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Existence of the FS-type renormalisation fixed point for unidirectionally-coupled pairs of maps

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Abstract

We give the first proof of the existence of a renormalisation fixed-point for period-doubling in pairs of maps of two variables lying in the so-called Feigenbaum-Summation (FS) universality class. The first map represents a subsystem that is unimodal with an extremum of degree two. The dynamics of the second map accumulates an integral characteristic of the dynamics of the first, via a particular form of unidirectional coupling. We prove the existence of the corresponding renormalisation fixed point by rigorous computer-assisted means and gain tight rigorous bounds on the associated universal constants. Our work provides the first step in establishing rigorously the picture conjectured by Kuznetsov *et al* of the birth, from the FS-type fixed point, of the so-called C-type two-cycle via a period doubling in the dynamics of the renormalisation group transformation itself.

Keywords: dynamical systems, renormalisation group, universality, period-doubling, bifurcations, computer-assisted proofs

(Some figures may appear in colour only in the online journal)

1. Introduction

The renormalisation group formalism has provided explanations for certain critical scaling phenomena observed in transitions to chaos in dynamical systems [1]. As external parameters

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are varied, causing bifurcations in the dynamics of the system, regularities can be observed in the dynamics at multiple scales—in time, in the parameter space, and in the state space—that are largely independent of the particular system under consideration, and are instead shared with other systems falling within a broad universality class.

The renormalisation group approach expresses these regularities in terms of a transformation that acts on systems to capture the relationship between the different scales present. Stationary orbits, in particular fixed-points, of the transformation characterise different types of scaling behaviour.

The number of parameters that need to be controlled in order to observe the universal behaviour in families of dynamical systems corresponds to the codimension of the stable manifold of the corresponding stationary orbit of the transformation, with behaviour that corresponds to low-codimension scenarios therefore being the most ubiquitous.

The significance of such low-codimension scenarios lies in the opportunity that they provide to understand, at a fundamental level, prototypical examples of qualitatively different critical scaling phenomena, and then to identify their presence across wide varieties of dynamical systems that include those derived from more complicated high-dimensional flows. Recently, a large number of such universality classes has been discovered.

The simplest example occurs in the accumulation of period-doublings for families of unimodal maps of the interval, $g_\mu : \mathbb{R} \to \mathbb{R}$, in which scaling regularities are observed corresponding to doubling the time-scale (that is, a doubling in the period of periodic orbits from period 2^k to 2^{k+1}), accompanied by a scaling in the dynamical (x) space by some factor α and in the parameter (μ) space by some factor δ.

Feigenbaum [2–4] provided an explanation for the observed universality in terms of a 'doubling operator', R_1, acting on a space of maps of a single variable. The operator encodes the relationship between the different scales present by rescaling the state variable by a suitable factor α about a critical point (thus $g(x) \mapsto \alpha g(x/\alpha)$, given a choice of coordinates for x that puts the critical point at the origin) together with composing the map with itself (thus $g(x) \mapsto \alpha g(g(x/\alpha))$), thereby halving the period of doubled orbits.

An explanation for the universal scaling observed is provided by the existence of a nontrivial fixed point g^* of the renormalisation operator. The nature of this fixed point, specifically the spectrum of $DR_1(g^*)$, the derivative of the renormalisation operator there, explains the universality. Essentially (up to coordinate changes) the fixed point has a one-dimensional unstable manifold with associated eigenvalue δ and codimension-one stable manifold.

Universality classes corresponding to higher codimension have been discovered in coupled maps. Kuznetsov *et al* [5–7] studied a natural generalisation of the doubling operator to pairs of maps of two variables, discovering nontrivial stationary orbits of the associated renormalisation transformation, R, each corresponding to a different type of universal behaviour that may be observed in families of dynamical systems. Two examples are the so-called 'FS-type' renormalisation fixed point and the 'C-type' period-two orbit.

The existence of the former (Feigenbaum-Summation (FS)-type) fixed point would explain universal behaviour observed in coupled systems in which the first subsystem undergoes the usual period-doubling route the chaos described above, and the dynamics of the second accumulates an integral characteristic of the dynamics of the first. The existence of the latter (C-type) period-two stationary orbit of the renormalisation operator is also of interest in a broad class of systems in which the variation of one experimental parameter gives rise to period doublings while the variation of another gives rise to a saddle-node bifurcation. The corresponding C-type critical behaviour may be observed in the synchronisation of period-doubling dissipative systems [5]. Another application is in characterising universal behaviour in alternating period-doubling cascades [8].

A concrete example of systems lying within the latter universality class is provided by certain biologically-plausible models of renal blood pressure autoregulation [9, 10].

The presence of multiple different universal scaling scenarios corresponding to a common renormalisation group transformation presents a challenge in understanding the organising principles connecting them. Kuznetsov *et al* [7] discovered a relationship between the scenarios described above and provided an explanation in terms of a bifurcation in the dynamics of the renormalisation group transformation R itself. Specifically, they provided strong evidence that a family of FS-type renormalisation fixed points undergoes a period-doubling bifurcation into a family of C-type period-two orbits as the degree d of the maximum of the unimodal maps is varied.

The goal of this paper is to take the first step in establishing this picture rigorously, by proving the existence of the FS-type fixed point. Historically, analytical proofs of the existence of stationary orbits of renormalisation operators have been very hard to come by. Many such questions were instead settled first via rigorous computer-assisted methods, whose subsequent success owes much to the pioneering work of Lanford [11] and Eckmann *et al* [12] (see, for example, [13–15]).

In this note, we provide a computer-assisted proof of the existence of the FS-type renormalisation fixed point relevant to families of coupled unimodal maps with degree $d = 2$ at the turning point.

1.1. Overview

In section 2, we introduce the doubling operator, R, acting on pairs of maps of two variables. In section 3, we will later establish the existence of a particular fixed point of the doubling operator essentially by proving that a variant, denoted Φ, of Newton's method for the fixed-point problem is a contraction mapping on a carefully-chosen ball around an approximate fixed point in a suitable space of pairs of maps. In order to prove contractivity of the relevant operator, we would require bounds on the tangent map $D\Phi(g,f)$ for pairs (g,f) lying in that ball. This means bounding the Fréchet derivative, $DR(g,f)$. In fact, we will first take an ansatz that restricts the fixed point to have a particular functional form (with unidirectional coupling) that will in-turn enable us to avoid working in the space of pairs of maps. However it is first necessary to compute the Fréchet derivative in the full space before applying the ansatz, which we therefore do first, in section 2.1. We then introduce the ansatz in section 2.2 and give the corresponding (simplified) Fréchet derivative in section 2.2.1.

