

Mahavier completeness and classifying diagrams

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Abstract

Generalised inverse limits of compacta were introduced by Ingram and Mahavier in 2006. The main difference between ordinary inverse limits and their generalised cousins is that the former concerns diagrams of singlevalued functions while the latter permits multivalued functions. However, generalised inverse limits are not merely limits in the Kleisli category of a hyperspace monad, a fact that independently motivated each of the authors of this article to come up with the same formalism which restores the link with category theory through the concept of Mahavier limit of an order diagram in an order extension of a category \mathcal{B} . Mahavier limits of diagrams in \mathcal{B} coincide with ordinary limits in \mathcal{B} , and so Mahavier limits are an extension of ordinary limits along the functor that views an ordinary diagram as a diagram in the extension. Within that context it is natural to consider Mahavier completeness, namely when all small diagrams admit Mahavier limits, as well as classifying diagrams, namely the existence of a right adjoint to the mentioned functor on diagrams. In this work we show that these two conditions are equivalent, and we study some of the properties of classifying diagrams and of the adjunction.

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1. Introduction

Generalised inverse limits of compacta were introduced by Ingram and Mahavier in 2006 in [1] and have since received much attention (e.g., [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]). Recall that an inverse limit of a sequence

$$\cdots \longrightarrow X_{n+1} \xrightarrow{f_n} X_n \xrightarrow{f_{n-1}} \cdots \longrightarrow X_2 \xrightarrow{f_1} X_1$$

of spaces and continuous functions is the space $X = \{\mathbf{x} \in \prod X_n \mid \mathbf{x}_n = f_n(\mathbf{x}_{n+1})\}$, viewed as a subspace of the product space. The passage to generalised inverse limits occurs by allowing the bonding functions $f_n: X_{n+1} \rightarrow X_n$ to be upper semicontinuous set-valued functions $f_n: X_{n+1} \rightsquigarrow X_n$, and by altering the definition of the space X to become $X = \{\mathbf{x} \in \prod X_n \mid \mathbf{x}_n \in f_n(\mathbf{x}_{n+1})\}$. The formal resemblance to inverse limits makes the generalised version very palatable. The hoard of interesting spaces that arise as generalised inverse limits of very simple diagrams with multivalued bonding functions of compacta (see [11, 25] for detailed examples), together with highly non-trivial ramification of the subtle change in definition from singlevaluedness to multivaluedness, and from equality to membership, contribute even more to the appeal of this relatively new area of research.

Of course, inverse limits of spaces are nothing but categorical limits in the category \mathbf{Top} of topological spaces and continuous mappings, and it is natural to ask whether the slogan generalises. Results addressing some categorical aspects of generalised inverse limits directly can be found in [4, 26], but they were only partially successful in fully restoring the link with category theory, and the difficulty can be traced to the following phenomenon. Consider the functor $T: \mathbf{Top} \rightarrow \mathbf{Top}$ which maps a space X to $T(X)$, the space of all subsets of X , endowed with the upper Vietoris topology. This hyperspace functor has a natural structure of a monad whose multiplication is given by taking unions. Let \mathbf{Top}_T be the Kleisli category of T , i.e., the objects of \mathbf{Top}_T are all spaces and a morphism $X \rightsquigarrow Y$ is a continuous function $X \rightarrow T(Y)$. It is easily seen that these are precisely the upper semicontinuous functions. In other words, the diagrams for generalised inverse limits of spaces are precisely diagrams in \mathbf{Top}_T . However, generalised inverse limits in \mathbf{Top} are not simply limits in \mathbf{Top}_T (an expected reality since limits in Kleisli categories are notoriously ill-behaved ([27]), while generalised inverse limits are much more tame).

The authors of this article independently found the same categorical formalism to fully restore the link between generalised inverse limits of spaces and category theory. In [18] the first named author developed a notion of limit in the category of compacta and upper semicontinuous set-valued functions in such a way that the slogan above is recovered. In [28] the second named author developed a formalism in full generality, allowing for generalised inverse limits to be considered beyond the scope of topology, which specialises to generalised inverse limits of spaces when interpreted in the context of $\mathbf{Top} \subseteq \mathbf{Top}_T$.

The aim of this work is summarised in the diagram

$$\begin{array}{ccc}
 [\mathcal{D}, \mathcal{B}] & \begin{array}{c} \xleftarrow{i_{\mathcal{D}}} \\ \xrightarrow{i_{\mathcal{D}}^*} \end{array} & [\mathcal{D}, \mathcal{C}]_{\mathcal{B}} \\
 \swarrow \Delta & \begin{array}{c} \xleftarrow{\text{lim}} \\ \xrightarrow{i_{\mathcal{D}} \circ \Delta} \end{array} & \searrow \text{lim}_{\mathcal{B}}^M \\
 & \mathcal{B} &
 \end{array}$$

which we briefly explain (all concepts are detailed below). Let \mathcal{B}, \mathcal{C} , and \mathcal{D} be categories, assume that \mathcal{B} is a subcategory of \mathcal{C} , that $\text{ob}(\mathcal{B}) = \text{ob}(\mathcal{C})$, and moreover that each hom-set in \mathcal{C} is endowed with an ordering, with some conditions. We call \mathcal{C} an *order extension* of \mathcal{B} . The ordering allows one to define order variants of functors and of natural transformations by suitably replacing $=$ by \leq . One obtains in this way the category $[\mathcal{D}, \mathcal{C}]$ of all order functors $\mathcal{D} \rightarrow \mathcal{C}$ and order natural transformations between them. An order natural transformation whose components are morphisms in \mathcal{B} is said to be an *order natural transformation relative to \mathcal{B}* , and we then denote by $[\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$ the subcategory of $[\mathcal{D}, \mathcal{C}]$ obtained by restricting to the relative order natural transformations. Let $[\mathcal{D}, \mathcal{B}]$ be the usual category of functors $\mathcal{D} \rightarrow \mathcal{B}$ and natural transformations. Since \mathcal{B} is a subcategory of \mathcal{C} there is an inclusion functor $i_{\mathcal{D}}: [\mathcal{D}, \mathcal{B}] \rightarrow [\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$, depicted at the top of the diagram above. On the left side of the diagram are the diagonal functor $\Delta: \mathcal{B} \rightarrow [\mathcal{D}, \mathcal{B}]$, mapping an object B to the constantly B functor, and its right adjoint, the functor \varprojlim , namely taking limits, provided \mathcal{D} -shaped limits in \mathcal{B} exist, e.g., if \mathcal{B} is complete.

On the right side of the diagram is the functor $i_{\mathcal{D}} \circ \Delta$ and its right adjoint $\varprojlim_{\mathcal{B}}^M$ which maps an order diagram $F: \mathcal{D} \rightarrow \mathcal{C}$ to $\varprojlim_{\mathcal{B}}^M F$, its *Mahavier limit relative to \mathcal{B}* , provided these limits exist, e.g., if \mathcal{C} is Mahavier complete relative to \mathcal{B} (i.e., Mahavier limits exist for all small order diagrams). Obviously, the triangle of left adjoints commutes. The result we prove below is that \mathcal{C} is Mahavier complete relative to \mathcal{B} if, and only if, \mathcal{B} is complete and $i_{\mathcal{D}}$ has a right adjoint $i_{\mathcal{D}}^*$, for all small categories \mathcal{D} . In that case, the triangle of right adjoints commutes up to a natural isomorphism, and thus $i_{\mathcal{D}}^*$ computes Mahavier limits in the sense that there is a natural isomorphism $\varprojlim_{\mathcal{B}}^M \cong \varprojlim \circ i_{\mathcal{D}}^*$.

The plan of the paper is as follows. Section 2 briefly introduces the terminology above, and Section 3 presents the main result. Properties of the adjunction are studied in Section 4, together with some applications. Finally, Section 5 revisits classical generalised inverse limits, exhibiting, in a rather informal fashion, how categorical Mahavier limit theory meshes with the existing interests and problems in the field.

2. Preliminaries

We briefly present the concepts required for the definition of Mahavier limits. For a much more detailed exposition, stressing motivation and applicability, the reader is referred to [28]. The reader more interested in applications to compacta is referred to [18].

