Multi-objective H_∞ Control for Vehicle Active Suspension Systems with Random Actuator Delay

Hongyi Li, Honghai Liu, Steve Hand and Chris Hilton

Abstract

This paper is concerned with the problem of multi-objective H_∞ control for vehicle active suspension systems with random actuator delay, which can be represented by signal probability distribution. First, the dynamical equations of a quarter-car suspension model are established for the control design purpose. Secondly, when taking into account vehicle performance requirements, namely, ride comfort, suspension deflection and the probability distributed actuator delay, we present the corresponding dynamic system, which will be transformed to the stochastic system for the problem of multi-objective H_∞ controller design. Thirdly, based on the stochastic stability theory, the state feedback controller is proposed to render that the closed-loop system is exponentially stable in mean-square while simultaneously satisfying H_∞ performance and the output constraint requirement. The presented condition is expressed in the form of convex optimization problems so that it can be efficiently solved via standard numerical software. Finally, a practical design example is given to demonstrate the effectiveness of the proposed method.

Keywords: Active suspension system; Actuator delay; Exponentially stability in mean-square; Multi-objective control; H_∞ control.

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I. Introduction

Performance requirements of modern vehicle suspensions will include ride comfort, (i.e., it is to isolate passengers from vibration and shock arising from road roughness) good road holding, (i.e., it is to suppress the hop of the wheels so as to maintain firm and uninterrupted contact of wheels to road) and suspension deflection, (i.e., it is to keep suspension strokes within an allowable maximum [1]). these requirements are conflicted in principle which means that a compromise of the requirements must be achieved for better combination performances [2], [3].

Recently, considerable attention has been paid to improving the performances of vehicle suspensions in the literature [4]–[6]. It is evident that active suspension systems is an effective way to improve suspension performance [1], [7]. Active suspension control research is aimed at improving the ride performance which is generally quantified by sprung mass acceleration, and to maintain an acceptable level of suspension stroke and tyre deflection. To reach the target, various kinds of control techniques have been applied to active suspension control to improve the performance, such as fuzzy logic and neural network control [8], gain scheduling control [9], linear optimal control [10], adaptive control [11] and H_∞ control [3], [12]. Among these methods, it has been proven that H_∞ control methods [13] of vehicle active suspensions [3], [12] is a feasible way to manage the trade-off between conflicting performance i.e. means that minimization of suspension travel cannot be accomplished simultaneously with maximization of the ride comfort. Therefore, more recently, many researchers have paid growing attention to the H_∞ control problem for active suspension vehicles and presented some results, for example, [3], [12], [14], [15].

It is evident that the time delay inherently existing in the vehicle suspension systems may be the source of poor performance and instability of the systems especially for the active control problem. Time delay or transportation lag is often encountered in many engineering systems [16], such as pneumatic and hydraulic systems, chemical processes, and long transmission lines. In general, the total delay in a control system is two fold. On one hand, time delay is caused by online data acquisition, filtering, calculating control forces, and transmitting the control force signals from a computer to the actuator. Specially, such time delay may be caused by the utilization of digital computers in control systems in that it takes finite time to carry out computation. However, the relevant computational delay can be modelled as constant and equal to the sampling period by using periodically driven sample and hold systems. Given today's computing power, such as digital-signal-processer-based system, this kind of delay will not play a substantial negative impact on the performance of a suspension control system since it can be estimated in advance. On the other hand, the delay known as floating time delay is taken by an actuator to build up the required control force. This kind of time delay usually depends on the particular dynamics of actuators [17], [18]. Due to the delay existing in actuators, the problem of controller design for active vehicle suspensions has been investigated to make the controller tolerate the time delay effect in [19], [20]. very recently, the authors in [3], [21] considered controller design for active suspensions with actuator delay.

Over the past years, there has been considerable research work reported to address the problems of H_∞ control of active suspensions in the context of robustness and disturbance attenuation [2], [3], [12], [14], [15]. In [12], the authors presented the constrained control H_∞ scheme for active suspensions with output and control constraint. The problem of H_∞ controller design for a class of uncertain vehicle suspension systems with sampling measurements was addressed in [2]. In addition, the authors in [3], [22] proposed the H_∞ controller design method for the vehicle suspension systems with actuator delay.

Motivated by the above observation, in this paper we focus on the problem of multi-objective H_∞ control for active vehicle suspension systems with random actuator delay occurred according to its probability distribution. The random actuator delay is assumed to be time-varying and vary in a given range. First, we establish the dynamical equations of a quarter-car suspension model. Secondly, the vehicle performance requirements such as ride comfort, suspension deflection and the probability distributed actuator delay are considered for the presented dynamic system. Thirdly, based on the stochastic stability theory, a state feedback controller is proposed to ensure that the closed-loop system is exponentially stable in mean-square with H_∞ performance and output constraint requirement simultaneously. The corresponding multi-objective H_∞ controller existence condition can be expressed by the form of linear matrix inequalities (LMIs), which can be efficiently solved via standard numerical software [23]. Finally, a design example is employed to present the effectiveness of the proposed method. The remaining part of the paper is organized as follows: The problem to be addressed is presented in Section II and multi-objective H_∞ controller design method is proposed in Section III, respectively. In Section IV, we present a design example to demonstrate the effectiveness of the developed approach and we conclude the paper in Section V.

Notation: The notation used in this paper is fairly standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ stands for the set of real $n \times m$ matrices. The superscript "T" is denoted as matrix transposition; The notation $P > 0$ (≥ 0) means that P is real symmetric and positive definite (semi-definite). In symmetric block matrices or complex matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry and diag$\{\ldots\}$ stands for a block-diagonal matrix. The notation sym$\{A\}$ is used to stand for $A + A^T$. $\mathcal{E}\{\cdot\}$ is used to denote the mathematical expectation. The space of square-integrable vector functions over $[0,\infty)$ is denoted by $L_2[0,\infty)$, and for $w=\{w(t)\}\in L_2[0,\infty)$, its norm is given by $\|w\|_2=\sqrt{\int_{t=0}^{\infty}|w(t)|^2\,\mathrm{d}t}$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation

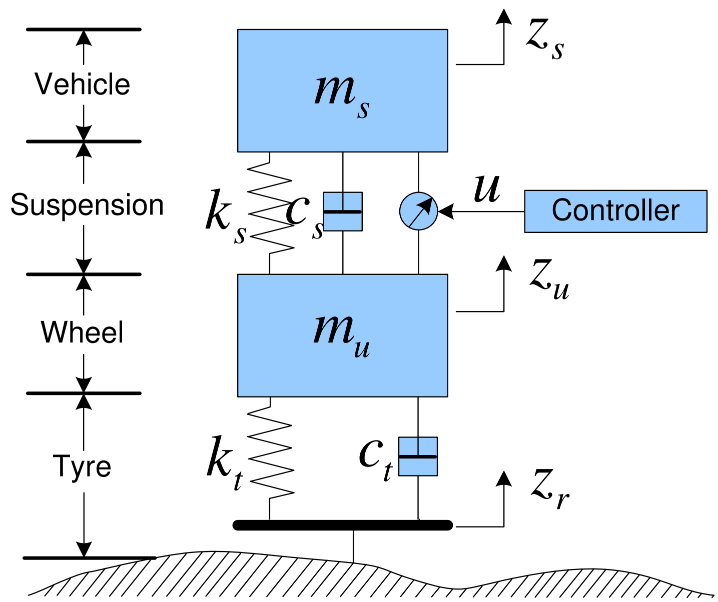

Fig. 1. A quarter-car model with an active suspension

The generalized quarter-car suspension model is shown in Fig. 1, in which z_s and z_u stands for the displacements of the sprung and unsprung masses, respectively; z_r denotes the road displacement input; u is the active input of the suspension system; m_s is the sprung mass, which represents the car chassis; m_u is the unsprung mass, which represents mass of the wheel assembly; c_s and k_s are damping and stiffness of the suspension system, respectively; k_t and c_t stand for compressibility and damping of the pneumatic tyre, respectively. Then, the equations of motion can be established as follows:

$$\begin{aligned} m_s\ddot{z}_s(t) &= c_s[\dot{z}_u(t)-\dot{z}_s(t)]+k_s[z_u(t)-z_s(t)]+u(t), \\ m_u\ddot{z}_u(t) &= c_s[\dot{z}_s(t)-\dot{z}_u(t)]+k_s[z_s(t)-z_u(t)]+k_t[z_r(t)-z_u(t)]+c_t[\dot{z}_r(t)-\dot{z}_u(t)]-u(t). \end{aligned} \quad (1)$$

Let us define the following state variables:

$x_1(t) = z_s(t) - z_u(t)$	denotes the suspension deflection,
$x_2(t) = z_u(t) - z_r(t)$	denotes the tire deflection,
$x_3(t) = \dot{z}_s(t)$	denotes the sprung mass speed,
$x_4(t) = \dot{z}_u(t)$	denotes the unsprung mass speed.