In section 3, we show that our desired fixed point pair (g,f) may be constructed from a solution $g(x)$ of the Cvitanović–Feigenbaum functional equation, together with a function of the form $(x,y) \mapsto y + f(x)$ in which f is an even analytic function. Rather than building this restriction into our function space, we instead (section 3.1) choose a modified operator denoted T_2 whose action on maps F corresponds to that of the renormalisation operator acting on suitable (even) functions. Our goal is then reduced to proving the existence of a fixed point of T_2. In section 3.2 we define the space of functions in which we will work in order to do this, and in section 3.3 we prove that the operator is well-defined and differentiable on a carefully-chosen ball in that space, giving the corresponding Fréchet derivative $DT_2(F)$ in section 3.4. In section 3.5 we show that DT_2 has expanding directions, and therefore that the operator T_2 is not itself contractive, at the desired fixed point. This motivates us to formulate the quasi-Newton operator Φ (section 3.6) and in section 3.7 we demonstrate how we establish that it is a uniform contraction on the ball, by carefully bounding the expressions derived previously for the derivative $DT_2(F)$. This is done via rigorous computer-assisted calculations and we provide the necessary bounds in section 3.8 and demonstrate how these may be tightened in order to assess, and improve considerably, the accuracy of previous numerical estimates for the relevant universal constants.

We conclude, in section 4, with suggestions for future work and provide, in the appendix, sufficient details of the location of the fixed point in order that the proof may be reproduced (the code is also provided in a Zenodo repository [16]).

2. The doubling operator, *R*

Consider the doubling operator on pairs of maps of two variables,

$$R:\begin{pmatrix} g(x,y) \\ f(x,y) \end{pmatrix} \mapsto \begin{pmatrix} \alpha g\big(g(x/\alpha,y/\beta),\, f(x/\alpha,y/\beta)\big) \\ \beta f\big(g(x/\alpha,y/\beta),\, f(x/\alpha,y/\beta)\big) \end{pmatrix}, \tag{1}$$

where $g,f:\mathbb{R}^2\to\mathbb{R}$, in which we define

$$\alpha := g\big(g(0,0),\, f(0,0)\big)^{-1}, \tag{2}$$

$$\beta := f\big(g(0,0),\, f(0,0)\big)^{-1}, \tag{3}$$

which preserves the normalisation $g(0,0)=1$ and $f(0,0)=1$.

For brevity, we adopt the following shorthand notation for the maps g,f applied to the rescaled variables x/α and y/β,

$$\tilde{g} := g(x/\alpha,\ y/\beta), \tag{4}$$

$$\tilde{f} := f(x/\alpha,\ y/\beta). \tag{5}$$

2.1. Fréchet derivative, $DR(g,f)$

In section 3, we will establish the existence of a particular fixed point of the doubling operator essentially by proving that a variant, denoted Φ, of Newton's method for the fixed-point problem is a contraction mapping. In order to prove contractivity of the relevant operator, we will require bounds on the tangent map $D\Phi(g,f)$. This means bounding the Fréchet derivative, $DR(g,f)$.

The operator R has Fréchet derivative $DR(g,f)$ given formally by

$$\begin{aligned}&[DR(g,f)]_1 : (\delta g(x,y), \delta f(x,y))\\ &\mapsto \delta\alpha\cdot g(\tilde{g},\tilde{f}) + \alpha\left(\delta g(\tilde{g},\tilde{f}) + \partial_1 g(\tilde{g},\tilde{f})\cdot\delta\tilde{g} + \partial_2 g(\tilde{g},\tilde{f})\cdot\delta\tilde{f}\right),\end{aligned} \tag{6}$$

and

$$\begin{aligned}&[DR(g,f)]_2 : (\delta g(x,y), \delta f(x,y))\\ &\mapsto \delta\beta\cdot f(\tilde{g},\tilde{f}) + \beta\left(\delta f(\tilde{g},\tilde{f}) + \partial_1 f(\tilde{g},\tilde{f})\cdot\delta\tilde{g} + \partial_2 f(\tilde{g},\tilde{f})\cdot\delta\tilde{f}\right),\end{aligned} \tag{7}$$

in which

$$\delta\tilde{g} = \delta g(x/\alpha,y/\beta) + \partial_1 g(x/\alpha,\ y/\beta)\cdot(-1/\alpha^2)\delta\alpha\cdot x + \partial_2 g(x/\alpha,\ y/\beta)\cdot(-1/\beta^2)\delta\beta\cdot y, \tag{8}$$

$$\delta\tilde{f} = \delta f(x/\alpha,y/\beta) + \partial_1 f(x/\alpha,\ y/\beta)\cdot(-1/\alpha^2)\delta\alpha\cdot x + \partial_2 f(x/\alpha,\ y/\beta)\cdot(-1/\beta^2)\delta\beta\cdot y, \tag{9}$$

and

$$\delta\alpha = -\alpha^2 \cdot \big(\delta g(g(0,0), f(0,0)) + \partial_1 g(g(0,0), f(0,0)) \cdot \delta g(0,0) + \partial_2 g(g(0,0), f(0,0)) \cdot \delta f(0,0)\big), \quad (10)$$

$$\delta\beta = -\beta^2 \cdot \big(\delta f(g(0,0), f(0,0)) + \partial_1 f(g(0,0), f(0,0)) \cdot \delta g(0,0) + \partial_2 f(g(0,0), f(0,0)) \cdot \delta f(0,0)\big). \quad (11)$$

We note that most previous works consider α, β to be constant (thus, in those works, the variations $\delta\alpha, \delta\beta$ vanish) but that this affects the spectral characteristics only up to inessential (coordinate-change) eigenvalues.

