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INTEGRABLE SYSTEMS, RANDOM MATRICES AND APPLICATIONS

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Abstract

Nonlinear integrable systems emerge in a broad class of different problems in Mathematics and Physics.

One of the most relevant characterisation of integrable systems is the existence of an infinite number of conservation laws, associated to integrable hierarchies of equations.

When nonlinearity is involved, critical phenomena may occur. A solution to a nonlinear partial differential equation may develop a gradient catastrophe and the consequent formation of a shock at the critical point. The approach of differential identities provides a convenient description of systems affected by phase transitions, identifying a suitable nonlinear equation for the order parameter of the system.

This thesis is aimed to give a contribution to the perspective offered by the approach of differential identities. We discuss how this method is particularly useful in treating mean-field theories, with some explicit application. The core of the work concerns the Hermitian matrix ensemble and the symmetric matrix ensemble, analysed in the context of integrable systems. They both underlie a discrete integrable structure in form of a lattice, satisfying a discrete integrable hierarchy. We have studied a particular reduction of both system and determined the continuum limit of the dynamics of the field variables at the leading order.

Particular emphasis has been given to the study of the symmetric matrix ensemble. We have unveiled an unobserved double-chain structure shared by the field variables populating the lattice structure associated to the ensemble. In the continuum limit of a particular reduction of the lattice, we have found a new hydrodynamic chain, a hydrodynamic system with infinitely many components. We have shown that the hydrodynamic chain is integrable and we have conjectured the form of the associated hierarchy. The new integrable hydrodynamic chain constitutes per se an interesting object of study. Indeed, it presents some properties that are different from those shared by the standard integrable hydrodynamic chains studied in literature.

Contents

Declaration

I declare that the work contained in this thesis has not been submitted for any other award and that it is all my own work. I also confirm that this work fully acknowledges opinions, ideas and contributions from the work of others.

Any ethical clearance for the research presented in this thesis has been approved. Approval has been sought and granted by the Faculty Ethics Committee on the *4/03/2019*.

I declare that the Word Count of this thesis is 46.925.

Name: Marta Dell'Atti

Signature:

Date: 11/10/2021

Introduction

Nonlinear integrable systems emerge in a plethora of phenomena pertaining to the realms of Physics and Mathematics. Concerning integrability, a general conventional definition is not given [33, 68]. Instead, depending on the context, one or more features commonly shared by integrable systems are considered as a suitable characterisation of it. Over time, several methods to approach integrability have been introduced, each of them focusing on one particular facet of the issue [64, 89, 4, 91, 93, 87]. A crucial step in the study of integrability is the discovery of infinitely many conservation laws [95], associated with hierarchies of nonlinear integrable equations. In this thesis we will encounter several integrable hierarchies, either associated with discrete systems (the Toda lattice and the Pfaff lattice) or with continuous ones (systems of hydrodynamic type).

Random matrix ensembles are typically introduced within the framework offered by random matrix theory [92], but they constitute an interesting object of study in the context of integrable systems as well. In this thesis we will deal with the Hermitian matrix ensemble [6] and the symmetric matrix ensemble [12], following the approach established by Adler and van Moerbeke in their prolific production on the topic (e.g. [13, 117, 7, 11]). The ensembles show two different underlying integrable structure, the Toda lattice for the Hermitian ensemble and the Pfaff lattice for the symmetric ensemble. These structures are introduced in terms of hierarchies in the Lax formulation and can be interpreted as emerging from an algebra splitting, in virtue of the Adler–Kostant–Symes theorem [10]. The connection with the matrix ensembles is realised via the introduction of suitably defined τ-functions, that in turns satisfy specific integrable hierarchies. In both the Hermitian and the symmetric case, the specific τ-function is proportional to the partition function given in terms of an integral on the real eigenvalues of the matri-

ces. The field variables composing the elements of the Lax operator for the two lattices are written in terms of functions of the sequence of τ-functions. We study suitable reductions of these structures, leading to the emergence of hierarchies for the continuum limit of the field variables, that will assume the form of two very different systems of hydrodynamic type.

In [50, 114], Dubrovin, Novikov and Tsarev give a geometric interpretation of hydrodynamic systems in finitely many components, describing the manifold spanned by their solutions. In the context of Hamiltonian formalism and Riemannian geometry, they relate integrability of hydrodynamic systems to geometric properties of the manifold, in terms of metrics, connections, and torsions. A geometric point of view is applied also in the case of hydrodynamic chains, a particular class of hydrodynamic systems with infinitely many components. In [60], Ferapontov and Marshall treat integrability of hydrodynamic chains within the geometric framework with the introduction of the Nijenhuis and Haantjes torsions.

One of the most relevant aspects of hydrodynamic type systems is the occurrence of critical phenomena, when solutions develop a gradient catastrophe as an effect of nonlinearity [120]. Since a discontinuity is generated, the solution exists in a weak sense only. This solution takes the name of a shock solution. The discontinuity can then be resolved by an appropriate mechanisms of regularisation, giving rise to either viscous shocks or dispersive shocks [55]. The first is modelled as a travelling wave solution to an ordinary differential equation, whereas the second give rise to a more complex structure, represented by a modulated periodic train wave.

The fact that the theory of nonlinear integrable systems offers a suitable tool to describe critical phenomena is the main underlying idea for the development of the method of differential identities [96]. The latter has its foundations in a new perspective to describe systems in the realm of Statistical Mechanics, typically affected by phase transitions. It is indeed possible to outline a proper correspondence between the typical features of Thermodynamics and those of nonlinear hydrodynamic systems. With some general assumptions on the properties of the thermodynamic system, the method of differential identities provides the equation of state as the solution to a nonlinear partial differential equation. The nonlinear character of the system induces a gradient catastro-

phe and gives rise to a shock solution [41, 22]. The approach has been successfully applied to several mean-field theories in a series of recent publications [67, 15, 90, 28]. In these cases, the suitable differential identities are defined at the partition function level and then the corresponding nonlinear equation for the order parameter is provided. The shock solution for the order parameter emerging in the context of mean-field theories is regularised by a viscous term.

A completely different phenomenon emerges in the study of the Hermitian matrix ensemble [23]. Here, a particular reduction of the Toda lattice is considered, obtained by the selection of the even coupling constants in the partition function defined for the ensemble. The resulting structure depends on one type of field variables only and at the leading order in the thermodynamic limit a quasilinear hierarchy is obtained. The behaviour of the solution for different scenarios in the space of parameters is then analysed. It develops oscillating patterns, observed in [78] and there interpreted as a chaotic behaviour. In [23] these patterns are instead qualitatively described as a manifestation of a dispersive regularisation mechanism, giving rise to a dispersive shock solution.

The aim of this work is to provide a contribution to the development of the new paradigma based on the approach of differential identities. In particular, we will focus on the study of the symmetric matrix ensemble and a suitable reduction of it. The original results collected in this thesis are part of a recent publication [24].

The thesis is organised as follows.

Part I - Background   The first part is devoted to the introduction of the general theories constituting the grounds of the objects of study of this work, i.e. integrable systems and random matrix ensembles.

In chapter 1 we provide an overview of the different perspectives that have been developed to approach integrability in nonlinear systems over time. We focus on the existence of infinitely many conservation laws associated with integrable systems and describe the related integrable hierarchies.

In chapter 2 we present the random matrix ensembles and the main tools that will be used in the core part of the thesis. We display the procedure leading to define the partition function for the Hermitian and symmetric matrix ensembles. These ensembles are

intrinsically related to the integrable structures of Toda lattice and Pfaff lattice respectively. We study how these structures emerge from an algebra splitting and investigate the realisation of the connection between the matrix ensembles and the lattices via the τ-function.

In chapter 3 we study the theory of integrable hydrodynamic systems. We introduce the Hamiltonian formalism and the generalised hodograph method to treat integrability in hydrodynamic systems with finitely many components. Then we define the hydrodynamic chains as a particular class of hydrodynamic systems with infinitely many components and discuss their integrability.

In chapter 4 we deal with critical phenomena, emerging from the occurrence of a gradient catastrophe dynamically induced by nonlinearity. We describe the breaking of the solutions to a quasilinear conservation law and analyse the consequent formation of a shock. Then we introduce the viscous and the dispersive regularisation of the shock solution and delineate the main features of their associated structures.