Then we define the disturbance input $w(t) = \dot{z}_r(t)$ and the state vector as

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T,$$

next the dynamic equations in (1) can be expressed as the following state-space form:

$$\dot{x}(t) = Ax(t) + B_1 w(t) + Bu(t), \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ \frac{k_s}{m_u} & -\frac{k_t}{m_u} & \frac{c_s}{m_u} & -\frac{c_s+c_t}{m_u} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_s} \\ -\frac{1}{m_u} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ \frac{c_t}{m_u} \end{bmatrix}. \tag{3}$$

When dealing with the controller design for the active suspension system, the main performance requirement conditions, namely, ride comfort, and suspension deflection should be taken into account. Since ride comfort can be generally quantified by the body acceleration in the vertical direction, it is essential to choose body acceleration as the first control output, i.e., $\ddot{z}_s(t)$. so when we design the controller, one of our main objectives is to minimize the vertical acceleration $\ddot{z}_s(t)$. The H_∞ norm is employed to measure the performance, whose value actually gives an upper bound of the root-mean-square gain. Hence, our goal is to minimize the H_∞ norm of the transfer function from the disturbance $w(t)$ to the control output $z_1(t) = \ddot{z}_s(t)$ in order to improve ride comfort. Besides, due to the mechanical structure, the suspension stroke should not exceed the allowable maximum, that is,

$$|z_s(t) - z_u(t)| \leq z_{\max}, \tag{4}$$

where $z_{\max}$ is the maximum suspension deflection. The controller should be able to ensure the car safety and prevent the suspension from hitting its travel limit in order to avoid ride comfort deterioration and mechanical structural damage.

Based on the above conditions, therefore, we select the H_∞ norm to measure the body acceleration $\ddot{z}_s(t)$ being as a performance output, and the suspension stroke $z_s(t) - z_u(t)$ as a constrained output.

Subsequently, the vehicle active suspension system can be described by the following state-space equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + Bu(t), \\ z_1(t) &= C_1 x(t) + D_1 u(t), \\ z_2(t) &= C_2 x(t), \end{aligned} \tag{5}$$

where A, B_1 and B are defined in (2), and

$$C_1 = \begin{bmatrix} -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \end{bmatrix}, \quad D_1 = \frac{1}{m_s}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

For the vehicle active suspension system in (5), we suppose that the input $u(t)$ may or may not experience input delay, and may be dependent on probability distribution. We can describe the following two random events:

$$\begin{cases} \text{Even 1}: & u(t) \text{ does not experience input delay,} \\ \text{Even 2}: & u(t) \text{ does experience input delay.} \end{cases}$$

Let the stochastic variable $\delta(t)$ be defined as

$$\delta(t) = \begin{cases} 1, & \text{if Even 1 occurs,} \\ 0, & \text{if Even 2 occurs.} \end{cases}$$

As discussed in [24]–[27], $\delta(t)$ is a Markovian process and can be assumed to follow an exponential distribution of switchings, which satisfies,

$$\begin{cases} \text{Prob}\{\delta(t) = 1\} = \mathcal{E}\{\delta(t)\} = \bar{\delta}, \\ \text{Prob}\{\delta(t) = 0\} = 1 - \mathcal{E}\{\delta(t)\} = 1 - \bar{\delta}, \end{cases} \tag{7}$$

where the constant $\bar{\delta} \in [0, 1]$ shows the occurred probability of the event of no input delays. In what follows, we assume that the input is described by

$$u(t) = \delta(t) Kx(t) + (1 - \delta(t)) Kx(t - d(t)), \tag{8}$$

where K is a constant matrix to be determined later. $d(t)$ is the time-varying delay in the input channel which may be piecewise continuous satisfies the following condition.

There exist scalars d_1, d_2 and μ with $d_2 \geq d_1 > 0$ such that

$$0 < d_1 \leq d(t) \leq d_2, \quad \dot{d}(t) \leq \mu. \tag{9}$$

It means that the time delay is a smooth function of $d(t)$ and its derivative is known to be the upper bounded by μ.

Under the controller in (8), the closed-loop system in (5) can be described as

$$\begin{aligned}\dot{x}(t) &= (A+\delta(t)BK)x(t)+(1-\delta(t))BKx(t-d(t))+B_1w(t),\\ z_1(t) &= (C_1+\delta(t)D_1K)x(t)+(1-\delta(t))D_1Kx(t-d(t)),\\ z_2(t) &= C_2x(t).\end{aligned} \tag{10}$$

We define a continuous initial function $x(t)=\phi(t), t\in[-2d_2,0]$ and we assume that $w(t)=0$ for $t\in[-2d_2,0]$. It is seen that system in (10) is a stochastic system with

$$\mathcal{E}\left\{\left(\delta(t)-\bar{\delta}\right)\right\}=0, \quad \mathcal{E}\left\{\left(\delta(t)-\bar{\delta}\right)^2\right\}=\bar{\delta}\left(1-\bar{\delta}\right).$$

Subsequently, we rewrite the closed-loop systems in (10) as the following form:

$$\begin{aligned}\dot{x}(t) &= \left(A+\bar{\delta}BK\right)x(t)+\left(1-\bar{\delta}\right)BKx(t-d(t))+B_1w(t)\\ &\quad+\left(\delta(t)-\bar{\delta}\right)(BKx(t)-BKx(t-d(t))),\\ z_1(t) &= \left(C_1+\bar{\delta}DK\right)x(t)+\left(1-\bar{\delta}\right)D_1Kx(t-d(t))\\ &\quad+\left(\delta(t)-\bar{\delta}\right)(D_1Kx(t)-D_1Kx(t-d(t))),\\ z_2(t) &= C_2x(t).\end{aligned} \tag{11}$$

For convenience, we have

$$\begin{aligned}f(t) &= \left(A+\bar{\delta}BK\right)x(t)+\left(1-\bar{\delta}\right)BKx(t-d(t))+B_1w(t),\\ g(t) &= BKx(t)-BKx(t-d(t)).\end{aligned}$$

Since $f(t)$ and $g(t)$ in (11) satisfy the local Lipschitz condition and the linear growth condition, the existence and uniqueness of solution to (11) is guaranteed [28]. Moreover, under $w(t)=0$, it admits a trivial solution (equilibrium) $x\equiv 0$. The following definitions of exponential stability in mean square and H_∞ performance requirements are given for the proof in the next section.

Definition 1: ([27]) The system in (11) is said to be exponentially stable in mean square, if, under $u(t)=0$ and $w(t)=0$, there exist positive constant α and β, such that for all $t\geq 0$, the following inequality holds,

$$\mathcal{E}\left\{\|x(\phi,t)\|^2\right\}\leq \alpha e^{-\beta t}\mathcal{E}\left\{\sup_{-2d_2\leq s\leq 0}\|\phi(s)\|^2\right\}. \tag{12}$$

Definition 2: Given a scalar $\gamma>0$, the system in (11) with $u(t)=0$ is said to be exponentially stable in mean square with disturbance attenuation γ if it is exponentially stable in mean square and under zero initial conditions,

$$\|z_1(t)\|_{E_2} < \gamma \|w(t)\|_2 \tag{13}$$

is satisfied for all nonzero $w(t) \in L_2[0, \infty)$, where

$$\|z_1(t)\|_{E_2} = \left(\mathcal{E}\left\{ \int_0^{\infty} |z_1(t)|^2 \, \mathrm{d}t \right\} \right)^{\frac{1}{2}}. \tag{14}$$

Without loss of generality, it is assumed that $w \in L_2[0, \infty)$, and then we have $\|w\|_2^2 \leq w_{\max} < \infty$. The objective of this paper is to determine a state controller in (8) such that

(1) the closed-loop system is exponentially stable in mean square;

(2) under zero initial condition, the closed-loop system guarantees that $\|z_1(t)\|_{E_2} < \gamma \|w(t)\|_2$ for all nonzero $w \in L_2[0, \infty)$, while the output constraint in (4) is satisfied

$$|z_2(t)| \leq z_{2\max}. \tag{15}$$

In the above proposed control strategy, the multiple requirements such as ride comfort, suspension deflection limit and random actuator delay are formulated in a unified framework, based on which the controller design is cast into a multiple-objective minimization problem.

III. Multi-objective H_∞ Performance Analysis

In this section, we will first address the multi-objective H_∞ performance analysis problem for the active suspension system in (11) in the following theorem.

