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A generalisation of the Pucci–Saccomandi model of rubber elasticity☆

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ABSTRACT

The Pucci–Saccomandi model of rubber elasticity, also known as the Gent–Gent model, is generalised in this manuscript for a more enhanced applicability to a wider range of finite deformation behaviours, rubber-like material types, and an increased modelling accuracy. The generalising of interest is developed by devising a higher order rational approximant, i.e., of order [1/1], of the response functions. The proposed model is shown to be a *parent* to some of the celebrated limiting chain extensibility, classical $W(I_1, I_2)$ and generalised neo-Hookean models. The application of the proposed generalised model to the finite deformation of various isotropic incompressible rubber-like materials is demonstrated, including natural unfilled and filled rubbers, commercial polymers and biomaterials. The considered datasets exhibit various deformation behaviours from the classical large deformation of natural rubbers to the downward concavity of filled rubbers, strain/shear softening behaviours of polymers and the softening behaviour of biomaterials up to the onset of failure. The modelling results are directly compared with those of the Gent–Gent model, where the higher accuracy of the proposed model and its capability to favourably capture the aforementioned challenging and varied behaviours is established. In particular, the Gent–Gent model is shown unsuitable for capturing such downward concavity, shear/strain softening and continuous softening to failure behaviours. Given the improved modelling predictions, simplicity of the functional form of the devised model, and its relatively low number of model parameters, the proposed model is presented as a more general, comprehensive and versatile modelling tool for application to the finite deformation of various isotropic incompressible soft materials.

1. Introduction

In a seminal paper in 2002, Pucci and Saccomandi [1] proposed an enhancement to the original Gent model [2] by adding a logarithmic I_2 term of the form:

$$g(I_2) = C_2 \ln\left(\frac{I_2}{3}\right), \tag{1}$$

to the generalised neo-Hookean part

$$f(I_1) = -\frac{\mu_0}{2} J_m \ln\left(1 - \frac{I_1 - 3}{J_m}\right), \tag{2}$$

rendering the well-known binomial strain energy function:

$$W_{GG} = f(I_1) + g(I_2) = -\frac{\mu_0}{2} J_m \ln\left(1 - \frac{I_1 - 3}{J_m}\right) + C_2 \ln\left(\frac{I_2}{3}\right). \tag{3}$$

The I_2 term in Eq. (1) is indeed that of Gent and Thomas [3], which is why the proposed model by Pucci and Saccomandi is also known as the Gent-Gent model (and hence the subscript GG in Eq. (3) for W).

The impetus behind the Pucci–Saccomandi proposition stemmed from beyond a mere addition of an I_2 term to an otherwise incomplete generalised neo-Hookean $W(I_1)$ model. This incompleteness had already been noted by Horgan and Saccomandi [4] in a prior work in reference to universal relationships of finite elasticity using, for example, the simple torsion deformation, and the insufficiency of generalised neo-Hookean models to capture the whole-range properties of deformation were discussed. A detailed treatise on universal relationships of finite elasticity was later presented by Saccomandi [5], and the succeeding works of Saccomandi and co-workers continued to illustrate the necessity of including an I_2 term in the strain energy function of incompressible hyperelastic materials for capturing nuanced behaviours such as the Poynting effect (see, e.g., [6]) or the two-dimensional motion of shear waves (see, e.g., [7]). But rather, their addition of the I_2 term was motivated by an astute observation that there exist deviations between the predictions of the Gent model and the data at lower ranges of uniaxial deformation, noticeable in the Mooney space [1].

The Mooney space, though explicitly termed as such by Destrade et al. [8], was first introduced by Rivlin and Saunders [9] for fitting the experimental data obtained from uniaxial tension tests on rubber specimens to the Mooney–Rivlin model. Instead of using the

☆ Dedicated to Professor Giuseppe Saccomandi; a great mentor and a *treasure of wisdom and knowledge*.

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traditional stress–stretch plots, Rivlin and Saunders transformed the data into a space where the abscissa is λ^{-1} and the ordinate is the reduced engineering stress; i.e., divided by a factor $2\left(\lambda-\lambda^{-2}\right)$. Such a transformation simplifies the problem of fitting to a linear regression – a clear advantage given the available computational power at the time, and was also used in other modelling endeavours such as that by Gent and Thomas [3]. The later work of Prof. Saccomandi was instrumental for generalising the application of the Mooney space to other models beyond that of Mooney–Rivlin; see [10] for the formulation of the generalised Mooney space. However, the point of interest for Pucci and Saccomandi [1] was the fact that in the Mooney space, instead of dealing with the classical deformation domain $1 \leq \lambda < \infty$, one deals only with the range $0 < \lambda^{-1} \leq 1$, which magnifies the small deformation range, and thereby the performance of a given model in that range. Using the canonical data of Treloar [11], they showed the inadequacy of the Gent model at small deformation ranges, and the failure of this model in capturing the 'upturn' in the experimental data. As a remedy, they proposed the addition of the logarithmic I_2 term, which has the "*slowest growth possible for* $\lambda^{-1} \to 0$" [1], so that the good performance of the Gent model in high strains (i.e. the strain stiffening region) remained untouched.

However, despite the elegance of the Pucci–Saccomandi proposition, the ensuing Gent-Gent model W_{GG} still suffers from other, nonetheless fundamental, shortcomings. For example, the author's previous works have shown that Gent and Gent-Gent models do not capture well the shear/strain softening behaviours (e.g., [12,13]). Similarly, capturing the deformation of filled rubbers has proved to be a challenge for the Gent and Gent-Gent models (see, e.g., [13–15]), as well as other polymeric materials that exhibit a downward concavity in their stress – deformation behaviours [13]. These drawbacks are not a shortcoming of the rationale of the Pucci–Saccomandi proposition *per se*, but are related to the functional forms of the Gent and Gent-Gent models which do not offer a versatile solution in relation to the foregoing deformations and material types. It may now therefore be in order to devise a more generalised limiting chain extensibility (LCE) model, *parent* to the same family which the Gent and Gent-Gent models belong to, for improved modelling results and a more universality of application to various soft solids. Such an undertaking is of particular interest given the increased application of constitutive models in the finite deformation of soft solids, and the growing demand for more accurate modelling results and applicability across various elastomers from natural and filled rubbers to more advanced soft materials such as hydrogels, biomaterials and liquid crystal elastomers etc.

Accordingly, my aim in this manuscript is to present a generalisation of the Gent-Gent model by way of showcasing a more *universal*, or better put a more *comprehensive*, LCE model of the same family of, but *parent* to, the Gent and Gent-Gent models. The application of this generalised model to a range of extant experimental data will be presented, and direct comparisons with the Gent-Gent model will be provided to illustrate the marked improvements. The method by which the generalisation of interest will be arrived at is based on that of a previous work by the author [16], namely by devising higher order rational approximants of the response functions $2W_1$ and $2W_2$ (where the subscripts 1 and 2 denote the partial derivative of the model W with respect to the first and second principal invariants of the Cauchy–Green deformation tensors; i.e., I_1 and I_2, respectively) compared with those of Gent and Gent-Gent models. This higher order consideration, as will be demonstrated here, will produce the *parent* model with more accurate predictions of the finite deformation behaviour of elastomers, exemplified by the datasets which will be considered in the following. In Section 2 the theoretical underpinnings of the generalised model will be presented. The application of the model to some archetypal datasets including Treloar's multiaxial data on natural rubber [11], filled rubber of Lahellec et al. [17], a commercial polymer (Flextec®FT 101) specimen under uniaxial and simple shear, and pure shear and simple shear, deformations due to Nunes and Moreira [18] and Moreira and Nunes [19], respectively, and the softening behaviours of (whey) protein gels due to Liu and Böl [20] and abdominal aortic aneurysm samples due to Volokh [21] under uniaxial loading will be considered in Section 3, and the results will be directly compared with those obtained on using the Gent-Gent model. Clear improvements in fitting results facilitated by the generalised model will be demonstrated. Concluding remarks will be conferred in Section 4.

2. The generalised model

The generalisation of interest here is pursued by constructing a strain energy function W of binomial form:

$$W\left(I_1, I_2\right) = f\left(I_1\right) + g\left(I_2\right) , \tag{4}$$

where $f(\bullet)$ and $g(\bullet)$ are functions of their respective arguments, I_1 and I_2. In the following we will arrive at the desired $W\left(I_1, I_2\right)$ model by proposing specific generalisations for each of the $f(\bullet)$ and $g(\bullet)$ functions, via developing higher order rational approximants of the response functions in respect of the Gent-Gent model.