2.2. *Ansatz for a particular unidirectional coupling*

Following [7, 8], we are interested in a particular form of unidirectional coupling. With a slight abuse of notation, we take the ansatz

$$g(x,y) := g(x), \quad (12)$$

$$f(x,y) := y + f(x), \quad (13)$$

which restricts the pair to be of the so-called 'Feigenbaum-Summation' (FS) type. The name comes from the observation that when the mapping $(x_j, y_j) \mapsto (x_{j+1}, y_{j+1}) = (g(x_j), y_j + f(x_j))$ is iterated, the first variable x undergoes independent dynamics under the iteration of g, while the second dynamical variable, y, accumulates an integral characteristic of the first, specifically it sums successive values of the 'feature' $f(x)$ along the orbit $(x_j)_{j\geqslant 0}$. If the map g belongs to a family of unimodal maps that displays the usual 'Feigenbaum' period-doubling universality, then the second component of the dynamics may be tuned to observe a particular class of universal behaviour for coupled two-dimensional maps (for a thorough exposition, see [5]).

Restricting the functional forms of g and f in this way, the doubling operator looks as follows

$$R : \begin{pmatrix} g(x) \\ y + f(x) \end{pmatrix} \mapsto \begin{pmatrix} \alpha g\big(g(x/\alpha)\big) \\ y + \beta\Big(f(x/\alpha) + f\big(g(x/\alpha)\big)\Big) \end{pmatrix}, \quad (14)$$

with

$$\alpha := 1/g(g(0)), \quad (15)$$

$$\beta := 1/(f(0) + f(g(0))), \quad (16)$$

which preserves the normalisation $g(0) = 1$ and $f(0) = 1$. In particular, note that the doubling operator (14) preserves the chosen functional form (12) and (13).

It is convenient to define the following auxiliary operators for the first component of R,

$$R_1 : g(x) \mapsto \alpha g(g(x/\alpha)), \quad (17)$$

and for the second component of R for a given fixed function g, for which we define $R_2 = R_2(g)$ by

$$R_2 : f(x) \mapsto \beta\,(f(x/\alpha) + f(g(x/\alpha))). \quad (18)$$

2.2.1. Fréchet derivative. The Fréchet derivative given in (6)–(11), when taken at $(g(x,y),f(x,y)) = (g(x),\, y+f(x))$ with perturbations $(\delta g(x,y),\, \delta f(x,y))$ that are not restricted to conform to the ansatz, becomes

$$\begin{aligned}&[DR(g,f)]_1 : (\delta g(x,y), \delta f(x,y))\\ &\mapsto \delta\alpha \cdot g(g(x/\alpha)) + \alpha\big[\delta g\big(g(x/\alpha), y/\beta + f(x/\alpha)\big)\\ &\quad + g'(g(x/\alpha)) \cdot \big(\delta g(x/\alpha, y/\beta) + g'(x/\alpha)\cdot(-1/\alpha^2)\delta\alpha \cdot x\big)\big],\end{aligned} \tag{19}$$

and

$$\begin{aligned}&[DR(g,f)]_2 : (\delta g(x,y), \delta f(x,y))\\ &\mapsto \delta\beta \cdot (y/\beta + f(x/\alpha) + f(g(x/\alpha))) + \beta\big[\delta f(g(x/\alpha), y/\beta + f(x/\alpha))\\ &\quad + f'(g(x/\alpha)) \cdot \big(\delta g(x/\alpha, y/\beta) + g'(x/\alpha)\cdot(-1/\alpha^2)\delta\alpha \cdot x\big)\\ &\quad + \big(\delta f(x/\alpha, y/\beta) + f'(x/\alpha)\cdot(-1/\alpha^2)\delta\alpha \cdot x + (-1/\beta^2)\delta\beta \cdot y\big)\big]\end{aligned} \tag{20}$$

In the above,

$$\delta\alpha = -\alpha^2 \cdot (\delta g(g(0), f(0)) + g'(g(0)) \cdot \delta g(0,0)), \tag{21}$$

$$\delta\beta = -\beta^2 \cdot (\delta f(g(0), f(0)) + f'(g(0)) \cdot \delta g(0,0) + \delta f(0,0)). \tag{22}$$

3. Existence of the FS-type fixed point

We prove the existence of a fixed point $(g^*(x,y), f^*(x,y))$ to the doubling operator, R, given in (1), that satisfies the above ansatz (12) and (13), i.e.

$$(g^*(x,y),\, f^*(x,y)) = (g^*(x),\; y + f^*(x)), \tag{23}$$

in which $g^*(x)$ is the (nontrivial) solution to the Cvitanović-Feigenbaum fixed-point equation

$$g(x) = \alpha g(g(x/\alpha)), \tag{24}$$

i.e. $g^* = R_1(g^*)$, and $f^*(x)$ is a solution to the functional equation

$$y + f(x) = y + \beta\left(f(x/\alpha) + f\big(g(x/\alpha)\big)\right). \tag{25}$$

One such solution was already found in [8], namely that with $f^*(x) = g^*(x) - x$ and $\beta = \alpha \simeq -2.502\,907\,87$. The existence of a second (even) solution was conjectured, and the solution approximated numerically, for which $\beta \simeq -4.586\,196\,71$. We prove the existence of this latter solution and gain tight rigorous bounds on the universal constant β.

3.1. Evenness

Fixing $g(x)$ to be the nontrivial fixed point g^* of the operator R_1, we therefore seek an even fixed point $f^*(x)$ of the corresponding operator R_2, given by

$$R_2 : f(x) \mapsto \beta\,(f(x/\alpha) + f(g(x/\alpha))). \tag{26}$$

Specifically, we write $g(x) = G(x^2)$, $f(x) = F(x^2)$, and $X = x^2$, and solve the functional equation

$$T_2(F(X)) = F(X), \tag{27}$$

where $T_2 = T_2(G)$ is given by

$$T_2 : F(X) \mapsto \beta \left(F(X/\alpha^2) + F\big(G(X/\alpha^2)^2\big)\right), \tag{28}$$

where

$$\alpha := G(G(0)^2)^{-1}, \quad \beta := \big(F(0) + F(G(0)^2)\big)^{-1}. \tag{29}$$

We note that if $G^*(X)$ is a fixed point of the operator

$$T_1 : G(X) \mapsto \alpha G\left(G(X/\alpha^2)^2\right), \tag{30}$$

that corresponds to the action of the doubling operator

$$R_1 : g(x) \mapsto \alpha g(g(x/\alpha)), \tag{31}$$

on even maps of a single variable, then $g^*(x) = G^*(x^2)$ is a fixed point of R_1. Further, if $F^*(X)$ is a fixed point of the corresponding operator $T_2 = T_2(G^*)$, then the pair $(g^*(x,y), f^*(x,y)) = (G^*(x^2), y + F^*(x^2))$ is a fixed point of R.