Theorem 1: Consider the closed-loop system in (11). For given scalars d_1, d_2, μ, $\bar{\delta}$ and a feedback gain K, if there exist matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_1 = \mathrm{diag}\{Z_{11}, Z_{11}\} > 0$, $Z_2 = \mathrm{diag}\{Z_{21}, Z_{21}\} > 0$, N_1, N_2 and N_3 with appropriate dimensions such that the following LMIs hold:

$$\begin{bmatrix} \Omega + \Omega_0 & \sqrt{d_2} N_1 [I_n, I_n] & \sqrt{d_{21}} N_2 [I_n, I_n] \\ * & -Z_1 & 0 \\ * & * & -Z_1 - Z_2 \end{bmatrix} < 0, \tag{16}$$

$$\begin{bmatrix} \Omega + \Omega_0 & \sqrt{d_2} N_1 [I_n, I_n] & \sqrt{d_{21}} N_3 [I_n, I_n] \\ * & -Z_1 & 0 \\ * & * & -Z_2 \end{bmatrix} < 0, \tag{17}$$

$$\begin{bmatrix} -I & \sqrt{\rho} C_2 \\ * & -z_{\max}^2 P \end{bmatrix} < 0, \tag{18}$$

where

$$\begin{aligned}
\Omega &= \operatorname{diag}\left\{Q_1+Q_2+Q_3,(\mu-1)\,Q_1,-Q_2,-Q_3,-\gamma^2\right\} \\
&\quad +\operatorname{sym}\left\{W_x^T P P_f + N_1 W_{N_1} + N_2 W_{N_2} + N_3 W_{N_3}\right\}, \\
\Omega_0 &= [P_f^T,\delta_0 P_g^T]\,[d_2 Z_1 + d_{21} Z_2]\,[P_f^T,\delta_0 P_g^T]^T + Z_{11w}^T Z_{11w} + Z_{12w}^T Z_{12w}, \\
W_x &= \begin{bmatrix} I_n & 0_{n,3n+1} \end{bmatrix}, P_f = \begin{bmatrix} A+\bar{\delta}BK & \left(1-\bar{\delta}\right)BK & 0_{n,2n} & B_1 \end{bmatrix}, \\
P_g &= \begin{bmatrix} BK & -BK & 0_{n,2n+1} \end{bmatrix}, W_{N_1} = \begin{bmatrix} I_n & -I_n & 0_{n,2n+1} \end{bmatrix}, \\
W_{N_2} &= \begin{bmatrix} 0_n & I_n & 0_n & -I_n & 0_{n,1} \end{bmatrix}, W_{N_3} = \begin{bmatrix} 0_n & -I_n & I_n & 0_{n,n+1} \end{bmatrix}, \\
Z_{11w} &= \begin{bmatrix} C_1+\bar{\delta}D_1 K & \left(1-\bar{\delta}\right)D_1 K & 0_{n,2n+1} \end{bmatrix}, \\
Z_{12w} &= \begin{bmatrix} \delta_0 D_1 K & -\delta_0 D_1 K & 0_{n,2n+1} \end{bmatrix}, \quad d_{21}=d_2-d_1, \quad \delta_0=\sqrt{\bar{\delta}\left(1-\bar{\delta}\right)}
\end{aligned}$$

then a stabilizing controller in the form of (8) exists, such that

(1) the system is exponentially stable in mean square

(2) under zero initial condition, the closed-loop system guarantees that $\|z_1\|_{E_2} < \gamma\,\|w\|_2$ for all nonzero $w \in L_2[0,\infty)$;

(3) the control output constraint in (15) is guaranteed with the disturbance energy under the bound $w_{\max} = (\rho - V(0,0))/\gamma^2$.

Proof. Firstly, we will show that the system in (10) with $w(t) = 0$ is exponentially stable in mean square. Then, we will develop multi-objective H_∞ performance analysis condition with the output constrain in (15). Let us consider the Lyapunov-Krasovskii functional as follows:

$$\begin{aligned}
V(x_t,t) &= x^T(t)\,Px(t) + \int_{t-d(t)}^{t} x^T(s)\,Q_1 x(s)\,\mathrm{d}s \\
&\quad + \int_{t-d_1}^{t} x^T(s)\,Q_2 x(s)\,\mathrm{d}s + \int_{t-d_2}^{t} x^T(s)\,Q_3 x(s)\,\mathrm{d}s \\
&\quad + \int_{-d_2}^{0}\int_{t+\theta}^{t} \begin{bmatrix} f^T(s) & \delta_0 g^T(s) \end{bmatrix} Z_1 \begin{bmatrix} f^T(s) & \delta_0 g^T(s) \end{bmatrix}^T \mathrm{d}s\mathrm{d}\theta \\
&\quad + \int_{-d_2}^{-d_1}\int_{t+\theta}^{t} \begin{bmatrix} f^T(s) & \delta_0 g^T(s) \end{bmatrix} Z_2 \begin{bmatrix} f^T(s) & \delta_0 g^T(s) \end{bmatrix}^T \mathrm{d}s\mathrm{d}\theta, \qquad (19)
\end{aligned}$$

where $P>0$, $Q_1>0$, $Q_2>0$, $Q_3>0$, $Z_1>0$ and $Z_2>0$ are to be determined, $x_t = x(t+\sigma), \sigma \in [-2d_2, 0]$. $\mathcal{L}V(x_t,t)$ is used to denote the infinitesimal operator of $V(x_t,t)$, which is defined as

$$\mathcal{L}V(x_t,t) = \lim_{\triangle\to 0+} \triangle^{-1}\left[\mathcal{E}\left\{V(x_{t+\triangle},t+\triangle)\,|(x_t,t)\right\} - V(x_t,t)\right]. \qquad (20)$$

Then, it is can be seen from (19) and (20) that

$$\begin{aligned}
\mathcal{L}V\left(x_t,t\right) &= 2x^T\left(t\right)Pf\left(t\right)+x^T\left(t\right)\left(Q_1+Q_2+Q_3\right)x\left(t\right)\\
&\quad -\left(1-d\left(t\right)\right)x^T\left(t-d\left(t\right)\right)Q_1x\left(t-d\left(t\right)\right)\\
&\quad -x^T\left(t-d_1\right)Q_2x\left(t-d_1\right)-x^T\left(t-d_2\right)Q_3x\left(t-d_2\right)\\
&\quad +\left[f^T\left(t\right),\delta_0g^T\left(t\right)\right]\left(d_2Z_1+d_{21}Z_2\right)\left[f^T\left(t\right),\delta_0g^T\left(t\right)\right]^T\\
&\quad -\int_{t-d_2}^{t}\left[f^T\left(s\right),\delta_0g^T\left(s\right)\right]Z_1\left[f^T\left(s\right),\delta_0g^T\left(s\right)\right]^T\mathrm{d}s\\
&\quad -\int_{t-d_2}^{t-d_1}\left[f^T\left(s\right),\delta_0g^T\left(s\right)\right]Z_2\left[f^T\left(s\right),\delta_0g^T\left(s\right)\right]^T\mathrm{d}s\\
&\leq 2x^T\left(t\right)Pf\left(t\right)+x^T\left(t\right)\left(Q_1+Q_2+Q_3\right)x\left(t\right)\\
&\quad -\left(1-\mu\right)x^T\left(t-d\left(t\right)\right)Q_1x\left(t-d\left(t\right)\right)\\
&\quad -x^T\left(t-d_1\right)Q_2x\left(t-d_1\right)-x^T\left(t-d_2\right)Q_3x\left(t-d_2\right)\\
&\quad +\left[f^T\left(t\right),\delta_0g^T\left(t\right)\right]\left(d_2Z_1+d_{21}Z_2\right)\left[f^T\left(t\right),\delta_0g^T\left(t\right)\right]^T\\
&\quad -\int_{t-d(t)}^{t}\left[f^T\left(s\right),\delta_0g^T\left(s\right)\right]Z_1\left[f^T\left(s\right),\delta_0g^T\left(s\right)\right]^T\mathrm{d}s\\
&\quad -\int_{t-d_2}^{t-d(t)}\left[f^T\left(s\right),\delta_0g^T\left(s\right)\right]\left(Z_1+Z_2\right)\left[f^T\left(s\right),\delta_0g^T\left(s\right)\right]^T\mathrm{d}s\\
&\quad -\int_{t-d(t)}^{t-d_1}\left[f^T\left(s\right),\delta_0g^T\left(s\right)\right]Z_2\left[f^T\left(s\right),\delta_0g^T\left(s\right)\right]^T\mathrm{d}s. \qquad (21)
\end{aligned}$$

For any appropriately dimensioned matrices N_1, N_2 and N_3, it is easily to see that the following equalities hold:

$$2\xi^T\left(t\right)N_1\left[x\left(t\right)-x\left(t-d\left(t\right)\right)-\int_{t-d(t)}^{t}f(s)\mathrm{d}s-\int_{t-d(t)}^{t}\left(\delta\left(s\right)-\bar{\delta}\right)g(s)\mathrm{d}s\right]=0, \qquad (22)$$

$$2\xi^T\left(t\right)N_2\left[x\left(t-d\left(t\right)\right)-x\left(t-d_2\right)-\int_{t-d_2}^{t-d(t)}f(s)\mathrm{d}s-\int_{t-d_2}^{t-d(t)}\left(\delta\left(s\right)-\bar{\delta}\right)g(s)\mathrm{d}s\right]=0, \qquad (23)$$

$$2\xi^T\left(t\right)N_3\left[x\left(t-d_1\right)-x\left(t-d\left(t\right)\right)-\int_{t-d(t)}^{t-d_1}f(s)\mathrm{d}s-\int_{t-d(t)}^{t-d_1}\left(\delta\left(s\right)-\bar{\delta}\right)g(s)\mathrm{d}s\right]=0, \qquad (24)$$

where

$$\xi\left(t\right)=\left[\begin{array}{cccc} x^T\left(t\right) & x^T\left(t-d\left(t\right)\right) & x^T\left(t-d_1\right) & x^T\left(t-d_2\right)\end{array}\right]^T.$$