2.1. Generalisation of the Gent model

For uniformity of presentation, let us adopt the notation by Beatty [22] where the representation formula for the Cauchy stress is given by:

$$\mathbf{T} = -p\,\mathbf{I} + \beta_1\,\mathbf{B} + \beta_{-1}\mathbf{B}^{-1} , \tag{5}$$

with $\mathbf{B}$ the left Cauchy–Green deformation tensor and $\mathbf{B}^{-1}$ its inverse, p the arbitrary Lagrange multiplier enforcing the condition of incompressibility, $\mathbf{I}$ the identity tensor, and β_1 and β_{-1} the so-called response functions of the chosen material law, defined as:

$$\beta_1 = 2W_1 = 2\frac{\partial W}{\partial I_1} , \quad \beta_{-1} = -2W_2 = -2\frac{\partial W}{\partial I_2} . \tag{6}$$

Note that $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^3$ and $I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}$ are the first and second principal invariants of $\mathbf{B}$, while $I_3 = 1$ due to incompressibility. It was shown by Beatty [22] that the response function β_1 for the Gent model is the simplest rational (Padé) approximant in I_1, of the order [0/1], in the following form:

$$\left(\beta_1\right)_G = \frac{a}{c + d\,I_1} , \tag{7}$$

where the subscript G denotes 'Gent' model, and a, c and d are arbitrary coefficients. Integrating $\left(\beta_1\right)_G$ in Eq. (7) and setting $a = \mu_0\,J_m$, $c = J_m+3$ and $d = -1$, will result in the Gent model:

$$W_G = \frac{1}{2}\int_{I_1=3}^{I_1} \left(\beta_1\right)_G d\,I_1 = -\frac{\mu_0}{2}\,J_m \ln\left(1 - \frac{I_1 - 3}{J_m}\right) . \tag{8}$$

However, as posited by the author in [16], a straightforward generalisation, and improvement to the accuracy of the Gent model for capturing the limiting chain extensibility behaviour, is obtained by considering β_1 as a higher order rational approximant of I_1, of the following [1/1] order:

$$\beta_1 = \frac{a + b\,I_1}{c + d\,I_1} , \tag{9}$$

where, again, a, b, c and d are arbitrary coefficients. On imposing the requirements that: (i) at a limiting I_1 value we must have $\beta_1 \to \infty$, and (ii) $\beta_1\left(I_1 = 3\right) = \mu_0$, and after some mathematical manipulations (see [16] for exact details and derivations) we arrive at:

$$\beta_1 = \frac{1}{n}\mu\frac{-3nN + I_1}{-3N + I_1} , \tag{10}$$

where N may be viewed as the number of links, Kuhn segments, in the molecular chain network and μ is a material parameter related to the infinitesimal shear modulus μ_0 as:

$$\mu = \mu_0\,\frac{n(1-N)}{1-nN} , \tag{11}$$

where we have here made use of Eq. (10) and the fact that $\beta_1 \left(I_1 = 3\right) = \mu_0$. Integrating the response function β_1 in Eq. (10), subject to the condition that $W\left(I_1 = 3\right) = 0$, leads to the following generalised neo-Hookean strain energy function $f\left(I_1\right)$:

$$f\left(I_1\right) = \frac{3\left(n-1\right)}{2n}\,\mu N\left[\frac{1}{3N\left(n-1\right)}\,\left(I_1-3\right)-\ln\left(\frac{I_1-3N}{3-3N}\right)\right]. \quad (12)$$

While in the original derivation it was required that $N > 1$ and $n \in \mathbb{N}$ [16], (see also the review by Puglisi and Saccomandi [23] for a microstructural interpretation of the parameters derived from the kinetic molecular theory of rubber elasticity), from a phenomenological point of view these restrictions can be relaxed to:

$$\begin{cases} N > 0\,, \\ n > 0\,, \end{cases} \quad (13)$$

with the empirical inequalities remaining intact and unviolated. See [12,13] for analysis and discussion.

The strain energy function in Eq. (12) is a more generalised form of the Gent model. It can be verified that at the limit $n \to \infty$ we will have:

$$\lim_{n\to\infty} f\left(I_1\right) = -\frac{1}{2}\left(3N-3\right)\mu_0\ln\left(\frac{I_1-3N}{3-3N}\right), \quad (14)$$

where we have substituted for μ in Eq. (12) using Eq. (11). By setting the limiting stretch parameter J_m in the Gent model as: $J_m = 3N-3$, Eq. (14) readily gives:

$$\lim_{n\to\infty} f\left(I_1\right) = -\frac{1}{2}J_m\mu_0\ln\left[\frac{I_1-\left(J_m+3\right)}{-J_m}\right] = -\frac{\mu_0}{2}J_m\ln\left(1-\frac{I_1-3}{J_m}\right), \quad (15)$$

i.e., $\lim_{n\to\infty} f\left(I_1\right) = W_G$. Thus, the $f\left(I_1\right)$ function in Eq. (12) is a *parent* to the Gent model. Similarly, by fixing the value of n to $n = 3$, one recovers the model by Anssari-Benam and Bucchi [24,25]:

$$W_{ABB} = \mu N\left[\frac{1}{6N}\left(I_1-3\right)-\ln\left(\frac{I_1-3N}{3-3N}\right)\right]. \quad (16)$$

The $f\left(I_1\right)$ function in Eq. (12) therefore appears a comprehensive representation of the generalised neo-Hookean LCE models.

2.2. Generalisation of the I_2 term

In the same manner as β_1 in Eq. (9), one may consider a [1/1] rational approximant for the response function β_{-1} in I_2 as:

$$\beta_{-1} = \frac{A+B\,I_2}{C+D\,I_2}. \quad (17)$$

However, using the experimental data of Dickie and Smith [26], Gent has demonstrated that the limiting extensibility is governed by the value of I_1 alone [27]. Subsequently, Beatty [22] made the case that an independent limit on I_2 will be at odds with the notion of a single value of limiting extensibility. Following this school of thought, one may consider $C = 0$ in Eq. (17). Integrating the ensuing expression of β_{-1} gives the function $g\left(I_2\right)$ as:

$$g\left(I_2\right) = -\frac{1}{2}\int\frac{A+B\,I_2}{D\,I_2}\,dI_2 = -\frac{1}{2D}\left[B\left(I_2-3\right)+A\ln\left(\frac{I_2}{3}\right)\right], \quad (18)$$

subject to the condition that $g\left(I_2 = 3\right) = 0$. Empirically, if one chooses:

$$A = 1\,, \quad B = \frac{1}{3\left(m-1\right)}\,, \quad D = -\frac{m}{3C_2\left(m-1\right)}\,, \quad (19)$$

we arrive at the following functional form for $g\left(I_2\right)$:

$$g\left(I_2\right) = \frac{3\left(m-1\right)}{m}\,C_2\left[\frac{I_2-3}{3\left(m-1\right)}+\ln\left(\frac{I_2}{3}\right)\right]. \quad (20)$$

Note that $C_2 \geq 0$ is a stress-like material constant, and m is a non-dimensional parameter subject to $m > 0$. The reason for the specific choice of A, B and D in Eq. (19) is two-fold: (i) the resulting functional form of $g\left(I_2\right)$ in Eq. (20) is harmonious with that of $f\left(I_1\right)$ in Eq. (12); and (ii) at the limits of $m \to 1$ and $m \to \infty$ two well-known sub-set I_2 terms are recovered:

$$\begin{cases} \lim_{m\to1} g\left(I_2\right) = C_2\left(I_2-3\right)\,, \\ \lim_{m\to\infty} g\left(I_2\right) = 3C_2\ln\left(\frac{I_2}{3}\right)\,, \end{cases} \quad (21)$$

which are, of course, the classical I_2 terms of Mooney–Rivlin [28] and Gent and Thomas [3] (vis-à-vis Pucci and Saccomandi [1]), respectively. Note that multiplier 3 can be absorbed into coefficient C_2 in Eq. $(21)_2$. It is therefore observed that the response function β_{-1} in Eq. (17), and equivalently the $g\left(I_2\right)$ function in Eq. (20), is a generalisation of, and a *parent* to, the Pucci–Saccomandi adjunct I_2 term proposition.

Remark 1. While the $C_2\left(I_2-3\right)$ term can be shown to have a structural molecular basis, derived from the topological tube model of rubber molecules (see, e.g., [29]) or by relaxing the assumption of affine deformation of the chains from the statistical molecular theory (see, e.g., [30]), the structural origins of the logarithmic I_2 term remain less well-known.

2.3. The functional form of the generalised model $W\left(I_1, I_2\right)$

With the functional forms of $f\left(I_1\right)$, given in Eq. (12), and $g\left(I_2\right)$, given in Eq. (20), now at hand, it is straightforward to present the generalisation of the Pucci–Saccomandi model as the following binomial strain energy function W:

$$W = \frac{3\left(n-1\right)}{2n}\,\mu N\left[\frac{1}{3N\left(n-1\right)}\left(I_1-3\right)-\ln\left(\frac{I_1-3N}{3-3N}\right)\right] + \frac{3\left(m-1\right)}{m}\,C_2\left[\frac{I_2-3}{3\left(m-1\right)}+\ln\left(\frac{I_2}{3}\right)\right], \quad (22)$$

with five model parameters, namely μ [Pa], N [-], n [-], C_2 [Pa] and m [-]. The infinitesimal shear modulus μ_0 for this model is:

$$\mu_0 = 2\left[W_1\left(I_1=3\right)+W_2\left(I_2=3\right)\right] \implies \mu_0 = \frac{1}{n}\,\mu\frac{1-nN}{1-N}+2C_2\,. \quad (23)$$

It is readily observed that the model in Eq. (22) is the *parent* to the Gent-Gent model, reducing to that model at the limit when:

$$\lim_{n\to\infty\,,\,m\to\infty} W = W_{GG}\,. \quad (24)$$

The $W\left(I_1, I_2\right)$ models considered in [31] are also recovered from the generalised model in Eq. (22) when $\{n = 3, m \to 1\}$ and $\{n = 3, m \to \infty\}$. The classical Mooney–Rivlin model is recovered from the W function in Eq. (22) too at the limit:

$$\lim_{N\to\infty\,,\,m\to1} W = W_{MR}\,. \quad (25)$$

It can be similarly verified that the generalised model in Eq. (22) is the *parent* to more existing $W\left(I_1\right)$ and $W\left(I_1, I_2\right)$ models in the literature.

2.4. The stress – deformation relationships

On using the representation formula in Eq. (5), and the model in Eq. (22), stress – deformation relationships for various boundary value problems may be derived. In accordance with the experimental data considered for this study, in the following these relationships are presented for uniaxial, equi-biaxial, pure and simple shear deformations.

