{v \in V : \exists u \in U \text{ such that } \{u, v\} \in E\}$ , and  $G[U]$  be the subgraph of  $G$  induced by  $U$ . A commitment structure in  $(G, w)$  is an ordered pair  $(I, N(I))$  of subsets of  $V$  such that

- (1)  $I$  is a nonempty independent set in  $G$ , and
- (2)  $N(I)$  is a minimum vertex cover for  $(G[I \cup N(I)], w)$ .

The importance of being able to identify a commitment structure  $(I, N(I))$  in  $(G, w)$  is contained in the following observation: each set of the form  $C := N(I) \cup C'$ , where  $C'$  is a minimum weighted vertex cover for  $(G[V \setminus (I \cup N(I))], w)$ , is a minimum weighted vertex cover for  $(G, w)$ . If a commitment structure  $(I, N(I))$  in  $(G, w)$  is identified, we can apply the *commitment reduction* to the MIN- $w$ -VC problem for  $(G, w)$ . That means, we commit ourselves to solutions that intersect  $I \cup N(I)$  exactly in  $N(I)$ , remove  $I \cup N(I)$  from  $G$ , and thus reduce the problem to the smaller induced subgraph  $(G[V \setminus (I \cup N(I))], w)$  of  $(G, w)$ . This procedure can be repeated while we are able to identify a commitment structure in the smaller graph.

Obviously, a commitment structure  $(I, N(I))$  in  $(G, w)$  always exists. In particular, any maximum independent set  $I$  for  $(G, w)$  determines such a structure. Unfortunately, the

problem of finding such a structure is NP-hard in general. To use this kind of reductions in a computationally efficient way, we have to confine ourselves to more restricted particular cases of commitment structures that can be found in polynomial time. For example, in [4] the restriction is put on the size of commitment structures that authors are looking for. In this paper we are rather focused on *structural* restrictions, under which the problem to find commitment structures, is polynomially solvable.

Historically, in kernelization techniques for MIN- $w$ -VC the role of the Linear Programming (LP) relaxation of the problem has been crucial. To explain how this relaxation can help to find a commitment structure in  $(G, w)$ , we start with the Integer Programming (IP) formulation of the MIN- $w$ -VC problem:

**IP formulation of Minimum Weighted Vertex Cover**

*Instance:* A graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ .

*Feasible solution:* A function  $x : V \rightarrow \{0, 1\}$  satisfying *edge constraints*  $x(u) + x(v) \geq 1$  for each edge  $\{u, v\} \in E$ .

*Goal:* To *minimize*  $w(x) := \sum_{u \in V} w(u) \cdot x(u)$  over all feasible solutions  $x$ .

There is a one-to-one correspondence between the set of vertex covers for  $G$  and the set of functions  $x : V \rightarrow \{0, 1\}$  satisfying all edge constraints; each such  $x$  is an indicator function of some vertex cover of  $G$ . The Linear Programming (LP) relaxation of the IP-formulation allows  $x(u) \in [0, 1]$  (or even  $x(u) \geq 0$ ). It is well known that there are polynomial time algorithms solving the LP-relaxation of this problem. They are generally based upon the fact that the relaxed problem is the dual of a particularly simple form of a matching problem.

An interesting underlying structure of the LP-relaxed problem is that any extreme point of the feasible region (and hence any extreme point of the solution region) is half-integral ([30]). Thus, instead of the LP-relaxation it is enough to study solutions of the Half-Integral (HI) relaxation of the MIN- $w$ -VC problem, that allows variables  $x(u) \in \{0, \frac{1}{2}, 1\}$  only.

**HI-relaxation of Minimum Weighted Vertex Cover**

*Instance:* A graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ .

*Feasible solution:* A *half-integral vertex cover* for  $G$ , i.e., a function  $x : V \rightarrow \{0, \frac{1}{2}, 1\}$  satisfying *edge constraints*  $x(u) + x(v) \geq 1$  for each edge  $\{u, v\} \in E$ .

*Goal:* To *minimize*  $w(x) := \sum_{u \in V} w(u) \cdot x(u)$  over all feasible solutions  $x$ .

An interesting and useful property of this HI-relaxation is that it can be solved faster than general linear programs. Namely, it can be reduced to the MIN- $w$ -VC problem in bipartite graphs, and such problems can be solved as the MAXIMUM FLOW problems. In the unweighted case the problem is essentially of the same complexity as the MAXIMUM MATCHING problem in bipartite graphs.

**Nemhauser-Trotter (NT) Reduction.** *A special case of the Commitment reduction, in which a commitment structure  $(I, N(I))$  in  $(G, w)$  is identified using a minimum half-integral vertex cover  $x : V \rightarrow \{0, \frac{1}{2}, 1\}$  for  $(G, w)$  distinct from  $x \equiv \frac{1}{2}$ , is called the Nemhauser-Trotter reduction. Setting  $V_i^x := \{u \in V : x(u) = i\}$  for each  $i \in \{0, \frac{1}{2}, 1\}$ , we observe that  $V_0^x$  is an independent set which is nonempty if  $x \not\equiv \frac{1}{2}$ , and that  $V_1^x = N(V_0^x)$  holds. The well known result of Nemhauser and Trotter states that there is a minimum vertex cover for  $(G, w)$  that contains all the vertices in  $V_1^x$  and none of the vertices in  $V_0^x$ . More precisely, their proof shows that  $V_1^x (= N(V_0^x))$  is a minimum vertex cover for  $(G[V_0^x \cup N(V_0^x)], w)$ , and hence  $(V_0^x, N(V_0^x))$  is a commitment structure in  $(G, w)$ , assuming  $x \not\equiv \frac{1}{2}$ . This special case of a commitment structure is present iff  $(G, w)$  admits a minimum half-integral vertex cover  $x$  distinct from  $x \equiv \frac{1}{2}$ . Moreover, it can be found in polynomial time ([31, p. 238]).*

In our search for a minimum vertex cover for  $(G, w)$  we now commit ourselves to solutions that intersect  $V_0^x \cup V_1^x$  exactly in  $V_1^x$ , remove  $V_0^x \cup V_1^x$  from  $G$ , and reduce the problem to the graph  $(G[V_{\frac{1}{2}}^x], w)$ . If a minimum half-integral vertex cover  $x$  is such that size of  $V_{\frac{1}{2}}^x$  is minimal among sets  $V_{\frac{1}{2}}^y$ , where  $y$  ranges over all minimum half-integral vertex covers for  $(G, w)$ , then  $(G[V_{\frac{1}{2}}^x], w)$  cannot be reduced yet in such a way. This problem kernel has the property that for every nonempty independent set  $I$  in it  $w(N(I)) > w(I)$  holds.

**Crown reductions: the unweighted case.** The notion of a reduction for the (UNWEIGHTED) VERTEX COVER problem that is based on the identification of a *crown structure* in a graph, was introduced by Chor, Fellows, and Juedes [14]. A *crown in a graph*  $G = (V, E)$  is a nonempty independent set  $I$  of vertices in  $G$  such that there is a matching of  $N(I)$  into  $I$ . If  $I$  is a crown in  $G$ , then a crown structure  $(I, N(I))$  is a special case of a commitment structure, as clearly  $N(I)$  is a minimum vertex cover in  $G[I \cup N(I)]$ . The corresponding commitment reduction is the so-called crown reduction.

The crown reductions were widely publicized in survey talks by M. Fellows (for example, [20]), and evaluated experimentally in the work by Abu-Khzam et al. [1]. This reduction method was studied also in [2], [16], [22], [29], [34], and [35]. It has turned out to be useful for a wide range of parametrized problems.

In [20] several questions regarding crown reductions were left open. In particular, the problem of finding a crown in a graph (if there exists one) in polynomial time. In that paper the crown reduction method is thought to be orthogonal to the one based on the Nemhauser-Trotter reduction. The connections of crown reductions to the older results on the LP-relaxation of the MINIMUM VERTEX COVER problem seem to be overlooked.

Firstly, the crown reduction method is not orthogonal to the Nemhauser-Trotter reduction, but it refines that. The pair  $(V_0^x, V_1^x)$ , identified by a minimum half-integral vertex cover  $x \not\equiv \frac{1}{2}$  in  $G$ , is always a crown structure. It follows immediately from the proof of Nemhauser-Trotter theorem ([31, p. 236] which shows that  $|N(J) \cap V_0^x| \geq |J|$  for every  $J \subseteq V_1^x$ , hence  $V_1^x$  is matched into  $V_0^x$  due to Hall's theorem (see, e.g. [23]).

Secondly, a crown in a graph (if there exists one) can be found in polynomial time. A graph  $G = (V, E)$  contains a crown if and only if there is a nonempty independent set  $I$  in it such that  $|N(I)| \leq |I|$ . (Not every such  $I$  but, for example, every inclusionwise minimal nonempty independent set  $I$  verifying this inequality is a crown, as easily follows using Hall's theorem.) This is known to be equivalent to the existence of a minimum half-integral vertex cover  $x$  in  $G$  that is distinct from  $x \equiv \frac{1}{2}$  ([31, p. 239], or [36, p. 95]). Moreover, one can find such  $x$  (if it exists) by a polynomial time algorithm based on bipartite maximum matchings. An algorithm given in [36, p. 98] provides such  $x$  with the number of vertices assigned  $\frac{1}{2}$  minimal possible. This in turn implies that an independent set  $V_0^x$  obtained this way is a crown in  $G$  which is optimal in a sense that the problem kernel  $G[V_{\frac{1}{2}}^x]$  contains no crown.

These connections were first mentioned in [13]. In the current paper we explain in detail the close relation of crown decompositions to the structure of solutions of the LP-relaxed problem. We introduce also a stronger notion of decompositions which are suitable for finding all minimum vertex covers, or a minimum vertex cover under certain side constraints. We demonstrate how results obtained about problem kernels can be used in applications: not only the size of the kernel, but also its special structure can be very useful.

**Crown reductions: the weighted case.** There is a natural way how the notion of a crown structure can be generalized to weighted graphs. Let a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  be given. A *crown* in  $(G, w)$  is a nonempty independent set  $I$  of  $G$

such that  $w(N(U) \cap I) \geq w(U)$  for every  $U \subseteq N(I)$ . This condition in the unweighted case ( $w \equiv 1$ ) is equivalent, due to Hall's theorem, that  $N(I)$  is matched into  $I$ . Thus this is really a generalization of a crown from unweighted version. A *crown structure*  $(I, N(I))$  is a special case of a commitment structure, i.e.,  $N(I)$  is a minimum vertex cover in  $(G[I \cup N(I)], w)$ . If a crown  $I$  in  $(G, w)$  is identified, the ordered triple  $(I, H, K)$  where  $H = N(I)$  (the *head* of the crown  $I$ ), and  $K = V \setminus (I \cup H)$  (the rest, or the kernel), is called a *crown decomposition*. The so-called *crown reduction*, that reduces the problem MIN- $w$ -VC for  $(G, w)$  to the one for  $(G[K], w)$  is again a special case of a commitment reduction.

Most of results about the crown reduction and its relation to the Nemhauser-Trotter reduction mentioned for the unweighted version can be proved for the weighted one as well. For example, there is a polynomial time algorithm that recognizes if  $(G, w)$  contains a crown, and if yes, finds a crown  $I$  such that graph  $(G[V \setminus (I \cup N(I))], w)$  obtained after reduction contains no crown. As the crown reduction method turns out to be a refinement of Nemhauser-Trotter reduction, one could hope that repetitively applied crown reductions provide in general smaller kernels than those relying on Nemhauser-Trotter reductions. However, this is not the case. We show that these two methods are basically equivalent, as they yield the same problem kernels. Applying either of the methods to a graph  $(G, w)$  repetitively while it can make progress and reduce the graph, we will end with the same problem kernel  $(G[K_{\min}], w)$ , where  $K_{\min}$  is the set of all vertices  $v \in V$  such that  $x(v) = \frac{1}{2}$  for all optimal solutions  $x$  of the relaxed (either HI- or LP-) problem.

Commitment reductions discussed above are suitable for the problem to find *one* minimum vertex cover for  $(G, w)$ . After identifying a commitment structure  $(I, N(I))$  in  $(G, w)$ , we know that there are minimum vertex covers for  $(G, w)$  that intersect  $I \cup N(I)$  exactly in  $N(I)$ , and only such minimum vertex covers are considered. However, this approach can be hardly used if the problem is to find (resp., to count) *all* minimum vertex covers, or to find one minimum vertex cover for  $(G, w)$  *under some additional constraints*. For such problems stronger reductions have to be introduced.

### ***Strong Commitment Reduction***

This is a particular case of the commitment reduction with the notion of a commitment structure strengthened. A commitment structure  $(I, H(I))$  in  $(G, w)$  is a *strong commitment structure*, if  $N(I)$  is the *only* minimum vertex cover for  $(G[I \cup N(I)], w)$ . It is easy to observe that then minimum vertex covers for  $(G, w)$  are *exactly* the sets  $C := N(I) \cup C'$ , where  $C'$  is a minimum vertex cover for  $(G[V \setminus (I \cup N(I))], w)$ . Unfortunately, in this generality it is NP-hard to decide if  $(G, w)$  contains a strong commitment structure. But the following particular case of *strong commitment structures* can be used as computationally efficient way of kernelization.

**Strong Crown Reduction.** A *strong crown* in a weighted graph  $(G, w)$  is a nonempty independent set  $I$  of  $G$  such that  $w(N(U) \cap I) > w(U)$  for every nonempty  $U \subseteq N(I)$ . If  $I$  is a *strong crown* in  $(G, w)$ , then  $N(I)$  is the only minimum vertex cover for  $(G[I \cup N(I)], w)$ . The strong crown decomposition and the strong crown reduction are defined in an obvious way.

It turns out that graphs containing a strong crown can be recognized efficiently. For such a graph  $(G, w)$  there is a unique strong crown decomposition  $(I, H, K)$  such that  $(G[K], w)$  contains no strong crowns. Moreover, this decomposition can be computed in polynomial time. Recall that minimum vertex covers for  $(G, w)$  are exactly the sets of the form  $C := H \cup C'$ , where  $C'$  is a minimum vertex cover for  $(G[K], w)$ . The corresponding

problem kernel  $K$  can be obtained also using Nemhauser-Trotter reductions only. Namely, there is a minimum half-integral vertex cover  $x$  for  $(G, w)$  such that  $I = V_0^x$ ,  $H = V_1^x$ , and  $K = V_{\frac{1}{2}}^x$ . Moreover,  $K$  equals  $K_{\max}(G, w)$ , the union of all sets  $V_{\frac{1}{2}}^y$ , where  $y$  ranges over all minimum half-integral vertex covers for  $(G, w)$ . These results about Nemhauser-Trotter reductions are closely related to those given in [24].

In the unweighted case the problem of determining the set  $V_0(G)$  of vertices belonging to *no* minimum vertex cover, and the set  $V_1(G)$  of vertices belonging to *all* minimum vertex covers, are also well studied NP-hard problems. Our results improve also those of [5] on the lower bound of  $|V_0(G)|$ . If  $G$  contains a strong crown, then the above mentioned strong crown decomposition  $(I, H, K)$  provides efficiently computable subsets  $I \subseteq V_0(G)$  and  $H \subseteq V_1(G)$ .