Part II - Case studies   This part is dedicated to the description of shock solutions emerging at the leading order in the thermodynamic limit in the context of mean-field theories and the Hermitian matrix ensemble. In the first case the shock solution is regularised by viscous corrections, in the second case by dispersive corrections.

In chapter 5 we introduce the method of differential identities as a suitable tool to describe phase transitions in thermodynamic systems. Equations of state are defined as solutions to nonlinear hydrodynamic type equations, after a redefinition of variables and a precise correspondence between Thermodynamics and nonlinear systems is outlined. The method of differential identities is explicitly applied to the Curie-Weiss model and we study the shock solution regularised by a viscous term, this being a typical feature observed in several mean-field theories.

In chapter 6 we study the Hermitian matrix ensemble and we present the construction of the associated integrable hierarchy, i.e. the Toda lattice hierarchy. This is shown in the Lax formulation of infinitely many commuting flows. We focus on a suitable reduction of the system, i.e. the Volterra lattice. The associated hierarchy will be composed of even flows only. We investigate the continuum limit of the lattice and at the leading order

we find a scalar nonlinear integrable hierarchy. We restrict our study to the case of the first three times and analyse the solution in the parameters' space, where we detect the occurrence of a dispersive shock.

Part III - Results   This part is aimed to present the original results of this work [24].

We consider the symmetric matrix ensemble and its related integrable structure via a suitable algebra splitting, i.e. the Pfaff lattice. We analyse the structure of the lattice in terms of the field variables, whose evolution is inspected for different flows of the associated hierarchy. We introduce a specific notation for the fields aimed at emphasising the underpinning observed double-chain structure.

We focus on a suitable reduction of the Pfaff lattice, for which the thermodynamic limit of the first flow is studied. At the leading order, this is represented by a new hydrodynamic chain. We investigate the diagonalisability and the integrability of the hydrodynamic chain and define the corresponding Gibbons–Tsarev system. The new hydrodynamic chain is interesting in itself since it presents more than just one seed, as in the case of standard integrable chains.

We verify that for the two next flows the form of the leading order in the thermodynamic limit is a chain as well. We conjecture that this is indeed the case for every flow of the suitable reduction of Pfaff, defining a new hydrodynamic chain hierarchy. Finally, we present a comparison with the Hermitian random ensemble.

Part IV - Explorative studies   This part collects some applications of the method of differential identities on systems describable in graph theory.

We introduce the basics aspects of simple graphs, their main features and the corresponding adjacency matrices. We study the specific example of the two-star model with a classical mean-field approach and with the method of differential identities.

We look for differential identities in the one-dimensional Ising model, for which we define a partition function in terms of the trace of the associated adjacency matrix. We analyse the form of the symmetric factors appearing in the partition function, encoding information about automorphisms of graphs. Lastly, we consider the case of the exponential random graph theory.

Part I

Background

Chapter 1

Nonlinear PDEs and integrability

This chapter is devoted to the introduction of the theory of integrable nonlinear systems. Firstly, we will approach the issue of integrability giving an insight into the different ways in which it has been studied. In section 1.1 we will briefly refer to the crucial steps in the development of the theory of nonlinear systems by introducing two of the equations that we will encounter in different context throughout this work, i.e. the Korteweg-de Vries equation and the Burgers' equation.

We will mention the various aspects of integrability, from the Inverse Scattering Transform methods, to the bi-Hamiltonian structure and the existence of infinitely many conservation laws (in section 1.2).

Finally, in section 1.3 we will introduce the concept of integrable hierarchies, with emphasis on those that we will encounter in the following chapters.

1.1   Nonlinear integrable systems

Partial differential equations (PDEs) are fundamental for the study of problems in the realm of mathematics and for the description of a plethora of phenomena in physics.

There is no general theory concerning the solvability of all PDEs, instead the research focuses on several particular cases that are relevant for applications in a broad variety of fields. The possible solvability of PDEs is related to their integrability and here too there is no a general conventional definition of what integrability is. Dating back to Poincaré, to integrate a differential equation means to find a general solution expressible in a finite

number of "elementary" functions [107]. The emphasis given to the word finite relates integrability to a general knowledge rather than a local knowledge of the solutions [68]. This is in some sense connected with the idea of the universality of nonlinear integrable systems.

Calogero describes this concept in [33], focusing on the fact that some integrable nonlinear PDEs share the aspects of universality and wide applicability. Indeed, a large class of nonlinear evolution equations can be mapped into certain universal nonlinear evolution PDEs via rescaling and asymptotic expansion. In particular, the focus is on PDEs of the form

$$D\,u(x,t) = F\left[u, u_x, u_t, u_{xx}, u_{tt}, \dots\right], \tag{1.1}$$

in terms of the field variable $u(x,t)$ with $x \in \mathbb{R}, t \in \mathbb{R}$ and its spatial and time derivatives. The left hand side (i.e. $D\,u$) corresponds to the linear part that is constructed to be dispersive and otherwise arbitrary. The right hand side (i.e. $F[\dots]$) is the nonlinear part, for which the only constraint is that it is an analytic function of the field variable and its derivatives. The universal equations obtained by the limiting procedure appear in several contexts and they are widely applicable. Moreover, this procedure generally preserves integrability, and the universal equations are likely to be integrable. An example of universal equation is the celebrated Korteweg-de Vries (KdV) equation [84] in its nondimensional form[1]

$$u_t + 6\,u\,u_x + u_{xxx} = 0, \tag{1.2}$$

introduced to describe the propagation of one-dimensional, long surface gravity waves with small amplitude in a shallow water channel. The KdV equation arises in many disparate contexts, such as stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics, gravity. The universal character of the equation is signaled by the fact that it emerges whenever the governing equation is affected by weak quadratic nonlinearity and weak dispersion [1, 2].

One of the basic features of integrable systems is their solvability and in [33, 68] a heuristic distinction between two procedures applied to solve those systems is given.

In the first approach, nonlinear systems can be reduced to a linear form (integrable)

---

[1]With the notation of the expression (1.1) for KdV $D\,u(x,t) = u_t + u_{xxx}$ and $F[u,\dots] = -6\,u\,u_x$.

via a specific change of variables. The archetype of this procedure is given by the integrability of Burgers' equation [32]

$$u_t + u\,u_x = \nu\,u_{xx}, \qquad 0 < \nu \ll 1, \tag{1.3}$$

that is linearised through the Cole-Hopf transformation

$$u(x,t) = -2\,\nu\,\partial_x \ln \phi(x,t), \tag{1.4}$$

giving the heat equation in the new field variable $\phi(x,t)$

$$\phi_t = \nu\,\phi_{xx}. \tag{1.5}$$

In the second approach, the system is linearised in terms of integro-differential equations through the method of the Inverse Scattering Transform (IST), discovered by Gardner–Green–Kruskal–Miura in [64] for KdV and generalised by Lax in [89]. In [4], the scheme describing the method is built mimicking the Fourier transform and the name IST is coined. The main idea of the procedure relies on the connection established between the KdV equation (1.2) and the linear time-independent Schrödinger problem

$$\psi_{xx} + u(x,t)\,\psi = \lambda\,\psi, \tag{1.6}$$

where $u(x,t)$ is solution to the KdV equation and here it plays the role of a potential, the time t is treated as a parameter and $\psi(x)$ is the eigenfunction of the scattering problem. The procedure of the inverse scattering is borrowed from the realm of Quantum Mechanics. This method leads to the reconstruction of the potential from the scattering data. The evolution of the function ψ is described by a second equation, i.e.

$$\psi_t = (\gamma + u_x)\,\psi + (4\lambda + 2\,u)\,\psi_x, \tag{1.7}$$

with γ being an arbitrary constant. We assume λ being a function of time $\lambda = \lambda(t)$. We derive (1.6) with respect to t and (1.7) twice with respect to x.

Imposing the compatibility condition

$$\psi_{txx} = \psi_{xxt},\tag{1.8}$$

the constraints on λ and $u(x,t)$ are $\partial_t \lambda = 0$ and $u(x,t)$ satisfies (1.2). Hence, the equations (1.6) and (1.7) are compatible if the eigenvalues are constant in time and the potential is a solution to the KdV equation.