Accordingly, for uniaxial deformation we have:

$$T_{uni} = \frac{\mu}{n}\,\frac{I_1-3nN}{I_1-3N}\left(\lambda^2-\lambda^{-1}\right)+\frac{2C_2}{m}\,\frac{I_2-3+3m}{I_2}\left(\lambda-\lambda^{-2}\right), \quad (26)$$

where $I_1 = \lambda^2 + 2\lambda^{-1}$ and $I_2 = 2\lambda + \lambda^{-2}$. In equi-biaxial deformation:

$$T_{eq-bi} = \frac{\mu}{n}\,\frac{I_1 - 3nN}{I_1 - 3N}\left(\lambda^2 - \lambda^{-4}\right) + \frac{2C_2}{m}\,\frac{I_2 - 3 + 3m}{I_2}\left(\lambda^4 - \lambda^{-2}\right), \qquad (27)$$

with $I_1 = 2\lambda^2 + \lambda^{-4}$ and $I_2 = \lambda^4 + 2\lambda^{-2}$. Pure shear deformation renders:

$$T_{ps} = \left[\frac{\mu}{n}\,\frac{I_1 - 3nN}{I_1 - 3N} + \frac{2C_2}{m}\,\frac{I_2 - 3 + 3m}{I_2}\right]\left(\lambda^2 - \lambda^{-2}\right), \qquad (28)$$

where $I_1 = I_2 = \lambda^2 + 1 + \lambda^{-2}$. Finally, for the case of simple shear deformation where the deformation gradient $\mathbf{F}$ is:

$$\mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (29)$$

the resulting $T - \gamma$ relationship becomes:

$$T_{ss} = 2\gamma\left(W_1 + W_2\right) \implies T_{ss} = \gamma\left[\frac{\mu}{n}\,\frac{I_1 - 3nN}{I_1 - 3N} + \frac{2C_2}{m}\,\frac{I_2 - 3 + 3m}{I_2}\right], \qquad (30)$$

with $I_1 = I_2 = 3 + \gamma^2$.

For the purposes of modelling comparison with the Gent-Gent model in Section 3, we shall require the (Cauchy)stress – deformation relationships for this model too. These are:

$$\begin{cases} \left(T_{uni}\right)_{GG} = \mu_0 J_m \frac{1}{I_1 - 3 - J_m}\left(\lambda^{-1} - \lambda^2\right) + \frac{2C_2}{I_2}\left(\lambda - \lambda^{-2}\right), \\ \left(T_{eq-bi}\right)_{GG} = \mu_0 J_m \frac{1}{I_1 - 3 - J_m}\left(\lambda^{-4} - \lambda^2\right) + \frac{2C_2}{I_2}\left(\lambda^4 - \lambda^{-2}\right), \\ \left(T_{ps}\right)_{GG} = \left[\mu_0 J_m \frac{1}{I_1 - 3 - J_m} - \frac{2C_2}{I_2}\right]\left(\lambda^{-2} - \lambda^2\right), \\ \left(T_{ss}\right)_{GG} = \gamma\left[-\mu_0 J_m \frac{1}{I_1 - 3 - J_m} + \frac{2C_2}{I_2}\right], \end{cases} \qquad (31)$$

where the subscript GG indicates that the relationships pertain to the Gent-Gent model. Note that I_1 and I_2 are specific to each mode of deformation.

Depending on the experimental data, the appropriate stress – deformation relationship(s) in Eqs. (26), (27), (28) and (30) for the generalised model, and those in Eq. (31) for the Gent-Gent model, will be employed for fitting with the data, via a fitting procedure described in the next section.

2.5. The fitting procedure

The appropriate Cauchy T stress – deformation relationships in Eqs. (26), (27), (28) and (30), or equivalently those in Eq. (31) for the Gent-Gent model, are simultaneously fitted to the relevant deformation modes from each considered dataset. The best fit is sought by minimising the residual sum of squares (RSS) function defined as: $RSS = \sum_i \left(T^{model} - T^{experiment}\right)_i^2$, where i is the number of data points. This minimisation is performed via an in-house developed code in MATLAB®, using the genetic algorithm (GA) function. The coefficient of determination R^2 values are reported as a measure of the goodness of the obtained fits, as well as the ensuing relative error (%) calculated as: $\left|\frac{T^{model} - T^{experiment}}{T^{experiment}}\right| \times 100$. This procedure has been employed in previous publications by the author, on using various measures of stress and/or RSS functions — see, e.g., [25,32,33].

3. Application and improved results

It is perhaps reasonable to start with Treloar's canonical dataset on natural unfilled rubber specimens [11], which the Pucci–Saccomandi improvement to the Gent model was originally based on. Eqs. (26), (27) and (28) for the generalised model, and $(31)_1$, $(31)_2$ and $(31)_3$ for the Gent-Gent model, were simultaneously fitted to the uniaxial, equi-biaxial and pure shear deformation datasets. The fitting results are presented in Fig. 1, and Table 1 summarises the identified model parameter values.

Table 1
Model parameter values for the Treloar [11] dataset.

The generalised model						
	μ [MPa]	N [–]	n [–]	C_2 [MPa]	m [–]	R^2
Uniaxial Equi-Biaxial Pure Shear	0.27	25.37	2.83	0.04	20.17_5	0.99_9 0.99_9 0.99_9
The Gent-Gent model						
	μ_0 [MPa]		J_m [–]		C_2 [MPa]	R^2
Uniaxial Equi-Biaxial Pure Shear	0.24		77.77		0.46	0.99_9 0.98_6 0.99_7

Table 2
Model parameter values for the filled rubber specimens of Lahellec et al. [17].

The generalised model						
	μ [MPa]	N [–]	n [–]	C_2 [MPa]	m [–]	R^2
Uniaxial Simple Shear	1.84_5	0.99	0.99	0.06	0.65	0.99_9 0.99_9
The Gent-Gent model						
	μ_0 [MPa]		J_m [–]		C_2 [MPa]	R^2
Uniaxial Simple Shear	1.72		99.15		0.84	0.99 0.92

The results indicate that while the Gent-Gent model captures the uniaxial data well (as was intended by Pucci and Saccomandi [1]), the equi-biaxial and pure shear simulations deviate from the data. In the case of equi-biaxial deformation, at both lower and higher ranges of deformation the relative error is increased, while for pure shear data the deviation is observed at low- and mid-ranges of deformation. The generalised model, on the other hand, provides much improved fits with higher accuracy — see the relative error plots in Fig. 1.

It is noted that the generalised model includes five model parameters, versus that of the Gent-Gent model with three. However, the reader is reminded that even the generalised neo-Hookean part of the proposed model here on its own, i.e., $f\left(I_1\right)$ in Eq. (12), with three parameters provides more accurate fits than the Gent-Gent model to this dataset; see the plots in Figure 1 of [13]. Therefore, it is *not* simply the case here that the generalised model is providing better fits merely due to the additional two model parameters. Rather, it is a case of the *parent* model having a more comprehensive functional form than the subset Gent-Gent model.

It may be informative to consider next the experimental data of Lahellec et al. [17] on uniaxial and simple shear deformations of a commercial *filled* rubber sample. Filled rubbers, similar to some polymeric materials, exhibit a pronounced downward concavity in their stress – deformation curves (as shown in Fig. 2). Capturing this downward concavity, as well as the rest of the deformation, has proved challenging for many of the existing constitutive models in the literature (e.g., see [13] for a comparison of various models). Indeed, apart from the family of models recently proposed by the author (e.g., [13,33,34]), the only other model directly known to me with this capability is that of Lopez-Pamies [14]. However, that is an I_1-based model and as such all the usual disadvantages of generalised neo-Hookean models may be extended to this model too (and also to those of [13,25]). Therefore, having a $W\left(I_1, I_2\right)$ model with this capability would be an advantage. In this regard, the plots of Fig. 2 illustrate the modelling results for the dataset of Lahellec et al. [17] on using both the generalised and Gent-Gent models. The obtained model parameter values have been presented in Table 2.

It is observed that the Gent-Gent model proves inadequate for capturing this dataset of filled rubber specimens, while the generalised model provides most favourable fits. Note again that the generalised neo-Hookean part of the proposed model here, i.e., $f\left(I_1\right)$ in Eq. (12),