It has been noticed in [5] that if  $\mathcal{F}$  is a hereditary (i.e., induced subgraph closed) family of graphs for which computing the cardinality  $vc(G)$  of a minimum vertex cover for  $G$  in  $\mathcal{F}$  is polynomial, then  $V_0(G)$  and  $V_1(G)$  can be computed efficiently for  $G \in \mathcal{F}$ . The question has been raised how to find  $V_0(G)$  and/or  $V_1(G)$  efficiently for some other classes of graphs. Our results contribute to this theory as well, proving that  $V_0(G, w)$  and  $V_1(G, w)$  can be computed efficiently for König-Egerváry graphs  $(G, w)$ . Recall that a weighted graph  $(G, w)$  is a König-Egerváry graph (KEG) for which the LP-relaxed (equivalently, HI-relaxed) problem has an integer optimum solution. (For unweighted graphs these are the graphs for which the size of a maximum matching equals the size of a minimum vertex cover.) Let us note that the set of KEG graphs is *not* a hereditary family. Our Theorem 10 in combination with Theorem 6 provides a decomposition of a weighted graph  $(G, w)$  into “irreducible parts”. It describes how *all* minimum vertex covers for  $(G, w)$  are structured in the König-Egerváry part  $(G[V \setminus K_{\min}], w)$  of  $(G, w)$ .

In Section 5 we show how reductions studied in this paper provide for some specific classes of graphs efficient approximation algorithm for the MIN-VC problem with a constant approximation factor  $r < 2$ . We show, for example, that for the matching number  $\nu(G)$  and the fractional matching number  $\nu^*(G)$  it holds  $\nu(G) \geq \frac{2}{3}\nu^*(G)$ , and for the class  $G_\delta := \{G : \nu(G) \leq (1 - \delta)\nu^*(G)\}$  of graphs (for any fixed constant  $\delta \in [0, \frac{1}{3}]$ ) we describe a simple  $2\frac{1-\delta}{1+\delta}$ -approximation algorithm. On the other hand, we conclude that for the graphs with perfect matchings, or for regular graphs, or for Hamiltonian connected graphs, the MIN-VC problem is as hard to approximate as for general graphs.

In Section 6 we demonstrate how strong crown reductions (or, strong NT-reductions) can be used for fixed-parameter tractable problems related to the MIN-VC problem (unweighted, for simplicity). It can be used as an efficient reduction to find (or to count) *all* minimum vertex covers in  $G$ , or to find one minimum vertex cover in  $G$  *under some additional constraints*.

The paper is essentially self-contained and in some cases it contains also new proofs of some previously known results about the minimum half-integral vertex covers. We include them because we believe that our approach provides a better insight.

**Preliminaries.** Let  $G = (V, E)$  be a graph with vertex weights  $w : V \rightarrow (0, \infty)$ . For a set of vertices  $U \subseteq V$ , let  $N(U) := \{v \in V : \exists u \in U \text{ such that } \{u, v\} \in E\}$  stand for the set of its neighbors, and  $G[U]$  denote the subgraph of  $G$  induced by  $U$ . The weight of a vertex subset  $U \subseteq V$  is defined by  $w(U) := \sum_{u \in U} w(u)$ .

Let  $VC(G, w)$  be the set of all minimum vertex covers for  $(G, w)$  and  $vc(G, w)$  stand for the weight of the minimum vertex cover for  $(G, w)$ . In the unweighted case we use shorter  $VC(G)$  and  $vc(G)$ .

Let  $VC^*(G, w)$  be the set of all minimum half-integral vertex covers  $x : V \rightarrow \{0, \frac{1}{2}, 1\}$ , and  $vc^*(G, w)$  stand for the weight of a minimum half-integral vertex cover for  $(G, w)$ . For a minimum half-integral vertex cover  $x$  for  $(G, w)$ , we denote  $V_i^x := \{u \in V : x(u) = i\}$  for each  $i \in \{0, \frac{1}{2}, 1\}$ .

Clearly,  $vc^*(G, w) \leq vc(G, w)$ , as for any vertex cover  $C$  its indicator function  $x^C$  is a feasible solution for the HI-relaxed problem with  $w(x^C) = w(C)$ . Further,  $vc^*(G, w) \leq \frac{1}{2}w(V)$ , as the function  $x \equiv \frac{1}{2}$  on  $V$  is always feasible solution for the HI-relaxation. A weighted graph  $(G, w)$  will be called a *König-Egerváry graph* (see, e.g. [27] and [6]), shortly KEG, if  $vc(G, w) = vc^*(G, w)$ .

MAXIMUM FRACTIONAL  $w$ -MATCHING (MAX- $w$ -FM)

*Instance:* A simple graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ .

*Feasible solution:* A fractional  $w$ -matching  $\lambda : E \rightarrow [0, \infty)$  such that  $\sum_{u \in N(v)} \lambda(\{u, v\}) \leq w(v)$  for every  $v \in V$ .

*Objective function:* The sum  $\lambda(E) := \sum_{\{u, v\} \in E} \lambda(\{u, v\})$  of the fractional  $w$ -matching.

Let  $\nu^*(G, w)$  denote the value of a maximum fractional  $w$ -matching for  $(G, w)$ . The MAXIMUM FRACTIONAL  $w$ -MATCHING problem is precisely the dual linear program of the LP relaxation of MIN- $w$ -VC for  $(G, w)$ , hence  $\nu^*(G, w) = vc^*(G, w) (\leq vc(G, w))$ .

## 2 Kernelization by minimum half-integral vertex covers

It is well known (see [31]) that the problem of finding a minimum half-integral vertex cover for a weighted graph  $(G, w)$  can be reduced to the problem of finding a minimum vertex cover in a related weighted bipartite graph  $(G^b, w^b)$ , defined by the following construction.

**Definition 1.** For a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  we define the weighted bipartite graph  $(G^b, w^b)$  with  $G^b = (V^b, E^b)$ , as follows: there are two copies  $u^L$  and  $u^R$  of each vertex  $u \in V$  of the same weight  $w^b(u^L) = w^b(u^R) = w(u)$  in  $(G^b, w^b)$ ,  $V^L := \{u^L : u \in V\}$ ,  $V^R := \{u^R : u \in V\}$ , and  $V^b := V^L \cup V^R$ . Each edge  $\{u, v\} \in E$  of  $G$  creates two edges in  $G^b$ , namely  $\{u^L, v^R\}$  and  $\{v^L, u^R\}$ . Hence  $E^b := \{\{u^L, v^R\}, \{v^L, u^R\} : \{u, v\} \in E\}$ . For  $U \subseteq V$  we use also  $U^L$ ,  $U^R$ , and  $U^b := U^L \cup U^R$  for the corresponding sets of vertices.

For any set  $C \subseteq V^L \cup V^R$  we associate a map  $x_C : V \rightarrow \{0, \frac{1}{2}, 1\}$  in the following way:  $x_C(u) = \frac{1}{2}|C \cap \{u^L, u^R\}|$  for any  $u \in V$ . Clearly,  $w(x_C) = \frac{1}{2}w^b(C)$  for any  $C \subseteq V^L \cup V^R$ .

**Lemma 1.** ([31]) *The mapping  $C \mapsto x_C$  maps the set of vertex covers in  $(G^b, w^b)$  onto the set of half-integral vertex covers in  $(G, w)$ . Moreover, it maps  $VC(G^b, w^b)$  onto  $VC^*(G, w)$ . Consequently,  $vc^*(G, w) = \frac{1}{2}vc(G^b, w^b)$ .*

*Remark 1.* In bipartite graphs the MINIMUM (WEIGHTED) VERTEX COVER problem can be solved in polynomial time. The optimal solution for MIN- $w$ -VC can be identified from the solution of the corresponding MINIMUM CUT problem, that can be found by efficient algorithms for the MAXIMUM FLOW problem on bipartite graphs (see Lawler [26]). For instance, the problem is solvable in time  $O(|E||V| \log \frac{|V|^2}{|E|})$  using Goldberg and Tarjan's algorithm [21]. When the problem is unweighted, Dinic's algorithm for the MAXIMUM FLOW problem runs in  $O(|E|\sqrt{|V|})$  time. Another approach in the unweighted case is based on the bipartite graph matching theory. A maximum matching of a bipartite graph can be constructed in time  $O(|E|\sqrt{|V|})$  by the algorithm of Hopcroft and Karp ([25]) (or even for general graphs by the algorithm of Micali and Varizani), and a minimum vertex cover for a bipartite graph can be constructed from a maximum matching in time  $O(|E|)$ .

As it follows from Lemma 1 and from efficient solvability of MIN- $w$ -VC in bipartite graphs,  $vc^*(G, w)$  for a graph  $(G, w)$  can be computed efficiently.

**Definition 2.** Let  $G = (V, E)$  be a graph with vertex weights  $w : V \rightarrow (0, \infty)$ . Denote by  $V_0(G, w)$  the set of vertices avoided by each minimum vertex cover for  $(G, w)$  and  $V_1(G, w)$  the set of vertices contained in each minimum vertex cover for  $(G, w)$ . Similarly, for  $i \in \{0, 1\}$ , denote by  $V_i^*(G, w)$  the set of vertices with value  $i$  in each minimum half-integral vertex cover for  $(G, w)$ .

*Remark 2.* The mapping that maps every  $u^L$  on  $u^R$  and vice versa, simultaneously for each  $u \in V$ , is an automorphism of  $(G, w)$ . For a fixed  $u \in V$  we obtain  $u^L \in V_0(G^b, w^b)$  iff  $u^R \in V_0(G^b, w^b)$  iff  $u \in V_0^*(G, w)$  (using Lemma 1);  $u^L \in V_1(G^b, w^b)$  iff  $u^R \in V_1(G^b, w^b)$  iff  $u \in V_1^*(G, w)$ . In other words, for each  $i \in \{0, 1\}$ ,  $V_i(G^b, w^b)$  is the union of sets  $\{u^L, u^R\}$  over all  $u \in V_i^*(G, w)$ .

*Remark 3.* If  $G = (V, E)$  is a bipartite graph with bipartition  $V = A \cup B$  and with vertex weights  $w : V \rightarrow (0, \infty)$ , then  $(G^b, w^b)$  consists of two disjoint copies of  $(G, w)$ , namely  $(G^b[A^L \cup B^R], w^b)$  and  $(G^b[A^R \cup B^L], w^b)$ . Therefore  $vc(G^b, w^b) = 2vc(G, w)$ , and  $vc^*(G, w) = vc(G, w)$  by Lemma 1. Moreover,  $u \in V_0(G, w)$  iff  $u^L, u^R \in V_0(G^b, w^b)$  iff  $u \in V_0^*(G, w)$ , hence  $V_0(G, w) = V_0^*(G, w)$ . In the same way we get  $V_1(G, w) = V_1^*(G, w)$ .

**Definition 3.** A minimum half-integral vertex cover  $x$  with the property that no  $y \in VC^*(G, w)$  satisfies both,  $V_0^y \subsetneq V_0^x$  and  $V_1^y \subsetneq V_1^x$ , is called a pivot.

The existence of a pivot is clear. For example,  $x \in VC^*(G, w)$  such that  $V_0^x$  is inclusion-wise minimal among sets  $\{V_0^y : y \in VC^*(G, w)\}$ , is a pivot. We will prove later that there is only one pivot in  $VC^*(G, w)$  and it can be found efficiently. The following lemma describes basic properties of decompositions generated by minimum half-integral vertex covers.

**Lemma 2.** Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  and a partition  $V = V_0^x \cup V_1^x \cup V_{\frac{1}{2}}^x$  according to a fixed minimum half-integral vertex cover  $x$  for  $(G, w)$ . Then the following statements hold:

- (i)  $V_0^x$  is an independent set, every isolated vertex of  $G$  (if any) belongs to  $V_0^x$ , and  $N(V_0^x) = V_1^x$ .
- (ii)  $vc^*(G[V_{\frac{1}{2}}^x], w) = \frac{1}{2}w(V_{\frac{1}{2}}^x)$
- (iii) For each  $U \subseteq V_1^x$ ,  $w(N(U) \cap V_0^x) \geq w(U)$ . If  $x$  is a pivot, then  $\emptyset \neq U \subseteq V_1^x$  implies  $w(N(U) \cap V_0^x) > w(U)$ .

*Proof.* (i) As  $x$  satisfies all edge constraints, there are no edges within  $V_0^x$  or between  $V_0^x$  and  $V_{\frac{1}{2}}^x$ . Hence  $V_0^x$  is an independent set, every isolated vertex of  $G$  (if any) belongs to  $V_0^x$ , and  $N(V_0^x) \subseteq V_1^x$ . If there is  $u \in V_1^x \setminus N(V_0^x)$ , then changing  $x$  on  $u$  to  $\frac{1}{2}$  results in a half-integral vertex cover  $\tilde{x}$  with  $w(\tilde{x}) < w(x)$ , a contradiction with minimality of  $x$ .

(ii) If there is  $y \in VC^*(G[V_{\frac{1}{2}}^x], w)$  with  $w(y) < \frac{1}{2}w(V_{\frac{1}{2}}^x)$ , then changing  $x$  on  $V_{\frac{1}{2}}^x$  to  $y$  implies a half integral vertex cover  $\tilde{x}$  for  $(G, w)$  with  $w(\tilde{x}) < w(x)$ , a contradiction.

(iii) Let us consider a fixed nonempty set  $U \subseteq V_1^x$ . We define a new half-integral vertex cover  $y$  changing  $x$  on  $U \cup (N(U) \cap V_0^x)$  to  $\frac{1}{2}$ . Clearly,  $w(y) = w(x) + \frac{1}{2}(w(N(U) \cap V_0^x) - w(U)) \geq w(x)$ , as  $x \in VC^*(G, w)$ . Hence  $w(N(U) \cap V_0^x) \geq w(U)$  and the equality occurs iff  $y \in VC^*(G, w)$  as well. Obviously,  $V_0^y \subsetneq V_0^x$  and  $V_1^y \subsetneq V_1^x$ . Therefore such  $y$  doesn't exist if  $x$  is a pivot and necessarily  $w(N(U) \cap V_0^x) > w(U)$  in that case.  $\square$

Given  $x \in VC^*(G, w)$ , from Lemma 2 it easily follows that  $V_1^x$  is a minimum vertex cover for  $(G[V_0^x \cup V_1^x], w)$ , and it is the only minimum vertex cover for  $(G[V_0^x \cup V_1^x], w)$ , if  $x$  is a pivot. From that Nemhauser-Trotter theorem [31] follows: each set  $C$  of the form  $C = V_1^x \cup C'$ , where  $C'$  is a minimum vertex cover for  $(G[V_{\frac{1}{2}}^x], w)$ , is a minimum vertex cover for  $(G, w)$ . If  $x$  is a pivot, we can conclude that all minimum vertex covers for  $(G, w)$  are of that form.

### 3 Kernelization by crown reductions

A new kernelization technique, called crown reduction, has been introduced in [1], [20] for the unweighted MIN-VC problem. In this section we study crown reductions and their properties in the more general case of vertex weighted graphs. We describe the relation of crown reductions to the LP-relaxation of the MIN- $w$ -VC problem and to the NT-reductions.

**Definition 4.** For an independent set  $I$  in  $G$  let  $G[I, N(I)]$  denote the bipartite graph obtained from  $G[I \cup N(I)]$  removing all edges within  $N(I)$  (if any).