2.1. Order extensions

A main ingredient in the categorical formalisation we consider for generalised inverse limits is the ordering on the hom-sets of \mathbf{Top}_T , turning it into an order-enriched category. An ordered category is thus a special form of 2-category, and thus the well-developed theory of 2-categories (see, e.g., [29]) can be applied. To site just a couple of examples where the 2-categorical machinery

80 works very well for particular order-enriched categories we mention [30, 31, 32],
 which involves a translation of a 2-categorical notion to a condition on a monad
 known as the Kock-Zöberlein condition, and [33] in the area of ordered universal
 algebra. However, as noted generally already in [34], the standard 2-categorical
 constructions yield the 'wrong' results in certain ordered categories arising in
 85 computer science. The situation with generalised inverse limits in topology is
 another case where the 2-categorical notions are inadequate in a particular scen-
 ario. Interestingly, even though the motivations are very different, there are
 some similarities between our notion of Mahavier limits and some of the material
 in [34], where the notion of near limit is introduced and various lax conditions
 90 are given, in the study of partial functions in computer science, capturing some
 aspects that go back to [35].

Since our motivation is in securing a categorical home for generalised in-
 verse limits in topology, we feel free to deviate from the 2-categorical doctrine.
 In particular, what we call 'ordered category' is the same as 'order enriched
 95 category' but our notion of 'order functor' is not the enriched notion. We make
 the more permissive choice in order to address even the most esoteric of gener-
 alised diagrams considered in the literature on generalised inverse limits. We
 mention that some aspects of the theory become more 2-categorical if one takes
 the enriched notion of functor, and there may be good reasons to prefer that.
 100 However, the main notion, that of Mahavier limit, remains non-2-categorical.
 For that reason, we simply spell out the relevant notions, rather than obtain
 some of them as special cases.

An *ordered category* is a category \mathcal{C} together with an ordering \leq on each
 hom-set $\mathcal{C}(C, C')$ such that composition is monotone in each variable, i.e.,
 105 the conditions $c_1 \leq c_2$ and $c_3 \leq c_4$ imply $c_1 \circ c_3 \leq c_2 \circ c_4$, for all mor-
 phisms c_1, c_2, c_3, c_4 for which the compositions are defined. An *order functor*
 $F: \mathcal{C} \rightarrow \mathcal{C}'$ between ordered categories consists of the same ingredients as a
 functor, namely an object part and a morphism part, but the preservation of
 composition is weakened to merely requiring that $F(c_1 \circ c_2) \leq F(c_1) \circ F(c_2)$,
 110 for all morphisms $c_1, c_2 \in \mathcal{C}$ for which the composition is defined (though we
 still demand that $F(\text{id}_C) = \text{id}_{FC}$). Every category \mathcal{D} shall be viewed as an
 ordered category by endowing each hom-set $\mathcal{D}(D, D')$ with the trivial ordering,
 namely the identity relation. An order functor $\mathcal{D} \rightarrow \mathcal{C}$ is also referred to as
 an *order diagram of shape* \mathcal{D} in \mathcal{C} . Given order functors $F_1, F_2: \mathcal{D} \rightarrow \mathcal{C}$, an
 115 *order natural transformation* $\alpha: F_1 \rightarrow F_2$ is a family $\{\alpha_D\}_{D \in \mathcal{D}}$ of morphisms
 in \mathcal{C} with the property that the inequality $\alpha_{D'} \circ F_1(d) \leq F_2(d) \circ \alpha_D$ holds for
 all morphisms $d: D \rightarrow D'$ in \mathcal{D} . It is easy to see that the usual vertical com-
 position of natural transformations extends to order natural transformations. In
 more detail, if $\alpha: F_1 \rightarrow F_2$ and $\beta: F_2 \rightarrow F_3$ are order natural transformations
 120 between order functors $F_1, F_2, F_3: \mathcal{D} \rightarrow \mathcal{C}$, then $\beta \circ \alpha: F_1 \rightarrow F_3$ is the order
 natural transformation whose component at D is $\beta_D \circ \alpha_D$. For a fixed category
 \mathcal{D} and an ordered category \mathcal{C} let $[\mathcal{D}, \mathcal{C}]$ denote the category of all order functors
 $F: \mathcal{D} \rightarrow \mathcal{C}$ as objects and all order natural transformations as morphisms.

Let \mathcal{B} be a category. An *order extension* of \mathcal{B} is an ordered category \mathcal{C} of
 125 which \mathcal{B} is a subcategory, with $\text{ob}(\mathcal{B}) = \text{ob}(\mathcal{C})$ and such that the ordering on

$\mathcal{B}(C, C')$ induced by the ordering on $\mathcal{C}(C, C')$ is the identity relation, for all objects C, C' . In the context of an order extension $\mathcal{B} \subseteq \mathcal{C}$, morphisms in \mathcal{C} are denoted by $C \rightsquigarrow C'$, and to stress that a morphism is in \mathcal{B} we write $C \rightarrow C'$. Further, if \mathcal{D} is a category, we say that an order natural transformation $\alpha \in [\mathcal{D}, \mathcal{C}]$ is an order natural transformation *relative to* \mathcal{B} if all of the components α_D belong to \mathcal{B} , and we denote the subcategory of $[\mathcal{D}, \mathcal{C}]$ consisting of the relative natural transformations by $[\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$. Since every functor $F: \mathcal{D} \rightarrow \mathcal{C}$ is automatically an order functor, and since any natural transformation in $[\mathcal{D}, \mathcal{B}]$ is automatically an order natural transformation, we obtain an inclusion functor

 135 $i_{\mathcal{D}}: [\mathcal{D}, \mathcal{B}] \rightarrow [\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$.

2.2. Mahavier limits

Let us fix an order extension $\mathcal{B} \subseteq \mathcal{C}$ and a category \mathcal{D} . Recall the diagonal functor $\Delta: \mathcal{B} \rightarrow [\mathcal{D}, \mathcal{B}]$, where $\Delta(B): \mathcal{D} \rightarrow \mathcal{B}$ maps every object D to B and every morphism d to id_B . Given a functor $F: \mathcal{D} \rightarrow \mathcal{B}$, a cone from B to F is precisely a natural transformation $\Delta(B) \rightarrow F$, and a limit of F is a universal cone. Extending to \mathcal{C} we consider the extended diagonal functor $i_{\mathcal{D}} \circ \Delta: \mathcal{B} \rightarrow [\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$. Then, given an order diagram $F: \mathcal{D} \rightarrow \mathcal{C}$, an *order cone* from an object B to F is an order natural transformation $(i_{\mathcal{D}} \circ \Delta)(B) \rightarrow F$ relative to \mathcal{B} , and a Mahavier limit is a universal such order cone (to stress the role of \mathcal{B} we may refer to an order cone relative to \mathcal{B} or a Mahavier limit relative to \mathcal{B}). In more detail, an order cone from B to F is a family $\{\pi_D: B \rightarrow F(D)\}_{D \in \mathcal{D}}$ of morphisms in \mathcal{B} as in the diagram

$$\begin{array}{ccc}
 F(D) & \overset{F(d)}{\rightsquigarrow} & F(D') \\
 \swarrow \pi_D & & \searrow \pi_{D'} \\
 & B &
 \end{array}$$

satisfying $\pi_{D'} \leq F(d) \circ \pi_D$, for all $d: D \rightarrow D'$ in \mathcal{D} . Such an order cone is universal if for any other order cone $\{\psi_D: B' \rightarrow F(D)\}_{D \in \mathcal{D}}$ from an object $B' \in \mathcal{B}$ to F , there exists a unique morphism $b: B' \rightarrow B$ in \mathcal{B} with $\psi_D = \pi_D \circ b$.

 140 It is obvious that if $F: \mathcal{D} \rightarrow \mathcal{B}$ is a diagram, then any cone to F is also an order cone to F , and that any limit of F in the usual sense is a Mahavier limit of $i_{\mathcal{D}} \circ F$ relative to \mathcal{B} , and vice versa. Of course a Mahavier limit need not exist, and if it exists it is easily seen to satisfy the same uniqueness up to isomorphism property that the usual limit satisfies. We thus write $\varprojlim_{\mathcal{D}}^{\text{M}}(F)$ to denote a

 145 Mahavier limit of an order diagram F , with the same ambiguity accepted by the notation $\varprojlim F$ for the limit of a digram. In particular, there is a natural isomorphism $\varprojlim \cong \varprojlim^{\text{M}} \circ i_{\mathcal{D}}$ as functors $[\mathcal{D}, \mathcal{B}] \rightarrow \mathcal{B}$.