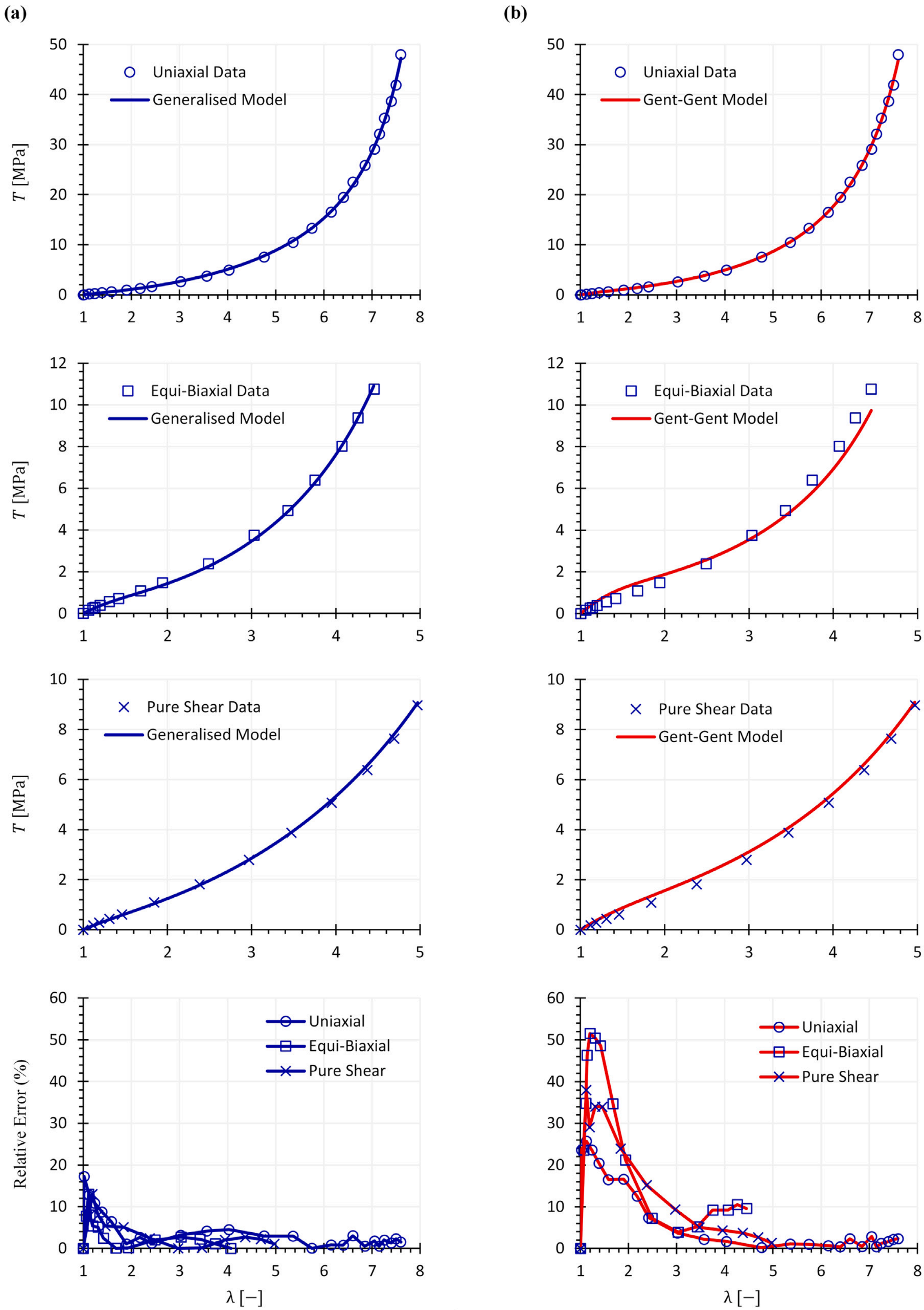

Fig. 1. Modelling results for the multiaxial data of Treloar [11] on natural unfilled rubber specimens: (a) the generalised model; and (b) the Gent-Gent model. The relative error plots (bottom panels) have been presented in the same scale for a better appreciation of the improved results.

on its own provides much favourable fits to this dataset than the Gent-Gent model; see Figure 5 of [13]. Therefore, this is another argument for the point made earlier in relation to the results obtained using Treloar's data [11] that it is *not* only the additional two model parameters that are resulting the better fits; but the functional form of the proposed *parent* model is richer than that of the Gent-Gent model.

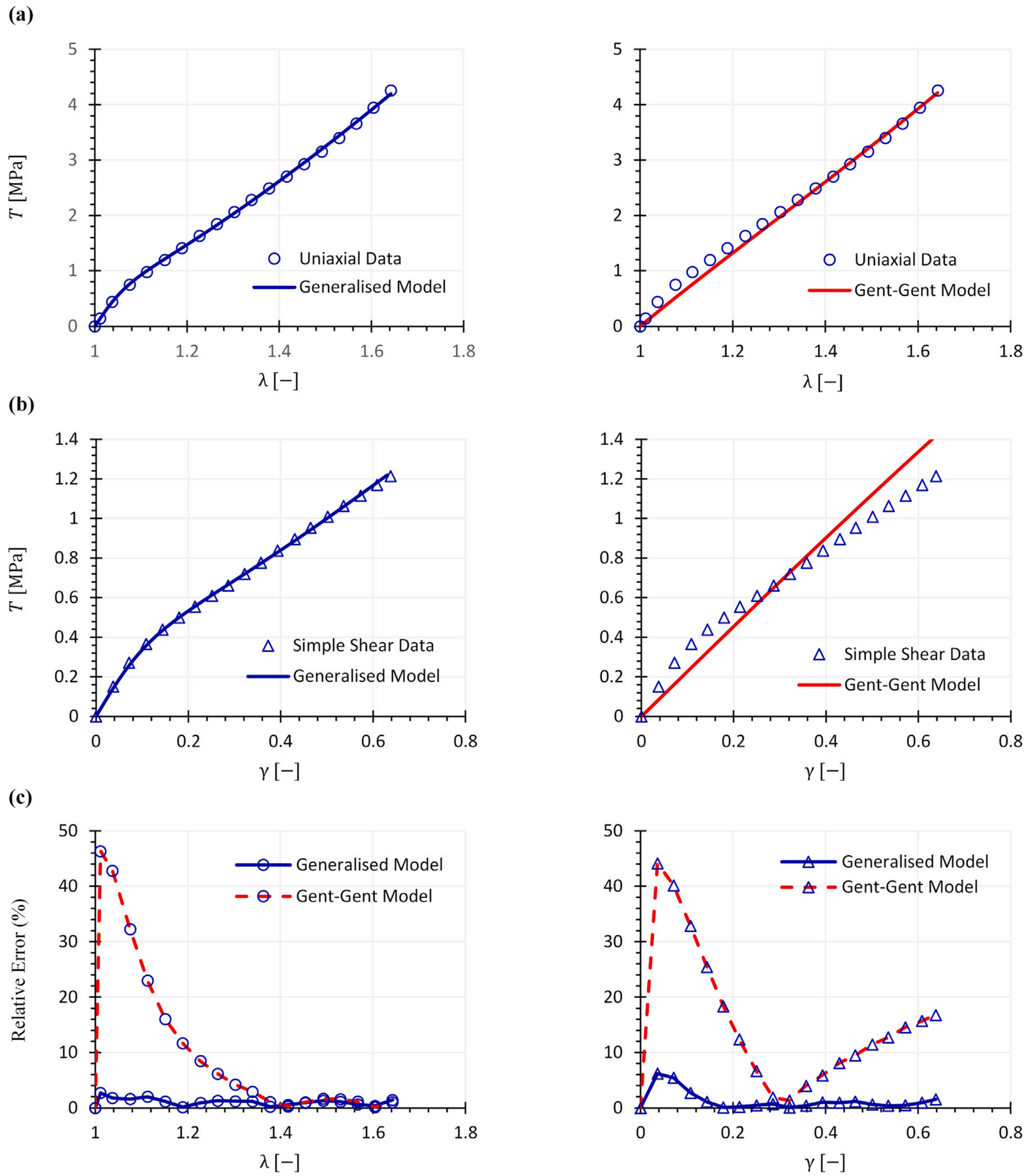

Fig. 2. Modelling results for *filled* rubber specimens due to Lahellec et al. [17] using both the generalised and Gent-Gent models: (a) uniaxial deformation; (b) simple shear deformation; and (c) the relative errors (left panel: uniaxial; right panel: simple shear).

Remark 2. The R^2 values for the fits provided by the Gent-Gent model are notable, particularly that of the uniaxial fit with $R^2 = 0.99$, while the poor quality of the fits is evident from the plots of Fig. 2. This perhaps is a reminder that R^2 values in nonlinear fits can be deceiving.[1] Indeed, following the works of Saccomandi and co-workers [1,8,10], it is always instructive to use additional measures such as the relative error [8], or alternative stress – deformation plots via Mooney [1,8] or the generalised Mooney spaces [10].

Another example of this 'downward concavity' behaviour is exhibited by the polymeric specimens of Nunes and Moreira [18], under uniaxial and simple shear deformations (Fig. 3). When both models are fitted with this experimental data, it is again observed that the Gent-Gent does not capture this behaviour well. As with the previous trend of results, the generalised model is seen to simulate the data favourably. Table 3 contains the identified model parameter values via simultaneous fits, for both models.

At this point I should also like to mention the dataset due to Moreira and Nunes [19] on pure and simple shear deformations of their polymeric specimens. For this particular dataset, the generalised model provides the best fit with a negative value of parameter N. See Fig. 4 and the parameter values in Table 4. While intuitively, and according to the condition set out in Eq. (13), one expects N to be positive, form a purely phenomenological point of view a negative N does not violate Beatty's [35] empirical inequalities, nor does it necessarily lead to non-convexity issues — see [12,13] for an analysis and discussion on

[1] I am grateful to Professor Michel Destrade for this insight.