The asymptotic behaviour (for $|x| \to \infty$) of eigenfunctions ψ and the set of their associated eigenvalues λ determine the the scattering data $S(\lambda, 0)$, which in turn depends on the potential $u(x, 0)$. The direct scattering problem consists in the mapping from the potential to the scattering data. The time evolution equation takes the initial scattering data $S(\lambda, 0)$ to $S(\lambda, t)$, whereas the inverse scattering problem is to reconstruct the potential from the scattering data [1].

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\;\text{Direct scattering}\;} & S(k,0) \\[2pt]
\Big\downarrow {\scriptstyle \text{Integrable nonlinear PDE}} & & \Big\downarrow {\scriptstyle \text{Linear time evolution of scattering data}} \\[2pt]
u(x,t) & \xleftarrow{\;\text{Inverse scattering}\;} & S(k,t)
\end{array}
$$
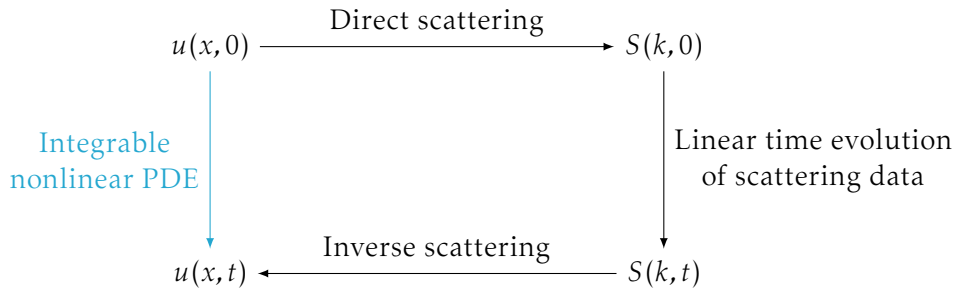
In the generalisation of the method provided by Lax, equations (1.6) and (1.7) are rewritten in terms of the linear operators L, M as

$$L\psi = \lambda \psi$$
$$\psi_t = M\psi. \tag{1.9}$$

The compatibility condition is expressed via the Lax equation

$$L_t = [M, L],\tag{1.10}$$

this becoming the key point for the treatment of integrable nonlinear PDEs. The operator L in equation (1.10) satisfies the isospectral property: its spectrum is preserved with the evolution in time.

Over time, several approaches to tackle integrability have flourished, each focusing on the latest features discovered in the context of integrable systems.

In [95], Miura discovered the existence of infinitely many conservation laws associated with the KdV equation, introducing nonlinear transformations, that are now known as Miura transformations and will be described in section 1.2. This feature has also been of crucial importance in the developing of the IST method described above.

Another milestone in the theory of integrable systems is the discovery of solitons, a kind of solution that emerges in many exactly solvable models. The presence of soliton solutions, intended as structures that interact elastically preserving the spectral portrait, was considered to unveil the integrability of the system[2]. They were introduced by Zabusky and Kruskal in [124] to address the solitary waves observed in the study of the continuum limit of the Fermi–Pasta–Ulam–Tsingou lattice [63]. The discrete model is a lattice of coupled anharmonic oscillators with fixed ends and its continuous limit is described by the KdV equation. The solitons preserve their shape and velocity upon nonlinear interactions with other solitons and they are solutions to the KdV equation. Then Hirota, in [71], proved the existence of solutions with an arbitrary number of solitons for KdV, developing the powerful formalism of the bilinear relations named after him. The Hirota bilinear formalism has a pivotal role in the representation of integrable hierarchies, as we will see in section 2.4.

In [112], Toda constructs the first example of nonlinear discrete integrable system, in contrast with the Fermi–Pasta–Ulam–Tsingou lattice, integrable in the continuum limit. He describes a one-dimensional chain of particles with an exponentially shaped first neighbours interaction, that is now known as Toda lattice. In [116] the integrable Toda lattice hierarchy is defined via the Hirota formalism [71] in terms of a suitable τ-function [75], that we will introduce in section 2.4.

In [93, 94], a symmetry approach is established, where nonlinear perturbations to linear equations are introduced. In particular, the conditions leading to the emerging of nontrivial groups of local symmetry transformations are studied for a class of PDEs. Also, the existence of a few symmetries implies that they are actually infinitely many.

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[2]We emphasise that soliton solutions have been later found in non-integrable systems as well, but in that case their interaction is not elastic anymore.

In another approach [91], the bi-Hamiltonian property is considered as the identifier for integrability. In particular, it concerns systems that can be formulated as a Hamiltonian dynamical system with respect to a Hamiltonian structure via a certain Poisson bracket. The bi-Hamiltonian property consists in the possibility of the system to be written in two different Hamiltonian structures. If these two structures are compatible, meaning that the sum of the Poisson brackets of the two structures is still a Poisson bracket, the system is integrable. Here, integrability is intended in the sense of the existence of infinitely many conserved quantities in involution with respect to both Poisson brackets.

Finally, we mention the approach of integrability involving to the study of monodromy, where the integrability of a system of PDEs is related to the study of the singularity structure of the solutions. The first observation in this context dates back to the end of 19th century, when Kovalevskaya [87] discussed the problem of the integrability of a top in a gravitational field. Motivated by this observation, she discovered that many integrable systems can be integrated in terms of elliptic functions, hence meromorphic functions that do not show movable critical points. These results were recovered several decades later and the coeval works by Dubrovin [46] and Matveev and Its [74] posed the basis for what now is called finite-gap theory.

In the following, we will encounter several integrable systems of different nature. We will consider the integrability of systems of hydrodynamic type [50]

$$u_t^i = v_j^i(u)\, u_x^i, \tag{1.11}$$

where the field variables $u^i(x,t)$ depend on the space coordinate x and time t both in the case of a finite number of components $i \in \{1,\dots,m\}$ and of an infinite number of components $i \in \mathbb{N}$. In the first case integrability is related to the semi-Hamiltonian property [113] satisfied by the characteristic speeds in the context of the treatment involving the Riemann invariants, as we will see in section 3.1. In the second case, the system takes the name of a hydrodynamic chain [60] and integrability is discussed introducing the concepts of the Nijenhuis and Haantjes tensors [86], as we will see in 3.2.

Moreover, we will study the discrete integrable systems of the Toda lattice in chap-

ter 6 and Pfaff lattice in chapter 7, the underlying structures of the Hermitian matrix ensemble [6] and symmetric matrix ensemble [12], respectively. We will see how these structures are intimately related to the hierarchies written in terms of τ-functions in the formalism of the Hirota bilinear relations. In particular, we will see how the Toda lattice is related to the KP hierarchy in section 6.1.2 and the Pfaff lattice to the so called Pfaff-KP hierarchy in section 7.1.2.

1.2   Conservation laws and Lax equation

One of the main properties of integrable systems is the existence of infinitely many conservation laws. This aspect was firstly discovered by Miura in [95], where some nonlinear transformations are applied to KdV allowing one to recursively construct the associated conservation laws.

In general, it is possible that with a PDE

$$G[x,t;u,u_x,u_t,u_{xx},u_{tt},\ldots] = 0, \tag{1.12}$$

is associated a conservation law [1, 97] of the form

$$\partial_t \rho^i + \partial_x q^i = 0, \tag{1.13}$$

satisfied by all the solutions to (1.12). In (1.13), $\rho^i(x,t;u)$ is called the conserved density and $q^i(x,t;u)$ the relative conserved flux. If the solution $u \to 0$ as $|x| \to \infty$ sufficiently rapidly and $q^i(x,t;u)$ belongs to the Schwartz class, the integration of (1.13) yields

$$\partial_t \int_{-\infty}^{\infty} \rho^i(x,t;u)\,dx = 0 \qquad \Longrightarrow \qquad \int_{-\infty}^{\infty} \rho^i(x,t;u)\,dx = c_i, \tag{1.14}$$

with c_i the conserved quantity. For KdV (1.2) the first conservation laws are

$$(u)_t + \left(3u^2 + u_{xx}\right)_x = 0$$
$$\left(u^2\right)_t + \left(4u^3 + 2u\,u_{xx} - u_x^2\right)_x = 0 \tag{1.15}$$
$$\left(u^3 - \frac{1}{2}u_x^2\right)_t + \left(\frac{9}{2}u^4 + 3u^2\,u_{xx} - 6u\,u_x^2 - u_x\,u_{xxx} + \frac{1}{2}u_{xx}^2\right)_x = 0,$$