We say that the order extension $\mathcal{B} \subseteq \mathcal{C}$ is *Mahavier complete* if every small order diagram $F: \mathcal{D} \rightarrow \mathcal{C}$ has a Mahavier limit relative to \mathcal{B} . We also say that

 150 \mathcal{C} is Mahavier complete relative to \mathcal{B} . It is immediate that if \mathcal{C} is Mahavier complete relative to \mathcal{B} , then \mathcal{B} is complete in the ordinary sense.

The following are the properties of Mahavier limits which we require below.

- If B and B' are Mahavier limiting objects of the same order cone F , then $B \cong B'$ as objects in \mathcal{B} , and there is a unique isomorphism factorising one limiting cone through the other.
- If \mathcal{C} is Mahavier complete relative to \mathcal{B} , then for every small category \mathcal{D} , any arbitrary choice of Mahavier limiting object $\varprojlim_{\mathcal{B}}^M F$, for each order functor $F: \mathcal{D} \rightarrow \mathcal{C}$, extends canonically to a functor $\varprojlim_{\mathcal{B}}^M: [\mathcal{D}, \mathcal{C}]_{\mathcal{B}} \rightarrow \mathcal{B}$.
- If $S: \mathcal{D}' \rightarrow \mathcal{D}$ is a functor between categories and $F: \mathcal{D} \rightarrow \mathcal{C}$ is an order diagram, then there is a canonical shape change morphism $\varprojlim_{\mathcal{B}}^M F \rightarrow \varprojlim_{\mathcal{B}}^M (F \circ S)$, assuming the Mahavier limits exist.

Proofs can be found in [28].

2.3. Classifying diagrams

Given an order extension $\mathcal{B} \subseteq \mathcal{C}$ and a category \mathcal{D} , if the functor $i_{\mathcal{D}}: [\mathcal{D}, \mathcal{B}] \rightarrow [\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$ has a right adjoint $i_{\mathcal{D}}^*$, then we say that the order extension admits classification of diagrams of shape \mathcal{D} . We refer to $i_{\mathcal{D}}^*(F)$, for an order diagram $F: \mathcal{D} \rightarrow \mathcal{C}$, as the *classifying diagram* of F . We say $\mathcal{B} \subseteq \mathcal{C}$ admits classification of diagrams if it admits classification of diagrams of all small shapes \mathcal{D} .

Expectedly, the classification of diagrams is related to the size of the order extension $\mathcal{B} \subseteq \mathcal{C}$, and in a sense their behaviour is a qualitative measurement of it. For a terminal category $\mathcal{D} = \star$, the requirement that $\text{ob}(\mathcal{B}) = \text{ob}(\mathcal{C})$ implies at once that i_{\star} is the identity, and thus classification of diagrams of shape \star is automatic in any order extension. More interestingly, for the free-living morphism $\mathcal{D} = \{\bullet \rightarrow \circ\}$, the category $[\{\bullet \rightarrow \circ\}, \mathcal{C}]_{\mathcal{B}}$ has as objects the morphisms c in \mathcal{C} , and as morphisms squares

$$\begin{array}{ccc} C_1 & \overset{c}{\rightsquigarrow} & C_2 \\ \downarrow f & & \downarrow g \\ C_3 & \overset{c'}{\rightsquigarrow} & C_4 \end{array}$$

satisfying $g \circ c \leq c' \circ f$. The category $[\{\bullet \rightarrow \circ\}, \mathcal{B}]$ is the usual category of morphisms of \mathcal{B} . Thus, for $\mathcal{B} \subseteq \mathcal{C}$ to admit classification of diagrams of shape $\{\bullet \rightarrow \circ\}$ entails that with every morphisms $c \in \mathcal{C}$ there is associated a morphism $i_{\{\bullet \rightarrow \circ\}}^*(c) \in \mathcal{B}$ such that there is, for all $b \in \mathcal{B}$ and $c \in \mathcal{C}$, a natural bijection

$$\begin{array}{ccc} \begin{array}{ccc} B & \xrightarrow{b} & B' \\ \downarrow f & & \downarrow g \\ C & \overset{c}{\rightsquigarrow} & C' \end{array} & \longleftrightarrow & \begin{array}{ccc} B & \xrightarrow{b} & B' \\ \downarrow f_{\#} & & \downarrow g_{\#} \\ \bullet & \xrightarrow{i_{\{\bullet \rightarrow \circ\}}^*(c)} & \bullet \end{array} \end{array}$$

between morphisms in $[\{\bullet \rightarrow \circ\}, \mathcal{C}]_{\mathcal{B}}$ on the left, i.e., $g \circ b \leq c \circ f$, and morphisms in $[\{\bullet \rightarrow \circ\}, \mathcal{B}]$ on the right, i.e., $g_{\#} \circ b = i_{\{\bullet \rightarrow \circ\}}^*(c) \circ f_{\#}$.

Example 1. For an illustrative example which underlies similar situations in order extensions of the same nature as $\mathbf{Top} \subseteq \mathbf{Top}_T$, consider the category \mathbf{Set} of sets and functions, and the category \mathbf{Set}_T , the Kleisli category of the covariant non-empty power set monad $T: \mathbf{Set} \rightarrow \mathbf{Set}$. In simple terms, the objects of \mathbf{Set}_T are all sets, and a morphism $f: C \rightsquigarrow C'$ is a function $C \rightarrow T(C')$, i.e., a multivalued (total) function from C to C' . Ordering the hom-sets in \mathbf{Set}_T pointwise, namely, for all $f, g: C \rightsquigarrow C'$, declare that $f \leq g$ if $f(x) \subseteq g(x)$ for all $x \in C$, yields an ordered category, and $\mathbf{Set} \subseteq \mathbf{Set}_T$ is an order extension. Given a multivalued function $c: C \rightsquigarrow C'$, let $\text{Gr}(c) = \{(c, c') \in C \times C' \mid c' \in f(c)\}$ be the graph of c , and let $\hat{c}: \text{Gr}(c) \rightarrow C'$ be the obvious projection. Suppose that functions $f: B \rightarrow C$ and $g: B' \rightarrow C'$ are given, satisfying $g \circ b \leq c \circ f$. Then $f_{\#}: B \rightarrow \text{Gr}(c)$, given by $f_{\#}(x) = (fx, gbx)$, is well-defined, and together with $g_{\#} = g$ it is easily seen that $i_{\{\bullet \rightarrow \circ\}}^*(c: C \rightarrow C') = \text{Gr}(c) \xrightarrow{\hat{c}} C'$ is a classification of diagrams of shape $\{\bullet \rightarrow \circ\}$. Loosely speaking, the graph of a multivalued function is its classifying diagram.

Remark 1. Given a functor $S: \mathcal{D}' \rightarrow \mathcal{D}$, consider the diagram

$$\begin{array}{ccc} [\mathcal{D}, \mathcal{B}] & \xrightarrow{S^*} & [\mathcal{D}', \mathcal{B}] \\ i_{\mathcal{D}} \uparrow i_{\mathcal{D}}^* & & i_{\mathcal{D}'} \uparrow i_{\mathcal{D}'}^* \\ [\mathcal{D}, \mathcal{C}]_{\mathcal{B}} & \xrightarrow{S^*} & [\mathcal{D}', \mathcal{C}]_{\mathcal{B}} \end{array}$$

where S^* denotes pre-composition with S . It is obvious that the square involving the left adjoints commutes. However, generally, the right adjoints, even when they exist, are not compatible along S , namely the square involving the right adjoints typically does not commute. This is seen by the example above for the simple case where $S: \star \rightarrow \{\bullet \rightarrow \circ\}$, with $S(\star) = \bullet$.

3. The main result

Recall that for a category \mathcal{D} and an object $D \in \mathcal{D}$ the slice category D/\mathcal{D} consists of all morphisms $d_0: D \rightarrow D_0$ as its objects (where D_0 ranges over all objects of \mathcal{D}) and with morphisms $d_1 \rightarrow d_2$, for $d_k: D \rightarrow D_k$, $k = 1, 2$, those morphisms $d: D_1 \rightarrow D_2$ with $d \circ d_1 = d_2$. Further, with every fixed morphism $d: D \rightarrow D'$ there is an associated functor $d^*: D'/\mathcal{D} \rightarrow D/\mathcal{D}$ given on objects $d_0: D' \rightarrow D_0$ by $d^*(d_0) = d_0 \circ d$, and trivially on morphisms. There is a forgetful functor $\pi_D: D/\mathcal{D} \rightarrow \mathcal{D}$, mapping an object to its codomain, and acting trivially on morphisms. Obviously, $\pi_D \circ d^* = \pi_{D'}$ holds for all $d: D \rightarrow D'$. Given an order functor $F: \mathcal{D} \rightarrow \mathcal{C}$ and an object $D \in \mathcal{D}$ we write $F_D: D/\mathcal{D} \rightarrow \mathcal{C}$ for the functor $F \circ \pi_D$, and we note that $F_D \circ d^* = F_{D'}$ holds for all $d: D \rightarrow D'$.