3.2. *Space of functions*

We now choose a convenient space to work in, on which the doubling operator has an analyticity-improving property and where bounds can readily be found on the relevant linear operators, in particular on the derivative of the doubling operator. We work with $G, F \in \mathscr{A}(\Omega)$, the Banach algebra of functions analytic on an open disc $\Omega = D(c,r) \subset \mathbb{C}$ and continuous on its closure, with finite ℓ_1-norm defined by

$$\|F\|_1 := \sum_{k=0}^{\infty} |F_k|, \tag{32}$$

where the F_k are the coefficients of the power series representation (convergent for $X \in \Omega$),

$$F(X) = \sum_{k=0}^{\infty} F_k \left(\frac{X-c}{r}\right)^k. \tag{33}$$

Correspondingly, we take the sequence of monomials $(e_k)_{k \geqslant 0}$ with $e_k : X \mapsto ((X-c)/r)^k$ as a Schauder basis.

We take the decomposition $\mathscr{A}(\Omega) \cong \mathbb{R}^{N+1} \oplus \ell_1$ in which we write $F \in \mathscr{A}(\Omega)$ as $F = F_P + F_H$, where $F_P = \mathcal{P}F$ contains the terms in the series (33) for $0 \leqslant k \leqslant N$, and $F_H = \mathcal{H}F$ contains the terms of order $k > N$. Thus $\mathcal{P}$ and $\mathcal{H}$ denote the canonical projection operators onto the polynomial and high-order parts of the space with respect to our chosen basis, respectively, and N denotes the truncation degree chosen for the polynomial part. In order to gain rigorous bounds, all computations use interval arithmetic with directed rounding modes together with

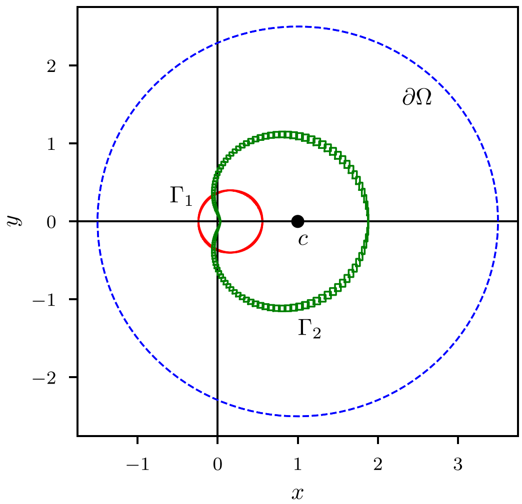

Figure 1. Illustration of the computer verification of the domain extension conditions for the operators T_1 and T_2 using a covering of the boundary $\partial\Omega$ of the disc domain $\Omega = D(1, 2.5)$ by 256 rectangles. Shown are the resulting coverings, Γ_1 and Γ_2, of $\overline{\Omega}/\alpha^2$ and $G(\overline{\Omega}/\alpha^2)^2$ respectively, valid for all $G \in B(G^0, \rho_G)$. (Here, $N = 40$ and $\rho_G = 10^{-23}$).

an implementation of the framework for rigorous computation in Banach spaces based on that of [12].

3.3. Domain extension

We first need to prove that the operators T_1 and T_2 are well-defined and differentiable on suitable balls in our space.

Given a ball of functions $B(G^0, \rho_G)$ of radius ρ_G around an approximate fixed point G^0 of T_1, we note that the domain extension ('analyticity improving') conditions [17] for this problem are that for all $G \in B(G^0, \rho_G)$ we have

$$\overline{\Omega}/\alpha^2 \subset \Omega, \tag{34}$$

$$G\left(\overline{\Omega}/\alpha^2\right)^2 \subset \Omega, \tag{35}$$

where the overline denotes the topological closure. Note that the universal quantifier is not vacuous for equation (34) because $\alpha = 1/G(G(0)^2)$. Note that these conditions are identical to those for the one-variable doubling operator T_1 (that corresponds to the action of R_1 on even functions) given in (30). We take $\Omega = D(1, 2.5)$ and establish first that these conditions hold for a certain closed ball of functions $B(G^0, \rho_G) \subset \mathscr{A}(\Omega)$ of radius ρ_G around an approximate fixed point $G^0 \in \mathcal{P}\mathscr{A}(\Omega)$ of T_1. See figure 1. Note, in particular, that the above conditions do not depend on F, either explicitly or via β, so that suitable choices for G^0 and ρ_G guarantee that T_2 is well-defined for all $F \in \mathscr{A}(\Omega)$. This establishes that both the one-variable doubling operator, T_1, and our operator T_2 are well-defined and differentiable, with compact derivatives, for all $G \in B(G^0, \rho_G)$ (such considerations are discussed in more detail in [14, 17, 18]). Then we prove that the ball contains a fixed-point, $G = G^*$, of the doubling operator T_1 by establishing that a quasi-Newton method for the fixed point problem is a contraction map on the ball [19] (we suppress the details of this previously-proven result, but provide details of the corresponding proof for F in what follows). This enables us to bound the action of the corresponding operator $T_2 = T_2(G)$ on $\mathscr{A}(\Omega)$. (We note that it is possible, although much less convenient, to work with $g(x)$ and R_1 directly, rather than $G(X)$ and T_1. In that case, there is no single disc domain that suffices, but one can find a suitable union of two discs and work with a pair of power series representing g).