17

related to the conservation of the mass, the energy, and the Hamiltonian of the system respectively. As anticipated above, Miura conjectured that these conservation laws are actually infinitely many. In [95], he studied the so called modified KdV (mKdV) equation

$$m_t - 6m^2 m_x + m_{xxx} = 0, \tag{1.16}$$

observing that if m is a solution to (1.16), the following expression for u

$$u = -m^2 - m_x, \tag{1.17}$$

satisfies the KdV. The equation (1.17) takes the name of Miura transformation. It is worth noticing that every solution to the mKdV equation leads to reconstruct a solution to the KdV equation, but the converse is not true. We consider now a generalisation of (1.17), given by

$$u = w - \varepsilon\, w_x - \varepsilon^2\, w^2. \tag{1.18}$$

The field u defined in this way is solution to the KdV equation if w satisfies

$$w_t + \left(3w^2 - 2\varepsilon^2\, w^3 + w_{xx}\right)_x = 0. \tag{1.19}$$

The solution u does not depend on ε, whereas the solution w depends on it. Given the arbitrariety of the choice of the parameter ε, we can consider the following formal series

$$w(x,t;\varepsilon) = \sum_{n=0}^{\infty} w_n(x,t)\,\varepsilon^n. \tag{1.20}$$

Since (1.19) is posed in a conservation form, we can write the equivalent of (1.14)

$$\int_{-\infty}^{\infty} w(x,t;\varepsilon)\,dx = c \qquad \Longrightarrow \qquad \int_{-\infty}^{\infty} w_n(x,t)\,dx = c_n. \tag{1.21}$$

With the substitution of (1.20) in the KdV equation for u obtained assuming (1.18) and

equating the coefficients of the powers of ε, we get

$$
\begin{aligned}
w_0 &= u \\
w_1 &= (w_0)_x = u_x \\
w_2 &= (w_1)_x + w_0^2 = u_{xx} + u^2 \\
w_3 &= (w_2)_x + w_0\,w_1 = u_{xxx} + 4u\,u_x.
\end{aligned}
\tag{1.22}
$$

Going further in powers of ε gives the infinitely many conservation laws.

We will now see how to construct the corresponding Lax equation (1.10) for KdV. The first consideration is that (1.17) can be seen as a Riccati equation for m in terms of u. It is known that the Riccati equation can be linearised via a change of variable, that will imply a new expression for u as well

$$
m = \frac{\psi_x}{\psi} \qquad \Longrightarrow \qquad u = -\frac{\psi_{xx}}{\psi},
\tag{1.23}
$$

and rewriting the second relation we obtain

$$
\psi_{xx} + u\,\psi = 0.
\tag{1.24}
$$

The KdV equation is invariant under a Galilean transformation

$$
(x, t, u(x, t)) \to (x - 6\lambda t, t, u(x, t) + \lambda),
\tag{1.25}
$$

for a constant λ. We then obtain the equation seen in the previous section (1.6) and (1.7)

$$
\begin{aligned}
\psi_{xx} + u(x, t)\,\psi &= \lambda\,\psi \\
\psi_t &= (\gamma + u_x)\,\psi + (4\lambda + 2\,u)\,\psi_x,
\end{aligned}
$$

whose compatibility condition $\psi_{txx} = \psi_{xxt}$, with the assumption that the eigenvalues are constant in time $\lambda_t = 0$, will lead to the KdV for the potential u and the introduction of suitable operators that will be the elements of the Lax equation (1.10). In particular, the

linear operators L and M for KdV are

$$L = \partial_x^2 + u$$
$$M = \gamma - 3u_x - 6u\,\partial_x - 4\partial_x^3.$$

$$(1.26)$$

As we have already mentioned, the equation (1.10) is obtained by the compatibility condition and imposing the isospectral property on L. The potential u, then, satisfies the KdV equation(1.2).

In the following section we will see how the expression (1.10) is of fundamental importance in one of the possible representations of hierarchies.

1.3   Integrable hierarchies

The nonlinear PDE representing an integrable system conceals an underlying associated integrable hierarchy. The latter is represented as a collection of equations commuting with each other, also known as commuting flows. This nomenclature refers to the fact that the hierarchies are displayed as infinitely many equations in terms of infinitely many "times". In particular, the infinitely many conservation laws associated with an integrable system can be thought as Hamiltonians generating time evolution in a multidimensional time space.

One way to represent the KdV hierarchy relies on the introduction of a so called pseudo-differential operator [43]

$$X = \partial + \sum_{n \geq 1} f_n \partial^{-n},$$

$$(1.27)$$

where $\partial := \partial_x$ and the negative powers of ∂ refers to a sort of formal integration. The pseudo-differential operator X, then represents a point on the infinite dimensional manifold $\mathcal{M}_L$ with coordinates given by the set of functions $\{f_1, f_2, \ldots\}$. Taking the operator L introduced in the previous section in (1.26), we consider its "square root" such that $X = L^{1/2}$. Evaluating X^2 yields

$$X^2 = \partial^2 + 2\sum_{n \geq 1} f_n \partial^{1-n} + \sum_{n \geq 1} (\partial f_n) \partial^{-n} + \sum_{m,n \geq 1, l \geq 0} \binom{-n}{l} f_n \left(\partial^l f_m\right) \partial^{-m-n-l}.$$

$$(1.28)$$

Comparing this expression with *L* we obtain

$$
\begin{aligned}
f_1 &= \frac{1}{2}u \\
f_2 &= -\frac{1}{4}u_x \\
f_3 &= -\frac{1}{8}\left(u^2 - u_{xx}\right) \\
f_4 &= -\frac{1}{16}u_{xxx} + \frac{3}{8}u\,u_x \\
&\;\vdots
\end{aligned}
\tag{1.29}
$$

The KdV hierarchy can be formulated by introducing the infinitely many parameters t_i in the Lax form

$$
\frac{\partial L}{\partial t_i} = \left[\left(L^{\frac{2i-1}{2}}\right)_+, L\right], \qquad i = 1, 2, \dots,
\tag{1.30}
$$

Given the explicit expressions for the first three flows

$$
\begin{aligned}
\partial_{t_1} u &= u_x \\
\partial_{t_2} u &= \frac{1}{4}u_{xxx} + \frac{3}{2}u\,u_x \\
\partial_{t_3} u &= \frac{1}{16}u^{(5)} + \frac{5}{4}\left(u_x\,u_{xx} + \frac{1}{2}u\,u_{xxx}\right) + \frac{15}{8}u^2\,u_x,
\end{aligned}
\tag{1.31}
$$

we can see that the first equation corresponds to the identification of t_1 with x, the second is the KdV equation[3] and other flows are the higher KdV flows.

The discovery of the KdV hierarchy is accompanied by that of many others [45]. The Kadomtsev-Petviashvili (KP) [36, 108] hierarchy has been found unifying all the generalised KdV hierarchies. These were then generalised involving matrix equations and generating the so called multi-component KdVs and KP. The latter are so called scalar hierarchies, generated by differential or pseudo-differential operators of arbitrary orders. Equations of another kind are generated by matrix first order differential operators with a linear dependence on a spectral parameter. The 2×2 matrix version is named after Ablowitz–Kaup–Newell–Segur (AKNS) [3] and their $n \times n$ generalisation is due to Dubrovin [47]. A further in generalisation is realised by Zakharov–Shabat (ZS) [126] for hierarchies generated by linear operators with a rational dependence on a parameter [44].

---

[3]The different coefficients compared to the form of the KdV previously mentioned can be obtained by a suitable rescaling of the variables.

In the following, we will run into several hierarchies represented in different ways. We will present the hierarchies associated to the Toda lattice in chapter 6 and the Pfaff lattice in chapter 7, respectively. The hierarchies will be written in form of Lax equations for commuting vector fields

$$\frac{\partial L}{\partial t_k} = \left[\left(L^k\right)_{\mathfrak{p}}, L\right], \qquad k = 1, 2, \dots,$$

where $\mathfrak{p}$ is a particular projection. These hierarchies can be described by an algebraic approach invoking the Adler–Kostant–Symes theorem [10], as we will see in section 2.3.