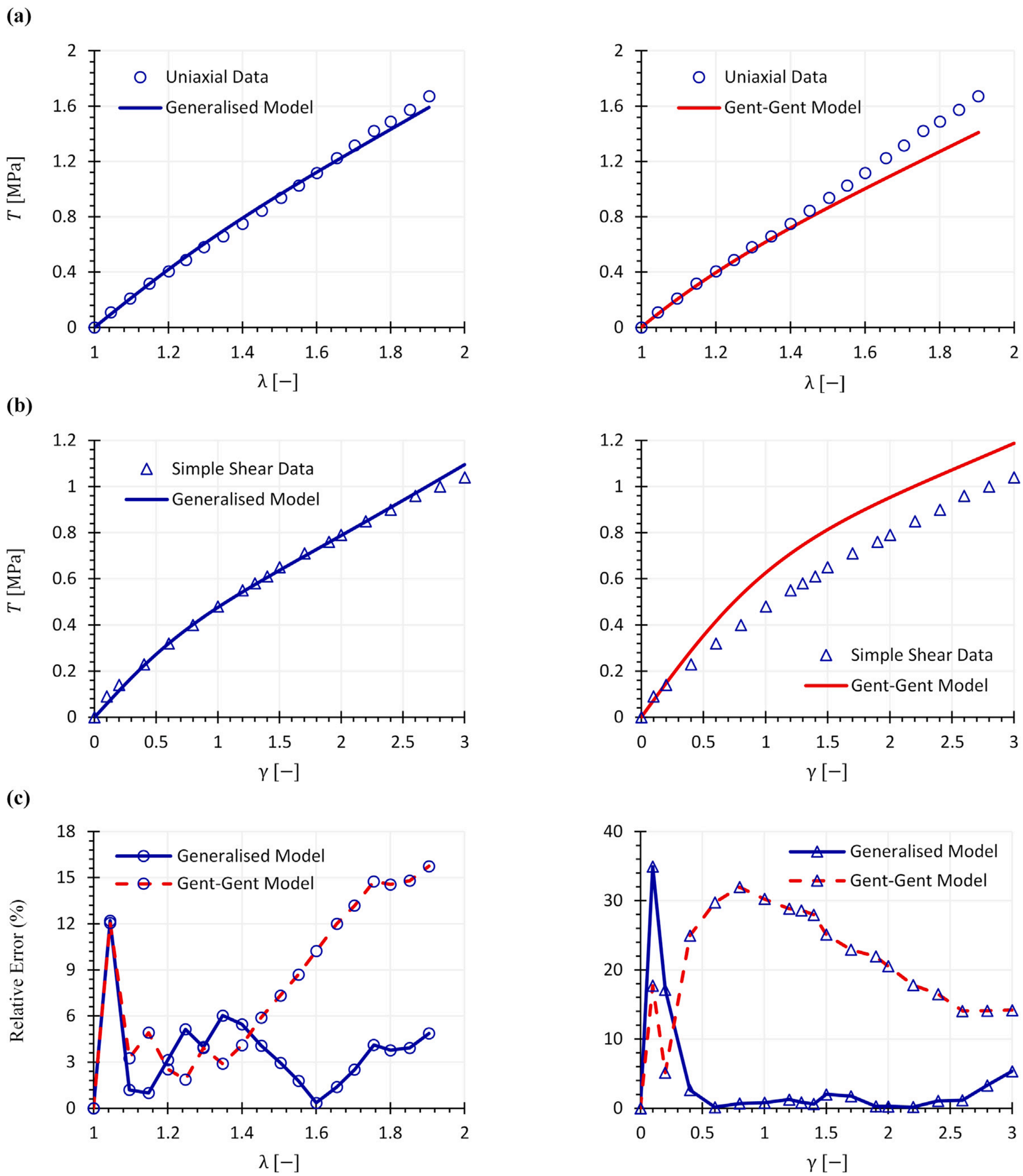

Fig. 3. Modelling results for the commercial polymer (Flextec®FT 101) specimens due to Nunes and Moreira [18] using both the generalised and Gent-Gent models: (a) uniaxial deformation; (b) simple shear deformation; and (c) the relative errors (left panel: uniaxial; right panel: simple shear).

Table 3
The identified model parameter values for the commercial polymer (Flextec®FT 101) specimens due to Nunes and Moreira [18].

The generalised model						
	μ [MPa]	N [–]	n [–]	C_2 [MPa]	m [–]	R^2
Uniaxial Simple Shear	0.03	0.48	0.08_5	0.04	100.05	0.99 0.99

The Gent-Gent model				
	μ_0 [MPa]	J_m [–]	C_2 [MPa]	R^2
Uniaxial Simple Shear	0.27	382.47	0.70_5	0.94 0.81

this point using the generalised neo-Hookean part of the model. This is a useful flexibility that is provided by the generalised model, and is not known to the author to have been exhibited by the Gent (and subsequently the Gent-Gent) model, as a negative value of N requires a negative value of J_m in Gent and Gent-Gent models, which has not yet been shown to be valid/viable.

Finally, we consider here two datasets which demonstrate continuous softening up to the onset of failure. The customary approach in the literature to model such softening behaviour is to either use Volokh's energy limiters method [36,37], or the continuum damage mechanics framework originally proposed by Simo [38]. However, in a recent contribution by the author, the concept of *hyperelasticity with intrinsic softening* was introduced [39], wherein it was posited that if the functional form of a basic hyperelastic model W is rich-enough and it

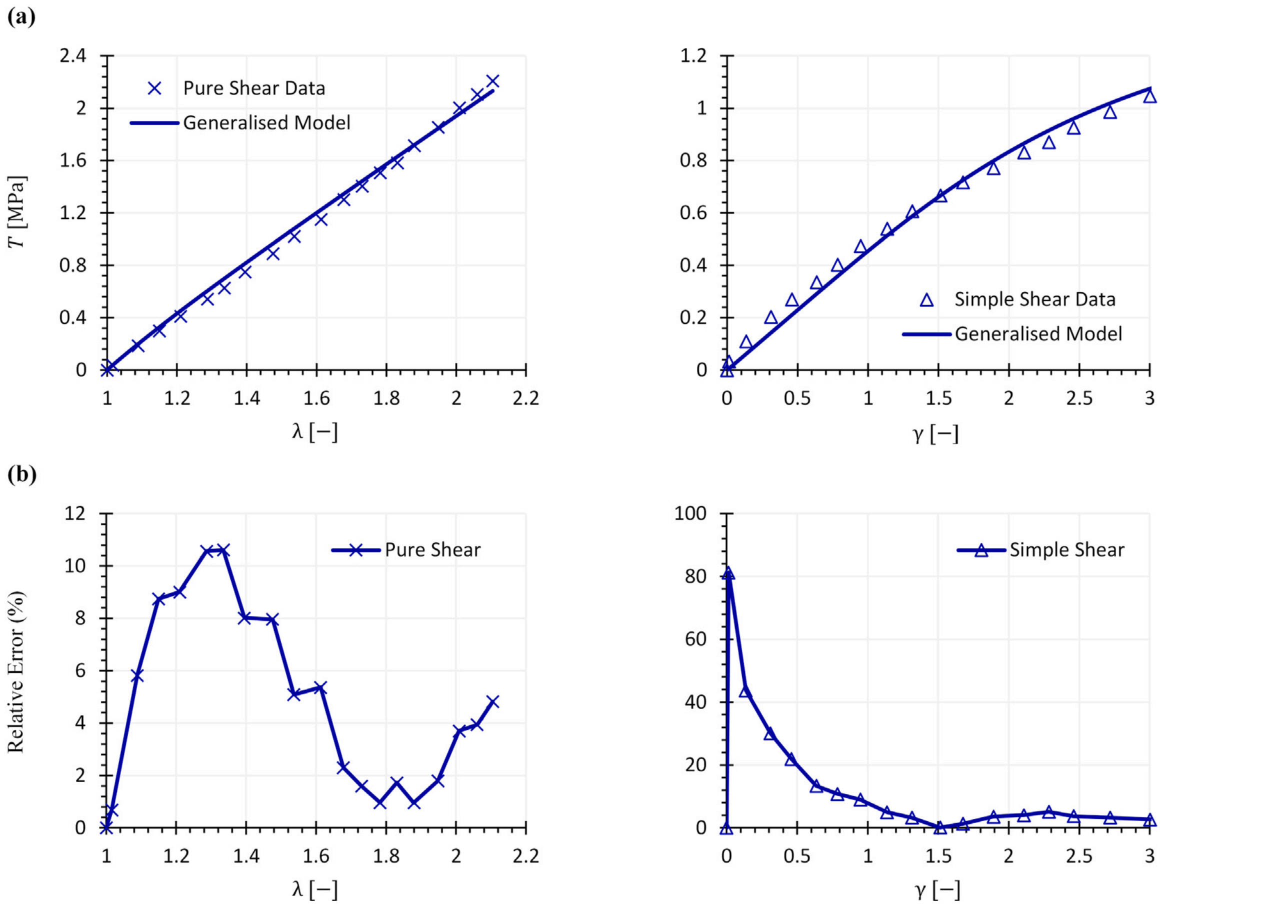

Fig. 4. Modelling results for the dataset due to Moreira and Nunes [19]: (a) pure and simple shear deformations; and (b) the ensuing relative errors (left panel: pure shear; right panel: simple shear).

Table 4
Model parameter values for the dataset due to Moreira and Nunes [19].

The generalised model						
	μ [MPa]	N [–]	n [–]	C_2 [MPa]	m [–]	R^2
Simple Shear Pure Shear	0.67	-3.01_5	10	0.03	0.97	0.99 0.99

Table 5
Model parameter values for the (whey) protein gels of Liu and Böl [20] and abdominal aortic aneurysm samples of Volokh [21].

The generalised model						
	μ [kPa]	N [–]	n [–]	C_2 [kPa]	m [–]	R^2
Data of Liu and Böl [20]	1.72	25.02	0.28	0.00_4	24.85	0.99_7
Data of Volokh [21]	1.36×10^3	1.13	0.75	1.48×10^3	0.02	0.99_6
The Gent-Gent model						
	μ_0 [kPa]	J_m [–]	C_2 [kPa]	R^2		
Data of Liu and Böl [20]	0.52	246.77	3.79	0.95		
Data of Volokh [21]	1.43×10^3	0.66	0.00	0.97		

incorporates an appropriate set of constitutive parameters, the hyperelastic model will be able to capture the softening behaviour without the need of external softening/damage parameters being imposed. The specific hyperelastic model considered therein was that of [34], which favourably captured the softening behaviour (up to the onset of failure) of various soft solids [39]. To showcase the basic capability of the proposed generalised model here to capture this softening behaviour, we consider the application of this model to two exemplar datasets from a protein gel and abdominal aortic aneurysm, due to Liu and Böl [20] and Volokh [21], respectively, under uniaxial deformation. The modelling results are shown in Fig. 5, and the model parameters are listed in Table 5. For comparison, the predictions of the Gent-Gent model have also been presented (dotted red lines in the plots of Fig. 5). The superiority of the generalised model reflects itself in the quality of fits and the reduced levels of the relative error. The Gent-Gent model appears not well-suited for simulating such softening behaviour.