We define a special version of commitment reduction (resp., strong commitment reduction) when the assumption on  $N(I)$  to be a minimum vertex cover (resp., the only minimum vertex cover) for  $(G[I \cup N(I)], w)$  is strengthened to be true even for its bipartite subgraph  $(G[I, N(I)], w)$ .

**Definition 5.** Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ . A crown (resp., a strong crown) in  $(G, w)$  is a nonempty independent set  $I$  of  $G$  such that  $w(N(U) \cap I) \geq w(U)$  (resp.,  $w(N(U) \cap I) > w(U)$ ) holds for every nonempty set  $U \subseteq N(I)$ .

If  $I$  is a crown (resp., a strong crown) in  $(G, w)$  then the ordered triple  $(I, H, K)$ , where  $H = N(I)$  (the head of the crown  $I$ ) and  $K = V \setminus (I \cup H)$  (the rest), is called a crown decomposition (resp., a strong crown decomposition) of  $(G, w)$ . All such crown decompositions (resp., strong crown decompositions) are called nontrivial, and the triple  $(\emptyset, \emptyset, V)$  is called a trivial crown decomposition (resp., strong crown decomposition).

In the unweighted case the condition for a crown is equivalent, due to Hall's theorem, that  $N(I)$  is matched into  $I$ . This is exactly the way how the crown reduction have been introduced in [1], [20].

Lemma 2 yields that for any  $x \in VC^*(G, w)$  the triple  $(V_0^x, V_1^x, V_{\frac{1}{2}}^x)$  is a crown decomposition of  $(G, w)$  which is trivial only if  $x \equiv \frac{1}{2}$ . (Especially, every Nemhauser-Trotter reduction is also a crown reduction.)  $(V_0^x, V_1^x, V_{\frac{1}{2}}^x)$  is even a strong crown decomposition, if  $x$  is a pivot. Such conclusions are proved in Lemmas 3 and 4 for general crown decompositions.

**Lemma 3.** Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ . For every nonempty independent set  $I$  in  $G$  the following conditions (i)–(iv) are equivalent and any of them implies that  $N(I) \in VC(G[I \cup N(I)], w)$ .

- (i)  $I$  is a crown in  $(G, w)$ ,
- (ii)  $N(I) \in VC(G[I, N(I)], w)$ ,
- (iii)  $x \in VC^*(G[I, N(I)], w)$  for  $x$  defined by  $x|_I \equiv 0$ ,  $x|_{N(I)} \equiv 1$ ,
- (iv)  $x \in VC^*(G[I \cup N(I)], w)$  for  $x$  defined by  $x|_I \equiv 0$ ,  $x|_{N(I)} \equiv 1$ .

*Proof.* (i) $\Rightarrow$ (ii): The set  $N(I)$  is a vertex cover in  $G[I, N(I)]$ . To prove its optimality, take another vertex cover  $C$  in  $G[I, N(I)]$  and show that  $w(N(I)) \leq w(C)$  as follows. Let  $U := N(I) \setminus C$ , then clearly  $N(U) \cap I \subseteq C \cap I$  and using (i) we get  $w(C \cap I) \geq w(N(U) \cap I) \geq w(U)$ .

As  $N(I) \cup (C \cap I) = C \cup U$  and both unions are disjoint,  $w(N(I)) + w(C \cap I) = w(C) + w(U)$ . That combined with the previous gives  $w(N(I)) \leq w(C)$ .

(ii) $\Rightarrow$ (iii): If  $x$  is defined as in (iii), then it is a half-integral vertex cover for  $G[I, N(I)]$  and  $w(x) = w(N(I))$ . Hence it suffices to prove that  $vc^*(G[I, N(I)], w) = w(N(I))$ . As  $G[I, N(I)]$  is bipartite,  $vc^*(G[I, N(I)], w) = vc(G[I, N(I)], w)$  (Remark 3), and this is equal to  $w(N(I))$ , by (ii).

(iii) $\Rightarrow$ (iv): As  $x$  is a half-integral vertex cover for  $(G[I \cup N(I)], w)$  that is optimal even in its subgraph  $G[I, N(I)]$ , it has to be in  $VC^*(G[I \cup N(I)], w)$ .

(iv) $\Rightarrow$ (i): It follows from Lemma 3(iii) applied to the graph  $(G[I \cup N(I)], w)$ .

Hence (i)–(iv) are equivalent. Moreover,  $N(I) \in VC(G[I \cup N(I)], w)$  is trivially implied by any of them.  $\square$

**Lemma 4.** *Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ . For every nonempty independent set  $I$  in  $G$  the following conditions (i)–(iv) are equivalent and any of them implies that  $N(I)$  is the only element of  $VC(G[I \cup N(I)], w)$ .*

- (i)  $I$  is a strong crown in  $(G, w)$ ,
- (ii) the only element of  $VC(G[I, N(I)], w)$  is  $N(I)$ ,
- (iii) the only element of  $VC^*(G[I, N(I)], w)$  is  $x$  defined by  $x|_I \equiv 0$ ,  $x|_{N(I)} \equiv 1$ ,
- (iv) the only element of  $VC^*(G[I \cup N(I)], w)$  is  $x$  defined by  $x|_I \equiv 0$ ,  $x|_{N(I)} \equiv 1$ .

*Proof.* In the following chain of implications we don't repeat arguments that are the same as in Lemma 3.

(i) $\Rightarrow$ (ii):  $N(I) \in VC(G[I, N(I)], w)$  is clear by Lemma 3. To prove the uniqueness, take any vertex cover  $C$  in  $G[I, N(I)]$  and prove that either  $C = N(I)$ , or  $w(C) > w(N(I))$  as follows. The case  $N(I) \subseteq C$  being clear, so we can assume that  $U := N(I) \setminus C \neq \emptyset$ . Now we can argue as in the proof in Lemma 3, but we have now the strict inequality  $w(C \cap I) \geq w(N(U) \cap I)w(U)$  assuming (i). It results in the strict inequality  $w(N(I)) < w(C)$ .

(ii) $\Rightarrow$ (iii): Again,  $x \in VC^*(G[I, N(I)], w)$  is clear. To prove the uniqueness, take any  $y \in VC^*(G[I, N(I)], w)$  and prove that  $y = x$  as follows. By Lemma 1,  $y$  is determined by some  $C \in VC(G[I, N(I)]^b, w^b)$  for which  $y(u) = \frac{1}{2}|\{u^L, u^R\} \cap C|$  for each  $u \in I \cup N(I)$ . As  $G[I, N(I)]$  is bipartite,  $(G[I, N(I)]^b, w^b)$  consists of two disjoint copies of  $(G[I, N(I)], w)$  (Remark 3). Assuming (ii),  $C$  has to choose  $N(I)^L$  from one copy and  $N(I)^R$  from the another one. Hence  $C = N(I)^L \cup N(I)^R$  and  $y = x$  follows.

(iii) $\Rightarrow$ (iv): Again,  $x \in VC^*(G[I \cup N(I)], w)$ . If  $y \in VC^*(G[I \cup N(I)], w)$ , then  $w(y) = w(x)$  and  $y$  is also a half-integral vertex cover in  $G[I, N(I)]$ . Consequently,  $y \in VC^*(G[I, N(I)], w)$ , and  $y = x$  follows by (ii).

(iv) $\Rightarrow$ (i): Consider  $U \subseteq N(I)$  and define a half-integral vertex cover  $y$  for  $(G[I \cup N(I)], w)$  changing  $x$  on  $U \cup (N(U) \cap I)$  to  $\frac{1}{2}$ . Then  $w(N(U) \cap I) - w(U) = 2(w(y) - w(x))$  that has to be positive as  $x$  was the only minimum, assuming (iv).

Hence (i)–(iv) are equivalent, and any of them implies that the only element of  $VC(G[I \cup N(I)], w)$  is  $N(I)$ .  $\square$

**Notation and terminology.** Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ . Let us denote  $K_{\max}(G, w) := \cup_{y \in VC^*(G, w)} V_{\frac{1}{2}}^y$ , and  $K_{\min}(G, w) := \cap_{y \in VC^*(G, w)} V_{\frac{1}{2}}^y$ . We use abbreviated forms  $K_{\max}$  and  $K_{\min}$  as well.

**Lemma 5.** *Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ , and a crown decomposition  $(I, H, K)$  of  $(G, w)$ .*

- (i) *Every (minimum) vertex cover for  $(G[K], w)$  together with  $H$  forms a (minimum) vertex cover for  $(G, w)$ . Every (minimum) half-integral vertex cover for  $(G[K], w)$  extended by 1 on  $H$  and by 0 on  $I$  forms a (minimum) half-integral vertex cover for  $(G, w)$ .*
- (ii) *For every minimum vertex cover  $C$  for  $(G, w)$ ,  $C \cap K$  and  $C \cap (I \cup H)$  are minimum vertex covers for  $(G[K], w)$  and  $(G[I \cup H], w)$ , respectively. For every minimum half-integral vertex cover  $y$  for  $(G, w)$ ,  $y|_K$  and  $y|_{(I \cup H)}$  are minimum half-integral vertex covers for  $(G[K], w)$  and  $(G[I \cup H], w)$ , respectively.*
- (iii) *If  $(I, H, K)$  is a strong crown decomposition then minimum vertex covers for  $(G, w)$  are exactly the sets  $C = H \cup C'$ , where  $C'$  is a minimum vertex cover for  $(G[K], w)$ . Minimum half-integral vertex covers for  $(G, w)$  are exactly the mappings  $x : V \rightarrow \{0, 1, \frac{1}{2}\}$  such that  $x|_I \equiv 0$ ,  $x|_H \equiv 1$ , and  $x|_K = x'$ , where  $x'$  is a minimum half-integral vertex cover for  $(G[K], w)$ . In particular,  $I \subseteq V_0^*(G, w) := \bigcap_{y \in VC^*(G, w)} V_0^y$ ,  $H \subseteq V_1^*(G, w) := \bigcap_{y \in VC^*(G, w)} V_1^y$ , and  $K \supseteq K_{\max}(G, w)$ .*

*Proof.* It easily follows from Lemma 3 and Lemma 4. The part (i) being trivial, let us give a sketch of proof of (ii) and (iii) for vertex covers. For half-integral vertex covers we can argue in the same way.

(ii) Consider  $C \in VC(G, w)$ . Then  $C \cap (I \cup H)$  is a vertex cover for  $(G[I \cup H], w)$ , and it is a minimum (i.e., of weight  $w(H)$ ) because otherwise  $C_1 := H \cup (C \cap K)$  is a vertex cover in  $G$  with  $w(C_1) < w(C)$ , a contradiction. Consequently,  $w(C_1) = w(C)$ . Now a vertex cover  $C \cap K = C_1 \cap K$  for  $(G[K], w)$  is a minimum, because otherwise  $C_2 := H \cup C'$  for any  $C' \in VC(G[K], w)$  is a vertex cover in  $G$  with  $w(C_2) < w(C)$ , a contradiction.

(iii) If  $I$  is a strong crown and  $C \in VC(G, w)$ , then (ii) implies that  $C \cap (I \cup H) \in VC(G[I \cup H], w)$ , hence  $C \cap (I \cup H) = H$  due to properties of a strong crown derived in Lemma 4. Clearly, any set  $C = H \cup C'$ , where  $C'$  is a (minimum) vertex cover for  $(G[K], w)$ , is a (minimum) vertex cover for  $(G, w)$ .  $\square$

The following theorem summarizes the most important properties of crown decompositions

**Theorem 1.** *Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  and  $(I, H, K)$  be a crown decomposition of  $(G, w)$ . Then*

- (i)  $vc(G[I \cup H], w) = vc^*(G[I \cup H], w) = w(H)$ ,
- (ii)  $vc(G, w) = vc(G[I \cup H], w) + vc(G[K], w)$ ,
- (iii)  $vc^*(G, w) = vc^*(G[I \cup H], w) + vc^*(G[K], w)$ ,
- (iv)  $VC^*(G[K], w)$  consists of restrictions to the set  $K$  of minimum half-integral vertex covers from  $VC^*(G, w)$ ,
- (v)  $VC(G[K], w)$  consists of intersections of  $K$  with minimum vertex covers from  $VC(G, w)$ .

*Proof.* Assertion (i) follows from Lemma 3, (ii), (iii), (iv), (v) from Lemma 5.  $\square$

### 3.1 Strong crown decompositions

To characterize efficiently graphs  $(G, w)$  that contain a strong crown we introduce a notion of *fractional deficiency*. Recall that the standard (i.e., integral and unweighted) definition of the deficiency is  $def(G) = |V(G)| - 2\nu(G)$ , where  $\nu(G)$  is the size of a maximum matching in an unweighted graph  $G$ .

**Definition 6.** Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ . The fractional deficiency of  $(G, w)$ , denoted by  $def^*(G, w)$ , is defined as  $def^*(G, w) = w(V) - 2vc^*(G, w)$ .

Clearly,  $def^*(G, w)$  can be computed in polynomial time and always  $def^*(G, w) \geq 0$ . Moreover,  $def^*(G, w) = 0$  iff  $x \equiv \frac{1}{2}$  is an element of  $VC^*(G, w)$ . For each  $x \in VC^*(G, w)$ ,  $vc^*(G, w) = w(x) = w(V_1^x) + \frac{1}{2}w(V_{\frac{1}{2}}^x)$  and hence  $def^*(G, w) = w(V_0^x) - w(V_1^x)$  follows. The following result can be viewed as the fractional version of Tutte-Berge Formula in our setting.

**Theorem 2.** For a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  the following holds:  $def^*(G, w) = \max\{w(I) - w(N(I)) : I \text{ is an independent set of } G\} = \max\{w(Z) - w(N(Z)) : Z \subseteq V\}$ . Moreover, an independent set  $I$  satisfies  $w(I) - w(N(I)) = def^*(G, w)$  if and only if  $I \in \{V_0^y : y \in VC^*(G, w)\}$ .

*Proof.* Put  $d = \max\{w(I) - w(N(I)) : I \text{ is an independent set of } G\}$ ,  $D = \max\{w(Z) - w(N(Z)) : Z \subseteq V\}$ . As it was observed above,  $w(V_0^x) - w(V_1^x) = def^*(G, w)$  holds for each  $x \in VC^*(G, w)$ . Hence  $d \geq def^*(G, w)$ .

Now assume that  $I$  is an independent set such that  $w(I) - w(N(I)) = d$  and prove that necessarily  $d = def^*(G, w)$  and  $I \in \{V_0^y : y \in VC^*(G, w)\}$ . Define  $x : V \rightarrow \{0, 1, \frac{1}{2}\}$  by  $x|_I \equiv 0$ ,  $x|_{N(I)} \equiv 1$ , and  $x|_{V \setminus (I \cup N(I))} \equiv \frac{1}{2}$ . Clearly,  $x$  is a half-integral vertex cover and  $2w(x) = w(V) - d$ . Hence  $d = w(V) - 2w(x) \leq w(V) - 2vc^*(G, w) = def^*(G, w) (\leq d)$  and both inequalities are, in fact, equalities. Thus  $def^*(G, w) = d$  and  $w(x) = vc^*(G, w)$ . That means  $x \in VC^*(G, w)$  and recall that  $I = V_0^x$ .