Theorem 1. Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension. Then \mathcal{C} is Mahavier complete relative to \mathcal{B} if, and only if, \mathcal{B} is complete and $\mathcal{B} \subseteq \mathcal{C}$ admits classification of diagrams.

PROOF. Assume that $\mathcal{B} \subseteq \mathcal{C}$ is Mahavier complete. We already noted that it is automatic that \mathcal{B} is then complete, and thus we turn to construct classifying diagrams for some fixed small category \mathcal{D} . To construct the functor $i_{\mathcal{D}}^*: [\mathcal{D}, \mathcal{C}]_{\mathcal{B}} \rightarrow [\mathcal{D}, \mathcal{B}]$, consider a fixed order functor $F: \mathcal{D} \rightarrow \mathcal{C}$. Let $F_*(D) = \varprojlim_{\mathcal{B}}^M(F_D)$ be an arbitrarily chosen Mahavier limiting object for F_D , which comes equipped with a Mahavier limiting cone $\{\pi_{d_0}: F_*(D) \rightarrow F_D(d_0)\}_{d_0 \in D/\mathcal{D}}$, where d_0 ranges over all morphisms in \mathcal{D} whose domain is D . In more detail, for every commuting triangle

$$\begin{array}{ccc}
 D_1 & \xrightarrow{d} & D_2 \\
 & \swarrow d_1 & \searrow d_2 \\
 & D &
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(D_1) & \xrightarrow{F(d)} & F(D_2) \\
 & \swarrow \pi_{d_1} & \searrow \pi_{d_2} \\
 & F_*(D) &
 \end{array}$$

205 on the left there corresponds an order commuting triangle on the right, and the universal property holds. For a morphism $d: D \rightarrow D'$ we obtain a morphism $d_*: F_*(D) \rightarrow F_*(D')$, the canonical shape change morphism $\varprojlim_{\mathcal{B}}^M F_D \rightarrow \varprojlim_{\mathcal{B}}^M (F_D \circ d^*)$, i.e., $d_*: F_*(D) \rightarrow F_*(D')$ is the unique morphism in \mathcal{B} with the property that $\pi_{d_0 \circ d} = \pi_{d_0} \circ d_*$ holds for all $d_0 \in D'/\mathcal{D}$. Defining $F_*(d) = d_*$ is easily seen to be functorial. We now define $i_{\mathcal{D}}^*(F) = F_*$, obtaining the object part of the functor $i_{\mathcal{D}}^*: [\mathcal{D}, \mathcal{C}]_{\mathcal{B}} \rightarrow [\mathcal{D}, \mathcal{B}]$, and we now tend to the morphism part of it.

210 Let $F, F': \mathcal{D} \rightarrow \mathcal{C}$ be order functors, and $\alpha: F \rightarrow F'$ an order natural transformation relative to \mathcal{B} , for which we are to construct a natural transformation $\alpha_*: F_* \rightarrow F'_*$. To obtain the component $(\alpha_*)_D$ at an object $D \in \mathcal{D}$, consider, for an arbitrary commuting triangle

$$\begin{array}{ccc}
 D_1 & \xrightarrow{d} & D_2 \\
 & \swarrow d_1 & \searrow d_2 \\
 & D &
 \end{array}$$

the diagram

$$\begin{array}{ccccc}
 & & F_*(D) & & \\
 & \swarrow \pi_{d_1} & & \searrow \pi_{d_2} & \\
 F(D_1) & \xrightarrow{F(d)} & & \xrightarrow{F(d)} & F(D_2) \\
 \downarrow \alpha_{D_1} & & & & \downarrow \alpha_{D_2} \\
 F'(D_1) & \xrightarrow{F'(d)} & & \xrightarrow{F'(d)} & F'(D_2) \\
 & \swarrow \pi'_{d_1} & & \searrow \pi'_{d_2} & \\
 & & F'_*(D) & &
 \end{array}$$

(A dashed arrow labeled $(\alpha_*)_D$ points from $F_*(D)$ to $F'_*(D)$.)

where the triangles are the respective Mahavier limiting cones. From the top triangle we have $\pi_{d_2} \leq F(d) \circ \pi_{d_1}$ and from the rectangle, since α is an order

215 natural transformation, we have $\alpha_{D_2} \circ F(d) \leq F'(d) \circ \alpha_{D_1}$. It follows that $\alpha_{D_2} \circ \pi_{d_2} \leq \alpha_{D_2} \circ F(d) \circ \pi_{d_1} \leq F'(d) \circ \alpha_{D_1} \circ \pi_{d_1}$, and thus that $\{\alpha_{D_i} \circ \pi_{d_i}\}_{d_i: D \rightarrow D_i}$ is an order cone to F'_D . The universal property of the Mahavier limiting cone in the bottom triangle yields the desired dashed morphism. It follows easily that α_* is a natural transformation.

220 Setting $i_{\mathcal{D}}^*(\alpha) = \alpha_*$, for all order natural transformations $\alpha \in [\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$, concludes the construction of the object and morphism part of $i_{\mathcal{D}}^*: [\mathcal{D}, \mathcal{C}]_{\mathcal{B}} \rightarrow [\mathcal{D}, \mathcal{B}]$. The functoriality of $i_{\mathcal{D}}^*$ is easily verified.

It now remains to show that $i_{\mathcal{D}}^*$ is right adjoint to $i_{\mathcal{D}}$, for which we construct a bijection $[\mathcal{D}, \mathcal{C}]_{\mathcal{B}}(G, F) \rightarrow [\mathcal{D}, \mathcal{B}](G, F_*)$, for some fixed functor $G: \mathcal{D} \rightarrow \mathcal{B}$ and order functor $F: \mathcal{D} \rightarrow \mathcal{C}$. Let $\alpha: G \rightarrow F$ be an order natural transformation, and we must construct a corresponding natural transformation $\alpha^\sharp: G \rightarrow F_*$. For each $D \in \mathcal{D}$ and every commuting triangle $d_2 = d \circ d_1$ as above, consider the diagram

$$\begin{array}{ccc}
 & G(D) & \\
 G(d_1) \swarrow & & \searrow G(d_2) \\
 G(D_1) & \xrightarrow{G(d)} & G(D_2) \\
 \downarrow \alpha_{D_1} & & \downarrow \alpha_{D_2} \\
 F(D_1) & \xrightarrow{F(d)} & F(D_2) \\
 \swarrow \pi_{d_1} & & \searrow \pi_{d_2} \\
 & F_*(D) &
 \end{array}$$

(Note: A dashed arrow labeled α_D^\sharp points from $G(D)$ to $F_*(D)$ in the original diagram.)

of solid arrows. Since G is a functor and α is an order natural transformation we obtain that $\alpha_{D_2} \circ G(d_2) = \alpha_{D_2} \circ G(d) \circ G(d_1) \leq F(d) \circ \alpha_{D_1} \circ G(d_1)$. Consequently, $\{\alpha_{D_i} \circ G(d_i): G(D) \rightarrow F(D_i)\}_{d_i: D \rightarrow D_i \in \mathcal{D}/\mathcal{D}}$ is an order cone to F_D , and thus, by the universal property of $F_*(D)$, the dashed morphism exists, and it is the unique morphism $\alpha_D^\sharp: G(D) \rightarrow F_*(D)$ in \mathcal{B} satisfying $\alpha_{D_i} \circ G(d_i) = \pi_{d_i} \circ \alpha_D^\sharp$, for all $d_i: D \rightarrow D_i$. Verifying that we thus obtain a natural transformation is straightforward.

In the other direction, if $\beta: G \rightarrow F_*$ is a natural transformation, then we construct an order natural transformation $\beta^b: G \rightarrow F$ by considering the diagram

$$\begin{array}{ccc}
 G(D) & \xrightarrow{G(d)} & G(D') \\
 \downarrow \beta_D & & \downarrow \beta_{D'} \\
 F_*(D) & \xrightarrow{d_*} & F_*(D') \\
 \downarrow \pi_{\text{id}_D} & \searrow \pi_d & \downarrow \pi_{\text{id}_{D'}} \\
 F(D) & \xrightarrow{F(d)} & F(D')
 \end{array}$$

(Note: Bent morphisms β_D^b and $\beta_{D'}^b$ connect $G(D)$ to $F(D)$ and $G(D')$ to $F(D')$ respectively.)