We will encounter equations belonging to the KP and Pfaff-KP hierarchies expressed in terms of the τ-functions for KP and Pfaffian τ-functions and related to Toda and Pfaff respectively. These will be introduced in section 2.4 in their formulation with the Hirota symbol

$$\left(s_{k+4}(\tilde{\partial}) - \frac{1}{2}\partial_{t_1}\partial_{t_{k+3}}\right)\tau_n(t) \circ \tau_n(t) = 0, \qquad k = 0, 1, 2, \dots,$$

$$\left(s_{k+4}(\tilde{\partial}) - \frac{1}{2}\partial_{t_1}\partial_{t_{k+3}}\right)\tau_{2n}(t) \circ \tau_{2n}(t) = s_k(\tilde{\partial})\,\tau_{2n-2}(t) \circ \tau_{2n+2}(t), \qquad k = 0, 1, 2, \dots.$$

In addition, we will deal with hierarchies in the context of hydrodynamic systems associated with the leading order in the thermodynamic limit for random matrix ensembles. We will see how the Hopf hierarchy

$$u_{t_{2k}} = c_k\,u^k\,u_x, \qquad k \in \mathbb{N},$$

will emerge in the context of Hermitian matrix ensemble in section 6.3.

Finally, we will define the hierarchy

$$u_{t_{2q}}^k = \sum_{p=-(q-1)}^{q} a_p^k\,u_x^p + \sum_{p=1}^{q}\left(a_{k-p}^k\,u_x^{k-p} + a_{k+p}^k\,u_x^{k+p}\right), \qquad k \in \mathbb{Z}, q \in \mathbb{N},$$

for the discovered hydrodynamic chain structure arising in the study of the symmetric matrix ensemble in section 7.5.

Chapter 2

Random Matrix Ensembles

This chapter is devoted to introduce random matrix ensembles, typically studied within the framework of random matrix theory [92]. Firstly, in section 2.1, we will define a general classification of matrix ensembles considering their general features. Then we will present the main tools that will be used in chapter 6 and in chapter 7, where we will follow the scheme proposed by Adler and van Moerbeke [6, 12] to describe the Hermitian matrix ensemble and the symmetric matrix ensemble in terms of their underlying integrable structure. The starting point of their approach is to determine the partition function for the ensemble, which is proportional to a suitable defined τ-function. The latter is defined in terms of a moments matrix constructed on a convenient inner product. The decomposition of the moments matrix leads to build the Lax operator L and the latter represents the underlying integrable lattice. The associated lattice hierarchy is given in terms of an infinite set of commuting vector fields

$$\frac{\partial L}{\partial t_k} = \left[\left(L^k \right)_{\mathfrak{p}}, L \right],$$ (2.1)

where $\mathfrak{p}$ is a particular projection. The fields composing the matrix L are expressed in terms of the above mentioned τ-functions, which in turn satisfy an integrable hierarchy.

In section 2.2, we will consider the random matrix ensembles described as tangent spaces to symmetric spaces [117] and we will give the expression for the associated partition function. We will then present the AKS theorem [10], that leads to the emergence of lattice hierarchies of the form (2.1) from an algebra splitting, in section 2.3.

In section 2.4, the τ-function will be introduced as the realisation of the connection between the lattices and the matrix ensembles. As mentioned above, the τ-function will be defined for the matrix ensembles in terms of a suitable moments matrix. The moments are defined considering the orthogonal (for the Hermitian ensemble) and skew-orthogonal (for the symmetric ensemble) polynomials, that we will present in section 2.5.

2.1 Wigner ensembles and rotational invariance

Random matrix ensembles consist of $n \times n$ matrices M with entries in the fields of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$) or quaternions ($\mathbb{Q}$), with real eigenvalues. By definition the Wigner ensemble consists of matrices whose elements M_{ij} are independent random variables. The joint probability density function takes the form

$$P(M) \propto \prod_{i=1}^{n} f_i(M_{ii}) \prod_{1 \le i < j \le n} f_{ij}\left(M_{ij}\right). \tag{2.2}$$

Assuming the ensembles exhibit a rotational invariance, for which any two matrices M and M' are related by the nonsingular similarity transformation $M \to M' = KMK^{-1}$ share the same probability

$$P(M)\,dM = P(M')\,dM', \tag{2.3}$$

condition (2.3) produces a constraint on the form of the joint probability density function $P(M)$ [92, 118]. The invariants of a $n \times n$ matrix under a similarity transformation $M \to M' = KMK^{-1}$ can be written in terms of the traces of the first n powers of M. Hence, the joint probability density function for a rotational invariant ensemble has the form

$$P(M) = f(\operatorname{tr} M, \operatorname{tr} M^2, \ldots, \operatorname{tr} M^n). \tag{2.4}$$

The Haar measure dM is invariant under the transformation $M \to M'$ by conjugation on K.

For $K \in U(n)$, we define the Hermitian matrix ensemble $\mathcal{H}_n$ (or Unitary ensemble), for $K \in O(n)$ the symmetric matrix ensemble $\mathcal{S}_n$ (or Orthogonal ensemble) and for $K \in Sp(n)$ the symplectic matrix ensemble $\mathcal{T}_{2n}$ (or Symplectic ensemble).

In these cases, in particular, we have

$$P(M \in dM) = c_n \, e^{-\text{tr } V(M)} \, dM \, , \tag{2.5}$$

where dM is the Haar measure respectively on $\mathcal{H}_n$, $\mathcal{S}_n$ and $\mathcal{T}_{2n}$ and $V(M)$ is the potential describing the specific ensemble, with derivative given by a rational function [117].

If the probability density $P(M)$ satisfies both the above conditions, then

$$P(M) = e^{-a\,\text{tr } M^2 + b\,\text{tr } M + c} \, , \qquad \text{with } a, b, c \in \mathbb{R} \, , \tag{2.6}$$

obtaining the Gaussian ensembles [92]: the Gaussian Unitary ensemble (GUE) for $K \in U(n)$, the Gaussian Orthogonal ensemble (GOE) for $K \in O(n)$ and the Gaussian Symplectic ensemble (GSE) for $K \in Sp(n)$.

2.2 Random matrix ensembles as tangent spaces to symmetric spaces

Hermitian, symmetric, and symplectic ensembles emerge as tangent spaces to symmetric spaces [117]. For the purpose of the present work we will focus on the Hermitian and symmetric ensembles.

Following [76], a symmetric space $\mathcal{M}$ can be defined as the quotient group G/K of the semi-simple Lie group G by the Lie subgroup K invariant under an involution $\sigma \colon G \to G$, i.e. $\sigma^2 = 1$

$$K = \{ g \in G, \, \sigma(g) = g \} \, , \tag{2.7}$$

so that for G/K we have

$$G/K \cong \{ g \, \sigma(g)^{-1} \, , \text{ with } g \in G \} \, . \tag{2.8}$$

The involution σ induces a map σ_* on the Lie algebra $\mathfrak{g}$ of the infinitesimal isometries on the symmetric space $\mathcal{M}$

$$\sigma_* \colon \mathfrak{g} \to \mathfrak{g} \, , \qquad \text{with } (\sigma_*)^2 = 1 \, . \tag{2.9}$$

The Lie algebra $\mathfrak{g}$ can be expressed as the direct sum[1]

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}, \tag{2.10}$$

where $\mathfrak{t}$ and $\mathfrak{p}$ can be interpreted as the eigenspaces corresponding to the eigenvalues of σ_*

$$\begin{aligned} \mathfrak{t} &= \{a \in \mathfrak{g} \mid \sigma_*(a) = a\} \\ \mathfrak{p} &= \{a \in \mathfrak{g} \mid \sigma_*(a) = -a\}, \end{aligned} \tag{2.11}$$

and since (2.9) these are ± 1. The Lie bracket for $\mathfrak{t}$ and $\mathfrak{p}$ are

$$[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{t}, \quad [\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t}. \tag{2.12}$$

Hence, $\mathfrak{t}$ is a Lie subalgebra of $\mathfrak{g}$, since it is a subset closed with respect to the Lie bracket and $\mathfrak{p}$, as a vector space, is isomorphic to $T_e \mathcal{M}$, the tangent space to the symmetric space $\mathcal{M}$ at the identity. The group K acts on $\mathfrak{p}$ by conjugation $k\mathfrak{p}k^{-1} \subset \mathfrak{p}$ and induces a root decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus \sum_{\alpha \in \Phi} \mathfrak{p}_\alpha \tag{2.13}$$

where $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$ and Φ is the set of roots of $\mathfrak{p}$ with respect to $\mathfrak{a}$

$$\mathfrak{p}_\alpha = \{x \in \mathfrak{p} \mid [a, x] = \alpha(a)x \text{ for all } a \in \mathfrak{a}\}. \tag{2.14}$$

In section 6.1 and 7.1, we will show how this approach is developed for $\mathcal{H}_n$ and $\mathcal{S}_n$ respectively. Taking into account the probability (2.5), this approach will lead to determine the partition functions for both ensembles

$$Z_n^{(\beta)}(t) = c_n \int_{\mathbb{R}^n} \prod_{1 \le i < j \le n} \left(z_i - z_j\right)^\beta \prod_{k=1}^n \rho_t(z_k) \, dz_k, \tag{2.15}$$

in terms of the eigenvalues z_k, the weight $\rho_t(z)$ and with $\beta = 1, 2$ for $\mathcal{S}_n$ and $\mathcal{H}_n$ respectively. We will se that the coupling constants $t = \{t_1, t_2, \ldots\}$ on which the partition func-

---

[1] Also called Cartan decomposition

tion depends via the weight $\rho_t(z)$ are called "times".