Now we will prove that  $d = D$ . Obviously,  $d \leq D$ . To prove  $d \geq D$  consider any  $Z \subseteq V$  such that  $w(Z) - w(N(Z)) = D$ . Let  $I$  be the set of isolated vertices of  $G[Z]$ . Clearly,  $Z \setminus I \subseteq N(Z)$ ,  $N(I) \subseteq N(Z)$ ,  $(Z \setminus I) \cap N(I) = \emptyset$ , and  $w(I) - w(N(I)) \leq d$ , hence  $w(Z) - D = w(N(Z)) \geq w(Z \setminus I) + w(N(I)) \geq w(Z \setminus I) + w(I) - d = w(Z) - d$ , and  $d \geq D$  follows.  $\square$

**Lemma 6.** Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  and a crown decomposition  $(I, H, K)$  of  $(G, w)$ . Then

- (i)  $def^*(G, w) = def^*(G[I \cup H], w) + def^*(G[K], w) = w(I) - w(H) + def^*(G[K], w)$ ,
- (ii)  $def^*(G[K], w) = 0$  if and only if  $(I, H, K) = (V_0^x, V_1^x, V_{\frac{1}{2}}^x)$  for some  $x \in VC^*(G, w)$ .

*Proof.* (i) It follows from Theorem 1(i) and (iii). (ii) By (i) we get that  $def^*(G[K], w) = 0$  iff  $w(I) - w(H) = def^*(G, w)$ . This is, due to Theorem 2, equivalent to  $I = V_0^x$  for some  $x \in VC^*(G, w)$ .  $\square$

**Theorem 3.** Let  $G = (V, E)$  be a graph with vertex weights  $w : V \rightarrow (0, \infty)$ . Then there is exactly one pivot  $x \in VC^*(G, w)$  and the corresponding strong crown decomposition  $(V_0^x, V_1^x, V_{\frac{1}{2}}^x)$  of  $(G, w)$  has the following properties:  $V_0^x = V_0^*(G, w) := \bigcap_{y \in VC^*(G, w)} V_0^y$ ,  $V_1^x = V_1^*(G, w) := \bigcap_{y \in VC^*(G, w)} V_1^y = N(V_0^x)$ , and  $V_{\frac{1}{2}}^x = K_{\max}(G, w)$ . Consequently,  $def^*(G[K_{\max}], w) = 0$  and  $def^*(G, w) = w(V_0^*(G, w)) - w(V_1^*(G, w))$ .  $V_0^*(G, w)$  is a strong crown in  $(G, w)$  if and only if  $def^*(G, w) > 0$ .

*Proof.* a) Assume first that  $def^*(G, w) = 0$ . Then  $x \equiv \frac{1}{2}$  belongs to  $VC^*(G, w)$  and it is clearly the only pivot. The corresponding decomposition is trivial.

b) Assume now that  $def^*(G, w) > 0$  and let  $x \in VC^*(G, w)$  be a pivot. Clearly,  $V_0^x \neq \emptyset$ , and by Lemma 2(iii),  $V_0^x$  is a strong crown in  $(G, w)$ . Using Lemma 5(iii),  $V_0^x \subseteq V_0^*(G, w)$

( $\subseteq V_0^x$ ) and  $V_1^x \subseteq V_1^*(G, w)$  ( $\subseteq V_1^x$ ), and  $V_{\frac{1}{2}}^x \supseteq K_{\max}(G, w)$  ( $\supseteq V_{\frac{1}{2}}^x$ ). Hence,  $(V_0^x, V_1^x, V_{\frac{1}{2}}^x) = (V_0^*(G, w), V_1^*(G, w), K_{\max}(G, w))$  holds for a pivot  $x$ , thus its uniqueness is obvious. Now  $\text{def}^*(G[K_{\max}], w) = 0$  and  $\text{def}^*(G, w) = w(V_0^*(G, w)) - w(V_1^*(G, w))$  follow by Lemma 6.  $\square$

The following theorem provides a simple characterization of graphs  $(G, w)$  containing no strong crown.

**Theorem 4.** *For a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  the following statements are equivalent:*

- (i)  $(G, w)$  contains no strong crown,
- (ii)  $\text{def}^*(G, w) = 0$ ,
- (iii)  $w(N(I)) \geq w(I)$  holds for each independent set  $I$  of  $G$ ,
- (iv)  $w(N(Z)) \geq w(Z)$  holds for each  $Z \subseteq V$ .

*Proof.* Assertions (ii)–(iv) are equivalent due to Theorem 2.  $\neg(\text{i}) \Rightarrow \neg(\text{iii})$  is trivial, as any strong crown  $I$  in  $(G, w)$  by its definition satisfies  $w(I) > w(N(I))$ .

$\neg(\text{ii}) \Rightarrow \neg(\text{i})$ : If  $\text{def}^*(G, w) > 0$  then, by Theorem 3,  $V_0^*(G, w) \neq \emptyset$  is a strong crown in  $(G, w)$ .  $\square$

**Theorem 5.** *Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ . There is only one strong crown decomposition  $(I, H, K)$  of  $(G, w)$  for which  $(G[K], w)$  contains no strong crown, namely the decomposition  $(V_0^*(G, w), V_1^*(G, w), K_{\max}(G, w))$ .*

*Proof.* The strong crown decomposition  $(V_0^*, V_1^*, K_{\max})$  has the property that  $(G[K_{\max}], w)$  contain no strong crown by Theorems 3 and 4. Let  $(I, H, K)$  be a fixed strong crown decomposition of  $(G, w)$  for which  $(G[K], w)$  contains no strong crown. By Theorem 4 it means that  $\text{def}^*(G[K], w) = 0$ . Using Lemma 6 it follows that  $(I, H, K) = (V_0^y, V_1^y, V_{\frac{1}{2}}^y)$  for some  $y \in VC^*(G, w)$ . Consequently,  $I \supseteq V_0^*$ . On the other hand, by Lemma 5(iii),  $I \subseteq V_0^*$ . Hence  $I = V_0^*$ , and  $(I, H, K) = (V_0^*, V_1^*, K_{\max})$  follows.  $\square$

Now we prove that the optimal strong crown decomposition  $(V_0^*, V_1^*, K_{\max})$  for a graph  $(G, w)$ , that is described in Theorem 5, can be computed efficiently.

**Lemma 7.** *Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ . The strong crown decomposition  $(V_0^*(G, w), V_1^*(G, w), K_{\max}(G, w))$  can be constructed in polynomial time; in the unweighted case in time  $O(|E|\sqrt{|V|})$ .*

*Proof.* Let  $G = (V, E)$  be a graph with vertex weights  $w : V \rightarrow (0, \infty)$ . Clearly, it suffices to compute  $V_0^*(G, w)$ , as then  $V_1^* = N(V_0^*)$  and  $K_{\max} = V \setminus (V_0^* \cup V_1^*)$  can be simply found.

By Remark 3, computing  $V_0^*(G, w)$  reduces to computing  $V_0(G^b, w^b)$ , where  $(G^b, w^b)$  is the corresponding bipartite graph.

It is easy to derive that in any graph  $(G^b, w^b)$  (not necessarily bipartite)  $V_0(G^b, w^b) = \{v \in V^b : \text{vc}(G^b \setminus v, w^b) + w^b(v) > \text{vc}(G^b, w^b)\}$ . But for the bipartite graph  $(G^b, w^b)$  one can compute the set  $V_0(G^b, w^b)$  in polynomial time, as  $\text{vc}(G^b, w^b)$  and  $\text{vc}(G^b \setminus v, w^b)$  are efficiently computable in such case using maximum flow techniques. The running time of the algorithm computing  $V_0^*(G, w)$  in the weighted case can be easily overestimated by  $O(|E||V|^2 \log \frac{|V|^2}{|E|})$ . In the unweighted case this estimate can be significantly improved as follows: If  $(G, w)$  is unweighted (i.e.,  $w \equiv 1$ ), then  $(G^b, w^b)$  is unweighted. Using the bipartite version of Gallai-Edmonds Structure theorem for graph  $G^b$  we can see that  $V_0(G^b)$  is exactly the set  $D$  of

vertices in  $G^b$  that are avoided (i.e., unmatched) by at least one maximum matching in  $G^b$ . The set  $D$  can be easily computed using Edmonds matching algorithm. It starts with any maximum matching  $M$  of  $G^b$  (it can be done in time  $O(|E|\sqrt{|V|})$  by algorithm from [25] or [28]). Let  $D_M$  be the set of vertices avoided by  $M$ . An alternating path with respect to  $M$  is a simple path  $\{v_1, v_2, \dots, v_r\}$  in  $G^b$  such that  $v_1 \in D_M$  and the edges  $\{v_{2k}, v_{2k+1}\}$  are in  $M$  for all integers  $k$  such that  $1 \leq k \leq \frac{r-1}{2}$ . The length of such a path is  $r - 1$  (possibly 0). One can prove that the set  $D$  is exactly the set of vertices in  $G^b$  that are reachable from at least one vertex in  $D_M$  via an alternating path of even length. The total complexity to construct  $D$  ( $= V_0^*$ ) is majorized by that of constructing the maximum matching  $M$ , hence  $O(|E|\sqrt{|V|})$ .  $\square$

We can summarize our previous results as follows:

**Theorem 6.** *There exists a polynomial time algorithm that partitions the vertex set  $V$  of a given graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  into three subsets  $V_0^*$ ,  $V_1^*$ ,  $K_{\max}$  with no edges between  $V_0^*$  and  $K_{\max}$  or within  $V_0^*$ , such that*

- (i)  $vc(G[K_{\max}], w) \geq vc^*(G[K_{\max}], w) = \frac{1}{2}w(G[K_{\max}])$ ,
- (ii) every minimum vertex cover  $C$  for  $(G, w)$  satisfies  $V_1^* \subseteq C \subseteq V_1^* \cup K_{\max}$  and  $C \cap K_{\max}$  is a minimum vertex cover for  $(G[K_{\max}], w)$ ,
- (iii) every (minimum) vertex cover for  $(G[K_{\max}], w)$  together with  $V_1^*$  forms a (minimum) vertex cover for  $(G, w)$ ,
- (iv)  $V_0^* = \bigcap_{y \in VC^*(G, w)} V_0^y$ ,  $V_1^* = \bigcap_{y \in VC^*(G, w)} V_1^y = N(V_0^*)$ , and  $K_{\max} = \bigcap_{y \in VC^*(G, w)} V_{\frac{1}{2}}^y$ .
- (v)  $\emptyset \neq U \subseteq V_1^*$  implies  $w(N(U) \cap V_0^*) > w(U)$ ,
- (vi)  $V_0^* = \emptyset$  if and only if  $def^*(G, w) = 0$ ,
- (vii)  $w(V_0^*) - w(V_1^*) = def^*(G, w)$ .

The parts (i)–(iii) yield to the strengthened version of Nemhauser-Trotter theorem proved already in [24].

*Remark 4.* In the unweighted case the question of whether there are vertices in a given graph  $G$  belonging to all maximum independent sets of  $G$ , is well studied. The problem is known to be NP-hard, so one of the questions studied is to provide lower bounds on  $|V_0(G)|$  for specific classes of graphs (see [5], and references therein). The best result of this kind in [5] is that if  $G$  is a graph without isolated vertices and with  $\alpha(G) > \nu(G)$ , then  $|V_0(G)| > \alpha(G) - \nu(G)$  (here  $\alpha$  is independence number,  $\nu$  is matching number).

Unfortunately, this bound depends on  $\alpha(G)$  that is NP-hard to compute. Using Theorem 6 one can provide results at least as strong, but with efficiently computable bounds. As  $V_0^*(G) \subseteq V_0(G)$ , one can provide a lower bound  $|V_0(G)| \geq |V_0^*(G)| = def^*(G) + |V_1^*(G)|$ .

By a bound given by Lorentzen (see Theorem 64.12 in [37]),  $def^*(G) \geq \alpha(G) - \nu(G)$ . Hence under a weaker and efficiently decidable assumption  $def^*(G) > 0$  we obtain at least as good lower bound  $|V_0(G)| \geq |V_0^*(G)| > def^*(G)$  for a graph  $G$  without isolated vertices. Moreover, the subset  $V_0^*(G)$  of  $V_0(G)$  of cardinality  $def^*(G) + |V_1^*(G)|$  can be efficiently computed, at the same time with the subset  $V_1^*(G) \neq \emptyset$  of  $V_1(G)$ .

### 3.2 Crown decompositions

We have seen that there is only one strong decomposition  $(I, H, K)$  of  $(G, w)$  for which  $(G[K], w)$  contains no strong crown, namely  $(V_0^*, V_1^*, K_{\max})$ . Now we deal with crown decompositions  $(I, H, K)$  of  $(G, w)$  for which  $(G[K], w)$  contains no crown.

The following theorem provides a simple characterization of graphs  $(G, w)$  containing no crown.

**Theorem 7.** *Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$ . Then the following statements are equivalent:*

- (i)  $(G, w)$  contains no crown,
- (ii)  $x \equiv \frac{1}{2}$  is the only element of  $VC^*(G, w)$ ,
- (iii)  $w(N(I)) > w(I)$  holds for each nonempty independent set  $I$  of  $G$ ,
- (iv)  $w(N(Z)) > w(Z)$  holds for each  $Z \subseteq V$  such that  $N(Z) \neq Z$ .

*Proof.* (ii) $\Leftrightarrow$ (iii) has been proved in [31],  $\neg$ (i) $\Rightarrow$  $\neg$ (iii) is trivial, as any crown  $I$  in  $(G, w)$  satisfies  $w(I) \geq w(N(I))$  by Definition 5.

$\neg$ (ii) $\Rightarrow$   $\neg$ (i) If  $x \in VC^*(G, w)$  and  $x \not\equiv \frac{1}{2}$ , then by Lemma 2  $(V_0^x, V_1^x, V_{\frac{1}{2}}^x)$  is a nontrivial crown decomposition of  $(G, w)$ .