230 where the bent morphisms are defined to be the composition of the vertical morphisms, the top square commutes by naturality of β , the top triangle commutes by definition of d_* , and the bottom triangle order commutes as it is part of the

order cone defining $F_*(D)$. A simple diagram chase shows that the morphisms β_D^b constitute the components of an order natural transformation, as required.

235 Verifying that each construction is the inverse of the other is routine, as is the verification of the naturality of the constructions.

For the converse, assume that \mathcal{B} is complete and that for each small category \mathcal{D} , the adjunction $i_{\mathcal{D}} \dashv i_{\mathcal{D}}^*$ exists. We then have the diagram

$$\begin{array}{ccc}
 & i_{\mathcal{D}} \circ \Delta & \\
 & \curvearrowright & \\
 \mathcal{B} & \xrightarrow{\Delta} & [\mathcal{D}, \mathcal{B}] \xrightarrow{i_{\mathcal{D}}} [\mathcal{D}, \mathcal{C}]_{\mathcal{B}} \\
 & \xleftarrow{\varprojlim} & \xleftarrow{i_{\mathcal{D}}^*} \\
 & \curvearrowleft & \\
 & \varprojlim \circ i_{\mathcal{D}}^* &
 \end{array}$$

240 in which $i_{\mathcal{D}} \dashv i_{\mathcal{D}}^*$ and $\Delta \dashv \varprojlim$. Since the compositions of left (respectively right) adjoints is a left (respectively right) adjoint it follows that $\varprojlim \circ i_{\mathcal{D}}^*$ is right adjoint to $i_{\mathcal{D}} \circ \Delta$, but this precisely defines Mahavier limits of shape \mathcal{D} , so that $\varprojlim_{\mathcal{B}}^M F \cong \varprojlim_{\mathcal{B}} (i_{\mathcal{D}}^*(F))$, and in particular \mathcal{C} is Mahavier complete relative to \mathcal{B} .

In fact, it is evident from the proof that the above result can be stated more accurately as follows.

245 **Theorem 2.** *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension and \mathcal{D} a small category. If \mathcal{B} has limits of shape \mathcal{D} , and $i_{\mathcal{D}}: [\mathcal{D}, \mathcal{B}] \rightarrow [\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$ has a right adjoint, then \mathcal{C} has all Mahavier limits relative to \mathcal{B} of shape \mathcal{D} . If \mathcal{C} has all Mahavier limits of shapes D/\mathcal{D} , for all $D \in \mathcal{D}$, then $i_{\mathcal{D}}$ has a right adjoint.*

For emphasis, we also make a note of the following.

250 **Theorem 3.** *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension and \mathcal{D} a small category. If $i_{\mathcal{D}}^*$ exists, then, as functors $[\mathcal{D}, \mathcal{C}]_{\mathcal{B}} \rightarrow \mathcal{B}$, there is a natural isomorphism $\varprojlim_{\mathcal{B}}^M \cong \varprojlim \circ i_{\mathcal{D}}^*$.*

We can now give a definition of the classifying diagram of an order diagram independently of the existence of the adjunction $i_{\mathcal{D}} \dashv i_{\mathcal{D}}^*$.

255 **Definition 1.** *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension and $F: \mathcal{D} \rightarrow \mathcal{C}$ an order diagram. The classifying diagram of F is the diagram F_* as constructed in the proof of the main theorem, provided the auxiliary Mahavier limits exist.*

Due to the choice of Mahavier limiting objects, a classifying diagram, if it exists, is defined up to isomorphism in $[\mathcal{D}, \mathcal{B}]$. Obviously, if F has a classifying diagram, then it computes the Mahavier limit of F in the evident sense.

260 Classifying diagrams are only of interest for order diagrams F that take at least one value outside of \mathcal{B} , as the following result clarifies.

Theorem 4. *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension and $F: \mathcal{D} \rightarrow \mathcal{B}$ a diagram. If the classifying diagram F_* of F exists, then $F \cong F_*$ in $[\mathcal{D}, \mathcal{B}]$.*

PROOF. When F takes values in \mathcal{B} all of the auxiliary Mahavier limits are computed in the degenerate order extension $\mathcal{B} \subseteq \mathcal{B}$, which thus reduce to ordinary categorical limits in \mathcal{B} . Then, $F_*(D) \cong \varprojlim^M(F_D) = \varprojlim(F_D)$, and since $\text{id}_D: D \rightarrow D$ is an initial object in D/\mathcal{D} , it follows that $F_*(D) \cong F_D(\text{id}_D) \cong F(D)$. The argument for the morphisms is straightforward, showing that $F_* \cong F$.

Remark 2. When $\mathcal{B} \subseteq \mathcal{C}$ is Mahavier complete, we thus obtain a reduction of the study of Mahavier limits to the study of the combination of ordinary limits and the adjunctions $i_{\mathcal{D}} \dashv i_{\mathcal{D}}^*$. However, as the proof of Theorem 1 reveals, the generic definition of $i_{\mathcal{D}}^*$ makes use of Mahavier limits, and so the reduction is more of a theoretical tool than an effective computational one. Moreover, the right adjoints $i_{\mathcal{D}}^*$ may be very complicated, even for relatively simple diagrams, and so we propose the following point-of-view. Studies of Mahavier complete order extensions $\mathcal{B} \subseteq \mathcal{C}$ (of which a prominent example is the study of generalised inverse limits of spaces) are, in disguise, studies of the adjunctions $i_{\mathcal{D}} \dashv i_{\mathcal{D}}^*$ within the usual category theoretic framework of limits. One may choose whether to work with these functors directly, if one can compute them, or whether to employ the techniques of [28] as a means to gain information about the functors of interest, and aid in their computation. This idea is reflected in the results given in the following section.

4. Properties and applications

For an order extension $\mathcal{B} \subseteq \mathcal{C}$ and a small category \mathcal{D} , it is trivial that $i_{\mathcal{D}}$ is faithful. The condition that the ordering on each hom-set $\mathcal{C}(C, C')$ induces the identity relation when restricted to $\mathcal{B}(C, C')$ implies that $i_{\mathcal{D}}$ is also full, and thus if the adjunction $i_{\mathcal{D}} \dashv i_{\mathcal{D}}^*$ exists, its unit $\eta: \text{id}_{[\mathcal{D}, \mathcal{B}]} \rightarrow i_{\mathcal{D}}^* \circ i_{\mathcal{D}}$ is a natural isomorphism. In other words, $[\mathcal{D}, \mathcal{B}]$ is a coreflective subcategory of $[\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$. As for the counit $\varepsilon: i_{\mathcal{D}} \circ i_{\mathcal{D}}^* \rightarrow \text{id}_{[\mathcal{D}, \mathcal{C}]_{\mathcal{B}}}$, if the second condition in Theorem 2 is met, then its component at an order diagram F is the order natural transformation $\varepsilon_F: i_{\mathcal{D}}^*(F) \rightarrow F$, whose components $(\varepsilon_F)_D: \varprojlim^M_{\mathcal{B}}(F_D) \rightarrow F(D)$ are given by π_{id_D} , the canonical projection from the Mahavier limit to $F_D(\text{id}_D) = F(D)$.

We now turn to consider some relations between properties of classifying diagrams and properties of the order extension.

4.1. Classifying diagrams as a measurement of the size of the order extension

We already remarked above that classifying diagrams are a proxy to the behaviour of the order extension in terms of size; the ability to classify an order diagram in the extension by an ordinary diagram is already an indication that the extension is not too wild. We now look at this phenomenon in more detail. Recall that we write $\{\bullet \rightarrow \circ\}$ for the free-living morphism category.

Theorem 5. For an order extension $\mathcal{B} \subseteq \mathcal{C}$, the following conditions are equivalent

1. $i_{\mathcal{D}}^*$ exists for all categories \mathcal{D} , and $i_{\mathcal{D}} \dashv i_{\mathcal{D}}^*$ is an adjoint equivalence.
2. $i_{\{\bullet \rightarrow \circ\}}^*$ exists, and $i_{\{\bullet \rightarrow \circ\}} \dashv i_{\{\bullet \rightarrow \circ\}}^*$ is an adjoint equivalence.
- 305 3. $i_{\{\bullet \rightarrow \circ\}}$ is essentially surjective.
4. $\mathcal{B} = \mathcal{C}$.