4. Concluding remarks

A generalisation of the Pucci–Saccomandi model of rubber elasticity was presented in this work. The approach used for this generalisation was based on developing higher order rational approximant; i.e., [1/1], response functions $2W_1$ and $2W_2$, compared with the typically [0/1] order used in Gent and Gent-Gent models. It was shown that the generalised model is the *parent* to a range of LCE and classical (I_1, I_2) models, as well as generalised neo-Hookean strain energy functions. By way of applying the model to a group of diverse experimental datasets, it was demonstrated that not only the proposed model: (i) provides marked improvement to capturing the classical behaviour of natural rubbers (as exemplified by the canonical Treloar [11] data), but also (ii) favourably captures the finite deformation of filled rubbers; as well as (iii) shear/strain softening and (iv) the softening behaviour up to the onset of failure. In addition, the model provides the flexibility of accommodating a wider plausible range for N, i.e., $N < 0$, which for a similar application using the Gent model requires the parameter J_m to be negative – a feature that has not yet been shown valid for the Gent model. While the proposed model contains five model parameters

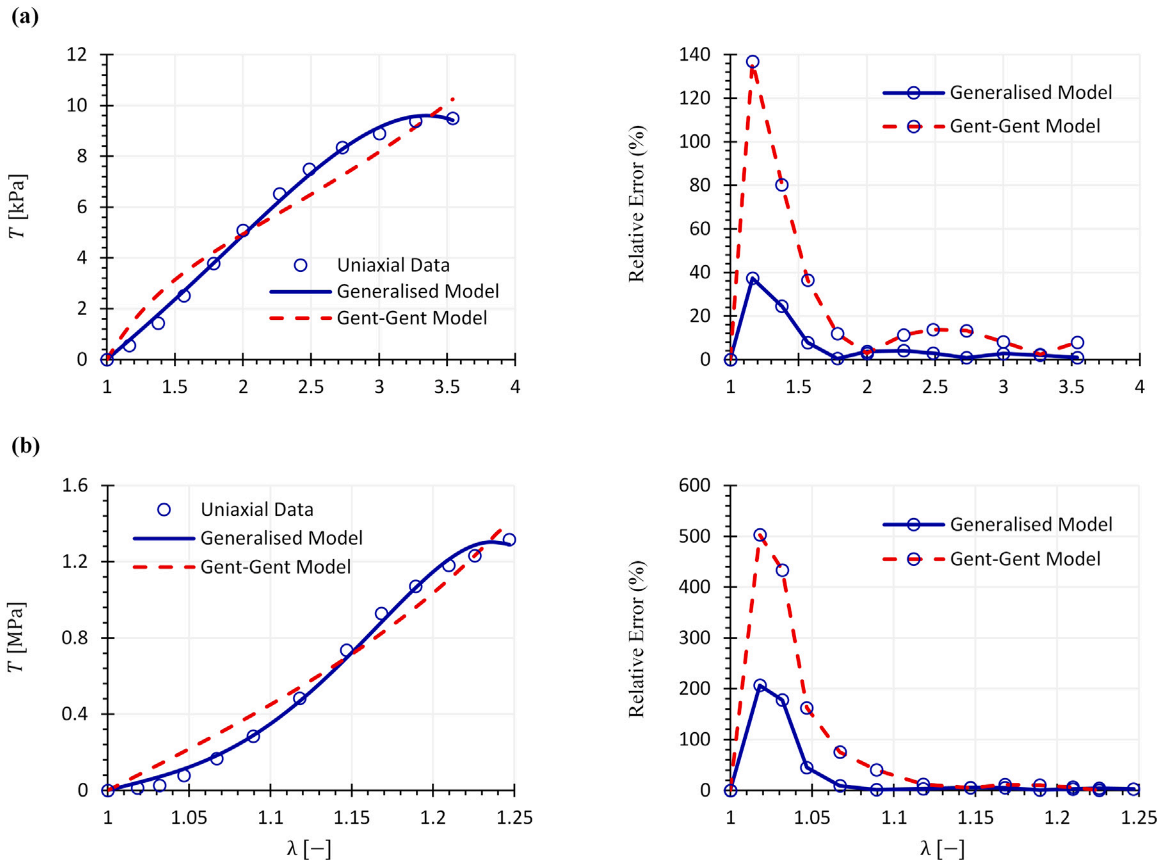

Fig. 5. Modelling results of the softening behaviour (up to the onset of failure) for: (a) (whey) protein gel samples due to Liu and Böl [20]; and (b) abdominal aortic aneurysm specimens due of Volokh [21], using both the generalised and Gent-Gent models. The dotted lines indicate the modelling results for the Gent-Gent model.

versus three in the Gent-Gent model, the increased quality of fits are not merely due to the additional number of parameters, given that the generalised neo-Hookean part of the model (i.e., $f(I_1)$ in Eq. (12)) with three parameters provides improved fits compared to the Gent-Gent model on its own (see [13] for direct comparisons). Instead, therefore, the improvements stem from the functional form of the proposed model, which at the very least is a higher order approximant.

While the aim of this manuscript was to provide a generalisation, and thereby improvement, to the Gent-Gent model, it may be instructive to present comparisons with other models of similar type including, for example, the cubic Yeoh [40] and Carroll [41] models. The motivation for choosing these two exemplar models is centred on the fact that similar to the $f(I_1)$ part of the proposed model, given in Eq. (12), those models have also three parameters, and as shown in [13], $f(I_1)$ in Eq. (12) on its own provides much improved fits compared with those two models. Therefore, the improvements portended by the proposed generalised model in this manuscript with the added $g(I_2)$ term in Eq. (20) cannot be merely attributed to the additional two model parameters. As a case in point, the plots in Fig. 6 provide a comparison between the three models for the filled rubber specimens of Lahellec et al. [17]. Note that the plots for the generalised model have been reproduced from Fig. 2. The superiority of fits provided by the generalised model of Eq. (22) is visibly clear, and therefore we refrain from showing the relative error plots.

It may be useful to also consider here the *predictive* capability of the proposed generalised model. In doing so, one must remain mindful of a common misapplication often encountered in the literature, namely calibrating a model with one deformation dataset; say by fitting a model to uniaxial data, and then use the obtained model parameter values to attempt 'predicting' another independent deformation behaviour, say the biaxial or pure shear etc defamations. However, in a fitting process, one is essentially trying to characterise, through a numerical minimisation scheme, the response function(s) of a model. A model such as the proposed generalised $W(I_1, I_2)$ function in Eq. (22) possesses two response functions, namely $\frac{\partial W}{\partial I_1}$ and $\frac{\partial W}{\partial I_2}$. Therefore, at least two independent datasets are required to solve for the two unknown response functions. This point has been originally explained by Holzapfel and Ogden [42]. Accordingly, here we first fit the model simultaneously to the uniaxial and pure shear datasets of Kawabata et al. [43] for unfilled rubber vulcanizates, and using the obtained model parameter values then predict the equi-biaxial behaviour. The results are presented in the plots of Fig. 7. Panels (a) and (b) illustrate the fitting results, and panel (c) shows the predicted equi-biaxial behaviour by the model versus the experimental data. As the relative error plots in panel (d) also indicate, the agreement between the predicted behaviour and the experimental data is favourable.

It is possible to also construct rational approximants of other orders than the [1/1] order considered here. Examples of such developments can be seen in the models of [33,34], particularly when used in their multi-term forms. However, such higher order generalisations give rise to more model parameter numbers, which in turn detract from the interpretability of the modelling results in terms of the uniqueness of the optimal fits and the obtained model parameter values; see, e.g., Ogden et al. [44]. Therefore, the proposed model here would be a versatile tool for applications where a $W(I_1, I_2)$ model with a simple functional form and a manageable number of parameters, but with a high degree of accuracy, is required. In particular, recent studies have further highlighted the role that an appropriate functional