Furthermore, we have

$$\begin{aligned}
& 2\xi^T(t)\,N_1\left(-\int_{t-d(t)}^{t} f(s)\mathrm{d}s-\int_{t-d(t)}^{t}\left(\delta(s)-\bar{\delta}\right)g(s)\mathrm{d}s\right)\\
= & \int_{t-d(t)}^{t}\left\{-2\xi^T(t)\,N_1\,[I_n,I_n]\left[f^T(s),\left(\delta(s)-\bar{\delta}\right)g^T(s)\right]^T\right\}\mathrm{d}s\\
\leq & \int_{t-d(t)}^{t}\xi^T(t)\,N_1\,[I_n,I_n]\,Z_1^{-1}\,[I_n,I_n]^T\,N_1^T\xi(t)\,\mathrm{d}s\\
& +\int_{t-d(t)}^{t}\left[f^T(s),\left(\delta(s)-\bar{\delta}\right)g^T(s)\right]Z_1\left[f^T(s),\left(\delta(s)-\bar{\delta}\right)g^T(s)\right]^T\mathrm{d}s\\
= & \; d(t)\,\xi^T(t)\,N_1\,[I_n,I_n]\,Z_1^{-1}\,[I_n,I_n]^T\,N_1^T\xi(t)\\
& +\int_{t-d(t)}^{t}\left[f^T(s),\left(\delta(s)-\bar{\delta}\right)g^T(s)\right]Z_1\left[f^T(s),\left(\delta(s)-\bar{\delta}\right)g^T(s)\right]^T\mathrm{d}s, \qquad (25)
\end{aligned}$$

similarly,

$$\begin{aligned}
& 2\xi^T(t)\,N_2\left(-\int_{t-d_2}^{t-d(t)} f(s)\mathrm{d}s-\int_{t-d_2}^{t-d(t)}\left(\delta(s)-\bar{\delta}\right)g(s)\mathrm{d}\right)\\
\leq & \left(d_2-d(t)\right)N_2\xi^T(t)\,[I_n,I_n]\left(Z_1+Z_2\right)^{-1}[I_n,I_n]^T\,N_2^T\xi(t)\\
& +\int_{t-d_2}^{t-d(t)}\left[f^T(s),\left(\delta(s)-\bar{\delta}\right)g^T(s)\right]\left(Z_1+Z_2\right)\left[f^T(s),\left(\delta(s)-\bar{\delta}\right)g^T(s)\right]^T\mathrm{d}s, \qquad (26)\\
& 2\xi^T(t)\,N_3\left(-\int_{t-d(t)}^{t-d_1} f(s)\mathrm{d}s-\int_{t-d(t)}^{t-d_1}\left(\delta(s)-\bar{\delta}\right)g(s)\mathrm{d}s\right)\\
\leq & \left(d(t)-d_1\right)\xi^T(t)\,N_3\,[I_n,I_n]\,Z_2^{-1}\,[I_n,I_n]^T\,N_3^T\xi(t)\\
& +\int_{t-d(t)}^{t-d_1}\left[f^T(s),\left(\delta(s)-\bar{\delta}\right)g^T(s)\right]Z_2\left[f^T(s),\left(\delta(s)-\bar{\delta}\right)g^T(s)\right]^T\mathrm{d}s. \qquad (27)
\end{aligned}$$

To develop the stability condition, adding (22)–(24) to the right hand side of (21), using the inequalities (25)–(27), and taking the expectation on the both sides of (21), after some simple manipulation, we obtain,

$$\begin{aligned}
\mathcal{E}\mathcal{L}V(x_t,t) \;\leq\; & \mathcal{E}\Big\{\xi^T(t)\Big[\hat{\Omega}+\hat{\Omega}_0+d_2N_1\,[I_n,I_n]\,Z_1^{-1}\,[I_n,I_n]^T\,N_1^T\\
& +\left(d_2-d(t)\right)N_2\,[I_n,I_n]\left(Z_1+Z_2\right)^{-1}[I_n,I_n]^T\,N_2^T\\
& +\left(d(t)-d_1\right)N_3\,[I_n,I_n]\,Z_2^{-1}\,[I_n,I_n]^T\,N_3^T\Big]\,\xi(t)\Big\}\\
= \; & \mathcal{E}\Big\{\xi^T(t)\Big[\frac{\left(d_2-d(t)\right)}{d_{21}}\Big(\hat{\Omega}+\hat{\Omega}_0+d_2N_1\,[I_n,I_n]\,Z_1^{-1}\,[I_n,I_n]^T\,N_1^T\\
& +d_{21}N_2\,[I_n,I_n]\left(Z_1+Z_2\right)^{-1}[I_n,I_n]^T\,N_2^T\Big)\\
& +\frac{\left(d(t)-d_1\right)}{d_{21}}\Big(\hat{\Omega}+\hat{\Omega}_0+d_2N_1\,[I_n,I_n]\,Z_1^{-1}\,[I_n,I_n]^T\,N_1^T\\
& +d_{21}N_3\,[I_n,I_n]\,Z_2^{-1}\,[I_n,I_n]^T\,N_3^T\Big)\Big]\,\xi(t)\Big\},
\end{aligned}$$

where

$$\begin{aligned}\hat{\Omega} &= \operatorname{diag}\left\{Q_1+Q_2+Q_3,(\mu-1)\,Q_1,-Q_2,-Q_3\right\} \\ &\quad +\operatorname{sym}\left\{\hat{W}_x^T P\hat{P}_f + N_1\hat{W}_{N_1} + N_2\hat{W}_{N_2} + N_3\hat{W}_{N_3}\right\}, \\ \hat{\Omega}_0 &= [\hat{P}_f^T,\delta_0\hat{P}_g^T]\,[d_2Z_1+d_{21}Z_2]\,[\hat{P}_f^T,\delta_0\hat{P}_g^T]^T, \\ \hat{W}_x &= \begin{bmatrix} I_n & 0_{n,3n}\end{bmatrix}, \hat{P}_f = \begin{bmatrix} A+\bar{\delta}BK & \left(1-\bar{\delta}\right)BK & 0_{n,2n}\end{bmatrix}, \\ \hat{P}_g &= \begin{bmatrix} BK & -BK & 0_{n,2n}\end{bmatrix}, W_{\hat{N}_1} = \begin{bmatrix} I_n & -I_n & 0_{n,2n}\end{bmatrix}, \\ W_{\hat{N}_2} &= \begin{bmatrix} 0_n & I_n & 0_n & -I_n\end{bmatrix}, W_{\hat{N}_3} = \begin{bmatrix} 0_n & -I_n & I_n & 0_n\end{bmatrix}.\end{aligned}$$

Applying Schur complement to (16) and (17), we can conclude that

$$\begin{aligned}\Xi_1 &= \hat{\Omega}+\hat{\Omega}_0+d_2N_1\left[I_n,I_n\right]Z_1^{-1}\left[I_n,I_n\right]^T N_1^T + d_{21}N_2\left[I_n,I_n\right](Z_1+Z_2)^{-1}\left[I_n,I_n\right]^T N_2^T < 0, \\ \Xi_2 &= \hat{\Omega}+\hat{\Omega}_0+d_2N_1\left[I_n,I_n\right]Z_1^{-1}\left[I_n,I_n\right]^T N_1^T + d_{21}N_3\left[I_n,I_n\right]Z_2^{-1}\left[I_n,I_n\right]^T N_3^T < 0.\end{aligned}$$

Therefore, the following inequality holds,

$$\mathcal{E}\mathcal{L}V\left(x_t,t\right) \le -a\left(\left\|x\left(t\right)\right\|^2 + \left\|x\left(t-d\left(t\right)\right)\right\|^2\right), \tag{28}$$

where

$$a=\lambda_{\min}\left\{-\left(\frac{\left(d_2-d\left(t\right)\right)}{d_{21}}\Xi_1 + \frac{\left(d\left(t\right)-d_1\right)}{d_{21}}\Xi_2\right)\right\} > 0.$$

From the definitions of $V\left(x_t,t\right)$, $f\left(t\right)$ and $g\left(t\right)$, there exist positive scalars η_1, η_2 and η_3 such that the following inequality holds

$$V\left(x_t,t\right) \le \eta_1\left\|x\left(t\right)\right\|^2 + \eta_2\int_{t-d_2}^{t}\left\|x\left(s\right)\right\|^2 \mathrm{d}s + \eta_3\int_{t-d_2}^{t}\left\|x\left(s-d\left(s\right)\right)\right\|^2\mathrm{d}s. \tag{29}$$

Choose $\beta>0$, such that

$$\beta\max\left\{\eta_1+\eta_2 d_2 e^{\beta d_2}, \eta_3 d_2 e^{\beta d_2}\right\} \le a. \tag{30}$$