PROOF. Condition 1 trivially implies condition 2, and condition 2 immediately implies condition 3. If condition 4 is met then obviously $i_{\mathcal{D}}$ is the identity, so the validity of condition 1 is clear. We now show that condition 3 implies condition

310 4. Let $c: C \rightsquigarrow C'$ be an arbitrary morphism in \mathcal{C} , thought of as an order diagram $F: \{\bullet \rightarrow \circ\} \rightarrow \mathcal{C}$. There is then a diagram $G: \{\bullet \rightarrow \circ\} \rightarrow \mathcal{B}$, corresponding to a morphism $\hat{c}: B \rightarrow B'$ in \mathcal{B} , and an order natural transformation $\alpha: G \rightarrow F$, together with its inverse, an order natural transformation $\beta: F \rightarrow G$. Since composition of order natural transformations is component-wise, it follows at

315 once that the components of β are the inverses of the components of α . In more detail, the components of α are thus isomorphisms $b: C \rightarrow B$ and $b': C' \rightarrow B'$ in \mathcal{B} satisfying $b' \circ \hat{c} \leq c \circ b$, from which $b' \circ \hat{c} \circ b^{-1} \leq c$ follows. Moreover, $(b')^{-1} \circ c \leq \hat{c} \circ b^{-1}$, from which $c \leq b' \circ \hat{c} \circ b^{-1}$ follows. We may now conclude that $c = (b') \circ \hat{c} \circ b^{-1}$, a composition of morphisms in \mathcal{B} , and thus that $c \in \mathcal{B}$,

320 completing the argument.

The above result stems from the following simple observation. While for ordinary natural transformations α , invertibility of α is equivalent to the invertibility of each of its components, the same does not hold for order natural transformation; if an order natural transformation is invertible, then each of its

325 components is too, but the converse may fail. To obtain a somewhat more refined result, we introduce the following concepts for an order extension $\mathcal{B} \subseteq \mathcal{C}$. We say that \mathcal{C} is *nearly equal* to \mathcal{B} if for every $c \in \mathcal{C}$ there is a unique $\hat{c} \in \mathcal{B}$ with $\hat{c} \leq c$. \mathcal{C} is *functorially nearly equal* to \mathcal{B} if \mathcal{C} is nearly equal to \mathcal{B} and the assignment $c \mapsto \hat{c}$ is functorial. Finally, we say that $i_{\mathcal{D}}$ is *nearly essentially surjective* if for all order diagrams $F: \mathcal{D} \rightarrow \mathcal{C}$, there exists a diagram

330 $G: \mathcal{D} \rightarrow \mathcal{B}$ and an order natural transformation $\alpha: i_{\mathcal{D}}(G) \rightarrow F$ with each component an isomorphism in \mathcal{B} , and such that, for all $d: D \rightarrow D' \in \mathcal{D}$, the inequality $\alpha_{D'} \circ x \circ \alpha_D^{-1} \leq F(d)$ has a unique solution in $\mathcal{B}(G(D), G(D'))$, and that solution is $G(d)$. The adjunction $i_{\mathcal{D}} \dashv i_{\mathcal{D}}^*$ is *nearly an equivalence* if $i_{\mathcal{D}}$ is

335 nearly essentially surjective.

Theorem 6. *For an order extension $\mathcal{B} \subseteq \mathcal{C}$, the following conditions are equivalent*

1. For all categories \mathcal{D} , there exists a right adjoint $i_{\mathcal{D}}^*$ such that $i_{\mathcal{D}} \dashv i_{\mathcal{D}}^*$ is nearly an adjoint equivalence.
- 340 2. There exists a right adjoint $i_{\{\bullet \rightarrow \circ\}}^*$ such that $i_{\{\bullet \rightarrow \circ\}} \dashv i_{\{\bullet \rightarrow \circ\}}^*$ is nearly an adjoint equivalence.
3. $i_{\{\bullet \rightarrow \circ\}}$ is nearly essentially surjective.
4. \mathcal{C} is nearly equal to \mathcal{B} .
5. \mathcal{C} is functorially nearly equal to \mathcal{B} .

345 PROOF. Condition 1 trivially implies condition 2, and condition 2 equally trivially implies condition 3. Assume now that condition 3 holds, and let $c: C \rightarrow C'$ be a morphism in \mathcal{C} , thought of as an order diagram $F: \{\bullet \rightarrow \circ\} \rightarrow \mathcal{C}$. Since $i_{\{\bullet \rightarrow \circ\}}$ is nearly essentially surjective, there exists a morphism $b: B \rightarrow B' \in \mathcal{B}$ and isomorphisms $\alpha_C: B \rightarrow C$ and $\alpha_{C'}: B' \rightarrow C'$, such that $\hat{c} = \alpha_{C'} \circ b \circ \alpha_C^{-1} \leq c$.
350 If y is any other morphism in \mathcal{B} with $y \leq c$, then $\alpha_{C'}^{-1} \circ y \circ \alpha_C$ is a solution of $\alpha_{C'} \circ x \circ \alpha_C^{-1} \leq c$, and thus $b = \alpha_{C'}^{-1} \circ y \circ \alpha_C$, namely $y = \hat{c}$. In other words, for all $c \in \mathcal{C}$ there is a unique $\hat{c} \in \mathcal{B}$ with $\hat{c} \leq c$, as required. Showing that condition 4 implies condition 5 is a simple observation which we do not detail. Assume now that condition 5 holds, so that $c \mapsto \hat{c}$ is a functor. Given
355 a category \mathcal{D} , if $F: \mathcal{D} \rightarrow \mathcal{C}$ is an order diagram, then let $\hat{F}: \mathcal{D} \rightarrow \mathcal{B}$ be given by $\hat{F}(D) = F(D)$ for all $D \in \mathcal{D}$, and $\hat{F}(d) = \widehat{F(d)}$ for all $d \in \mathcal{D}$. Noting that generally $c_1 \leq c_2$ in \mathcal{C} implies $\hat{c}_1 = \hat{c}_2$, it follows easily that \hat{F} is a functor. Similarly, the components of an order natural transformation $\alpha: F_1 \rightarrow F_2$ are also the components of a natural transformation $\hat{\alpha}: \widehat{F_1} \rightarrow \widehat{F_2}$. In short, we may
360 define $i_{\mathcal{D}}^*(F) = \hat{F}$ and $i_{\mathcal{D}}^*(\alpha) = \hat{\alpha}$, which is then easily seen to be a right adjoint of $i_{\mathcal{D}}$. Further, for any order diagram $F: \mathcal{D} \rightarrow \mathcal{C}$, taking $\alpha: i_{\mathcal{D}}(i_{\mathcal{D}}^*(F)) \rightarrow F$ to have components $\alpha_D = \text{id}_D$, for all $D \in \mathcal{D}$, shows that $i_{\mathcal{D}}$ is nearly essentially surjective, and completing the proof.

4.2. Classifying diagrams and initial functors

365 For ordinary categorical limits recall that if $S: \mathcal{D}_0 \rightarrow \mathcal{D}$ is an initial functor (which sometimes, confusingly, is also called a final functor, see, e.g., [36], also for further details if needed) then the shape change morphism $\varprojlim S: \varprojlim F \rightarrow \varprojlim (F \circ S)$ is an isomorphism. This foundational result of category theory, one that is used extensively in applications of inverse limits in topology and
370 algebra prior to the formulation of category theory, is well-known not to hold for generalised inverse limits of spaces. Phrased in the context of diagrams indexed by the integers, the problem was coined as the “subsequence theorem problem” and is one of the earliest driving forces of research efforts in the theory of generalised inverse limits of spaces, calling for conditions under which the shape
375 change morphism as above between the generalised limits is an isomorphism. The subsequence theorem problem is discussed in [37]. Recently, Greenwood and Youl ([10]) presented a subsequence theorem for generalised inverse limits of compacta with a single multivalued bonding function, when the latter is constructed out of a finite family of singlevalued functions satisfying some rather
380 strong fixed-point conditions.