2.3 Integrable systems emerging from algebra splitting

The ensembles $\mathcal{H}_n$ and $\mathcal{S}_n$ are deeply related to the integrable systems called respectively Toda lattice and Pfaff lattice, as we will see in detail in chapters 6 and 7. Here, we will briefly review how these integrable structures emerge from an algebraic point of view and the next paragraph will be devoted to describe their connection to matrix ensembles via the τ–functions.

The Adler–Kostant–Symes (AKS) theorem states that vector space decompositions of Lie algebras into subalgebras lead to integrable systems [14, 20]. We will briefly review this theorem in the version presented in [10], where the starting point is a Lie algebra $\mathfrak{g}$ for which $\mathfrak{g} \cong \mathfrak{g}^*$, via an Ad-invariant non-degenerate bilinear form, that is $\langle \cdot, \cdot \rangle_{\mathfrak{g} \times \mathfrak{g}} \to \mathbb{C}$ such that

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle, \qquad X, Y, Z \in \mathfrak{g}. \tag{2.16}$$

We introduce $\nabla F(L) \in \mathfrak{g}$ the gradient of F at L for functions F on $\mathfrak{g}^* \cong \mathfrak{g}$

$$dF(L) = \langle \nabla F(L), dL \rangle, \tag{2.17}$$

and the Kostant–Kirillov Poisson structure[2] on $\mathfrak{g}^* \cong \mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$

$$\{F, H\}(L) = \langle L, [\nabla F(L), \nabla H(L)] \rangle. \tag{2.18}$$

The Hamiltonian vector fields χ_H on $\mathfrak{g}^* \cong \mathfrak{g}$ take the Lax form

$$\chi_H(L) = \{H, L\} = [\nabla H(L), L]. \tag{2.19}$$

Let us consider a vector space decomposition of the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \tag{2.20}$$

---

[2]The Kostant–Kirillov Poisson structure on $\mathfrak{g}^*$ is such that it mimics the Lie structure on $\mathfrak{g}$. Given a basis $\{\varepsilon^a\}$ the Lie structure on $\mathfrak{g}$ is $\left[\varepsilon^a, \varepsilon^b\right] = \sum_c f_c^{ab} \varepsilon^c$, with f_c^{ab} structure constants. The corresponding Kostant–Kirillov Poisson structure on $\mathfrak{g}^*$ is $\{\varepsilon^a, \varepsilon^b\} = \sum_c f_c^{ab} \varepsilon^c$ (see e.g. [17]).

and, due to the non-degeneracy of $\langle\,\cdot\,,\cdot\,\rangle$, we have

$$\mathfrak{g}^* \cong \mathfrak{g} \cong \mathfrak{g}_+^\perp \oplus \mathfrak{g}_-^\perp, \qquad \mathfrak{g}_\pm^\perp \cong \mathfrak{g}_\mp^*, \tag{2.21}$$

with $\mathfrak{g}_\pm^\perp$ the orthogonal complement with respect to $\langle\,\cdot\,,\cdot\,\rangle$ of $\mathfrak{g}_\pm$. The restriction of the Hamiltonian vector fields on $\mathfrak{g}_\mp^\perp$ is then given by

$$\chi_H(L)\big|_{\mathfrak{g}_\mp^\perp} = \hat{P}_\mp\left[\nabla_\pm H(L), L\right], \qquad L \in \mathfrak{g}_\mp^\perp, \tag{2.22}$$

with $\hat{P}_\mp$ projections onto $\mathfrak{g}_\mp^\perp$ along $\mathfrak{g}_\pm^\perp$. Analogously, we have for the Lie group G associated with the Lie algebra $\mathfrak{g}$ the decomposition in groups $G_\pm$. In addition, with the decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, we introduce the projections $P_\pm \mathfrak{g} \to \mathfrak{g}_\pm$. We define $R = P_+ - P_-$ and the Lie algebra

$$[L_1, L_2]_R = \frac{1}{2}\left([RL_1, L_2] + [L_1, RL_2]\right). \tag{2.23}$$

We can then state the AKS theorem on $\mathfrak{g}$.

Theorem 2.3.1 *Suppose that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is a Lie algebra splitting and that $\langle\,\cdot\,,\cdot\,\rangle$ is an Ad-invariant non-degenerate bilinear form on $\mathfrak{g}$, leading to a vector space splitting*

$$\mathfrak{g} = \mathfrak{g}_+^\perp \oplus \mathfrak{g}_-^\perp \simeq \mathfrak{g}_-^* \oplus \mathfrak{g}_+^*. \tag{2.24}$$

The Hamiltonian vector fields $\chi_H := \{\,\cdot\,, H\}_R$ are given by

$$\chi_H(L) = -\frac{1}{2}\left[L, R\left(\nabla H(L)\right)\right] = \pm\left[L, P_\pm(\nabla H(L))\right]. \tag{2.25}$$

For the purpose of this work, we are interested in the discrete integrable systems of Toda and Pfaff lattice. Each system arises from a particular decomposition of the general linear algebra $\mathfrak{gl}(\infty) = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, with $\langle\,\cdot\,,\cdot\,\rangle$ the Frobenius inner product

$$\langle A, B\rangle = \mathrm{tr}\,(AB). \tag{2.26}$$

Applying the AKS theorem to the specific algebra splitting, the Hamiltonian vector fields

are

$$\frac{\partial L}{\partial t_k} := \chi_{H^{(k)}} = \pm \left[P_\pm \nabla H_k, L \right],$$ (2.27)

with $H_k \propto \dfrac{\operatorname{tr} L^{k+1}}{k+1}$, conserved quantities in involution [5, 85]. We obtain

$$\frac{\partial L}{\partial t_k} = \pm \left[P_\pm \left(L^k \right), L \right].$$ (2.28)

The matrix L is given by $L = K \Lambda K^{-1}$, with $K \in G_+$ and $\Lambda = \{\delta_{i,j-1}\}_{1 \le i,j < \infty}$ the shift operator.

We will see how the Toda lattice emerges from the splitting $\mathfrak{gl}(\infty) = \mathfrak{s} \oplus \mathfrak{b}$, with $\mathfrak{s}$ skew-symmetric and $\mathfrak{b}$ lower triangular projections, in section 6.1.2. Section 7.1.2 will be devoted to the study of Pfaff lattice from $\mathfrak{go}(\infty) = \mathfrak{t} \oplus \mathfrak{p}$, with $\mathfrak{t}$ the projection on lower triangular matrices with 2×2 blocks along the diagonal proportional to the identity and $\mathfrak{p} = \mathfrak{sp}(\infty)$.