Therefore,

$$\begin{aligned}\mathcal{L}\left[e^{\beta t}V\left(x_t,t\right)\right] &\le e^{\beta t}\left[\left(\beta\eta_1-a\right)\left\|x\left(t\right)\right\|^2 - a\left\|x\left(t-d\left(t\right)\right)\right\|^2\right. \\ &\quad +\eta_2\int_{t-d_2}^{t}\left\|x\left(s\right)\right\|^2\mathrm{d}s + \beta\eta_3\int_{t-d_2}^{t}\left\|x\left(s-d\left(s\right)\right)\right\|^2\mathrm{d}s.\end{aligned} \tag{31}$$

By using Dynkin's formula, for $T>0$, we have

$$\begin{aligned}\mathcal{E}\left(e^{\beta T}V\left(x_T,T\right)\right) &\le J_1+\left(\beta\eta_1-a\right)\mathcal{E}\left\{\int_0^T e^{\beta t}\left\|x\left(t\right)\right\|^2\mathrm{d}t\right\}\\ &\quad -a\mathcal{E}\left\{\int_0^T e^{\beta t}\left\|x\left(t-d\left(t\right)\right)\right\|^2\mathrm{d}t\right\}\\ &\quad +\beta\eta_2\mathcal{E}\left\{\int_0^T\int_{t-d_2}^t e^{\beta t}\left\|x\left(s\right)\right\|^2\mathrm{d}s\mathrm{d}t\right\}\\ &\quad +\beta\eta_3\mathcal{E}\left\{\int_0^T\int_{t-d_2}^t e^{\beta t}\left\|x\left(s-d\left(s\right)\right)\right\|^2\mathrm{d}s\mathrm{d}t\right\}\end{aligned}\tag{32}$$

where

$$J_1=\left[\eta_1+d_2\eta_2+d_2\eta_3\right]\sup_{-2d_2\le s\le 0}\mathcal{E}\left\|\phi\left(s\right)\right\|^2.\tag{33}$$

Consequently, by changing the integration sequence, the following inequalities hold,

$$\begin{aligned}\int_0^T\int_{t-d_2}^t e^{\beta t}\left\|x\left(s\right)\right\|^2\mathrm{d}s\mathrm{d}t &\le \int_{-d_2}^T\left(\int_{s\vee 0}^{(s+d_2)\wedge T}e^{\beta t}\mathrm{d}t\right)\left\|x\left(s\right)\right\|^2\mathrm{d}s\\ &\le \int_{-d_2}^T d_2e^{\beta(s+d_2)}\left\|x\left(t\right)\right\|^2\mathrm{d}t\\ &\le d_2e^{\beta d_2}\int_0^T e^{\beta t}\left\|x\left(t\right)\right\|^2\mathrm{d}t+d_2e^{\beta d_2}\int_{-d_2}^0\left\|\phi\left(s\right)\right\|^2\mathrm{d}s\\ &\le d_2e^{\beta d_2}\int_0^T e^{\beta t}\left\|x\left(t\right)\right\|^2\mathrm{d}t+d_2^2e^{\beta d_2}\sup_{-d_2\le s\le 0}\left\|\phi\left(s\right)\right\|^2,\\ &\le d_2e^{\beta d_2}\int_0^T e^{\beta t}\left\|x\left(t\right)\right\|^2\mathrm{d}t+d_2^2e^{\beta d_2}\sup_{-2d_2\le s\le 0}\left\|\phi\left(s\right)\right\|^2,\end{aligned}\tag{34}$$

$$\int_0^T\int_{t-d_2}^t e^{\beta t}\left\|x\left(s-d\left(s\right)\right)\right\|^2\mathrm{d}s\mathrm{d}t\le d_2e^{\beta d_2}\int_0^T e^{\beta t}\left\|x\left(t-d\left(t\right)\right)\right\|^2\mathrm{d}t+d_2^2e^{\beta d_2}\sup_{-2d_2\le s\le 0}\left\|\phi\left(s\right)\right\|^2.\tag{35}$$

After substituting (34)–(35) into the right side of (32) and the using (30), we can obtain

$$\mathcal{E}\left(e^{\beta T}V\left(x_T,T\right)\right)\le J_1+J_2,$$

where

$$J_2=\left(\beta\eta_2d_2^2e^{\beta d_2}+\beta\eta_3d_2^2e^{\beta d_2}\right)\sup_{-2d_2\le s\le 0}\mathcal{E}\left\|\phi\left(s\right)\right\|^2.\tag{36}$$

So,

$$\mathcal{E}\left\{\left\|x\left(\phi,T\right)\right\|^2\right\}\le\frac{J_1+J_2}{\lambda_{\min}\left(P\right)}e^{-\beta T},$$

then it can be shown that for any $T>0$,

$$\mathcal{E}\left\{\left\|x\left(\phi,T\right)\right\|^2\right\}\le\alpha e^{-\beta T}\sup_{-2d_2\le s\le 0}\mathcal{E}\left\|\phi\left(s\right)\right\|^2,$$

where

$$\alpha = \frac{1}{\lambda_{\min}(P)}\left[\eta_1 + d_2\eta_2 + d_2\eta_3 + \beta\eta_2 d_2^2 e^{\beta d_2} + \beta\eta_3 d_2^2 e^{\beta d_2}\right].$$

Consequently, according to definition 1, the system in (11) is exponentially stable in mean square.

Furthermore, we will establish the H_∞ performance analysis condition for the system in (11) under zero initial conditions. First, we define the Lyapunov-Krasovskii functional as in (19). For

$$\bar{\xi}(t) = \left[\begin{array}{ccccc} x^T(t) & x^T(t-d(t)) & x^T(t-d_1) & x^T(t-d_2) & w^T(t) \end{array}\right]^T,$$

by following the similar lines as in the above proof, one has

$$\begin{aligned}
\mathcal{E}\mathcal{L}V(x_t,t) \leq\ & \mathcal{E}\Big\{\bar{\xi}^T(t)\Big[\check{\Omega} + \check{\Omega}_0 + d_2 N_1 [I_n, I_n] Z_1^{-1} [I_n, I_n]^T N_1^T \\
& + (d_2 - d(t)) N_2 [I_n, I_n] (Z_1 + Z_2)^{-1} [I_n, I_n]^T N_2^T \\
& + (d(t) - d_1) N_3 [I_n, I_n] Z_2^{-1} [I_n, I_n]^T N_3^T\Big]\bar{\xi}(t)\Big\} \\
=\ & \mathcal{E}\Big\{\bar{\xi}^T(t)\Big[\frac{(d_2 - d(t))}{d_{21}}\Big(\check{\Omega} + \check{\Omega}_0 + d_2 N_1 [I_n, I_n] Z_1^{-1} [I_n, I_n]^T N_1^T \\
& + d_{21} N_2 [I_n, I_n] (Z_1 + Z_2)^{-1} [I_n, I_n]^T N_2^T\Big) \\
& + \frac{(d(t) - d_1)}{d_{21}}\Big(\check{\Omega} + \check{\Omega}_0 + d_2 N_1 [I_n, I_n] Z_1^{-1} [I_n, I_n]^T N_1^T \\
& + d_{21} N_3 [I_n, I_n] Z_2^{-1} [I_n, I_n]^T N_3^T\Big)\Big]\bar{\xi}(t)\Big\},
\end{aligned}$$

where

$$\begin{aligned}
\check{\Omega} &= \text{diag}\{Q_1 + Q_2 + Q_3, (\mu - 1) Q_1, -Q_2, -Q_3, 0\} \\
&\quad + \text{sym}\left\{W_x^T P P_f + N_1 W_{N_1} + N_2 W_{N_2} + N_3 W_{N_3}\right\}, \\
\check{\Omega}_0 &= [P_f^T, \delta_0 P_g^T]\,[d_2 Z_1 + d_{21} Z_2]\,[P_f^T, \delta_0 P_g^T]^T.
\end{aligned}$$

Thus, we can show that

$$\begin{aligned}
& \mathcal{E}\left\{z_1^T(t) z_1(t) - \gamma^2 w^T(t) w(t) + \mathcal{L}V(x_t,t)\right\} \\
\leq\ & \mathcal{E}\Big\{\bar{\xi}^T(t)\Big[\frac{(d_2 - d(t))}{d_{21}}\Big(\Omega + \Omega_0 + d_2 N_1 [I_n, I_n] Z_1^{-1} [I_n, I_n]^T N_1^T \\
& + d_{21} N_2 [I_n, I_n] (Z_1 + Z_2)^{-1} [I_n, I_n]^T N_2^T\Big) \\
& + \frac{(d(t) - d_1)}{d_{21}}\Big(\Omega + \Omega_0 + d_2 N_1 [I_n, I_n] Z_1^{-1} [I_n, I_n]^T N_1^T \\
& + d_{21} N_3 [I_n, I_n] Z_2^{-1} [I_n, I_n]^T N_3^T\Big)\Big]\bar{\xi}(t)\Big\}.
\end{aligned}$$

By using Schur complement to (16)–(17), we have

$$\mathcal{E}\left\{z_1^T(t)\,z_1(t)-\gamma^2 w^T(t)\,w(t)+\mathcal{L}V(x_t,t)\right\}<0, \tag{37}$$

for all nonzero $w \in L_2[0,\infty)$. Because $V(\phi(t),0)=0$ under zero initial condition, that is, $\phi(t)=0$ for $t\in[-2d_2,0]$. Thus, by Itô's formula, after integrating both sides of (37), we obtain $\|z_1(t)\|_{E_2} < \gamma\|w(t)\|_2$ for all nonzero $w\in L_2[0,\infty)$, and the H_∞ performance is established.