Let us phrase the problem in the context of an arbitrary order extension $\mathcal{B} \subseteq \mathcal{C}$. Let \mathcal{F} be a family of initial functors $S: \mathcal{D} \rightarrow \mathcal{D}'$, where \mathcal{D} and \mathcal{D}' are allowed to range over all small categories. We shall say that $\mathcal{B} \subseteq \mathcal{C}$ is \mathcal{F} -
385 *conservative* if for all order diagrams $F: \mathcal{D} \rightarrow \mathcal{C}$ such that the Mahavier limits exist, the shape change morphism $\varprojlim_{\mathcal{B}}^M(F) \rightarrow \varprojlim_{\mathcal{B}}^M(F \circ S)$ is an isomorphism.

Two extreme cases where conservativeness is guaranteed are the following. Firstly, if \mathcal{F} consists only of isomorphisms, then any order extension $\mathcal{B} \subseteq \mathcal{C}$ is \mathcal{F} -conservative (trivially so). On the other hand, if $\mathcal{B} = \mathcal{C}$, in which case Mahavier

limits are simply ordinary limits, then \mathcal{F} -conservatism holds for all collections \mathcal{F}
 390 as above. This is nothing but a re-iteration of the opening line of this subsection,
 which is well-known, but not quite as trivial as the former condition.

Theorem 7. *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension. If \mathcal{C} is nearly equal to \mathcal{B} , then
 $\mathcal{B} \subseteq \mathcal{C}$ is \mathcal{F} -conservative for all families \mathcal{F} of initial functors.*

PROOF. Referring to Theorem 6, let $c \mapsto \hat{c}$ be the unique functor $\mathcal{C} \rightarrow \mathcal{B}$
 with $\hat{c} \leq c$ for all $c \in \mathcal{C}$. Let $S: \mathcal{D}' \rightarrow \mathcal{D}$ be an initial functor, and $F: \mathcal{D} \rightarrow$
 \mathcal{C} an arbitrary order diagram, and assume the Mahavier limits $\varprojlim_{\mathcal{B}}^M(F)$ and
 $\varprojlim_{\mathcal{B}}^M(F \circ S)$ exist. Recall from the proof of Theorem 6 that $i_{\mathcal{D}}^*(F) = \hat{F}$ and
 that $i_{\mathcal{D}'}^*(F \circ S) = \widehat{F \circ S} = \hat{F} \circ S$. By Theorem 3 we have the vertical canonical
 isomorphisms in the diagram

$$\begin{array}{ccc} \varprojlim_{\mathcal{B}}^M(F) & \dashrightarrow & \varprojlim_{\mathcal{B}}^M(F \circ S) \\ \uparrow \cong & & \uparrow \cong \\ \varprojlim(\hat{F}) & \xrightarrow{\cong} & \varprojlim(\hat{F} \circ S) \end{array}$$

while the horizontal arrow is the shape change morphism for ordinary limits,
 395 which, since S is initial, is an isomorphism. Composing the three morphisms
 yields the shape change morphism for the Mahavier limits, which is thus an
 isomorphism, as required.

Theorem 8. *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension, and \mathcal{F} a collection of initial
 functors containing the functor $S: \star \rightarrow \{\bullet \rightarrow \circ\}$ with $S(\star) = \bullet$. Then if $\mathcal{B} \subseteq \mathcal{C}$
 400 has Mahavier limits of shape $\{\bullet \rightarrow \circ\}$ and is \mathcal{F} -conservative, then \mathcal{C} is nearly
 equal to \mathcal{B} .*

PROOF. Let $c: C \rightarrow C'$ be a morphism, thought of as an order diagram $F: \{\bullet \rightarrow$
 $\circ\} \rightarrow \mathcal{C}$ with $F(\bullet) = C$. By assumption, the canonical morphism $\varprojlim_{\mathcal{B}}^M(F) \rightarrow$
 $\varprojlim_{\mathcal{B}}^M(F \circ S)$ is an isomorphism. Obviously, we may take $\varprojlim_{\mathcal{B}}^M(F \circ S) = C$,
 405 and thus also $\varprojlim_{\mathcal{B}}^M(F) = C$. The universal order cone for that Mahavier limit
 consists of morphisms $\text{id}_C: C \rightarrow C$ and $\hat{c}: C \rightarrow C'$, and its order commutativity
 is the claim that $\hat{c} \leq c$. Any other morphism b with $b \leq c$ yields, together with
 id_C , another order cone to F , which thus factorises through the universal one,
 necessarily trivially, and thus $b = \hat{c}$.

410 Qualitatively, given an order extension $\mathcal{B} \subseteq \mathcal{C}$, the largest class \mathcal{F} of initial
 functors with respect to which the extension is conservative may be called the
conservatism degree of the extension. The collection of all adjunctions $i_{\mathcal{D}} \dashv i_{\mathcal{D}}^*$,
 for all small categories \mathcal{D} , may be called the *classifying degree* of the extension.
 These degrees measure different aspects of the extension, and the results above
 415 reveal that there is a tension between these two aspects: an order extension
 with a large conservatism degree (in the sense that \mathcal{F} is a large collection of
 initial functors) tends to have a small classifying degree (in the sense that the
 adjunctions exhibit simple behaviour), and vice versa.

Remark 3. *Phrased in the formalism we developed, a very general formulation of the subsequence theorem problem in the theory of generalised inverse limits of spaces can be stated as follows. Identify an intermediate order extension $\mathbf{Top} \subseteq \mathbf{X} \subseteq \mathbf{Top}_T$ which is \mathcal{F} -conservative for a sufficiently interesting class of initial functors \mathcal{F} . Classically, \mathcal{F} consists of all initial functors between categories isomorphic to the poset of natural numbers, or to more general posets. Of course, one would like \mathbf{X} to be a significant portion of \mathbf{Top}_T , namely $\mathbf{Top} \subseteq \mathbf{X}$ should have a large classifying degree, and, since \mathcal{F} should be a useful collection of initial functors, at the same time have a large conservatism degree. As seen above, attaining both degrees to be large is impossible, thus explaining the difficulty in resolving the subsequence theorem problem. This observation, and the results above, are of importance in further framing the subsequence theorem problem of generalised inverse limits of spaces, and, perhaps most importantly, in setting realistic expectations from any possible solution of it.*

4.3. Classifying diagrams and Mahavier limits in terms of ordinary limits and colimits

This final subsection is an interesting consequence of Theorem 3, though we are unaware of practical applications of it. It is well-known that small limits can be constructed from small products and equalisers, and a version of that result for Mahavier limits is given in [28]. It is also well-known that limits can sometimes be constructed in terms of colimits. For instance, a join complete lattice is automatically meet complete. A systematic approach is to consider, given categories \mathcal{B}, \mathcal{D} , the diagonal functor $\Delta: \mathcal{B} \rightarrow [\mathcal{D}, \mathcal{B}]$, and construct its right adjoint by means of (any particular version of) the adjoint functor theorem.

Obviously, given an order extension $\mathcal{B} \subseteq \mathcal{C}$, the same approach can be applied to $i_{\mathcal{D}} \circ \Delta$, but another approach is also possible, namely to apply the adjoint functor theorem to the functor $i_{\mathcal{D}}$. When we constructed $i_{\mathcal{D}}^*$ above, Mahavier limits were explicitly used in the construction, and thus that proof is of limited use in computing Mahavier limits. But, if the solution set condition can be established, then the adjoint functor theorem can circumvent the need to directly use Mahavier limits. We phrase this observation using Freyd's adjoint functor theorem.

Theorem 9. *Let $\mathcal{B} \subseteq \mathcal{C}$ be an order extension. If \mathcal{B} is complete and cocomplete, and for each small category \mathcal{D} the functor $i_{\mathcal{D}}: [\mathcal{D}, \mathcal{B}] \rightarrow [\mathcal{D}, \mathcal{C}]_{\mathcal{B}}$ is cocontinuous and satisfies the solution set condition, then \mathcal{C} is Mahavier complete relative to \mathcal{B} .*

PROOF. When \mathcal{B} is cocomplete, so is $[\mathcal{D}, \mathcal{B}]$, and thus, by the adjoint functor theorem, $i_{\mathcal{D}}^*$ exists. Since \mathcal{B} is complete, $i_{\mathcal{D}}^*$ computes Mahavier limits of shape \mathcal{D} , so, in particular, they exist.

In as much as the solution set condition can be established without recourse to any explicit Mahavier limits, we obtain a construction of Mahavier limits in terms of limits and colimits in \mathcal{B} .