3.4. Fréchet derivative, $DT_2(F)$

We note that the Fréchet derivative of the operator T_2 (for general functions $G, F : \mathbb{R} \to \mathbb{R}$) is given formally by

$$\begin{aligned}
&DT_2(F) : (\delta G(X),\ \delta F(X)) \\
&\mapsto \delta\beta \cdot \Big(F(X/\alpha^2) + F\big(G(X/\alpha^2)^2\big)\Big) \\
&\quad + \beta\Big[\delta F(X/\alpha^2) + F'(X/\alpha^2)\cdot(-2/\alpha^3)\delta\alpha \cdot X \\
&\qquad + \delta F\big(G(X/\alpha^2)^2\big) + F'\big(G(X/\alpha^2)^2\big)\cdot 2G(X/\alpha^2) \\
&\qquad \cdot \Big(\delta G(X/\alpha^2) + G'(X/\alpha^2)\cdot(-2/\alpha^3)\delta\alpha \cdot X\Big)\Big].
\end{aligned} \tag{36}$$

For the normalisation $\alpha := 1/G(G(0)^2)$ and $\beta := 1/(F(0) + F(G(0)^2))$, we obtain

$$\delta\alpha = -\alpha^2 \cdot \Big(\delta G(G(0)^2) + G'(G(0)^2)\cdot 2G(0)\cdot \delta G(0)\Big), \tag{37}$$

$$\delta\beta = -\beta^2 \cdot \Big(\delta F(0) + \delta F(G(0)^2) + F'(G(0)^2)\cdot 2G(0)\cdot \delta G(0)\Big). \tag{38}$$

In what follows, we fix G in the definition of T_2 at the nontrivial fixed point of T_1 that corresponds to the usual Cvitanović–Feigenbaum fixed point of R_1. This implies that $G(0) = 1$, $\alpha = 1/G(1)$, $\beta = 1/(F(0) + F(1))$ and also $\delta G \equiv 0$ so that $\delta\alpha = 0$ and

$$\delta\beta = -\beta^2 \cdot \big(\delta F(0) + \delta F(1)\big), \tag{39}$$

and at a general function F, we thus have

$$DT_2(F) : \delta F(X) \mapsto \delta\beta \cdot \Big(F(X/\alpha^2) + F\big(G(X/\alpha^2)^2\big)\Big) + \beta\Big(\delta F(X/\alpha^2) + \delta F\big(G(X/\alpha^2)^2\big)\Big). \tag{40}$$

3.5. Spectrum of $DT_2(F)$

We note [7] that the spectrum of $DR(g(x,y), f(x,y))$ as given by (6)–(11) at the FS-type fixed point has four essential (i.e. non-coordinate change) eigenvalues lying outside the unit disc: $\lambda_1 = 2\beta \simeq -9.172$, $\lambda_2 = \delta \simeq 4.669$, $\lambda_3 = 2$, and $\lambda_4 = \delta/\beta \simeq -1.018$, in which λ_1 corresponds to a perturbation of the f-component only with eigenfunction $\delta f(x,y) = 1 - f^*(x)$, λ_2

is the familiar 'Feigenbaum delta' corresponding to a perturbation of both g and f, λ_3 corresponds to changing the strength of the y-dependence in the f-component at the fixed point, and λ_4 corresponds to altering the nature of the coupling by introducing bidirectionality.

Eigenvalues corresponding to perturbations of G^* or g^* and those that violate the ansatz (12) and (13) thereby changing the nature of the unidirectional coupling (including λ_2, λ_3, and λ_4), together with any that destroy the evenness of f, are no longer present in the spectrum of $DT_2(F^*(X))$ when restricted as above (39) and (40).

However, a straightforward derivation, using the fact that $T_2(F^*) = F^*$, $F^*(0) = 1$, and $F^*(1) = 1/\beta - 1$, together with the relevant formal expressions (39) and (40), confirms that the function $\delta F(X) = 1 - F^*(X)$ is an eigenfunction of $DT_2(F^*(X))$, with eigenvalue $\lambda_1 = 2\beta$, corresponding to the eigenfunction $\delta f(x) = 1 - f^*(x)$ of $DR_2(f^*(x))$ as given by (19)–(22), and to the eigenfunction pair

$$\big(\delta g(x,y),\ \delta f(x,y)\big) = \big(0,\ 1 - f^*(x)\big), \tag{41}$$

of $DR(g^*(x,y), f^*(x,y))$ as given by (6)–(11).

3.6. Quasi-Newton operator for the fixed-point problem

From the above, the linearisation $DT_2(F)$ of the operator T_2 has a strongly expanding direction with corresponding eigenvalue $2\beta \simeq -9.172$ at the desired fixed point and T_2 itself is therefore not contractive there. Instead, we consider a quasi-Newton operator for the fixed-point problem, that shares the same fixed points as T_2, given by

$$\Phi : F \mapsto F - \Lambda(T_2(F) - F), \tag{42}$$

in which Λ is chosen to be a fixed (that is, not varying with F) invertible linear operator approximating $(DT_2(F) - I)^{-1}$ at the desired fixed point. Specifically, we choose $\Lambda = (\Delta - I)^{-1}$ in which Δ is a fixed linear block-diagonal operator chosen with polynomial part that approximates $DT_2(F)$ at the fixed point and with action zero on the high-order part of the space. Choosing Δ (and hence Λ) to be fixed avoids the need to take second Fréchet derivatives, and inverses of expressions involving Fréchet derivatives, in what follows.

We note that the Fréchet derivative of this quasi-Newton operator is given by

$$D\Phi(F) : \delta F \mapsto \delta F - \Lambda(DT_2(F)\delta F - \delta F), \tag{43}$$

for any $\delta F \in \mathscr{A}(\Omega)$. Note that our choice for Δ, above, means that Λ has action $-I$ on the high-order part of the space and, thus, that for high-order perturbations $\delta F \in \mathcal{H}\mathscr{A}(\Omega)$, we may reduce the number of explicit occurrences of δF in the expression (43), and thereby avoid dependency problems [18] in the rigorous computations for such high-order perturbations δF, via

$$D\Phi(F) : \delta F \mapsto -\Lambda(DT_2(F)\delta F). \tag{44}$$

3.7. Contraction mapping

We take a closed ball $B(F^0, \rho_F) \subset \mathscr{A}(\Omega)$ of radius ρ_F around a good approximate fixed point $F^0 \in \mathcal{P}\mathscr{A}(\Omega)$ of T_2, obtained by truncating to degree N an approximate fixed point found by iterating the Newton operator numerically (using polynomial operations up to degree at least $2N$).