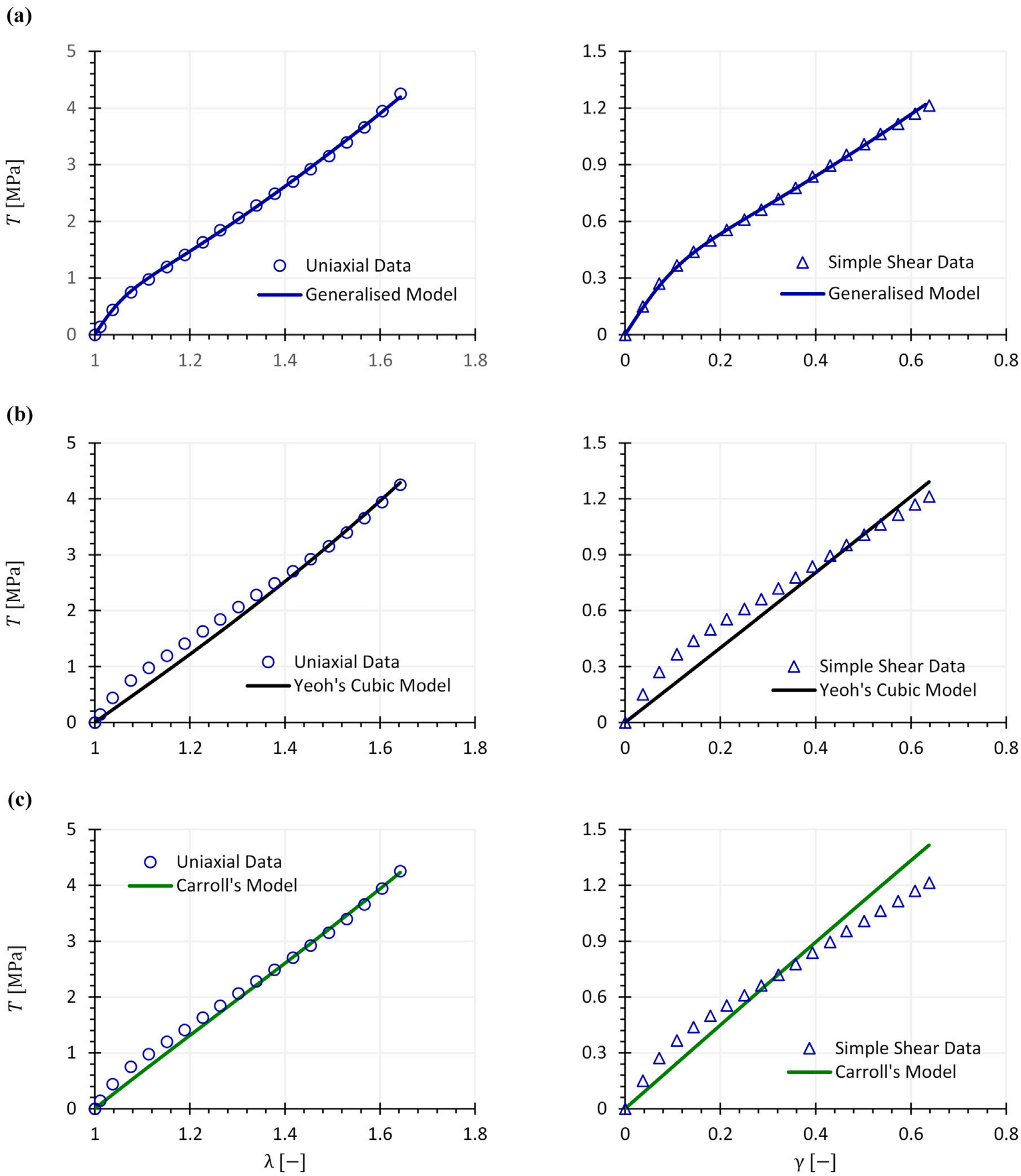

Fig. 6. Modelling results for the *filled* rubber specimens of Lahellec et al. [17] using: (a) the generalised model (reproduced from Fig. 2; (b) the cubic Yeoh model $W(I_1) = c_1(I_1-3) + c_2(I_1-3)^2 + c_3(I_1-3)^3$ with $c_1 = 1.00$ MPa, $c_2 = 0.015$ MPa and $c_3 = 0$ Pa; and (c) Carroll's model $W(I_1, I_2) = AI_1 + BI_1^4 + CI_2^{0.5}$ with $A = 0.87$ MPa, $B = 0.27$ Pa and $C = 0.87$ MPa.

dependence on the I_2 term can play in improving the model predictions (see, e.g., [45]). The generalised I_2 term constructed in this study not only appears to have improved the quality of fits in the low(er) range of deformation, but the relative errors remained generally low at higher levels of deformation too, i.e., in the so-called strain stiffening region, further making the case for the application of this model to the full-range typical finite deformation behaviour of incompressible soft materials. The need for such more accurate and universal hyperelastic models is continuously highlighted in the literature, either through the reviews of the state of the art (e.g., [46,47]) or via the need for improved predictions under more intricate boundary value problems (e.g., [48,49]) and complex material behaviours (e.g., [50]).

Declaration of competing interest

The author has no competing interests to declare.

Data availability

All experimental data used here were collated from the cited studies within the manuscript.

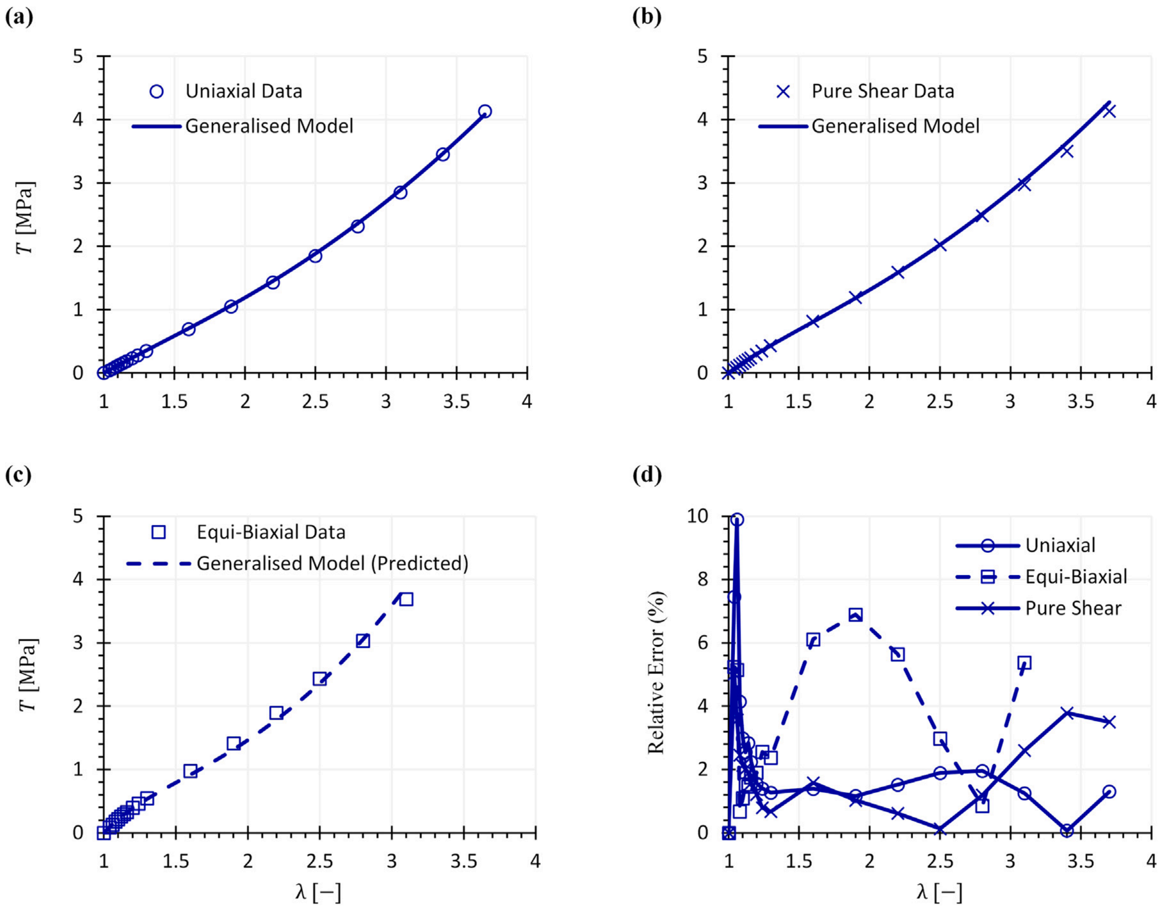

Fig. 7. Modelling *prediction* on using the unfilled rubber vulcanizate specimens of Kawabata et al. [43]: The model was first simultaneously fitted to the (a) uniaxial and (b) pure shear datasets. The obtained model parameter values are $\mu = 0.17$ MPa, $N = 0.40$ [–], $n = 0.60$ [–], $C_2 = 0.02$ [MPa], and $m = 4.01$ [–]. Using these parameters, the (c) equi-biaxial deformation was then *predicted*. The ensuing relative errors are shown in panel (d).

References

[1] E. Pucci, G. Saccomandi, A note on the gent model for rubber-like materials, Rubber Chem. Technol. 75 (2002) 839–852, http://dx.doi.org/10.5254/1.3547687.

[2] A.N. Gent, A new constitutive relation for rubber, Rubber Chem. Technol. 69 (1996) 59–61, http://dx.doi.org/10.5254/1.3538357.

[3] A.N. Gent, A.G. Thomas, Forms for the stored (strain) energy function for vulcanized rubber, J. Polym. Sci. 28 (1958) 625–628, http://dx.doi.org/10.1002/pol.1958.1202811814.

[4] C.O. Horgan, G. Saccomandi, Simple torsion of isotropic, hyperelastic, incompressible materials with limiting chain extensibility, J. Elasticity 56 (1999) 159–170, http://dx.doi.org/10.1023/A:1007606909163.

[5] G. Saccomandi, Universal results in finite elasticity, in: Y. Fu, R.W. Ogden (Eds.), Nonlinear Elasticity: Theory and Applications, Cambridge University Press, Cambridge, UK, 2001, pp. 97–134, http://dx.doi.org/10.1017/CBO9780511526466.004.

[6] M. Destrade, M.D. Gilchrist, J.G. Murphy, B. Rashid, G. Saccomandi, Extreme softness of brain matter in simple shear, Int. J. Non-Linear Mech. 75 (2015) 54–58, http://dx.doi.org/10.1016/j.ijnonlinmec.2015.02.014.

[7] M. Destrade, E. Pucci, G. Saccomandi, Generalization of the Zabolotskaya equation to all incompressible isotropic elastic solids, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 475 (2019) 20190061, http://dx.doi.org/10.1098/rspa.2019.0061.

[8] M. Destrade, G. Saccomandi, I. Sgura, Methodical fitting for mathematical models of rubber-like materials, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 473 (2017) 20160811, http://dx.doi.org/10.1098/rspa.2016.0811.

[9] R.S. Rivlin, D. Saunders, Large elastic deformations of isotropic materials. vii. Experiments on the deformation of rubber, Philos. Trans. R. Soc. A, Math. Phys. Eng. Sci. 243 (1951) 251–288, http://dx.doi.org/10.1098/rsta.1951.0004.

[10] A. Anssari-Benam, A. Bucchi, M. Destrade, G. Saccomandi, The generalised mooney space for modelling the response of rubber-like materials, J. Elasticity 151 (2022) 127–141, http://dx.doi.org/10.1007/s10659-022-09889-1.

[11] L.R.G. Treloar, Stress-strain data for vulcanised rubber under various types of deformation, Trans. Faraday Soc. 40 (1944) 59–70, http://dx.doi.org/10.1039/TF9444000059.