2.4 The connection between lattices and matrix ensembles via τ-functions

The matrix $L = K \Lambda K^{-1}$ introduced above, in the case of the Toda lattice, satisfies

$$L \psi(t, z) = z \psi(t, z),$$ (2.29)

with times $t = \{t_1, t_2, \dots\}$, eigenvalues z, and where $\psi(t, z)$ is a wave vector constructed from the operator K

$$\psi(t, z) = K \, e^{\frac{1}{2} \sum_{i=1}^{\infty} t_i z^i} \zeta(z), \qquad \zeta(z) = (\zeta(z))_{n \in \mathbb{Z}} = (z^n)_{n \in \mathbb{Z}}.$$ (2.30)

The wave vector admits a representation in terms of a vector of τ−functions $\tau = (\tau_n)_{n \in \mathbb{Z}}$

$$\psi(t, z) = e^{\frac{1}{2} \sum_{i=1}^{\infty} t_i z^i} \left(z^n \, \frac{\tau_n(t - [z^{-1}])}{\sqrt{\tau_n(t) \tau_{n+1}(t)}} \right)_{n \in \mathbb{Z}},$$ (2.31)

as discussed by Adler and van Moerbeke in [6], having the form of a Baker–Akhiezer function expressed in terms of the τ-functions via the so called Sato formula [20, 42], with

$$t - \left[z^{-1} \right] = \left\{ t_k - \frac{1}{k} z^{-k} \right\}. \tag{2.32}$$

In the case of the Pfaff lattice [9], it is necessary to introduce two wave vectors $\psi_1(t, z)$, $\psi_2(t, z)$, that admit a representation in terms of τ-functions as well.

What is a τ-function? As pointed out in [98], there are different definitions of τ-functions, but all of them are related to a specific realization of the following idea: a τ-function is a generating functional of all the matrix elements of some group in a particular representation. One of the main aspects shared by τ-functions relevant in this context is that they satisfy a set of bilinear equation, the Hirota bilinear relations.

The τ-function has been introduced by Jimbo–Miwa–Ueno in [75], following the lead dating back to Riemann regarding the concept of deformations preserving monodromy properties in the context of linear ODEs. In particular, the τ-function is presented as an analogue of the Riemann θ-function associated to nonlinear deformations of ODEs. In their paper, the authors also discuss the emergence of a connection with the AKNS hierarchy. Two years later, Sato [108] proposed a geometrical interpretation of the τ-functions, establishing a connection with the infinite dimensional Grassmannian[3]. The τ-function in this context coincides with the so called Plücker coordinates of the Grassmannian. The latter are not independent and they satisfy the Plücker relation. This can be written in terms of the Hirota bilinear formalism and gives rise to the KP hierarchy.

As previously mentioned, in the discrete cases of the Toda and Pfaff lattice, the wave function admits a representation in terms of a sequence of suitably defined τ-functions $\tau_n(t)$ (see [116, 6] for Toda and [9] for Pfaff). The corresponding hierarchies written in terms of τ-functions are produced requiring that

$$\mathrm{Res}_{z=\infty} \left(\psi(z, t) \, \psi^*(z, t') \right) = 0 \qquad \forall t, t', \tag{2.33}$$

from which the set of infinite differential equations is written in the compact formalism

---

[3]In [121] the Sato theory is presented in pedagogical terms, starting from the simplest non-trivial case of the $Gr(2; 4)$ and showing the generalisation for the construction of the Sato theory for infinite dimension.

of the Hirota bilinear identity [20, 49, 121].

The sequence of $\tau_n(t)$ defined for Toda satisfy the KP hierarchy

$$\left(s_{k+4}(\tilde{\partial}) - \frac{1}{2}\partial_{t_1}\partial_{t_{k+3}}\right)\tau_n(t) \circ \tau_n(t) = 0, \qquad k = 0, 1, 2, \dots, \tag{2.34}$$

as it will be shown in section 6.1.2. Analogously, the sequence of $\tau_n(t)$ defined for the Pfaff lattice satisfy the Pfaff-KP hierarchy (following the nomenclature by Adler and van Moerbeke)

$$\left(s_{k+4}(\tilde{\partial}) - \frac{1}{2}\partial_{t_1}\partial_{t_{k+3}}\right)\tau_{2n}(t) \circ \tau_{2n}(t) = s_k(\tilde{\partial})\,\tau_{2n-2}(t) \circ \tau_{2n+2}(t), \qquad k = 0, 1, 2, \dots, \tag{2.35}$$

that we will study in section 7.1.2. It is worth mentioning that in the literature the Pfaff-KP is also called BKP hierarchy (introduced in [39] and see also [37, 72, 115, 21, 80]).

The expressions (2.34) and (2.35) involve the Hirota operator

$$\partial_{x_1}^{m_1} \dots \partial_{x_n}^{m_n} f(x) \circ g(x) = \left(\partial_{\varepsilon_1}^{m_1} \dots \partial_{\varepsilon_n}^{m_n}\right) f(x_1 + \varepsilon_1, \dots, x_n + \varepsilon_n)\, g(x_1 - \varepsilon_1, \dots, x_n - \varepsilon_n)\Big|_{\varepsilon_i = 0\,\forall i}, \tag{2.36}$$

the operator $\tilde{\partial} = \left(\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \dots\right)$ and the Schur polynomials, defined as

$$e^{\sum_{n=1}^{\infty} t_n z^n} = \sum_{j=0}^{\infty} s_j(t) z^j. \tag{2.37}$$

The hierarchy (2.34) is written in terms of the so called KP τ-functions, while the one of (2.35) in terms of Pfaffian τ-functions. This allows to establish the connection with the matrix ensembles $\mathcal{H}_n$ and $\mathcal{S}_n$. In particular, the KP τ-function is proportional to the partition function defined for $\mathcal{H}_n$

$$\tau_n^{\mathrm{KP}} \propto Z_n^{(2)}, \tag{2.38}$$

introduced in (2.15) with $\beta = 2$, as it will be described in section 6.1.1. On the other side, the Pfaffian τ-function proportional to the partition function defined for $\mathcal{S}_n$

$$\tau_n^{\mathrm{pf\text{-}KP}} \propto Z_n^{(1)}, \tag{2.39}$$

i.e. (2.15) with $\beta = 1$, as we will see in section 7.1.1.

In the context of the matrix theory, for $\mathcal{H}_n$ the τ-function is introduced as the determinant of a Hänkel moments matrix with respect to a symmetric measure, for $\mathcal{S}_n$ is given by the Pfaffian of a skew-symmetric moments matrix with respect to a skew-symmetric measure, as we will study in sections 6.1.2 and 7.1.2, respectively. The moments matrix is of significant importance in the approach that we will present in the following, leading to define the elements in the matrix L representing the systems of Toda and Pfaff lattice in terms of sequences of the respective τ-functions. Finally, it is worth mentioning that both Toda and Pfaff lattices can be seen as reductions of the 2−Toda lattice, where the initial condition is given by a moments matrix that is a Hänkel matrix for the Toda lattice and a skew-symmetric matrix for the Pfaff lattice [12, 13].

The τ-function approach has also lead Witten to elaborate his conjecture in [122] (generalised in [123]) and then proved in [83] for which the generating functional of correlators in the model of 2-dimensional gravity coincide with the τ-function of a matrix model and obey to the KdV hierarchy.

2.5   Orthogonal and skew-orthogonal polynomials

Orthogonal and skew-orthogonal polynomials are an established tool in the theory of random matrix models. The theory of orthogonal polynomials [111] is well known and has applications in many areas, while it is not the same for the theory of skew-orthogonal polynomials, emerging in the context of symmetric and symplectic matrix ensembles and deeply connected with the underlying Pfaffian structure. We will briefly mention the main aspects of orthogonal polynomials and then provide the standard introduction of orthogonal and skew-orthogonal polynomials in the context of the random matrix theory.

A sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is orthogonal in the interval $]a, b[$ with respect to the positive weight function $\rho(x)$ if

$$\int_a^b p_n(x) p_m(x) \rho(x)\, dx = \begin{cases} 0 & n \neq m \\ h_n \neq 0 & n = m. \end{cases} \tag{2.40}$$

The interval $]a, b[$ is defined interval of orthogonality and it can be either finite or infinite,

provided that the convergence of the integral is ensured. The weight function $\rho(x)$ is continuous and positive on the interval, so that the moments μ_n exist, with μ_n given by

$$\mu_n = \int_a^b \rho(x)\, x^n\, dx. \tag{2.41}$$

It is worth emphasising that the sequence of polynomials is uniquely defined up to normalization and they can be determined starting from initial conditions with the Gram-Schmidt orthogonalisation procedure. A fundamental property of the orthogonal polynomials is the fact that they satisfy a three-term recurrence relation of the form

$$p_{n+1}(x) = (a_n x + b_n)\, p_n(x) - c_n\, p_{n-1}(x) \qquad n = 0, 1, \dots. \tag{2.42}$$

Notable examples of orthogonal polynomials are the Hermite polynomials and Laguerre polynomials.