(iv) $\Rightarrow$ (iii) trivial, (iii)  $\Rightarrow$  (iv) If  $Z \subseteq V$  such that  $N(Z) \neq Z$ , the same is true for the restriction of  $Z$  to at least one of connected components of  $G$ . Hence to prove (iv) from (iii) we can assume that  $G$  is connected. Let  $Z \subseteq V$  with  $N(Z) \neq Z$  be fixed and  $I$  be the set of isolated vertices of  $G[Z]$ . Clearly  $Z \setminus I$  and  $N(I)$  are disjoint subsets of  $N(Z)$ , hence  $w(N(Z)) \geq w(Z \setminus I) + w(N(I))$ . If  $I \neq \emptyset$ , then  $w(N(I)) > w(I)$  by (iii), and  $w(N(Z)) > w(Z \setminus I) + w(I) = w(Z)$  follows. If  $I = \emptyset$ , then clearly  $Z \subsetneq N(Z)$  and  $w(N(Z)) > w(Z)$  follows as well.  $\square$

**Theorem 8.** *Given a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  and a crown decomposition  $(I, H, K)$  of  $(G, w)$  for which  $(G[K], w)$  contains no crown. Then  $(I, H, K) = (V_0^x, V_1^x, V_{\frac{1}{2}}^x)$  for some  $x \in VC^*(G, w)$ , and  $K = K_{\min}(G, w)$ .*

*Proof.* Using Theorem 4 we get  $def^*(G[K], w) = 0$  and, by Lemma 6(ii),  $(I, H, K) = (V_0^x, V_1^x, V_{\frac{1}{2}}^x)$  for some  $x \in VC^*(G, w)$ . By Theorem 7(ii),  $z \equiv \frac{1}{2}$  on  $K$  is the only element of  $VC^*(G[K], w)$ . It implies, by Lemma 5(ii), that  $y|_K \equiv \frac{1}{2}$  holds for every  $y \in VC^*(G, w)$ , and hence  $K \subseteq K_{\min}(G, w)$ . On the other hand, as  $K = V_{\frac{1}{2}}^x$  for some  $x \in VC^*(G, w)$ ,  $K \supseteq K_{\min}(G, w)$ . Consequently,  $K = K_{\min}(G, w)$ .  $\square$

**General crown reduction strategy.** Let us consider an algorithm  $\mathcal{A}$  that for a given input  $(G, w)$  provides a nontrivial crown decomposition, or reports that  $(G, w)$  contains no crown. Given  $G = (V, E)$  with weights  $w : V \rightarrow (0, \infty)$ , we write  $K_0 := V$  and by induction we define for  $i = 1, 2, \dots$  the following decompositions: if  $K_{i-1} \neq \emptyset$  and  $(G[K_{i-1}], w)$  has a crown, an algorithm  $\mathcal{A}$  provides a nontrivial crown decomposition  $(I_i, H_i, K_i)$  of  $(G[K_{i-1}], w)$ . If  $K_i \neq \emptyset$  and  $(G[K_i], w)$  has a crown, we define  $(I_{i+1}, H_{i+1}, K_{i+1})$  a similar way, and so on. We stop when  $K_k = \emptyset$ , or  $(G[K_k], w)$  contains no crown. If we denote  $I = \cup_{i=1}^k I_i$ ,  $H = \cup_{i=1}^k H_i$ , and  $K = K_k$ , then  $(I, H, K)$  is a crown decomposition of  $(G, w)$  for which  $(G[K], w)$  contains no crown. The previous theorem implies that  $(I, H, K) = (V_0^x, V_1^x, V_{\frac{1}{2}}^x)$  for some  $x \in VC^*(G, w)$ , and  $K = K_{\min}(G, w)$ . This shows that despite the fact that the crown reduction technique refines Nemhauser-Trotter reductions, it cannot provide smaller problem kernels. Both methods are essentially equivalent in a sense that they reduce an instance  $(G, w)$  to the same problem kernel  $(G[K_{\min}], w)$ .

The condition (ii) in Theorem 7 for an input graph  $(G, w)$  can be decided in polynomial time (see [31]). Hence the problem to decide of whether  $(G, w)$  has a crown is solvable in

polynomial time. Moreover, a crown decomposition  $(I, H, K)$  of  $(G, w)$  for which  $(G[K], w)$  contains no crown, can be found efficiently. By Theorem 8 and its proof it reduces to finding  $x \in VC^*(G, w)$  for which  $z \equiv \frac{1}{2}$  on  $V_{\frac{1}{2}}^x$  is the only element of  $VC^*(G[V_{\frac{1}{2}}^x], w)$ . Nemhauser and Trotter ([31, p. 238]) were the first who described how to find such  $x$  efficiently. It is possible that there are many such  $x \in VC^*(G, w)$  but all have the same  $V_{\frac{1}{2}}^x$ , as was proven first by Picard and Queyranne [33]. Theorem 8 shows that also the crown reduction method always leads to this set,  $K_{\min}(G, w)$ , as to the unique problem kernel for the MIN- $w$ -VC problem. More comprehensive treatment on this topic in the unweighted case was given by Bourjolly and Pulleyblank [7].

#### 4 Decomposition into Irreducible Subgraphs

We have seen that  $K_{\min}(G, w)$  is a natural problem kernel for the problem MIN- $w$ -VC, where the task is to find *one* minimum vertex cover for  $(G, w)$ . Finding an optimal solution for this kernel  $(G[K_{\min}], w)$  allows to identify *one* optimal solution for  $(G, w)$ , if information was stored what crowns have been removed to obtain a problem kernel. This kind of reduction doesn't preserve information about *all* minimum vertex covers for  $(G, w)$ . On the other hand, one can obtain efficiently a strong crown decomposition  $(V_0^*, V_1^*, K_{\max})$  of  $(G, w)$  with the property that *every* minimum vertex cover for  $(G, w)$  contains all vertices of  $V_1^*$  and none of  $V_0^*$ . In this section we will describe how all minimum vertex covers for  $(G, w)$  are structured inside the set  $K_{\max} \setminus K_{\min}$ . Therefore we can always pass from  $(G, w)$  to  $(G[K_{\max}], w)$  due to a close relation between sets of all minimum vertex covers for these two instances. In this way, we can confine ourselves to graphs  $(G, w)$  that contain no strong crowns. That means, to graphs with  $def^*(G, w) = 0$  or, equivalently said, to graphs satisfying  $w(N(I)) \geq w(I)$  for each independent set  $I$  of  $G$ .

**Definition 7.** Let  $G = (V, E)$  be a graph with vertex weights  $w : V \rightarrow (0, \infty)$  such that  $def^*(G, w) = 0$ . A crown  $I$  in  $G$  is called *irreducible*, if the bipartite graph  $(G[I, N(I)], w)$  has exactly two minimum vertex covers, namely  $I$  and  $N(I)$ . Define the set  $\mathcal{I} = \{I : I \text{ is a nonempty independent set such that } w(N(I)) = w(I)\}$ .

**Lemma 8.** Let  $G = (V, E)$  be a graph with vertex weights  $w : V \rightarrow (0, \infty)$  such that  $def^*(G, w) = 0$ . Then

- (i)  $I$  is a crown in  $(G, w)$  if and only if  $I \in \mathcal{I}$ ,
- (ii) a crown  $I$  in  $(G, w)$  is irreducible if and only if  $I$  is an inclusionwise minimal set in  $\mathcal{I}$ .

*Proof.* (i) If  $I$  is a crown in  $(G, w)$  then  $I$  is a nonempty independent set of  $G$  satisfying  $w(N(I)) \leq w(I)$  (Definition 5). The opposite inequality also holds, by Theorem 4, and  $I \in \mathcal{I}$  follows.

Assume that  $I \in \mathcal{I}$ . We want to show that  $w(N(U) \cap I) \geq w(U)$  for every  $U \subseteq N(I)$ . Keep  $U \subseteq N(I)$  fixed, and put  $S = I \setminus N(U)$ . As  $S$  is independent,  $w(N(S)) \geq w(S)$  by Theorem 4. Clearly  $N(S) \subseteq N(I) \setminus U$ , hence  $w(N(I) \setminus U) \geq w(N(S)) \geq w(S)$ .

As  $w(I) = w(N(I))$ ,  $w(N(U) \cap I) = w(I) - w(S) = w(N(I)) - w(S) \geq w(N(I)) - w(N(I) \setminus U) = w(U)$ .

(ii) Assume that  $I \in \mathcal{I}$  is not inclusionwise minimal, i.e., there exists nonempty independent set  $T \subsetneq I$ , such that  $w(N(T)) = w(T)$ . Then  $(I \setminus T) \cup N(T)$  is a minimum vertex cover for  $(G[I, N(I)], w)$  distinct from  $I$  and  $N(I)$ , and  $I$  is not irreducible.

If  $I$  is not irreducible, there is a minimum vertex cover  $C$  for  $(G[I, N(I)], w)$  such that  $\emptyset \neq I \setminus C \subsetneq I$ . Clearly,  $N(I \setminus C) \subseteq C \cap N(I)$ , hence we obtain  $w(N(I \setminus C)) \leq w(C \cap N(I))$  and  $w(N(I \setminus C)) \geq w(I \setminus C)$  (as  $\text{def}^*(G, w) = 0$ ). As  $w(C) = w(I)$ , we get  $w(I \setminus C) = w(C \cap N(I))$  and  $w(I \setminus C) = w(C \cap N(I)) \geq w(N(I \setminus C)) \geq w(I \setminus C)$ , and  $I \setminus C \in \mathcal{I}$  follows, hence  $I$  is not inclusionwise minimal.  $\square$

If  $I$  is an irreducible crown, then the MIN- $w$ -VC problem for  $(G[I \cup N(I)], w)$  has at most two solutions:  $N(I)$  is always a solution and, if there are no edges within  $N(I)$ , then  $I$  is a solution as well. One can recognize an irreducible crown  $I$  (more precisely its image in  $(G^b, w^b)$ ) using so called *allowed* and *forbidden* edges of this bipartite graph. This terminology comes from decomposition theorems related to maximal matchings in unweighted graphs [27].

**Definition 8.** Let  $G = (V, E)$  be a bipartite graph with bipartition  $V = L \cup R$  and vertex weights  $w : V \rightarrow (0, \infty)$  such that  $\text{vc}(G, w) = \frac{1}{2}w(V)$ . An edge  $\{u, v\} \in E$  is called *allowed* for  $(G, w)$ , if every minimum vertex cover  $C$  in  $(G, w)$  contains only one of the vertices  $u$  and  $v$ , otherwise it is called *forbidden*. Removing forbidden edges from  $G$  we obtain a graph whose connected components are called *elementary blocks*. Further,  $(G, w)$  is called *elementary*, if it has exactly two minimum vertex covers, namely  $L$  and  $R$ .

Clearly, if a bipartite graph  $(G, w)$  is elementary, then  $G$  is connected and every edge is allowed. If  $w \equiv 1$ , then we deal exactly with bipartite graphs with perfect matching. In this context an edge of  $G$  is called *allowed* if it is contained in some perfect matching of  $G$ , and  $G$  is called *elementary* if it is connected and every edge of  $G$  is allowed. (See [27, Thm. 4.1.1] for the proof that these notions, and those given in Definition 8, are equivalent in the case of unweighted bipartite graphs with perfect matching.)

Now we need to prove the generalization of the classical bipartite Dulmage-Mendelsohn Decomposition theorem to weighted bipartite graphs, with focus on minimum vertex covers instead of maximum matchings.

**Theorem 9.** Let  $G = (V, E)$  be a bipartite graph with bipartition  $V = L \cup R$  and vertex weight  $w : V \rightarrow (0, \infty)$  such that  $\text{vc}(G, w) = \frac{1}{2}w(V)$ . The subgraph of  $G$  which contains all allowed edges for  $(G, w)$  consists of elementary blocks,  $B_i = G[L_i \cup R_i]$  for  $i = 1, 2, \dots, r$  (here  $\cup_{i=1}^r L_i = L$  and  $\cup_{i=1}^r R_i = R$  are partitions). The ordering  $B_1, B_2, \dots, B_r$  can be chosen with the following property: every edge in  $G$  between two blocks  $B_i$  and  $B_j$  with  $i < j$  must have its  $R$ -vertex in  $B_i$  and  $L$ -vertex in  $B_j$ . The decomposition into blocks and their admissible ordering can be constructed in polynomial time; in the unweighted case in time  $O(|E|\sqrt{|V|})$ .

*Proof.* (a) Let us start with the structural part of the theorem.

If  $\emptyset \neq A \subsetneq L$  implies  $w(N(A)) > w(A)$ , then  $(G, w)$  is elementary and the decomposition trivially consists of one block. Otherwise we can find  $L_1, \emptyset \neq L_1 \subsetneq L$ , such that  $w(N(L_1)) = w(L_1)$ . We can take such  $L_1$  which is inclusionwise minimal with respect to this property, i.e.,  $\emptyset \neq A \subsetneq L_1$  implies  $w(N(A)) > w(A)$ . Put  $R_1 := N(L_1)$ . Clearly,  $(G[L_1 \cup R_1], w)$  is elementary. It is not difficult to see that for every  $C \in \text{VC}(G, w)$   $C \cap (R_1 \cup L_1)$  is a minimum vertex cover for  $(G[L_1 \cup R_1], w)$  and  $C \setminus (R_1 \cup L_1)$  is a minimum vertex cover for  $(G[V \setminus (L_1 \cup R_1)], w)$ . Hence,  $C \cap (L_1 \cup R_1)$  is either  $L_1$  or  $R_1$ . It implies that each edge between  $L_1$  and  $R_1$  is allowed for  $(G, w)$ .

Now we can do the same with the graph  $G_1 := G[V \setminus (L_1 \cup R_1)]$ , as  $(G_1, w)$  satisfies the same structural assumptions as  $(G, w)$ . Continuing by induction we will end in finitely

many steps with  $\emptyset \neq L_i \subseteq L$ ,  $\emptyset \neq R_i \subseteq R$  such that  $V = \cup_{i=1}^r (L_i \cup R_i)$  is a partition in  $G$  with the following properties:

- (i) for  $i < j$  there is no edge between  $L_i$  and  $R_j$ ,
- (ii) for every  $i \in \{1, 2, \dots, r\}$  and for every  $C \in VC(G, w)$   $C \cap (L_i \cup R_i)$  is either  $L_i$  or  $R_i$ ,
- (iii) for every  $i \in \{1, 2, \dots, r\}$   $(G[L_i \cup R_i], w)$  is an elementary graph.

The property (ii) implies that all edges inside a fixed block  $B_i = G[L_i \cup R_i]$  are allowed for  $(G, w)$ . The property (i) implies that all edges between two blocks  $B_j$  and  $B_i$  ( $j > i$ ) are forbidden; namely (for a fixed  $i = 1, \dots, r-1$ )  $C_i := \cup_{k=1}^i R_k \cup \cup_{k=i+1}^r L_k$  is an element of  $VC(G, w)$  containing both vertices from each edge between blocks  $B_{j'}$  and  $B_{i'}$ , where  $i' \in \{1, \dots, i\}$ ,  $j' \in \{i+1, \dots, r\}$ .

It easily follows that each  $B_i$  is a component of the subgraph of  $G$  obtained by restricting  $G$  to the edges allowed for  $(G, w)$ . This completes the proof of the structural part of the theorem.

(b) Let us focus on the complexity of the problem to construct the above decomposition into blocks. Identifying blocks basically reduces to identifying edges which are allowed for  $(G, w)$ . But  $\{u, v\} \in E$  is allowed for  $(G, w)$  iff  $vc(G \setminus (\{u\}, \{v\}), w) + w(u) + w(v) > vc(G, w)$ , which can be tested efficiently as  $G$  is bipartite. It easily follows that the above decomposition can be constructed in polynomial time.

Let us briefly discuss time complexity of the following algorithm that provides decomposition into blocks and their admissible ordering in the unweighted case:

**Step 1:** Find a perfect matching  $M$  of  $G$ .

**Step 2:** Build the directed graph  $G_M$  from  $G$  by replacing each edge  $\{u, v\}$  in  $M$  by two arcs  $\vec{uv}$  and  $\vec{vu}$ , and by orienting all other edges from  $R$  to  $L$ .

**Step 3:** Compute the strongly connected components of  $G_M$ , each of them corresponds to a block of  $G$  (independently of  $M$  chosen).

**Step 4:** Find an admissible ordering of blocks, build the reduced digraph from  $G_M$  by contracting each block of  $G_M$  in a vertex. The resulting digraph is acyclic, that induces a partial order between blocks. Hence a compatible total order of blocks can be obtained by any topological sorting of that acyclic graph.