We then establish via [16] the following rigorous bounds, valid for all $G \in B(G^0, \rho_G)$ and therefore, in particular, valid for $G = G^*$, the Cvitanović–Feigenbaum fixed point function,

$$\|\Phi(F^0) - F^0\| \leqslant \varepsilon_F, \tag{45}$$

$$\|\Phi(J) - \Phi(K)\| \leqslant \kappa_F \|J - K\| \quad \text{for all } J, K \in B(F^0, \rho_F), \tag{46}$$

where $\kappa_F < 1$, and we confirm the inequality

$$\varepsilon_F < \rho_F(1 - \kappa_F). \tag{47}$$

Equations (45)–(47) together establish that Φ is a (uniform) contraction map on $B(F^0, \rho_F)$. The contraction mapping theorem yields existence and local uniqueness of the fixed point $F^* \in B(F^0, \rho_F)$ of Φ and hence of T_2 itself and, in turn, the existence of the fixed point f^* of R_2, and thus of the fixed-point pair $(g^*(x), y + f^*(x))$ of the doubling operator R on pairs of maps of two variables.

The bound in (46) is established via the mean value theorem, by noting that for our choice of norm, $\|\cdot\| = \|\cdot\|_1$,

$$\|D\Phi(F)\|_{\text{op}} := \sup_{\|\delta F\|=1} \|D\Phi(F)\delta F\| \leqslant \sup_{k \geqslant 0} \|D\Phi(F)e_k\|, \tag{48}$$

and we may therefore work by bounding $\|D\Phi(F)e_k\| \leqslant \kappa_F$ for $0 \leqslant k \leqslant N$ using (43) and $\|D\Phi(F)E\| \leqslant \kappa_F$ using (44) where E is the 'high-order unit ball', i.e. $E := \mathcal{H}B(0,1)$, for which $E \ni e_k$ for all $k > N$. In both cases, we gain bounds on the subexpression $DT_2(F)\delta F$ in (43) and (44) by bounding the expressions (39) and (40) for all $G \in B(G^0, \rho_G)$ and for all $F \in B(F^0, \rho_F)$. Fully documented code, output files, and unit tests are provided in [16], building on the rigorous computational framework implemented in [20] for [18].

3.8. *Rigorous bounds*

The lowest truncation degree at which we are able to obtain a proof is $N = 20$, by working to a precision of $P = 20$ significant figures (empirically, we find that the choice $P = N$ provides sufficient precision in what follows) and using a ball of functions of radius $\rho_G = 10^{-12}$ proven previously [19] to contain the fixed point, G^*, of the Cvitanović–Feigenbaum operator (which, in this exposition, we therefore regard as an input). The bounds established for F are $\varepsilon_F = 1.6 \times 10^{-8}$, $\rho_F = 10^{-7}$, and $\kappa_F = 4.2 \times 10^{-2}$. The resulting bounds on the universal constants β and $b := 1/\beta$ are

$$b \in [-0.218\,046,\ -0.218\,045], \tag{49}$$

$$\beta \in [-4.586\,20,\ -4.586\,19]. \tag{50}$$

Appendix provides details of the power series coefficients for F^0 sufficient to reproduce the proof of existence of F^* with the above parameters.

Table 1. Parameters and resulting bounds valid for the proof of existence of the fixed-point function F: the columns give the truncation degree N, the equivalent number of digits of precision in the significand P, the radius ρ_G of the ball $B(G^0, \rho_G)$ proven to contain the nontrivial fixed point G of the doubling operator T_1 and proven to satisfy the domain extension conditions, the bound ε_F on the error in the approximate fixed point F^0 of the operator T_2, the radius ρ_F of the ball containing F, and a uniform bound κ_F on the contractivity of Φ valid for all $F \in B(F^0, \rho_F)$. (The bounds were obtained by using interval arithmetic and rigorous directed rounding with decimal endpoints for the intervals.) We note that, as these are ℓ_1-bounds, the radius ρ_F provides an upper bound on the error in each power series coefficient of F^0: we have $\|\mathcal{P}(F^* - F^0)\| \leqslant \rho_F$ with $|F^*_k - {F^0}_k| \leqslant \rho_F$ for all $0 \leqslant k \leqslant N$, and $\|\mathcal{H}F^*\| \leqslant \rho_F$.

N	P	ρ_G	ε_F	ρ_F	κ_F
20	20	10^{-11}	1.6×10^{-8}	10^{-7}	4.2×10^{-2}
40	40	10^{-23}	2.6×10^{-20}	10^{-19}	5.5×10^{-5}
80	80	10^{-48}	4.8×10^{-45}	10^{-43}	4.7×10^{-11}
160	160	10^{-98}	9.1×10^{-95}	10^{-93}	1.7×10^{-23}

However, we are able to improve the above bounds significantly by increasing the truncation degree and precision achieving, for $N = 80$ (working to $P = 80$ significant figures) and using a ball of radius $\rho_G = 10^{-48}$ proven to contain the fixed point G^*, the bounds $\varepsilon_F = 4.8 \times 10^{-45}$, $\rho_F = 10^{-43}$, and $\kappa_F = 4.7 \times 10^{-11}$. This yields

$$\begin{aligned} b \in [&-0.2180455971\,2282201368\,6061522743\,1296138035\,09, \\ &-0.2180455971\,2282201368\,6061522743\,1296138035\,08], \end{aligned} \tag{51}$$

$$\begin{aligned} \beta \in [&-4.5861967092\,9064476823\,4114397284\,0585953000\,4, \\ &-4.5861967092\,9064476823\,4114397284\,0585953000\,3]. \end{aligned} \tag{52}$$

For $N = 160$ (with $P = 160$), we obtain (with $\rho_G = 10^{-98}$ and $\rho_F = 10^{-93}$),

$$\begin{aligned} b \in -0.&2180455971\,2282201368\,6061522743\,1296138035 \\ &0866420577\,7015280180\,0113429830\,8042562308 \\ &1209372932\,7 - [1,2) \times 10^{-92}, \\ \beta \in -4.&5861967092\,9064476823\,4114397284\,0585953000 \\ &3383644172\,0205026501\,8619608373\,5020375014 \\ &5765904864 - [3,4) \times 10^{-91}. \end{aligned} \tag{53}$$

Table 1 provides details of parameters valid for the existence proof using various different truncation degrees for the polynomial part of the space. Figure 2 illustrates rigorous coverings of the graphs of the functions $F^*(X)$ and $f^*(x)$ and of their analytic extensions to larger domains.