In the following section, we consider the problems of the output constraints. From (37), it can be seen that

$$\mathcal{E}\mathcal{L}V(x_t,t)-\gamma^2 w^T(t)w(t)<0.$$

After integrating both sides of the above inequality from zero to any $t>0$, we obtain

$$\mathcal{E}V(x_t,t)-V(0,0)<\gamma^2\int_0^t w^T(\tau)w(\tau)\mathrm{d}\tau<\gamma^2\|w\|_2^2.$$

From the definition of the Lyapunov functional in (19), we obtain $x^T(t)Px(t)<\rho$, with $\rho=\gamma^2 w_{\max}+V(0,0)$. Consider

$$\begin{aligned}\max_{t>0}|z_2(t)|^2 &= \max_{t>0}\left\|x^T(t)C_2{}^T C_2 x(t)\right\|_2\\ &= \max_{t>0}\left\|x^T(t)P^{\frac{1}{2}}P^{-\frac{1}{2}}C_2^T C_2 P^{-\frac{1}{2}}P^{\frac{1}{2}}x(t)\right\|_2\\ &< \rho\cdot\theta_{\max}(P^{-\frac{1}{2}}C_2^T C_2 P^{-\frac{1}{2}}),\end{aligned}$$

where $\theta_{\max}(\cdot)$ represents maximal eigenvalue. From the above inequality, we know that the constraint (15) is guaranteed, if

$$\rho\cdot P^{-\frac{1}{2}}C_2{}^T C_2 P^{-\frac{1}{2}}<z_{\max}^2 I. \tag{38}$$

By Schur complement, (18) are equivalent to (38). We complete the proof. ■

IV. Multi-objective H_∞ Controller Design

Based on the H_∞ performance analysis condition proposed in Theorem 1, the H_∞ controller existence condition for the active suspension system in (10) is developed in the following theorem.

Theorem 2: Consider the closed-loop system in (11). For given constants d_1, d_2, μ and $\bar{\delta}$, if there exist matrices Y, $X>0$, $\bar{Q}_1>0$, $\bar{Q}_2>0$, $\bar{Q}_3>0$, $\bar{Z}_1=\mathrm{diag}\left\{\bar{Z}_{11},\bar{Z}_{11}\right\}>0$, $\bar{Z}_2=\mathrm{diag}\left\{\bar{Z}_{21},\bar{Z}_{21}\right\}>0$,

$\bar{N}_1$, $\bar{N}_2$, and $\bar{N}_3$ with appropriate dimensions such that the following LMIs hold:

$$\begin{bmatrix} \bar{\Omega} & \sqrt{d_2}\bar{N}_1 [I_n, I_n] & \sqrt{d_{21}}\bar{N}_2 [I_n, I_n] & \Upsilon_1 \\ * & \bar{Z}_1 - 2\hat{X} & 0 & 0 \\ * & * & \bar{Z}_1 + \bar{Z}_2 - 4\hat{X} & 0 \\ * & * & * & \Upsilon_2 \end{bmatrix} < 0, \tag{39}$$

$$\begin{bmatrix} \bar{\Omega} & \sqrt{d_2}\bar{N}_1 [I_n, I_n] & \sqrt{d_{21}}\bar{N}_3 [I_n, I_n] & \Upsilon_1 \\ * & \bar{Z}_1 - 2\hat{X} & 0 & 0 \\ * & * & \bar{Z}_2 - 2\hat{X} & 0 \\ * & * & * & \Upsilon_2 \end{bmatrix} < 0, \tag{40}$$

$$\begin{bmatrix} -I & \sqrt{\rho}C_2 X \\ * & -z_{\max}^2 X \end{bmatrix} < 0, \tag{41}$$

where

$$\begin{aligned} \bar{\Omega} &= \text{diag}\left\{\bar{Q}_1 + \bar{Q}_2 + \bar{Q}_3, (\mu - 1)\bar{Q}_1, -\bar{Q}_2, -\bar{Q}_3, -\gamma^2\right\} \\ &\quad + \text{sym}\left\{W_x^T \bar{P}_f + \bar{N}_1 W_{N_1} + \bar{N}_2 W_{N_2} + \bar{N}_3 W_{N_3}\right\}, \\ \Upsilon_1 &= \begin{bmatrix} \sqrt{d_2}\left[\bar{P}_f^T, \delta_0 \bar{P}_g^T\right] & \sqrt{d_{21}}\left[\bar{P}_f^T, \delta_0 \bar{P}_g^T\right] & \bar{Z}_{11w}^T & \bar{Z}_{12w}^T \end{bmatrix}, \\ \Upsilon_2 &= \text{diag}\left\{-\bar{Z}_1, -\bar{Z}_2, -1, -1\right\} \\ \bar{P}_f &= \begin{bmatrix} AX + \bar{\delta}BY & \left(1 - \bar{\delta}\right)BY & 0_{n,2n} & B_1 \end{bmatrix}, \\ \bar{P}_g &= \begin{bmatrix} BY & -BY & 0_{n,2n+1} \end{bmatrix}, W_{N_1} = \begin{bmatrix} I_n & -I_n & 0_{n,2n+1} \end{bmatrix}, \\ W_{N_2} &= \begin{bmatrix} 0_n & I_n & 0_n & -I_n & 0_{n,1} \end{bmatrix}, W_{N_3} = \begin{bmatrix} 0_n & -I_n & I_n & 0_{n,n+1} \end{bmatrix}, \\ \bar{Z}_{11w} &= \begin{bmatrix} C_1 X + \bar{\delta}D_1 Y & \left(1 - \bar{\delta}\right)D_1 Y & 0_{n,2n+1} \end{bmatrix}, \\ \bar{Z}_{12w} &= \begin{bmatrix} \delta_0 D_1 Y & -\delta_0 D_1 Y & 0_{n,2n+1} \end{bmatrix}, \hat{X} = \text{diag}\left\{X, X\right\}, \end{aligned}$$

then a stabilizing controller in the form of (8) exists, such that

(1) the closed-loop system is exponentially stable in mean square;

(2) under zero initial condition, the closed-loop system guarantees that $\|z_1\|_{E_2} < \gamma \|w\|_2$ for all nonzero $w \in L_2[0, \infty)$;

(3) the control output constraint in (15) is guaranteed with the disturbance energy under the bound $w_{\max} = (\rho - V(0,0))/\gamma^2$.

Moreover, if inequalities (39)–(41) have a feasible solution, then the control gain K in (8) is given by $K = YX^{-1}$.

Proof. Due to the relationship

$$\left(\bar{Z}_e - \hat{X}\right) \bar{Z}_e^{-1} \left(\bar{Z}_e - \hat{X}\right) \geq 0, \quad e = 1, 2, \tag{42}$$

we know

$$-\hat{X}\bar{Z}_e^{-1}\hat{X} \leq \bar{Z}_e - 2\hat{X}, \quad e = 1, 2.$$

From (39)–(40), we have

$$\begin{bmatrix} \bar{\Omega} & \sqrt{d_2}\bar{N}_1 \left[I_n, I_n\right] & \sqrt{d_{21}}\bar{N}_2 \left[I_n, I_n\right] & \Upsilon_1 \\ * & -\hat{X}\bar{Z}_1^{-1}\hat{X}_1 & 0 & 0 \\ * & * & -\hat{X}\left(\bar{Z}_1^{-1} + \bar{Z}_2^{-1}\right)\hat{X}_1 & 0 \\ * & * & * & \Upsilon_2 \end{bmatrix} < 0, \tag{43}$$

$$\begin{bmatrix} \bar{\Omega} & \sqrt{d_2}\bar{N}_1 \left[I_n, I_n\right] & \sqrt{d_{21}}\bar{N}_3 \left[I_n, I_n\right] & \Upsilon_1 \\ * & -\hat{X}\bar{Z}_1^{-1}\hat{X}_1 & 0 & 0 \\ * & * & -\hat{X}\bar{Z}_2^{-1}\hat{X}_1 & 0 \\ * & * & * & \Upsilon_2 \end{bmatrix} < 0. \tag{44}$$

Now, introduce the following matrices

$$\Theta = \text{diag}\left\{\Theta_1, \Theta_2, \Theta_3, \Theta_4, 1, 1\right\},$$

where

$$\begin{aligned} \Theta_1 &= \text{diag}\left\{X^{-1}, X^{-1}, X^{-1}, X^{-1}, 1\right\}, \Theta_2 = \text{diag}\left\{\hat{X}^{-1}, \hat{X}^{-1}\right\}, \\ \Theta_3 &= \text{diag}\left\{I_n, I_n\right\}, \Theta_4 = \text{diag}\left\{I_n, I_n\right\}, \Theta_5 = \text{diag}\left\{X^{-1}, X^{-1}, X^{-1}\right\}. \end{aligned}$$

After setting

$$\begin{aligned} P &= X^{-1}, \quad Z_e = \bar{Z}_e^{-1} > 0, \quad (e = 1, 2, h = 1, 2, 3) \\ Q_h &= X^{-1}\bar{Q}_h X^{-1}, \quad \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \Theta_1 \begin{bmatrix} \bar{N}_1 & \bar{N}_2 & \bar{N}_3 \end{bmatrix} \Theta_5. \end{aligned}$$

Pre- and post multiplying (43) and (44) by Θ^T and Θ respectively, we know that conditions in (16) and (17) hold by using Schur complement.