Steps 2-4 can be computed in time  $O(|V| + |E|)$ , e.g., by depth first search. Hence running time of the whole algorithm is dominated by the time complexity  $O(|E| \sqrt{|V|})$  of the search for a maximum matching.  $\square$

*Remark 5.* As follows from the proof of Theorem 9,  $L$  and  $R$  are always in  $VC(G, w)$ , but if  $(G, w)$  is not elementary, there are also “intermediate” minimum vertex covers. Namely, each set  $C_i := \cup_{k=1}^i R_k \cup \cup_{k=i+1}^r L_k$ ,  $i = 0, 1, \dots, r$  is a minimum vertex cover for  $(G, w)$ . Moreover, for each  $C \in VC(G, w)$  and each  $i \in \{1, 2, \dots, r\}$ ,  $C \cap (L_i \cup R_i)$  is either  $L_i$  or  $R_i$ .

In the following theorem we provide an efficient decomposition of  $(G, w)$  into irreducible parts, that nicely describes the structure of all minimum vertex covers in the subgraph  $(G[K_{\max} \setminus K_{\min}], w)$ . In some cases this theorem allows to find at least some vertices of  $V_i(G, w) \setminus V_i^*(G, w)$ , for  $i = 0, 1$ . Therefore it can be useful also for the problem of finding all minimum vertex covers for  $(G, w)$  to reduce further the kernel  $(G[K_{\max}], w)$  obtained in Theorem 6.

**Theorem 10.** *Let  $G = (V, E)$  be a graph with vertex weights  $w : V \rightarrow (0, \infty)$  and such that  $\text{def}^*(G, w) = 0$ . Then there exists a polynomial time algorithm (running in time  $O(|E|\sqrt{|V|})$  in the unweighted case) that constructs a partition*

$$V = K_{\min} \cup \bigcup_{i=1}^s I_i \cup \bigcup_{i=1}^s H_i$$

with the following properties:

- (i) For each  $i \in \{1, 2, \dots, s\}$  the following holds true:
  - (a)  $H_i = N(I_i) \setminus \bigcup_{j=1}^{i-1} H_j$ ,
  - (b)  $w(I_i) = w(H_i) = \text{vc}(G[I_i \cup H_i], w)$ ,
  - (c)  $\emptyset \neq T \subsetneq I_i$  implies  $w(N(T) \cap H_i) > w(T)$ ,
  - (d) for every  $C \in \text{VC}(G, w)$   $C \cap (I_i \cup H_i)$  is either  $I_i$  or  $H_i$ , and if  $H_i$  is not an independent set then  $C \cap (I_i \cup H_i) = H_i$ .
- (ii) There is a minimum half-integral vertex cover  $x$  for  $(G, w)$  such that  $V_0^x = \bigcup_{i=1}^s I_i$ ,  $V_1^x = \bigcup_{i=1}^s H_i$ , and  $V_{\frac{1}{2}}^x = K_{\min} := \bigcap_{y \in \text{VC}^*(G, w)} V_{\frac{1}{2}}^y$ .
- (iii) If  $K_{\min} \neq \emptyset$  then  $z \equiv \frac{1}{2}$  on  $K_{\min}$  is the unique element of  $\text{VC}^*(G[K_{\min}], w)$  and  $\text{vc}(G[K_{\min}], w) > \text{vc}^*(G[K_{\min}], w) = \frac{1}{2}w(K_{\min})$ .

*Proof.* First, we apply Theorem 9 to the bipartite graph  $(G^b, w^b)$  of  $(G, w)$ . For any  $i = 1, \dots, r$  a block  $B_i = G^b[L_i \cup R_i]$  of  $(G^b, w^b)$  has  $L_i = I_i^L$ ,  $R_i = H_i^R$  for some  $I_i, H_i \subseteq V$  with  $w(I_i) = w(H_i)$ . The structure of allowed edges and blocks of  $(G^b, w^b)$  now additionally reflects the presence of the automorphism of  $(G^b, w^b)$  described in Remark 2. For any  $\{u, v\} \in E$ ,  $\{u^L, v^R\}$  is an allowed edge of  $(G^b, w^b)$  iff  $\{v^L, u^R\}$  is allowed. Hence vertices  $I_i^L \cup H_i^R$  induce a block of  $(G^b, w^b)$  iff  $H_i^L \cup I_i^R$  do. It follows, in particular, that for any such block either  $H_i \cap I_i = \emptyset$  (a block is called *simple*), or  $H_i = I_i$  (a block is called *hard*). Simple blocks of  $(G^b, w^b)$  appear in pairs,  $G^b[I_i^L \cup H_i^R]$  together with  $G^b[H_i^L \cup I_i^R]$ . Hard blocks are of the form  $G^b[H_i^L \cup H_i^R]$ . Let there be exactly  $s$  pairs of simple blocks and  $(r - 2s)$  hard blocks. Now we can change the order of blocks guaranteed by Theorem 9 to fulfill in addition the following symmetry related to that automorphism: simple blocks form initial and end segments of  $B_1, B_2, \dots, B_r$ , with  $(B_i, B_{r+1-i})$  being a pair of twin simple blocks for each  $i = 1, 2, \dots, s$ , i.e., if  $B_i = G^b[I_i^L \cup H_i^R]$  then  $B_{r+1-i} = G^b[H_i^L \cup I_i^R] =: \tilde{B}_i$ .

Let us explain how any admissible ordering  $B_1, B_2, \dots, B_r$  of blocks can be converted to a desired symmetric one. If  $s = 0$  there is nothing to prove, hence suppose  $s \geq 1$ . Assume  $B_1$  is hard,  $B_1 = G^b[H_1^L \cup H_1^R]$ , it follows that there are no edges in  $G^b$  between  $H_1^L$  and  $V^b \setminus H_1^R$  (hence no edges in  $G$  between  $H_1$  and  $V \setminus H_1$ ). Consequently,  $B_1$  is a component of  $G^b$  and we can move it freely and still preserve admissibility of the ordering. The same applies to any block from initial segment (if any) of hard blocks, hence we can completely ignore them in our symmetric conversion (and we can place them later to the middle group of hard blocks).

Now take the first simple block, say  $B_i$ . Place it at the beginning, and its twin  $\tilde{B}_i$  place at the end, and relabel. It is easy to check that admissibility of the ordering is preserved. Now we can forget also this pair of blocks and apply the same rule for the rest. This procedure leads simply to a desired symmetric admissible ordering of blocks of  $(G^b, w^b)$ ,  $B_1, B_2, \dots, B_r$ , with  $B_i = G^b[I_i^L \cup H_i^R]$ ,  $B_1, B_2, \dots, B_s$ , being simple  $B_{r+1-i} = \tilde{B}_i$  for  $i = 1, 2, \dots, s$  as well and  $B_i = G^b[H_i^L \cup H_i^R]$  for  $s+1 \leq i \leq r-s$  being hard.

Now (i) follows directly from Theorem 9 and we prove the properties (ii) and (iii).

Let  $K := \bigcup_{i=s+1}^{r-s} H_i$ . In what follows we show that  $K = K_{\min}(G, w)$  and that there is  $x \in VC^*(G, w)$  for which  $V_{\frac{1}{2}}^x = K$ .

As any  $C \in VC(G^b, w^b)$  contains from each hard block  $B_i$  exactly one of sets  $H_i^L, H_i^R$ , it easily follows that for each  $y \in VC^*(G, w)$   $y \equiv \frac{1}{2}$  on  $K$ . Consequently,  $K \subseteq K_{\min}$ .

Now we will show that, in fact, the equality holds here, finding  $x \in VC^*(G, w)$  for which  $V_{\frac{1}{2}}^x = K$  holds.

As was mentioned in Remark 5, each  $C_i := \bigcup_{k=1}^i H_k^R \cup \bigcup_{k=i+1}^r I_k^L \in VC(G^b, w^b)$  for  $i = 0, 1, \dots, r$ , is a minimum vertex cover for  $(G, w)$ . If  $0 \leq i \leq s$ , then an element  $x_i \in VC^*(G, w)$  satisfies  $V_0^{x_i} = \bigcup_{k=1}^i I_k$ ,  $V_1^{x_i} = \bigcup_{k=1}^i H_k$ ,  $V_{\frac{1}{2}}^{x_i} = K \cup \bigcup_{k=i+1}^s (I_k \cup H_k)$ . In particular, for  $x := x_s \in VC^*(G, w)$  we have  $V_0^x = \bigcup_{k=1}^s I_k$ ,  $V_1^x = \bigcup_{k=1}^s H_k$ , and  $V_{\frac{1}{2}}^x = K$  ( $\subseteq \bigcap_{y \in VC^*(G, w)} V_{\frac{1}{2}}^y \subseteq V_{\frac{1}{2}}^x$ ). Hence  $V_{\frac{1}{2}}^x = K_{\min} = \bigcap_{y \in VC^*(G, w)} V_{\frac{1}{2}}^y$ .

If  $K_{\min} \neq \emptyset$ , from the structure of hard blocks it follows that  $z \equiv \frac{1}{2}$  on  $K_{\min}$  is the only element of  $VC^*(G[K_{\min}], w)$ . By Theorem 7, for a maximum weight independent set  $I$  for  $(G[K_{\min}], w)$  it holds  $w(N(I)) > w(I)$ , and  $vc(G[K_{\min}], w) = w(N(I)) > \frac{1}{2}w(K_{\min})$  follows.  $\square$

*Remark 6.* Under the assumptions of Theorem 10, we have  $V_0^*(G, w) = V_1^*(G, w) = \emptyset$  and we cannot say much, in general, about  $V_0(G, w)$  and  $V_1(G, w)$ . But in many cases Theorem 10 gives very useful information about  $V_0(G, w)$  and  $V_1(G, w)$ . Let the corresponding partition  $V = K_{\min} \cup \bigcup_{i=1}^s I_i \cup \bigcup_{i=1}^s H_i$  be fixed. We will say that  $i \in \{1, 2, \dots, s\}$  is *determined* if for every  $C \in VC(G, w)$ ,  $C \cap (I_i \cup H_i) = H_i$  (i.e.,  $H_i \subseteq C$  and  $I_i \cap C = \emptyset$ ). Clearly, if  $H_i$  is not an independent set, then  $i$  is *determined*. Further, if  $i$  is *determined* and  $j < i$  is such that there exists an edge between  $I_i$  and  $H_j$ , then  $j$  is *determined*. Also, if  $i$  is such that for some  $k > i$  there exists an edge between  $H_i$  and  $H_k$  as well as an edge between  $H_i$  and  $I_k$ , then  $i$  is *determined*. These and similar observations allow in some cases to further reduce the kernel  $(G[K_{\max}], w)$  obtained in Theorem 6 for the problem of *all* minimum vertex covers, as we can tell a priori for some  $i$  that  $I_i \subseteq V_0(G, w)$  and  $H_i \subseteq V_1(G, w)$ .

Let us mention some of the consequences of Theorem 10. For a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  we firstly apply Theorem 6 and reduce the problem to the graph  $(G[K_{\max}], w)$  satisfying assumptions of Theorem 10. Applying Theorem 10, the MIN- $w$ -VC problem for  $(G, w)$  reduces to the one for  $(G[K_{\min}], w)$ , for which  $x \equiv \frac{1}{2}$  on  $K_{\min}$  is the unique element of  $VC^*(G[K_{\min}], w)$ . Moreover, the difference  $vc(G, w) - vc^*(G, w)$  is preserved during this reduction. It is the same as  $vc(G[K_{\min}], w) - vc^*(G[K_{\min}], w)$ , which is zero iff  $K_{\min} = \emptyset$ . Hence we have obtained as a byproduct a new polynomial time algorithm that recognizes weighted KEG graphs and solves MIN- $w$ -VC problem on them.

Moreover, for a weighted KEG graph  $(G, w)$  we can find  $V_0(G, w)$  and  $V_1(G, w)$  in polynomial time. This problem reduces, by Theorem 6, to the one for  $(G[K_{\max}], w)$ . Hence we can assume that  $G = G[K_{\max}]$ . Applying Theorem 10 we see that  $K_{\min} = \emptyset$  and to find  $V_0(G, w)$  and  $V_1(G, w)$  means to find the set  $J$  of all  $i \in \{1, 2, \dots, s\}$  that are *determined* (with the meaning introduced in Remark 6). Clearly,  $V_0(G, w) = \bigcup_{i \in J} I_i$  and  $V_1(G, w) = \bigcup_{i \in J} H_i$ . We will proceed by induction. We start with  $i = 1$  and try to find if  $i$  is *determined*, or it is not. Clearly, 1 is not *determined* iff there is  $C \in VC(G, w)$  such that  $I_1 \subseteq C$  and  $H_1 \cap C = \emptyset$ . Consider one such  $C$  fixed, we can derive some properties about the sets  $J_I := \{i : I_i \subseteq C\}$ ,  $J_H := \{i : H_i \subseteq C\}$ . We either get a contradiction  $J_I \cap J_H \neq \emptyset$  showing that 1 is determined, or find such  $C$ .

We will build up  $J_I, J_H$  starting from  $J_I := \{1\}, J_H := \emptyset$ , as follows:

- Step 1.** If  $i \in J_I$  and  $j \notin J_I$  are such that there exists an edge between  $H_i$  and  $I_j$ , add  $j$  to  $J_I$ . Repeat this step while  $J_I$  can be enlarged this way. Then go to Step 2.
- Step 2.** If  $i \in J_I$  and  $j \in \{1, 2, \dots, s\}$  are such that there exists an edge between  $H_i$  and  $H_j$  (i.e., within  $H_i$ , if  $j = i$ ), add  $j$  to  $J_H$  if  $j \notin J_I$ , else go to Step 5. Repeat while  $J_H$  can be enlarged this way. Then go to Step 3.
- Step 3.** If  $i \in J_H$  and  $j < i$  are such that there exists an edge between  $I_i$  and  $H_j$ , add  $j$  to  $J_H$  if  $j \notin J_I$ , else go to Step 5. Repeat while  $J_H$  can be enlarged this way. Then go to Step 4.
- Step 4.** 1 is not *determined*. There is  $C \in VC(G, w)$  such that  $I_1 \subseteq C$ . Any such  $C$  contains each  $I_i, i \in J_I$ , and each  $H_i, i \in J_H$ . If there are still some  $i \in \{1, 2, \dots, s\} \setminus (J_I \cup J_H)$ , one can define one such  $C$ , taking  $C \cap (I_i \cap H_i) = H_i$  for any such  $i$ .
- Step 5.** 1 is *determined*.

Once the question of whether 1 is *determined* was answered, we remember the answer, remove  $I_1 \cup H_1$  from  $G$  and continue with the rest of the graph and with  $i = 2$ , and so on. The minimum vertex covers in the graph after each removal are exactly restrictions of  $VC(G, w)$  to that smaller vertex set. Hence the set  $J$  of all  $i \in \{1, 2, \dots, s\}$  that are *determined*, can be computed inductively.

**Corollary 1.** *There is a polynomial time algorithm (of time complexity  $O(|E|\sqrt{|V|})$  in the unweighted case) that for a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  decides whether  $vc(G, w) = vc^*(G, w)$ , and if the equality holds, finds one minimum vertex cover for  $(G, w)$ . Moreover, for graphs  $(G, w)$  for which the equality holds, i.e., for König-Egerváry graphs,  $V_0(G, w)$  and  $V_1(G, w)$  can be computed in polynomial time.*

#### 4.1 Fractional $w$ -matchings

In this subsection we explain how the results of previous sections can be interpreted using the duality between fractional vertex covers for  $(G, w)$  and fractional  $w$ -matchings in  $G$ .