[12] A. Anssari-Benam, C.O. Horgan, On modelling simple shear for isotropic incompressible rubber-like materials, J. Elasticity 147 (2021) 83–111, http://dx.doi.org/10.1007/s10659-021-09869-x.

[13] A. Anssari-Benam, C.O. Horgan, A three-parameter structurally motivated robust constitutive model for isotropic incompressible unfilled and filled rubber-like materials, Eur. J. Mech. A Solids 95 (2022) 104605, http://dx.doi.org/10.1016/j.euromechsol.2022.104605.

[14] O. Lopez-Pamies, A new I_1-based hyperelastic model for rubber elastic materials, C.R. Mecanique 338 (2010) 3–11, http://dx.doi.org/10.1016/j.crme.2009.12.007.

[15] M. Pelliciari, S. Sirotti, A.M. Tarantino, A strain energy function for large deformations of compressible elastomers, J. Mech. Phys. Solids 176 (2023) 105308, http://dx.doi.org/10.1016/j.jmps.2023.105308.

[16] A. Anssari-Benam, On a new class of non-Gaussian molecular based constitutive models with limiting chain extensibility for incompressible rubber-like materials, Math. Mech. Solids 26 (2021) 1660–1674, http://dx.doi.org/10.1177/10812865211001094.

[17] N. Lahellec, F. Mazerolle, J.C. Michel, Second-order estimate of the macroscopic behavior of periodic hyperelastic composites: Theory and experimental validation, J. Mech. Phys. Solids 52 (2004) 27–49, http://dx.doi.org/10.1016/S0022-5096(03)00104-2.

[18] L.C.S. Nunes, D.C. Moreira, Simple shear under large deformation: Experimental and theoretical analyses, Eur. J. Mech. A Solids 42 (2013) 315–322, http://dx.doi.org/10.1016/j.euromechsol.2013.07.002.

[19] D.C. Moreira, L.C.S. Nunes, Comparison of simple and pure shear for an incompressible isotropic hyperelastic material under large deformation, Polym. Test. 32 (2013) 240–248, http://dx.doi.org/10.1016/j.polymertesting.2012.11.005.

[20] J. Liu, M. Böl, Experimental characterisation of the mechanical failure behaviour of whey protein gel treated with sodium hydroxide, Food Bioprod. Process 129 (2021) 94–104, http://dx.doi.org/10.1016/j.fbp.2021.07.007.

[21] K.Y. Volokh, Loss of ellipticity in elasticity with energy limiters, Eur. J. Mech. A Solids 63 (2017) 36–42, http://dx.doi.org/10.1016/j.euromechsol.2016.10.003.

[22] M.F. Beatty, On constitutive models for limited elastic, molecular based materials, Math. Mech. Solids 13 (2008) 375–387, http://dx.doi.org/10.1177/1081286507076405.
[23] G. Puglisi, G. Saccomandi, Multi-scale modelling of rubber-like materials and soft tissues: An appraisal, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 472 (2016) 20160060, http://dx.doi.org/10.1098/rspa.2016.0060.
[24] A. Anssari-Benam, A. Bucchi, Modelling the deformation of the elastin network in the aortic valve, J. Biomech. Eng. 140 (2018) 011004, http://dx.doi.org/10.1115/1.4037916.
[25] A. Anssari-Benam, A. Bucchi, A generalised Neo-Hookean strain energy function for application to the finite deformation of elastomers, Int. J. Non-Linear Mech. 128 (2021) 103626, http://dx.doi.org/10.1016/j.ijnonlinmec.2020.103626.
[26] R.A. Dickie, T.L. Smith, Viscoelastic properties of a rubber vulcanizate under large deformations in equal biaxial tension, pure shear, and simple tension, Trans. Soc. Rheol. 15 (1971) 91–110, http://dx.doi.org/10.1122/1.549231.
[27] A.N. Gent, Extensibility of rubber under different types of deformation, J. Rheol. 49 (2005) 271–275, http://dx.doi.org/10.1122/1.1835343.
[28] R.S. Rivlin, Large elastic deformations of isotropic materials IV. further developments of the general theory, Philos. Trans. R. Soc. Lond. Ser. A 241 (1948) 379–397, http://dx.doi.org/10.1098/rsta.1948.0024.
[29] A. Anssari-Benam, A. Bucchi, G. Saccomandi, On the central role of the invariant I_2 in nonlinear elasticity, Internat. J. Engrg. Sci. 163 (2021) 103486, http://dx.doi.org/10.1016/j.ijengsci.2021.103486.
[30] E. Fried, An elementary molecular-statistical basis for the Mooney and Rivlin–Saunders theories of rubber elasticity, J. Mech. Phys. Solids 50 (2002) 571–582, http://dx.doi.org/10.1016/S0022-5096(01)00086-2.
[31] A. Anssari-Benam, A. Bucchi, C.O. Horgan, G. Saccomandi, Assessment of a new isotropic hyperelastic constitutive model for a range of rubberlike materials and deformations, Rubber Chem. Technol. 95 (2022) 200–217, http://dx.doi.org/10.5254/rct.21.78975.
[32] A. Anssari-Benam, Comparative modelling results between a separable and a non-separable form of principal stretches–based strain energy functions for a variety of isotropic incompressible soft solids: Ogden model compared with a *parent* model, Mech. Soft Mater. 5 (2023) 2, http://dx.doi.org/10.1007/s42558-023-00050-z.
[33] A. Anssari-Benam, A generalised $W\left(I_1, I_2\right)$ strain energy function of binomial form with unified applicability across various isotropic incompressible soft solids, Acta Mech. (2023) http://dx.doi.org/10.1007/s00707-023-03677-1.
[34] A. Anssari-Benam, Large isotropic elastic deformations: On a comprehensive model to correlate the theory and experiments for incompressible rubber-like materials, J. Elasticity 153 (2023) 219–244, http://dx.doi.org/10.1007/s10659-022-09982-5.
[35] M.F. Beatty, Topics in finite elasticity: Hyperelasticity of rubber, elastomers, and biological tissues - with examples, Appl. Mech. Rev. 40 (1987) 1699–1734, http://dx.doi.org/10.1115/1.3149545.
[36] K.Y. Volokh, Hyperelasticity with softening for modeling materials failure, J. Mech. Phys. Solids 55 (2007) 2237–2264, http://dx.doi.org/10.1016/j.jmps.2007.02.012.
[37] K.Y. Volokh, On modeling failure of rubber-like materials, Mech. Res. Commun. 37 (2010) 684–689, http://dx.doi.org/10.1016/j.mechrescom.2010.10.006.
[38] J.C. Simo, On a fully three-dimensional finite-strain viscoelastic damage model: Formulation and computational aspects, Comput. Methods Appl. Mech. Engrg. 60 (1987) 153–173, http://dx.doi.org/10.1016/0045-7825(87)90107-1.
[39] A. Anssari-Benam, Continuous softening up to the onset of failure: A hyperelastic modelling approach with *intrinsic* softening for isotropic incompressible soft solids, Mech. Res. Commun. 132 (2023) 104183, http://dx.doi.org/10.1016/j.mechrescom.2023.104183.
[40] O.H. Yeoh, Some forms of the strain energy function for rubber, Rubber Chem. Technol. 66 (1993) 754–771, http://dx.doi.org/10.5254/1.3538343.
[41] M.M. Carroll, A strain energy function for vulcanized rubbers, J. Elasticity 103 (2011) 173–187, http://dx.doi.org/10.1007/s10659-010-9279-0.
[42] G.A. Holzapfel, R.W. Ogden, On planar biaxial tests for anisotropic nonlinearly elastic solids: A continuum mechanical framework, Math. Mech. Solids 14 (2009) 474–489, http://dx.doi.org/10.1177/1081286507084411.
[43] S. Kawabata, M. Matsuda, K. Tei, H. Kawai, Experimental survey of the strain energy density function of isoprene rubber vulcanizate, Macromolecules 14 (1981) 154–162, http://dx.doi.org/10.1021/ma50002a032.
[44] R.W. Ogden, G. Saccomandi, I. Sgura, Fitting hyperelastic models to experimental data, Comput. Mech. 34 (2004) 484–502, http://dx.doi.org/10.1007/s00466-004-0593-y.
[45] G. Kumar, L. Brassart, On tube models of rubber elasticity: Fitting performance in relation to sensitivity to the invariant I_2, Mech. Soft Mater. 5 (2023) 6, http://dx.doi.org/10.1007/s42558-023-00054-9.
[46] P. Steinmann, M. Hossain, G. Possart, Hyperelastic models for rubber-like materials: Consistent tangent operators and suitability for Treloar's data, Arch. Appl. Mech. 82 (2012) 1183–1217, http://dx.doi.org/10.1007/s00419-012-0610-z.
[47] H. Dal, K. Açkgöz, Y. Badienia, On the performance of isotropic hyperelastic constitutive models for rubber-like materials: A state of the art review, Appl. Mech. Rev. 73 (2021) 020802, http://dx.doi.org/10.1115/1.4050978.
[48] M. Pelliciari, S. Sirotti, A. Aloisio, A.M. Tarantino, Analytical, numerical and experimental study of the finite inflation of circular membranes, Int. J. Mech. Sci. 226 (2022) 107383, http://dx.doi.org/10.1016/j.ijmecsci.2022.107383.
[49] S. Sirotti, M. Pelliciari, A. Aloisio, A.M. Tarantino, Analytical pressure–deflection curves for the inflation of pre-stretched circular membranes, Eur. J. Mech. A Solids 97 (2023) 104831, http://dx.doi.org/10.1016/j.euromechsol.2022.104831.
[50] K.M. Moerman, B. Fereidoonnezhad, J.P. McGarry, Novel hyperelastic models for large volumetric deformations, Int. J. Solids Struct. 193–194 (2020) 474–491, http://dx.doi.org/10.1016/j.ijsolstr.2020.01.019.