On the other hand, (41) is equivalent to (18) by performing a simple congruence transformation with diag $\left\{I, X^{-1}\right\}$. Therefore, all the conditions in Theorem 1 are satisfied. The proof is completed. ■

Remark 1: The random input delay idea used in this paper is motivated by the random sensor delay and random input delay proposed by [24]–[27]. In detail, the authors in [24] presented a controller design strategy for discrete-time networked systems with random communication delays. In [25], [26], the stability and stabilization problems for networked control systems and Takagi-Sugeno fuzzy systems with stochastic input delays were investigated respectively. The authors in [27] designed observer-based output feedback controller for continuous-time networked systems with random sensor delay, while the stability and stabilization conditions for the systems with random input delay have been developed. There exist few results on multi-objective H_∞ control for the systems with random input delay. Based on the observation, the multi-objective H_∞ control problem has been investigated for a quarter-car active suspension system with random input delay in this paper. Simulation results will be given in the following section to illustrate the effectiveness of the proposed control strategy.

Remark 2: It is can be seen from Theorem 2 that the condition is presented in the framework of LMIs both over the matrix variables and the objective scalar γ, which implies that γ can be included as an optimization variable to obtain a lower bound of the guaranteed H_∞ performance. Based on the condition in Theorem 2, the robust H_∞ controller can be obtained with the minimal γ by solving the following convex optimization problem:

$$\begin{aligned} \min \gamma \quad \text{s.t.} \quad (39) - (41) \\ X > 0, \bar{Q}_1 > 0, \bar{Q}_2 > 0, \bar{Q}_3 > 0, \bar{Z}_1 > 0, \bar{Z}_2 > 0, \bar{K}, \bar{N}_1, \bar{N}_2, \bar{N}_2. \end{aligned} \tag{45}$$

V. A Design Example

In this section, a design example is provided to demonstrate the effectiveness of the proposed method. A quarter-car model parameters borrowed from [3] is listed in Table I for the following controller design.

TABLE I

Quarter-car model parameters

m_s	m_u	k_s	k_t	c_s	c_t
973kg	114kg	42720N/m	101115N/m	1095Ns/m	14.6Ns/m

The objective of this paper is to design the desired controller to ensure that the sprung mass acceleration $z_1(t)$ is as small as possible and the suspension deflection is below the maximum allowable suspension stroke $z_{\max} = 0.08$ m in (15). It is assumed that the delay $d(t) = 7.5 + 2.5\sin(t/25)$ ms satisfies $d_1 = 5$ ms, $d_2 = 10$ ms and $\mu = 0.1$. Here, we choose $\rho = 1$ as discussed in [12]. Firstly, for $\bar{\delta} = 0$, by solving the convex optimization problem formulated in (45), it is found that the minimum guaranteed closed-loop H_∞ performance obtained is $\gamma_{\min} = 5.9996$, and admissible control gain matrix is given

$$K = 10^4 \times \begin{bmatrix} -2.8881 & 0.2487 & -1.8410 & 0.1826 \end{bmatrix}. \tag{46}$$

Moreover, to present more detailed results on H_∞ controller design for different $\bar{\delta}$, Table II lists the admissible controller gain matrices and the corresponding minimum guaranteed closed-loop H_∞ performance indexes.

TABLE II

H_∞ PERFORMANCE INDEX γ_{min} AND K FOR DIFFERENT $\bar{\delta}$

$\bar{\delta}$	γ_{min}	K
0.1	5.8982	$K = 10^4 \times \begin{bmatrix} -4.6681 & 1.1413 & -2.3007 & 0.1979 \end{bmatrix}$
0.2	5.8226	$K = 10^4 \times \begin{bmatrix} -4.3687 & 2.4833 & -2.3089 & 0.2760 \end{bmatrix}$
0.3	5.7563	$K = 10^4 \times \begin{bmatrix} -5.8424 & 1.7648 & -2.6021 & 0.2063 \end{bmatrix}$
0.4	5.7067	$K = 10^4 \times \begin{bmatrix} -7.1058 & 2.5891 & -2.9293 & 0.2187 \end{bmatrix}$
0.5	5.6527	$K = 10^4 \times \begin{bmatrix} -9.3258 & 3.5727 & -3.4869 & 0.2214 \end{bmatrix}$
0.6	5.6149	$K = 10^5 \times \begin{bmatrix} -1.1894 & 0.4943 & -0.4138 & 0.0229 \end{bmatrix}$
0.7	5.5940	$K = 10^5 \times \begin{bmatrix} -1.3334 & 0.7146 & -0.4561 & 0.0283 \end{bmatrix}$
0.8	5.5728	$K = 10^5 \times \begin{bmatrix} -1.3770 & 1.0997 & -0.4844 & 0.0426 \end{bmatrix}$
0.9	5.5562	$K = 10^5 \times \begin{bmatrix} -0.8263 & 1.9170 & -0.3995 & 0.0875 \end{bmatrix}$
1	5.5517	$K = 10^5 \times \begin{bmatrix} -0.0543 & 1.5399 & -0.2004 & 0.0844 \end{bmatrix}$

To further show the effectiveness of the proposed method, we consider the constant delay, that is $d_2 = d_1$ and $\mu = 0$ for $\bar{\delta} = 0$, $\bar{\delta} = 0.5$ and $\bar{\delta} = 1$. By solving the convex optimization problem formulated in (45), the minimum guaranteed closed-loop H_∞ performance index and admissible controller gain matrices are listed in Tables III-V for $\bar{\delta} = 0$, $\bar{\delta} = 0.5$ and $\bar{\delta} = 1$ respectively. From Table II, we can see that the closed-loop system is exponentially stable in mean square and synchronously satisfies the guaranteed H_∞ performance $\gamma_{\min}$ which is decreasing with $\bar{\delta}$ increasing. In addition, it can be seen from Tables III-V that the minimum guaranteed H_∞ performance $\gamma_{\min}$ decreases when $\bar{\delta}$ enlarges for the same constant delay, while under the same $\bar{\delta}$, the closed-loop performance $\gamma_{\min}$ is decreased with the constant delay increasing.

TABLE III

H_∞ PERFORMANCE INDEX γ_{min} AND K FOR DIFFERENT $\bar{\delta} = 0$

$d_2 = d_1$	γ_{min}	K
5 ms	5.4376	$K = 10^5 \times \begin{bmatrix} -3.1526 & 0.3409 & -0.8660 & -0.0056 \end{bmatrix}$
10 ms	5.7417	$K = 10^4 \times \begin{bmatrix} -5.5911 & 0.2482 & -2.4751 & 0.1443 \end{bmatrix}$

TABLE IV

H_∞ PERFORMANCE INDEX γ_{min} AND K FOR DIFFERENT $\bar{\delta} = 0.5$

$d_2 = d_1$	γ_{min}	K
5 ms	5.4156	$K = 10^5 \times \begin{bmatrix} -7.0378 & 0.3057 & -1.7584 & -0.0652 \end{bmatrix}$
10 ms	5.5339	$K = 10^5 \times \begin{bmatrix} -1.6413 & 0.3250 & -0.5115 & 0.0104 \end{bmatrix}$

In order to evaluate the suspension characteristics with respect to ride comfort and working space of the suspension, the variability of the road profiles is taken into account. In the context of active suspension performance, road disturbances can be generally assumed as shocks. Shocks are discrete events of relatively short duration and high intensity, caused by, for example, a pronounced bump or pothole on an otherwise smooth road surface. In this work, this case of road profile is considered first to reveal the transient response characteristic, which is given by

$$z_r(t) = \begin{cases} \frac{A}{2}(1 - \cos(\frac{2\pi V}{L} t)), & \text{if } 0 \le t \le \frac{L}{V}, \\ 0, & \text{if } t > \frac{L}{V}, \end{cases} \tag{47}$$

where A and L are the height and the length of the bump. Assume $A = 60$ mm, $L = 5$ m and the vehicle forward velocity as $V = 35$ (km/h).