**Definition 9.** *Let  $G = (V, E)$  be a graph with vertex weights  $w : V \rightarrow (0, \infty)$  and  $\lambda : E \rightarrow [0, \infty)$  be a fractional  $w$ -matching. The vertex  $v \in V$  is said to be saturated by  $\lambda$ , when  $\sum_{u \in N(v)} \lambda(\{u, v\}) = w(v)$ . A fractional  $w$ -matching  $\lambda$  is called perfect, if all vertices are saturated by  $\lambda$ . Further, an edge  $\{u, v\} \in E$  is active in  $\lambda$  if  $\lambda(\{u, v\}) > 0$ , otherwise it is called passive.*

The MAXIMUM  $w$ -MATCHING problem is the integral version of the MAX- $w$ -FM problem, i.e., all weights  $w$  are integral together with the additional constraints that each  $\lambda(\{u, v\})$  is integral. Let  $\nu(G, w)$  denote the optimal value of the MAXIMUM  $w$ -MATCHING problem. (We drop the acronym  $w$  in case  $w \equiv 1$ ).

Clearly,  $\nu(G, w) \leq \nu^*(G, w)$ . It is known that  $\nu(G, w) = \nu^*(G, w)$  whenever  $w$  is integral and  $vc(G, w) = vc^*(G, w)$ . Hence, for integral weights  $w$  a weighted graph  $(G, w)$  is KEG if and only if  $\nu(G, w) = vc(G, w)$ .

**Fractional Tutte-Berge and Gallai-Edmonds sets.** Let a graph  $G = (V, E)$  with vertex weights  $w : V \rightarrow (0, \infty)$  be given and consider a subset  $S$  of  $V$ . The *trivial  $S$ -components* are isolated vertices of the graph  $G \setminus S$ ; the set  $I_S$  denotes their union, which

is clearly an independent set of  $G$ . The *nontrivial  $S$ -components* are the other connected components of  $G \setminus S$ . The quantity  $def^*(S) := w(I_S) - w(S)$  denotes the *fractional deficiency of  $S$  in  $(G, w)$* . Observe that  $N(I_S) \subseteq S$  and if  $N(I_S) \neq S$  then  $S' := N(I_S)$  satisfies  $I_{S'} = I_S$ ,  $N(I_{S'}) = S'$ , and  $def^*(S') > def^*(S)$ .

Now Theorem 2 implies that  $\max_{S \subseteq V} def^*(S) = def^*(G, w)$ . Moreover,  $S \subseteq V$  satisfies  $def^*(S) = def^*(G, w)$  if and only if  $S \in \{V_1^y : y \in VC^*(G, w)\}$ .

Given  $S \subseteq V$ , it is important to notice that

- (1) every fractional  $w$ -matching  $\lambda$  in  $(G, w)$  satisfies

$$\begin{aligned} w(V) - 2\lambda(E) &= \sum_{v \in V} \left( w(v) - \sum_{u \in N(v)} \lambda(\{u, v\}) \right) \\ &\geq \sum_{v \in I_S} \left( w(v) - \sum_{u \in N(v)} \lambda(\{u, v\}) \right) = w(I_S) - \sum_{v \in I_S} \sum_{u \in N(v)} \lambda(\{u, v\}) \geq w(I_S) - w(S) = def^*(S), \end{aligned}$$

- (2) if a fractional  $w$ -matching  $\lambda$  in  $(G, w)$  satisfies  $w(V) - 2\lambda(E) = def^*(S)$ , then  $def^*(S) = def^*(G, w)$ ,  $\lambda$  is a maximum fractional matching for  $(G, w)$ , it contains a perfect fractional  $w$ -matching of each nontrivial  $S$ -component, and it saturates all vertices of  $S$ . In particular, all edges within  $S$ , or between  $S$  and  $V \setminus (S \cup I_S)$ , are passive in  $\lambda$ .

We generalize some notions that are standard in the literature on matching theory to the fractional  $w$ -matchings theory for vertex weighted graphs  $(G, w)$ . The books [27] and [37] provide comprehensive treatment of such theory, including various methods suitable for weighted problems as well.

A set  $S \subseteq V$  is called a *fractional Tutte-Berge set for  $(G, w)$*  if a (maximum) fractional  $w$ -matching  $\lambda$  for  $(G, w)$  exists, that satisfies (2). These are exactly the sets  $\{V_1^y : y \in VC^*(G, w)\}$ . Further,  $S \subseteq V$  is called a *fractional Gallai-Edmonds set for  $(G, w)$*  if

- (a) nontrivial  $S$ -components, if any, have a perfect fractional  $w$ -matching,  
(b) if  $S \neq \emptyset$ , then  $S$  satisfies generalized Hall's condition with positive surplus in  $G[I_S, S]$ , i.e.,

$$w(N(U) \cap I_S) > w(U) \text{ for every nonempty set } U \subseteq S.$$

It is easy to see that for a fractional Gallai-Edmonds set  $S$  for  $(G, w)$  the underlying vertex set  $I_S$  of trivial  $S$ -components is exactly the set of vertices left unsaturated by at least one maximum fractional  $w$ -matching and  $S = N(I_S)$ . Hence if such a set  $S$  exists, it is unique.

By linear programming duality,  $def^*(G, w) = 0$  iff  $G$  has a perfect fractional  $w$ -matching. This together with Lemma 2 implies that  $V_1^x$  is a fractional Gallai-Edmonds set for  $(G, w)$  if  $x$  is a pivot. This is another proof of uniqueness of pivot for  $(G, w)$ . Hence, by Theorem 5,  $V_1^*(G, w)$  is the (unique) fractional Gallai-Edmonds set for  $(G, w)$ .

## 5 Applications

The best known application of Nemhauser-Trotter reduction is that it provides a simple 2-approximation algorithm for the MIN- $w$ -VC problem. Consider the following algorithm for a given instance  $(G, w)$ : Find  $x \in VC^*(G, w)$  such that  $V_{\frac{1}{2}}^x = K_{\min}(G, w)$ , pick *any* vertex cover  $C'$  of  $G[K_{\min}]$ , and return a vertex cover  $C := V_1^x \cup C'$  of  $G$ .

As  $vc(G, w) = w(V_1^x) + vc(G[V_{\frac{1}{2}}^x], w) \geq w(V_1^x) + \frac{1}{2}w(V_{\frac{1}{2}}^x)$ , we obtain that the approximation factor of this algorithm is at most  $\frac{w(V_1^x) + w(V_{\frac{1}{2}}^x)}{w(V_1^x) + \frac{1}{2}w(V_{\frac{1}{2}}^x)} \leq 2$ . This factor is even strictly less than 2 on instances for which  $V_1^x$  is at least a fixed fraction of  $V_{\frac{1}{2}}^x$ . More precisely, if  $\delta|V_{\frac{1}{2}}^x| \leq 2(1 - \delta)|V_1^x|$ , then its approximation factor is at most  $2 - \delta$ .

Up to now only the property that  $vc(G[V_{\frac{1}{2}}^x], w) \geq \frac{1}{2}w(V_{\frac{1}{2}}^x)$  was used. But the fact that  $vc^*(G[V_{\frac{1}{2}}^x], w) = \frac{1}{2}w(V_{\frac{1}{2}}^x)$  (or, equivalently, that  $G[V_{\frac{1}{2}}^x]$  has a perfect fractional  $w$ -matching) can be used more efficiently. The stronger condition, that  $y \equiv \frac{1}{2}$  on  $V_{\frac{1}{2}}^x$  is the only element of  $VC^*(G[V_{\frac{1}{2}}^x], w)$  if  $x$  was such that  $V_{\frac{1}{2}}^x = K_{\min}(G, w)$ , can be used as well.

### The unweighted case

In what follows we will discuss the unweighted case, i.e., the case when  $w \equiv 1$ . For a fixed  $x \in VC^*(G)$  we obtain from Lemma 2 that

$$\begin{aligned} \nu(G) &= \nu(G[V_0^x \cup V_1^x]) + \nu(G[V_{\frac{1}{2}}^x]), \\ \nu^*(G) &= \nu^*(G[V_0^x \cup V_1^x]) + \nu^*(G[V_{\frac{1}{2}}^x]), \\ \nu(G[V_0^x \cup V_1^x]) &= \nu^*(G[V_0^x \cup V_1^x]) = |V_1^x|, \text{ and } \nu^*(G[V_{\frac{1}{2}}^x]) = \frac{1}{2}|V_{\frac{1}{2}}^x|. \end{aligned}$$

(Recall that by the LP duality  $vc^*(G) = \nu^*(G) \geq \nu(G)$ .)

Moreover, it is clear that for any maximum matching  $M$  in  $G$  all vertices of  $V_1^x$  are matched, and no edge within  $V_1^x$  belongs to  $M$ . It is well known that in the unweighted case the extremal maximum fractional matchings are half-integral. Hence  $G[V_{\frac{1}{2}}^x]$  has a perfect half-integral matching, and therefore the vertex set of  $G[V_{\frac{1}{2}}^x]$  can be covered by a set of vertex disjoint edges and odd cycles. This in turn shows that  $\nu(G[V_{\frac{1}{2}}^x]) \geq \frac{1}{3}|V_{\frac{1}{2}}^x|$ , with the equality iff all components of  $G[V_{\frac{1}{2}}^x]$  are triangles. In conjunction with inequalities above it implies that  $\nu(G) \geq \frac{2}{3}\nu^*(G)$ , with the equality iff all nontrivial components of  $G$  are triangles.

Consider the following simple approximation algorithm for MIN-VC for a given input instance  $G = (V, E)$ : find  $x \in VC^*(G)$  with  $V_{\frac{1}{2}}^x = K_{\min}(G)$  and pick any inclusionwise maximal matching  $M$  in  $G[V_{\frac{1}{2}}^x]$ . Let  $V_M$  be the set of vertices matched by  $M$ . Return a vertex cover  $C := V_1^x \cup V_M$  of  $G$ . Clearly,  $|C| \leq |V_1^x| + 2\nu(G[V_{\frac{1}{2}}^x])$ . Using Lorentzen bound  $vc(G) \geq 2\nu^*(G) - \nu(G)$  (applied to the graph  $G[V_{\frac{1}{2}}^x]$ ) we get

$$vc(G) = |V_1^x| + vc(G[V_{\frac{1}{2}}^x]) \geq |V_1^x| + |V_{\frac{1}{2}}^x| - \nu(G[V_{\frac{1}{2}}^x]).$$

For  $\delta \in [0, \frac{1}{3}]$  let  $\mathcal{G}_\delta = \{G : \nu(G) \leq (1 - \delta)\nu^*(G)\}$ . Hence  $\mathcal{G}_0$  is the set of all graphs, and  $\mathcal{G}_{\frac{1}{3}} = \{G : \text{each nontrivial component of } G \text{ is a triangle}\}$ . We have proved in [12] that the approximation threshold for the MIN-VC problem restricted to graphs with perfect matching is the same as for the problem in general graphs. This suggests that instances  $G = (V, E)$  with  $\nu^*(G) = \nu(G) = \frac{|V|}{2}$  are the hardest to approximate. We will show that the algorithm described above performs on graphs from  $\mathcal{G}_\delta$  with approximation factor at most  $2\frac{1-\delta}{1+\delta}$ . Notice that this varies from 2 to 1 when  $\delta$  varies from 0 to  $\frac{1}{3}$ .

Indeed,  $G \in \mathcal{G}_\delta$  means  $|V_1^x| + \nu(G[V_{\frac{1}{2}}^x]) \leq (1-\delta)(|V_1^x| + \frac{1}{2}|V_{\frac{1}{2}}^x|)$ , or equivalently  $\nu(G[V_{\frac{1}{2}}^x]) \leq \frac{1-\delta}{2}|V_{\frac{1}{2}}^x| - \delta|V_1^x|$ . Using that in the above inequalities for  $|C|$  and  $vc(G)$  we get  $|C| \leq (1-2\delta)|V_1^x| + (1-\delta)|V_{\frac{1}{2}}^x|$ , and  $vc(G) \geq (1+\delta)|V_1^x| + \frac{(1+\delta)}{2}|V_{\frac{1}{2}}^x|$ . Thus the approximation factor of this algorithm is at most

$$\frac{(1-2\delta)|V_1^x| + (1-\delta)|V_{\frac{1}{2}}^x|}{(1+\delta)|V_1^x| + \frac{(1+\delta)}{2}|V_{\frac{1}{2}}^x|} \leq 2\frac{1-\delta}{1+\delta}.$$

The left hand side provides even better estimate, if  $|V_1^x|$  is a significant fraction of  $|V_{\frac{1}{2}}^x|$ .

**Definition 10.** A graph  $G = (V, E)$  is called *regularizable* if it is possible to replace each edge  $e \in E$  with  $n(e) \geq 1$  multiple edges so that the resulting multigraph is regular. A graph  $G$  is called *Hamiltonian-connected* if every two distinct vertices are connected in  $G$  by a Hamiltonian path.

Recall that the problem kernel  $G[K_{\min}]$  has the property that  $y \equiv \frac{1}{2}$  on  $K_{\min}$  is the only element of  $VC^*(G[K_{\min}])$ . This in turn implies that instances  $G = (V, E)$  of MIN-VC for which  $y \equiv \frac{1}{2}$  on  $V$  is the only minimum half-integral vertex cover in  $G$ , are as hard to approximate as the general ones. Additionally to Theorem 7, the following characterization of these graphs was given by Berge [3] in the unweighted case. The solution  $y \equiv \frac{1}{2}$  on  $V$  is the only minimum half-integral vertex cover in  $G$  if and only if  $G$  is regularizable and each component of  $G$  is non-bipartite. It is easy to make such regularization of a regularizable graph  $G = (V, E)$  in polynomial time, and using standard multiplication techniques convert this regular multigraph to a regular graph in such way that the cardinality of a minimum vertex cover increases by a multiplicative factor that is easy to compute. This shows that regular instances of MIN-VC are as hard to approximate as the general ones.

A similar result can be obtained for Hamiltonian-connected graphs, as well. Let  $G = (V, E)$  be a graph for which  $y \equiv \frac{1}{2}$  on  $V$  is the only element of  $VC^*(G)$ . Then for some  $k \leq |V|$  the graph  $G[k]$  is Hamiltonian-connected, where  $G[k]$  is obtained from  $G$  by replacing each vertex with an independent set of size  $k$  (see [9]). But MIN-VC for  $G$  and  $G[k]$  are equally hard to approximate. This shows that Hamiltonian-connected instances of MIN-VC are as hard to approximate as the general ones.

**Corollary 2.** *The threshold on polynomial time approximability of the MIN-VC problem is the same as the one for this problem restricted to regular graphs or, alternatively, to Hamiltonian-connected graphs.*

## 6 Parametrized Complexity and Vertex Covers

The MINIMUM VERTEX COVER problem and its variants play a very special role among fixed-parameter tractable problems. To describe some applications of strong crown reductions for the parametrized version of the MINIMUM VERTEX COVER problem, we will confine ourselves to the unweighted MIN-VC problem in this section. But the results can be generalized in a straightforward way to the case of real weights  $w \geq 1$ .