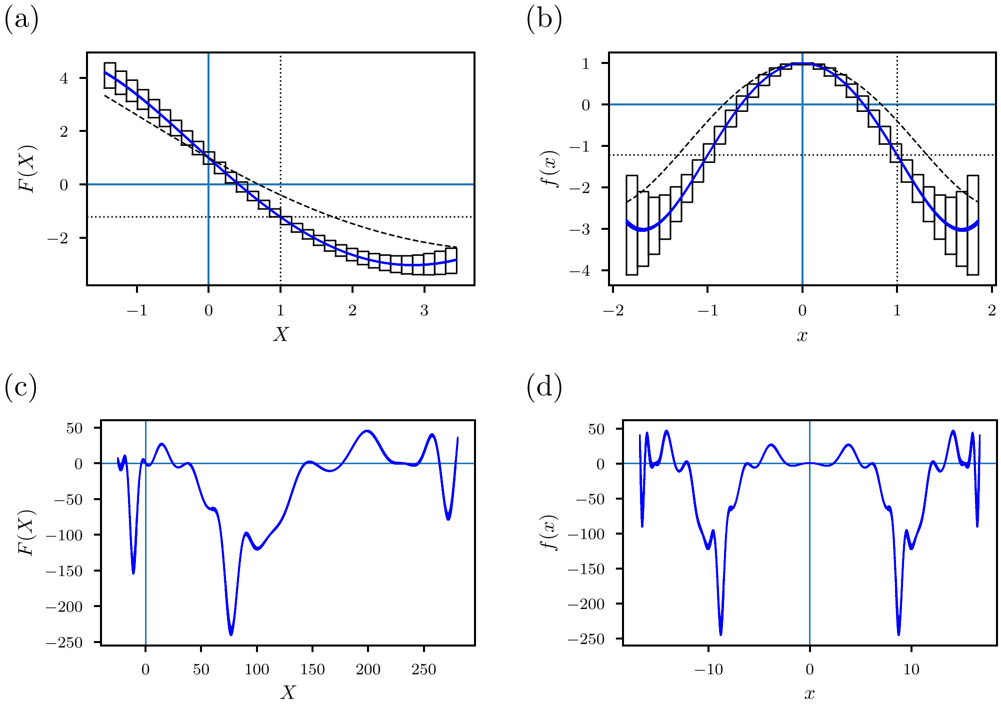

Figure 2. Illustration of rigorous coverings of the graphs of the functions (a) $F^*(X)$ and (b) $f^*(x)$, computed by bounding the value $F^*(X)$ on a covering of a subinterval of $\Omega \cap \mathbb{R} \ni X$ by 32 (shown as rectangles) and 1024 (thick line) subintervals, valid for all functions in the ball $B(F^0, \rho_F) \ni F^*$ proven to contain the fixed point with $N = 40$. The Feigenbaum fixed-point functions $G^*(X)$ and $g^*(x)$ are shown for comparison (dashed lines). The dotted lines indicate the location of the value $b - 1 = F(1) = f(1) \simeq -1.218\,045\,59$ where $b = 1/\beta$. Rigorous coverings of analytic extensions of (c) $F^*(X)$ and (d) $f^*(x)$, were computed using 2048 subintervals, evaluated by making use of the corresponding fixed-point equations and domain-extension conditions (34) and (35) recursively together with the balls of functions $B(G^0, \rho_G) \ni G^*$ and $B(F^0, \rho_F) \ni F^*$ as base cases.

4. Conclusions and future work

We have provided the first proof of the existence of the FS-type fixed point of the doubling operator acting on pairs of maps of two variables, via rigorous computer-assisted means. This provides the first step in verifying rigorously the picture described by Kuznetsov *et al* [7] that we outline further below.

Kuznetsov *et al* [7] observed that the spectrum of the derivative of the doubling operator at the FS-type renormalisation fixed point has an eigenvalue ($\lambda_4 = \delta/\beta < -1$) close to -1. This indicates that the dynamics of the renormalisation operator itself might undergo a bifurcation 'nearby' (if we allow some feature that is fixed in our present formulation to instead vary continuously). Specifically, we may augment the space by embedding it in a parameterised family of spaces in which the eigenvalue passes through -1 as the parameter is varied.

In this case, the degree d (of the maximum of the unimodal maps), set at the constant value $d = 2$ throughout this paper, was identified as just such a bifurcation parameter and the bifurcation was shown numerically [7] to be a period doubling that takes place at a critical value of the degree, $d = d_c < 2$, resulting in the birth of the previously-identified C-type period-two orbit [5] of the doubling operator (that exists for $d > d_c$ and, in particular, for $d = 2$).

A logical next step would be to prove the remaining conjectures rigorously, starting with the existence of the corresponding period-2 pair of functions for the case $d = 2$ that bifurcates from the family of FS-type fixed points that passes through the fixed point obtained in this paper.

Further work to establish more of the picture conjectured above would involve adapting the method used in section 3.1 in which the desired evenness of the solution (in x) was built into the operator itself by means of the ansatz $g(x,y) = G(x^2)$ and $f(x,y) = y + F(x^2)$ allowing us to work with the corresponding operator acting on the pairs (G,F). Specifically, rather than allowing the space to vary continuously with the degree d and then dealing with the original operator, R, we would instead express the problem in such a way that d appears explicitly in the form of the operator: we take an ansatz of the form

$$(g(x,y), f(x,y)) = \left(G(|x|^d),\ y + F(|x|^d)\right), \tag{54}$$

and hence define a one-parameter family of operators, parameterised by d, acting on the pairs (G,F) in our original space, that corresponds to the action of R on suitable pairs (g,f). Care would need to be taken with the domains of definition of the maps in the case of non-integer degree and also with correct handling of the modulus operation in the rigorous computational framework.