5. Revisiting classical generalised inverse limits

We conclude this work with a fresh look at the elements of generalised inverse limits in topology through the lens of the formalism presented above. We keep the discussion somewhat informal, and we only touch upon a few topics, for the sake of brevity. Where details and proofs can be found in [18] or [28], we will simply omit any arguments.

The ambient order extension for considering classical generalised inverse limits is $\mathbf{Top} \subseteq \mathbf{Top}_T$, where \mathbf{Top}_T is the category of all topological spaces with morphisms the upper semicontinuous functions, with each hom-set ordered by inclusion. An important order sub-extension is $\mathbf{Comp} \subseteq \mathbf{Comp}_T$, where \mathbf{Comp} is the full subcategory of \mathbf{Top} spanned by the compact Hausdorff spaces, and \mathbf{Comp}_T has the same objects but restricts to the closed valued mappings.

The first examples of generalised inverse limits were of sequences of spaces indexed by \mathbb{N} or \mathbb{Z} . Later on, more general shapes were allowed, but all still made use of *thin* diagrams, namely having at most one bonding function between any two spaces. We note first that Mahavier limits of the most general (small) shapes in $\mathbf{Top} \subseteq \mathbf{Top}_T$ exist.

Theorem 10. *The order extension $\mathbf{Top} \subseteq \mathbf{Top}_T$ is Mahavier complete.*

In particular, for different indexing categories \mathcal{D} , all of the notions of generalised inverse limits of spaces considered in the literature are obtained. In conjunction with the order extension $\mathbf{Top} \subseteq \mathbf{Top}_T$ it is natural to consider the order extension $\mathbf{Set} \subseteq \mathbf{Set}_T$, where \mathbf{Set}_T is the Kleisli category of the covariant power set monad on \mathbf{Set} , in other words, the category of all sets and whose morphisms are the multivalued functions, endowed with the evident order structure induced by set inclusion. Obviously, $\mathbf{Set} \subseteq \mathbf{Set}_T$ is isomorphic (in a suitable category of order extensions) to the full sub-order extension of $\mathbf{Top} \subseteq \mathbf{Top}_T$ spanned by the discrete spaces.

Theorem 11. *The order extension $\mathbf{Set} \subseteq \mathbf{Set}_T$ is Mahavier complete.*

PROOF. Discretise the details of any proof of Theorem 10.

The next result addresses the compacta oriented needs of the theory.

Theorem 12. *The order extension $\mathbf{Comp} \subseteq \mathbf{Comp}_T$ is Mahavier complete.*

PROOF. The requirement that the morphisms in \mathbf{Comp}_T are closed valued allows for an adaptation of the proof of Theorem 10.

One now obtains the diagram

$$\begin{array}{ccccc}
 \mathbf{Set} & \xrightleftharpoons{\quad} & \mathbf{Top} & \longleftarrow & \mathbf{Comp} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{Set}_T & \xrightleftharpoons{\quad} & \mathbf{Top}_T & \longleftarrow & \mathbf{Comp}_T
 \end{array}$$

495 exhibiting the three main order extensions of interest, together with forgetful
 functors leading from right to left, and, where depicted, their left and right
 adjoints.

Theorem 13. *A right adjoint $G: \mathcal{B} \rightarrow \mathcal{B}'$ which extends, as in the above dia-
 500 gram, to a right adjoint of order extensions $G: \mathcal{C} \rightarrow \mathcal{C}'$ is Mahavier continuous,
 i.e., it preserves all Mahavier limits that exist in \mathcal{C} : $G(\varprojlim_{\mathcal{B}}^M(F)) \cong \varprojlim_{\mathcal{B}}^M(G \circ F)$,
 for all order diagrams $F: \mathcal{D} \rightarrow \mathcal{C}$.*

In particular computations in topology one often finds it convenient to
 change perspective and move around the top part of the diagram, e.g., ignoring
 the topology and concentrating on the underlying sets. With the above result
 the same tools are at one's disposal when considering generalised inverse limits,
 505 allowing the use of standrd arguments to deduce various properties, for instance
 surjectivity of induced mappings between generalised inverse limits by applying
 the forgetful functor to sets.

Obviously, the categorical formalism we consider is a unifying mechanism,
 but in a somewhat stronger manner than the immediate labour saving conse-
 510 quence of treating different notions of generalised inverse limits as instances
 of a single concept. To see how, recall that the *graph* of a multivalued func-
 tion $f: X \rightarrow Y$ is $\text{Gr}(f) = \{(x, y) \in X \times Y \mid y \in f(x)\}$, endowed with
 the subspace topology (in case topologies are involved). Unlike the case of
 singlevalued functions, the graph of f is typically not homeomorphic, or even
 515 in bijection, with the domain of f . More generally, given any finite sequence
 $\mathbf{f} = X_1 \xrightarrow{f_1} X_2 \rightarrow \cdots \rightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n$, define its graph to be $\text{Gr}(\mathbf{f}) = \{\mathbf{x} \in$
 $X_1 \times \cdots \times X_n \mid \mathbf{x}_{k+1} \in f_k(\mathbf{x}_k), \forall 1 \leq k < n\}$. We notice at once that these
 graphs are nothing but an instance of Mahavier limit.

Proposition 1. *The graph $\text{Gr}(\mathbf{f})$ of a finite sequence of functions, with the
 520 evident projections, is the Mahavier limit of \mathbf{f} considered as a finite diagram:
 $\text{Gr}(\mathbf{f}) \cong \varprojlim_{\mathbf{Top}}^M (X_1 \xrightarrow{f_1} X_2 \rightarrow \cdots \rightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n)$.*

PROOF. Verification of the universal property is immediate.

Graphs appear prominently in the study of generalised inverse limits of se-
 525 quences indexed by \mathbb{N} , in fact as an instance of the main result of this arti-
 cle. In more detail, let \mathcal{D} be the category \mathbb{N} , i.e., the natural numbers with
 morphisms corresponding to $m \geq n$, and let $F: \mathcal{D} \rightarrow \mathbf{Top}_T$ be an order
 diagram, i.e., a generalised inverse system \mathbf{f} . According to the main result
 $\varprojlim_{\mathbf{Top}}^M(F) \cong \varprojlim(i_{\mathcal{D}}^*(F))$, and the Mahavier limits arising in the proof cor-
 respond to slices of \mathcal{D} . But all such slices are finite sequences, yielding the
 530 formula $\varprojlim_{\mathbf{Top}}^M(F) \cong \varprojlim \text{Gr}(\mathbf{f})$, where $\text{Gr}(\mathbf{f}): \mathcal{D} \rightarrow \mathbf{Top}$ is the functor sending
 n to the graph of the initial segment of \mathbf{f} of length n .

Further along this line, but no longer under the restriction on the shape
 of the diagrams to be sequential, the following is a trivial observation, indeed
 merely a tautology. Let P be a property applicable to a diagram $\mathcal{D} \rightarrow \mathbf{Top}$ of

535 spaces with ordinary singlevalued continuous functions. We say that an order
 diagram $F: \mathcal{D} \rightarrow \mathbf{Top}_T$ is Mahavier P if the classifying diagram $i_{\mathcal{D}}^*(F)$ is P .
 Note that by Theorem 4 if a diagram in \mathbf{Top} is P , then when viewed as a
 sequence in \mathbf{Top}_T it is automatically Mahavier P .

Theorem 14. *Let P be a property of diagrams in \mathbf{Top} and Q a property of*
 540 *spaces. If it is true that whenever a diagram $\mathcal{D} \rightarrow \mathbf{Top}$ is P the limit $\varprojlim(F)$ is*
 Q , then it also holds that whenever an order diagram $\mathcal{D} \rightarrow \mathbf{Top}_T$ is Mahavier
 P the Mahavier limit $\varprojlim^M(F)$ is Q .

Obviously, the challenge for a fruitful application of this principle is in identi-
 fying, for a given property P , conditions verifiable directly on an order diagram
 545 $\mathcal{D} \rightarrow \mathbf{Top}_T$ that render it Mahavier P . But even in the absence of such criteria,
 ad-hoc criteria can be obtained. It is precisely this principle that is applied,
 e.g., when studying generalised inverse limits of a sequence of spaces by means
 of graphs.

As a final note, the discussion above is meant to extract the essence of some
 550 of the most fundamental tools and techniques of generalised inverse limits and
 portray them categorically, focusing on the relationship between Mahavier limits
 and classifying diagrams. Although much more can be said, with [18] and [28]
 already containing significant theory and detail, we remain brief and conclude
 the work here.

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