TABLE V

H_∞ PERFORMANCE INDEX γ_{min} AND K FOR DIFFERENT $\bar{\delta} = 1$

$d_2 = d_1$	γ_{min}	K
5 ms	5.4038	$K = 10^6 \times \begin{bmatrix} -4.1738 & -0.2627 & -0.9622 & -0.0685 \end{bmatrix}$
10 ms	5.4641	$K = 10^6 \times \begin{bmatrix} -1.0251 & 0.2554 & -0.2610 & -0.0045 \end{bmatrix}$

In this paper, the bump responses of the open-loop ($u(t) = 0$, passive mode) and closed-loop (active mode) system with the controller K listed in Table II for $\bar{\delta} = 0$, $\bar{\delta} = 0.5$ and $\bar{\delta} = 1$ are showed Fig. 2. These figures plot the body vertical accelerations, suspension strokes, tyre deflections and active force for the open-loop and closed-loop system, respectively. They demonstrate that the closed-loop system is exponentially stable in mean square and has a better performance than the open-loop system, while the closed-loop system can guarantee the required performances such as the sprung mass acceleration $z_1(t)$ is as small as possible, the suspension deflection is below the maximum allowable suspension stroke $z_{\max} = 0.08$ m. In addition, we can see from Fig. 2 that the active control forces of the closed-loop systems, which are confined within a reasonable range which can be generated by hydraulic or electrorheological actuators in practice.

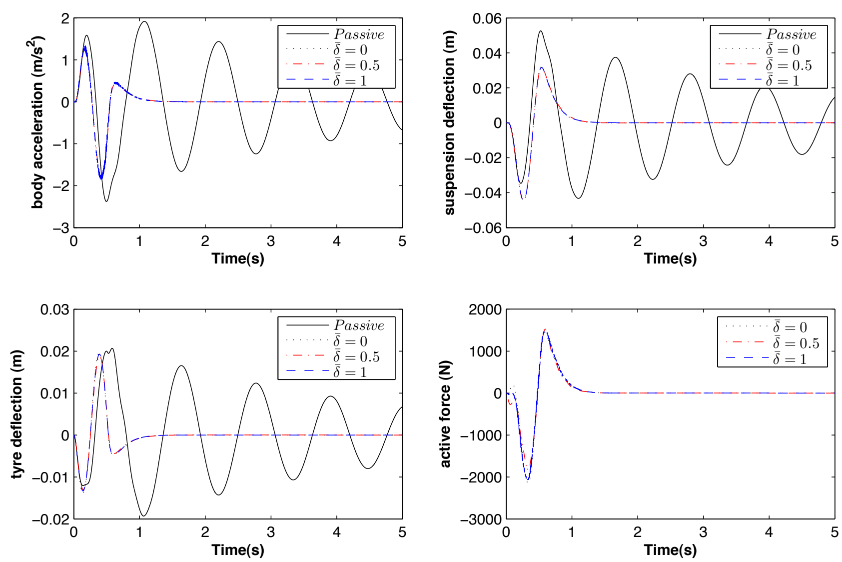

Fig. 2. Vertical accelerations, suspension strokes, tyre deflection and active force for system

Finally, by using the control gain matrices obtained in Table III–V, the responses of body vertical accelerations, suspension strokes, tyre deflection and active force for the closed-loop systems under different constant delays are illustrated in Fig. 3–5, from which it can be seen that the closed-loop system performance can be guaranteed. From the above figures, we know that the effectiveness of the proposed controller design method is confirmed.

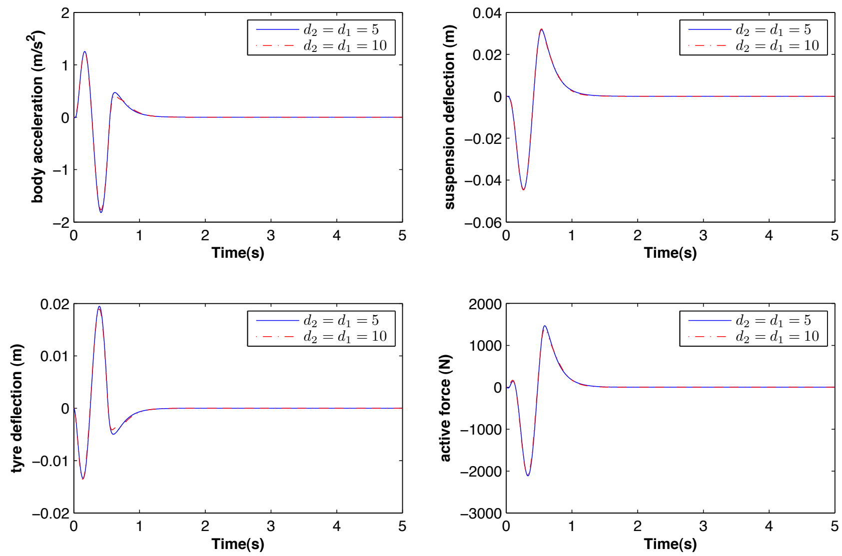

Fig. 3. Vertical accelerations, suspension strokes, tyre deflection and active force for system ($\bar{\delta} = 0$)

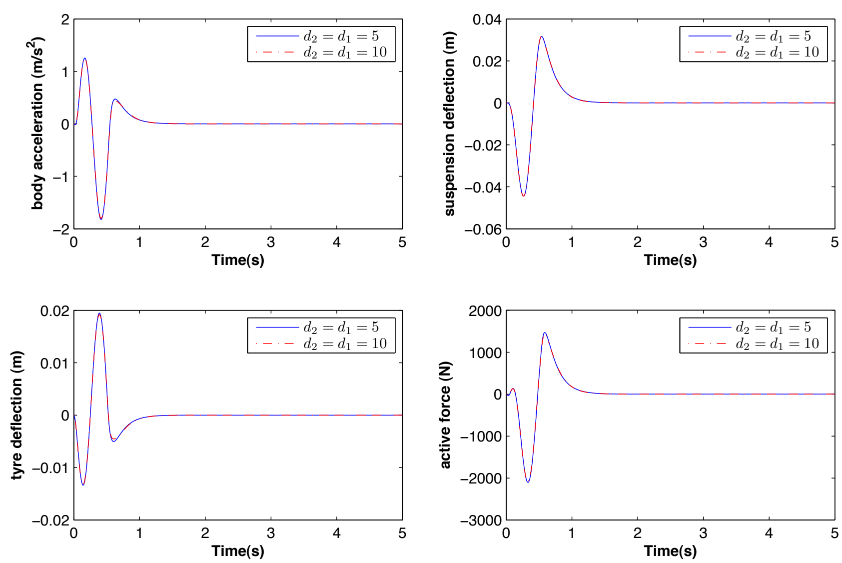

Fig. 4. Vertical accelerations, suspension strokes, tyre deflection and active force for system ($\bar{\delta} = 0.5$)

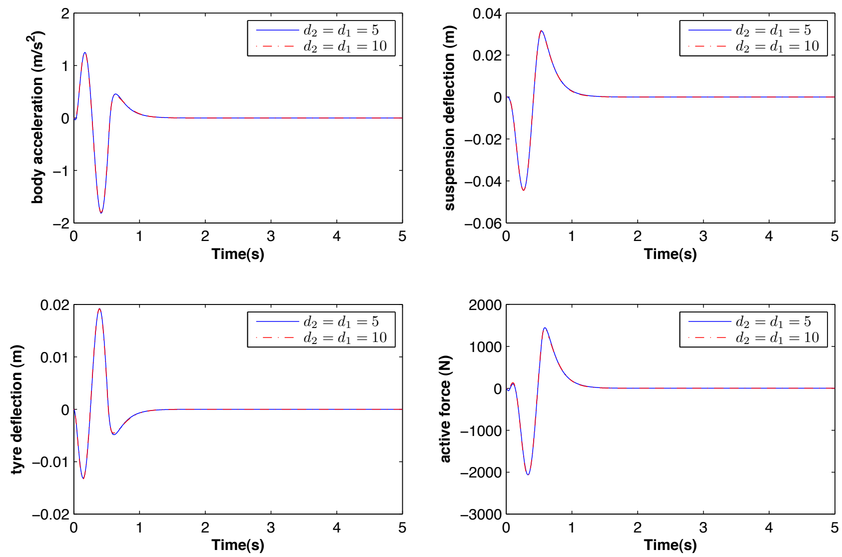

Fig. 5. Vertical accelerations, suspension strokes, tyre deflection and active force for system ($\bar{\bar{\delta}} = 1$)

VI. Concluding Remarks

This paper has investigated the problem of multi-objective H_∞ for active vehicle suspension systems with random actuator delay. The random delay is assumed to be probability distribution and time-varying. The dynamic equations of a quarter-car suspension model have been set up for the control design aim. Based on the stochastic stability theory, we have developed the multi-objective H_∞ performance analysis and controller synthesis conditions, which have been cast into a convex optimization problem with LMI constraints via some algebraic manipulations. Then the desired controller has been achieved by solving the corresponding LMIs. Finally, a design example has been given to demonstrate the effectiveness of the proposed controller design approach.

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