Data availability statement

The data that support the findings of this study are openly available at the following URL/DOI: 10.5281/zenodo.7139006.

Conflict of interest

The authors have no conflicts of interest to disclose.

Appendix. Power series coefficients

Table 2. Power series coefficients a_k of F^*, expanded with respect to the disc $\Omega = D(1, 2.5)$ used in the proof of existence.

k	a_k
00	$-1.2180455971\ 2282201368\ 6061522743\ 1296138035\ldots$
01	$-4.7257024072\ 6838035070\ 3398928822\ 1948306340\ldots$
02	$+2.5874313239\ 4454290688\ 8629433985\ 2645902721\ldots$
03	$+1.0733003037\ 2773592849\ 7726452227\ 6374324376\ldots$
04	$-6.3874265688\ 8450275509\ 4312889134\ 2842611996\ldots \times 10^{-1}$
05	$+1.2589711175\ 3382129118\ 5049676789\ 1824422777\ldots \times 10^{-1}$
06	$+2.0251654260\ 3341749270\ 6713826134\ 0960154321\ldots \times 10^{-2}$
07	$-1.4939816701\ 3869536487\ 3857910542\ 2241333161\ldots \times 10^{-2}$
08	$+1.0378745728\ 3087211816\ 3672122416\ 6953353553\ldots \times 10^{-3}$
09	$+9.2119005056\ 0476931057\ 9956358958\ 4218546687\ldots \times 10^{-4}$
10	$-2.5922913466\ 7925848337\ 8481373404\ 0123644849\ldots \times 10^{-4}$
11	$-5.3420626585\ 2185402841\ 8150417455\ 3536262606\ldots \times 10^{-6}$
12	$+1.9248611768\ 4118338144\ 6195950377\ 0360043132\ldots \times 10^{-5}$
13	$-4.5888240247\ 0344317981\ 8695261711\ 6770160840\ldots \times 10^{-6}$
14	$-8.8806273328\ 9415659231\ 3438234589\ 0645883498\ldots \times 10^{-8}$
15	$+3.1208105751\ 8578744421\ 0926729828\ 4360828898\ldots \times 10^{-7}$
16	$-7.0073245508\ 5381965482\ 9406885576\ 1651841086\ldots \times 10^{-8}$
17	$-2.1130001625\ 1031171597\ 5430139025\ 6591576595\ldots \times 10^{-9}$
18	$+4.5449057106\ 3281320788\ 3605016163\ 3407926163\ldots \times 10^{-9}$
19	$-8.6787157225\ 0484832645\ 4104153355\ 7771245898\ldots \times 10^{-10}$
20	$-7.6439297170\ 3906519458\ 6993302877\ 4848702403\ldots \times 10^{-11}$

Coefficients a_k of $F^*(X) = \sum_{k\geqslant 0} a_k((X-c)/r)^k$, i.e. $F(X)$ expanded with respect to the domain $\Omega = D(c,r) = D(1,2.5)$, used in the proof, are provided up to degree 20 in $X = x^2$ in table 2. Note that truncating these to $P = 20$ digits in the significand is sufficient to reproduce the proof of existence of the fixed-point function F^* with $N = 20$ and the bounds shown in the first row of table 1. Note also that $a_0 = F^*(1) = 1/\beta - 1$. Coefficients b_k of $F^*(X) = \sum_{k\geqslant 0} b_k X^k$, computed to enable a comparison with equation (12) of [7], in which the b_k were estimated using a collocation method rather than via operations on power series, are provided in table 3.

Table 3. Power series coefficients b_k of F^*, expanded about the origin, to allow direct comparison with earlier computations in which they were estimated using collocation methods.

k	b_k
00	$+1.0000000000\ 0000000000\ 0000000000\ 0000000000\ldots$
01	$-2.4410041974\ 9345277317\ 9309507381\ 6526462484\ldots$
02	$+9.8680068544\ 7852343986\ 2702072474\ 9441493473\ldots\times 10^{-2}$
03	$+1.4445994720\ 0928288658\ 3529447194\ 8959358943\ldots\times 10^{-1}$
04	$-2.0685041173\ 7635155301\ 7725625410\ 0403035445\ldots\times 10^{-2}$
05	$+2.7624593031\ 7336156176\ 7711987133\ 5672241881\ldots\times 10^{-4}$
06	$+2.4779681612\ 6879362518\ 4548481257\ 1760227323\ldots\times 10^{-4}$
07	$-1.8342801364\ 5021275088\ 3526366202\ 3399551100\ldots\times 10^{-5}$
08	$-2.4831525463\ 7684439703\ 5222504269\ 9401811367\ldots\times 10^{-6}$
09	$+4.1039031762\ 3860397185\ 4200917817\ 0274725140\ldots\times 10^{-7}$
10	$+3.9405890626\ 5284589373\ 2549704738\ 3217846147\ldots\times 10^{-9}$
11	$-5.8416825647\ 6499947074\ 2653285801\ 7008921931\ldots\times 10^{-9}$
12	$+5.0338630799\ 8993269195\ 4308948759\ 9812479260\ldots\times 10^{-10}$
13	$+2.0523022068\ 8831853790\ 0183620224\ 5611919748\ldots\times 10^{-11}$
14	$-7.4656847207\ 4888170826\ 8407364572\ 2797086179\ldots\times 10^{-12}$
15	$+4.5018079026\ 3157657220\ 7401895438\ 5046279619\ldots\times 10^{-13}$
16	$+3.6436154318\ 6010319538\ 6598057786\ 3680473325\ldots\times 10^{-14}$
17	$-7.2091258718\ 8717365293\ 7598629706\ 0845803407\ldots\times 10^{-15}$
18	$+1.9191994769\ 8723108146\ 7181454276\ 7759407111\ldots\times 10^{-16}$
19	$+5.4115740410\ 1366708946\ 9140550682\ 7565945293\ldots\times 10^{-17}$
20	$-5.4207692729\ 2923252252\ 2639000619\ 1251838005\ldots\times 10^{-18}$

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