Recall that for the parametrized *decision* version of the vertex cover problem the reduction to a problem kernel means to apply an efficient preprocessing on the instance  $(G, k)$  to construct another instance  $(G_1, k_1)$ , where  $k_1 \leq k$ , and  $G_1$  has a vertex cover with at

most  $k_1$  vertices iff  $G$  has a vertex cover with at most  $k$  vertices. As observed in [11], the Nemhauser-Trotter theorem allows to find efficiently a linear size problem kernel for MIN-VC. Namely, there is an algorithm of running time  $O(k|V| + k^3)$  that, given an instance  $(G = (V, E), k)$ , constructs another instance  $(G' = (V', E'), k')$  with the following properties:  $G'$  is an induced subgraph of  $G$ ,  $|V'| \leq 2k'$ ,  $k' \leq k$ , and  $G$  admits a vertex cover of size  $k$  iff  $G'$  admits a vertex cover of size  $k'$ . From our results it follows that this can be strengthened to  $|V'| \leq 2k' - 1$ , as the problem kernel  $G' = (V', E')$  ( $= G[K_{\min}]$ ) satisfies  $vc(G') > \frac{|V'|}{2}$ , by Theorem 10(iii).

Clearly, using the same technique one can modify this kernelization technique to the parametrized *search* version of the vertex cover problem: to find a minimum vertex cover of  $G$  if  $vc(G) \leq k$ , or report that  $vc(G) > k$ . We have  $(2k - 1)$  kernelization technique for those problems.

Unlike the Nemhauser-Trotter theorem, Theorem 6 can be used as efficient reduction to linear size problem kernel for the following problem: to find *all minimum vertex covers* if  $vc(G) \leq k$  or report that  $vc(G) > k$ .

PARAMETRIZED ALL-MIN-VC PROBLEM

*Parameter:*  $k > 0$  is a fixed constant

*Instance:* A graph  $G = (V, E)$

*Search version:* Either find all minimum vertex covers for  $G$  if  $vc(G) \leq k$ , or report that  $vc(G) > k$ .

**Theorem 11.** *There is an algorithm of running time  $O(k|V| + k^3)$  that for a given instance  $(G = (V, E), k)$  either reports that  $vc(G) > k$ , or finds a partition  $V = N \cup Y \cup V'$  such that  $G' := G[V']$ ,  $k' := k - |Y|$ ,  $vc(G') \geq \frac{1}{2}|V'|$ , and  $|V'| \leq 2k'$ . Moreover,  $vc(G) \leq k$  iff  $vc(G') \leq k'$ , and assuming  $vc(G) \leq k$ :*

(i) *for every minimum vertex cover  $C'$  for  $G'$ :  $C' \cup Y \in VC(G)$ , and*

(ii) *for every minimum vertex cover  $C$  for  $G$ :  $Y \subseteq C \subseteq Y \cup V'$  and  $C \cap V' \in VC(G')$ .*

*Proof.* Let an instance  $(G = (V, E), k)$  be given. Clearly, every vertex  $v \in V$  of degree at least  $k + 1$  has to belong to every vertex cover of size at most  $k$ , provided  $vc(G) \leq k$ . Denote by  $Y''$  the set of vertices of  $G$  of degree at least  $(k + 1)$ , by  $N''$  the set of isolated vertices of  $G \setminus Y''$ ,  $V'' := V \setminus (Y'' \cup N'')$ , and  $k'' = k - |Y''|$ . Firstly, in running time  $O(k|V|)$  we can construct a graph  $G'' = (V'', E'') := G[V'']$  (see, e.g., Buss [8] for such simple algorithm). Clearly,  $vc(G) \leq k$  iff  $vc(G'') \leq k''$ , and assuming  $vc(G) \leq k$ : (i) for every  $C'' \in VC(G'')$ ,  $C'' \cup Y'' \in VC(G)$ , and (ii) for every  $C \in VC(G)$ ,  $Y'' \subseteq C \subseteq Y'' \cup V''$  and  $C \cap V'' \in VC(G'')$ .

Each vertex of  $G''$  has degree at most  $k$ . Hence  $vc(G'') \leq k''$  is only possible if  $|E''| \leq kk''$ . If  $|E''| > kk''$ , we can report that  $vc(G) > k$  and the algorithm terminates. Otherwise, we have  $|E''| \leq k \cdot k'' (\leq k^2)$ , and since  $G''$  does not contain isolated vertices, it follows that  $|V''| \leq 2|E''| \leq 2k^2$ . Now we apply Theorem 6 to the graph  $G''$  (with  $w \equiv 1$ ). Namely, we partition the vertex set  $V''$  into three subsets  $V_0, V_1, V_{\frac{1}{2}}$  in time  $O(|E''|\sqrt{|V''|}) = O(k^3)$ . Further, we put  $Y := Y'' \cup V_1$ ,  $N := N'' \cup V_0$ ,  $V' := V_{\frac{1}{2}}$ ,  $G' := G[V']$ , and  $k' := k'' - |V_1| = k - |Y|$ . Obviously,  $vc(G) \leq k$  iff  $vc(G') \leq k'$ , and from Theorem 6 also  $vc(G') \geq \frac{1}{2}|V'|$ . Thus if  $|V'| > 2k'$ , we obtain that  $vc(G') > k'$ , hence  $vc(G) > k$ , and the algorithm terminates with reporting that. Otherwise  $|V'| \leq 2k'$  holds, as was required. All other properties follow directly from Theorem 6.  $\square$

Theorem 11 can be used to many other parametrized problems related to MIN-VC as reduction to linear size problem kernel. The typical example is the problem, whose task is to find one minimum vertex cover for  $G$  under some additional constraints.

## PARAMETRIZED CONSTRAINED-MIN-VC PROBLEM

*Instance:*  $(G = (V, E), k)$ ,  $k$  a nonnegative integer, and finitely many linear constraints  $P_1, P_2, \dots, P_r$  of the form  $P_i: \sum_{v \in V} a_i(v)x(v) \leq b_i$ ,  $i = 1, 2, \dots, r$ , where  $a_i(v), b_i \in \mathbb{R}$ .

*Task:* If  $vc(G) \leq k$  find  $C$  from  $VC(G)$ , whose indicator function  $x = x^C$  satisfies all constraints  $P_1, P_2, \dots, P_r$ , otherwise report that no such minimum vertex cover exists.

The most natural case is when each  $a_i(v)$  is either 0 or 1, and  $b_i$  are nonnegative integers. Then the constraint  $P_i$  says, that  $|C \cap A_i| \leq b_i$  for a set  $A_i := \{v \in V : a_i(v) = 1\}$  and for a vertex cover  $C \in VC(G)$  to be found. The problem has received considerable attention even in its very simplified version, when  $G = (V, E)$  is a bipartite graph with bipartition  $(L, R)$ , and two nonnegative integers  $k_L$  and  $k_R$  (with  $k = k_L + k_R$ ) are given as an input. The  $k_L$  and  $k_R$  represent constraints  $|C \cap L| \leq k_L, |C \cap R| \leq k_R$  on  $C \in VC(G)$  to be found. This problem arises from the extensively studied fault coverage problem for reconfigurable memory arrays in VLSI design, see [11] and references therein.

Theorem 11 clearly allows efficient reduction to the linear size problem kernel for PARAMETRIZED CONSTRAINED MIN-VC. Namely,  $(G = (V, E), k)$  with constraints  $P_i: \sum_{v \in V} a_i(v)x(v) \leq b_i$ ,  $i = 1, 2, \dots, r$  is reduced using Theorem 11 to  $(G' = (V', E'), k')$  with  $|V'| \leq 2k' (\leq 2k)$ , and with constraints  $P'_i: \sum_{v \in V'} a_i(v)x(v) \leq b'_i$  ( $:= b_i - \sum_{v \in Y} a_i(v)$ ),  $i = 1, 2, \dots, r$ .

## Concluding remarks

This paper provides systematic study of crown decompositions in weighted graphs. The crown reduction technique can be applied to several combinatorial problems (see, e.g., [14]). Our new decomposition theorems for minimum vertex covers have connections to problems of “parametrized enumerations” in the sense of listing all minimal solutions, as discussed in [18]. The technique developed in this paper may be a powerful tool in kernelization for other optimization problems with similar structure as the MINIMUM VERTEX COVER problem.

## References

1. F. N. Abu-Khazam, R. L. Collins, M. R. Fellows, M. A. Langston, W. H. Sutters and C. T. Symons, Kernelization Algorithms for the Vertex Cover Problem: Theory and Experiments, *Proceeding of Workshop on Algorithm Engineering and Experiments (ALENEX)*, New Orleans, Louisiana, January, 2004, pp. 62–69.
2. F. N. Abu-Khazam, M. A. Langston and W. H. Sutters, Fast, Effective Vertex Cover Kernelization: A Tale of Two Algorithms, *Proceedings, ACS/IEEE International Conference on Computer Systems and Applications*, Cairo, Egypt, January, 2005.
3. C. Berge, Regularizable graphs, *Advances in graph theory, Ann. Discrete Math.* 3(1978), 11–19.
4. P. Berman and T. Fujito, On Approximation Properties of the Independent Set Problem for Low Degree Graphs, *Theory Comput. Systems* 32(1999), 115–132.
5. E. Boros, M. C. Golumbic and V. E. Levit, On the number of vertices belonging to all maximum stable sets of a graph, *Discrete Applied Mathematics* 124(1-3)(2002), 17–25.
6. J.-M. Bourjolly, P. L. Hammer and B. Simeone, Node-weighted graphs having the König-Egerváry property, *Math. Programming Study* 22(1984), 44–63.
7. J. M. Bourjolly and W. R. Pulleyblank, König-Egerváry graphs, 2-bicritical graphs and fractional matchings, *Discrete Applied Mathematics* 24(1989), 63–82.
8. J. F. Buss and J. Goldsmith, Nondeterminism within  $P$ , *SIAM Journal on Computing* 22(3)(1993), 560–573.
9. G. J. Chang and X. Zhu, Pseudo-Hamiltonian-connected graphs, *Discrete Applied Mathematics* 100(2000), 145–153.
10. J. Chen, I. A. Kanj and W. Jia, Vertex cover: further observations and further improvements, *Journal of Algorithms* 41(2001), 280–301.
11. J. Chen and I. A. Kanj, On constrained minimum vertex covers of bipartite graphs: improved algorithms, *Journal of Computer Systems Sci.* 67(2003), 833–847.

12. M. Chlebík and J. Chlebíková, On Approximation Hardness of the Minimum 2SAT-DELETION Problem, Proceedings of 29th Mathematical Foundations of Computer Science (MFCS'04), *LNCS* 3153, 2004, pp. 263–273, to appear in *Discrete Applied Mathematics*.
13. M. Chlebík and J. Chlebíková, Optimal version of Nemhauser-Trotter Theorem, Proc. of the 9th Scandinavian Workshop on Algorithm Theory (SWAT'04), *LNCS* 3111, 2004, pp. 174–186.
14. B. Chor, M. Fellows, and D. Juedes, Linear kernels in linear time, or how to save  $k$  colors in  $O(n^2)$  steps. Proceedings of 30th International Workshop on Graph Theoretic Concepts in Computer Science (WG'04), *LNCS* 3353, 2004, pp. 257–269.
15. F. Dehne, M. Fellows, and F. Rosamond, An FPT algorithm for set splitting, Proceedings of 29th International Workshop on Graph Theoretic Concepts in Computer Science (WG'03), *LNCS* 2880, 2003, pp. 180–191.
16. F. Dehne, M. Fellows, F. Rosamond, and P. Shaw, Greedy Localization, Iterative Compression, and Modeled Crown Reductions: New FTP Techniques, and Improved Algorithm for Set Splitting, and a Novel  $2k$  Kernelization for Vertex Cover, Proceedings of the International Workshop on Parametrized and Exact Computation (IWPEC'04), *LNCS* 3162, 2004, pp. 271–280.
17. I. Dinur and S. Safra, The importance of being biased, *Proceedings of the 34th ACM Symposium on Theory of Computing* (STOC'02), 2002, pp. 33–42.
18. R. G. Downey and M. R. Fellows, *Parametrized Complexity*, Springer-Verlag, 1999.
19. H. Fernau, On parameterized enumeration, Proceedings of the 8th Annual International Computing and Combinatorics Conference (COCOON'02), *LNCS* 2383, 2002, pp. 564–573.
20. M. R. Fellows, Blow-Ups, Win/Win's and Crown Rules: Some New Directions in FPT, Proceedings of 29th International Workshop on Graph Theoretic Concepts in Computer Science (WG'03), *LNCS* 2880, 2003, pp. 1–12.
21. A. V. Goldberg and R. E. Tarjan, A new approach to the maximum-flow problem, *Journal of ACM* 35(1988), 921–940.
22. M. Fellows, P. Heggernes, F. Rosamond, C. Sloper, and J. A. Telle, Exact Algorithms for Finding  $k$  Disjoint Triangles in an Arbitrary Graph, Proceedings of the 30th Workshop on Graph Theoretic Concepts in Computer Science (WG'04), *LNCS* 3353, 2004, pp. 235–244.
23. P. Hall, On representatives of subsets, *Journal of London Math. Soc.* 10(1935), 26–30.
24. P. L. Hammer, P. Hansen, and B. Simeone, Vertices belonging to all or to no maximum stable sets of a graph, *SIAM J. Alg. Disc. Meth.* 3(4)(1982), 511–522.
25. J. Hopcroft and R. M. Karp, An  $O(n^{2.5})$  algorithm for maximum matching in bipartite graphs, *SIAM J. Computing* 2(1973), 225–231.
26. E. L. Lawler, *Combinatorial Optimization: Networks and Matroids*, Holt, Rinehart, and Winston, 1976.
27. L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland, Amsterdam, 1986.
28. S. Micali and V. V. Vazirani, An  $O(\sqrt{|V||E|})$  algorithm for finding maximum matching in general graphs, *Proc. of 21st IEEE Symposium on Foundation of Computer Science*, Syracuse, New York, 1980, pp. 17–27.
29. L. Mathieson, E. Prieto, and P. Shaw, Packing Edge Disjoint Triangles: A parametrized View, Proceedings of the International Workshop on Parametrized and Exact Computation (IWPEC'04), *LNCS* 3162, 2004, pp. 127–137.
30. G. L. Nemhauser and L. E. Trotter, Properties of vertex packing and independence system polyhedra, *Math. Programming* 6(1974), 48–61.
31. G. L. Nemhauser and L. E. Trotter, Vertex packings: structural properties and algorithms, *Math. Programming* 8(1975), 232–248.
32. R. Niedermeier and P. Rossmanith, On efficient fixed parameter algorithms for weighted vertex cover, *Journal of Algorithms* 47(2)(2003), 63–77.
33. J. C. Picard and M. Queyranne, On the integer valued variables in the linear vertex packing problem, *Math. Programming* 12(1977), 97–101.
34. E. Prieto and C. Sloper, Either/Or: Using Vertex Cover Structure in Designing FTP Algorithms – the Case of  $k$ -Internal Spanning Tree, Proceedings of the Workshop on Algorithms and Data Structures (WADS'03), *LNCS* 2748, 2003, pp. 474–483.
35. E. Prieto and C. Sloper, Looking at the Stars, Proceedings of the International Workshop on Parametrized and Exact Computation (IWPEC'04), *LNCS* 3162, 2004, pp. 138–148.
36. W. R. Pulleyblank, Minimum node covers and 2-bicritical graphs, *Math. Programming* 17(1979), 91–103.
37. A. Schrijver, *Combinatorial Optimization*, Springer, 2003.