▷ The Hermite polynomials H_n are defined by the generating function

$$e^{2xt - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!}, \tag{2.43}$$

and have the explicit form

$$H_n(x) = \sum_{k=0}^{n/2} \frac{(-1)^k\, n!}{k!\,(n - 2k)!} \,(2x)^{n-2k}. \tag{2.44}$$

The orthogonality property of $H_n(x)$ is

$$\int_{-\infty}^{\infty} H_n(x)\, H_m(x)\, e^{-x^2}\, dx = 2^n\, n!\, \sqrt{n}\, \delta_{nm}, \tag{2.45}$$

and they satisfy the recurrence relation

$$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x), \qquad n \geq 1. \tag{2.46}$$

▷ The Laguerre polynomials L_n are defined by the generating function

$$(1-t)^{-\alpha-1}e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{\alpha}(x)\,t^n, \qquad (2.47)$$

and have the explicit form

$$L_n(x) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k\,x^k}{(\alpha+1)_k\,k!}(2x)^{n-2k}, \qquad (2.48)$$

with $(a)_b = a(a+1)\dots(a+b-1)$. The orthogonality property of $L_n(x)$ is

$$\int_{-\infty}^{\infty} L_n^{\alpha}(x)\,L_m^{\alpha}(x)\,x^{\alpha}\,e^{-x}\,dx = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{nm}, \qquad (2.49)$$

and they satisfy the recurrence relation

$$(n+1)L_{n+1}^{\alpha}(x) = (1+2n+\alpha-x)L_n^{\alpha}(x) - (n+\alpha)L_{n-1}^{\alpha}(x), \qquad n \geq 1. \qquad (2.50)$$

In the context of random matrix theory, orthogonal and skew-orthogonal polynomials were introduced by Mehta [92] in relation to the partition functions for Gaussian ensembles defined at the end of section 2.1: the GUE with orthogonal polynomials, the GOE and GSE with the skew-orthogonal polynomials. The connection has been extended (especially for the orthogonal polynomials [27, 122]) and has led to a consistent description of the Hermitian, symmetric, and symplectic ensembles combining random matrix theory, τ-functions and the theory of orthogonal and skew-orthogonal polynomials [6, 12, 57, 26, 29] associated with the aforementioned integrable structures. In particular, as recalled in [10], the Toda lattice is the natural integrable system underpinning the deformation of GUE of random matrix theory as well as constituting the natural deformation class of orthogonal polynomials. Analogously, the Pfaff lattice is associated with the natural deformations for GOE (and GSE) and provides the natural deformation for skew-orthogonal polynomials.

In this context, we introduce the t-deformed weight $\rho_t(z)$

$$\rho_t(z) = \rho(z)e^{\sum_k t_k z^k}. \qquad (2.51)$$

Orthogonal and skew-orthogonal polynomials

We consider the symmetric inner product defined on $\rho_t(z)$

$$(f(z), g(z))_t := \int_{\mathbb{R}} f(z) g(z) \rho_t(z) \, dz \qquad (2.52)$$

and the sequence of polynomials $\{p_n(z,t)\}_{n+0}^{\infty}$ orthogonal with respect to $\rho_t(z)$

$$\left(p_j(z,t), p_k(z,t) \right)_t = \int_{\mathbb{R}} p_j(z,t) p_k(z,t) \rho_t(z) \, dz = \delta_{jk} h_k, \qquad (2.53)$$

where

$$p_n(z,t) = \gamma_n(t) z^n + \gamma_{n-1}(t) z^{n-1} + \dots, \qquad (2.54)$$

that is called monic if $\gamma_n(t) = 1$. For a monic sequence of orthogonal polynomials with respect to a positive measure $\rho_t(z) \, dz$, there exists the recurrence relation

$$p_{n+1}(z,t) = (z - a_n(t)) \, p_n(z,t) - b_n(t) p_{n-1}(z,t), \qquad n = 0, 1, \dots, \qquad (2.55)$$

with initial conditions $p_{-1}(z,t) = 0$, $p_0(z,t) = 1$. The recurrence coefficients are given by

$$a_n(t) = \frac{(z p_n(z,t), p_n(z,t))_t}{(p_n(z,t), p_n(z,t))_t}, \qquad b_n(t) = \frac{(z p_n(z,t), p_{n-1}(z,t))_t}{(p_{n-1}(z,t), p_{n-1}(z,t))_t}. \qquad (2.56)$$

Recurrence coefficients can be collected in a tridiagonal matrix J_n, known as Jacobi matrix [27]

$$J_n = \begin{pmatrix} a_1 & \sqrt{b_1} & 0 & 0 & 0 & \cdots \\ \sqrt{b_1} & a_2 & \sqrt{b_2} & 0 & 0 & \\ 0 & \sqrt{b_2} & a_3 & \sqrt{b_3} & 0 & \\ 0 & 0 & \sqrt{b_3} & a_4 & \sqrt{b_4} & \\ \vdots & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \sqrt{b_{n-1}} \\ & & & & & \sqrt{b_{n-1}} & a_n \end{pmatrix}. \qquad (2.57)$$

We note that this matrix coincides with the Lax operator $L(t)$ of the Toda lattice, which

we will build in section 6.1.2. The recursion relation can be formulated as

$$L(t)p(z,t) = z\,p(z,t), \qquad p(z,t) = (p_n(z,t))_{n\in\mathbb{N}}. \tag{2.58}$$

The orthogonal polynomials can be therefore interpreted as eigenvectors of the Toda lattice. They admit an integral representation [111] and for what is stated in the previous section, they can be expressed in terms of KP τ-functions [6, 117]. It is worth noticing that in presence of an even weight function, the term $a_n(t)$ in the recurrence relation (2.55) vanishes.

Similarly, the connection between the Pfaff lattice and the skew-orthogonal polynomials is established in [12]. The skew-orthogonal polynomials are defined with respect to a skew-symmetric weight $\tilde{\rho}_t(y,z) = -\tilde{\rho}_t(z,y)$, for which the corresponding inner product is

$$\langle f(y), g(z) \rangle_t := \int\int_{\mathbb{R}^2} f(y)\,g(z)\,\tilde{\rho}_t(y,z)\,dz\,dy. \tag{2.59}$$

A family of monic polynomials $\{q_n(z,t)\}_{n=0}^{\infty}$ is skew-orthogonal with respect to $\tilde{\rho}_t(z)$ if [11]

$$\langle q_{2m}(y,t), q_{2n+1}(z,t) \rangle_t = -\langle q_{2n+1}(z,t), q_{2m}(y,t) \rangle_t = \delta_{nm}\,r_m$$
$$\langle q_{2m}(y,t), q_{2n}(z,t) \rangle_t = \langle q_{2m+1}(y,t), q_{2n+1}(z,t) \rangle_t = 0. \tag{2.60}$$

It is worth noticing that the relations (2.60) are invariant under the transformation

$$q_{2m+1}(z,t) \mapsto q_{2m+1}(z,t) + \alpha_{2m}\,q_{2m}(z,t), \tag{2.61}$$

for an arbitrary α_{2m}, hence the skew-orthogonal transformations are not unique up to this mapping.

As in the case of the Toda lattice, for the skew-orthogonal polynomials a recurrence relation is established [106], that can be written as

$$L(t)q(z,t) = z\,q(z,t), \qquad (q_n(z,t))_{n\in\mathbb{N}}, \tag{2.62}$$

with $L(t)$ a lower triangular matrix with non-zero elements on the above diagonal. This matrix coincides with the Pfaff lattice and the skew-orthogonal polynomials are eigen-

vectors of Pfaff [117, 12], as we will see in section 7.1.2. Finally, also in this case, they admit a representation in terms of the Pfaffian τ-functions.

It is worth noting that, at this stage, the main difference between the Toda lattice and the Pfaff lattice relies on the number of recurrence coefficients necessary for their representation. The Toda lattice is represented by a symmetric tridiagonal matrix, completely describable by defining two recurrence coefficients uniquely determined. Instead, in the Pfaff lattice case, the form of the matrix leads to consider infinitely many recurrence coefficients, that are not uniquely determined because of (2.61).

In chapter 6 and chapter 7, the recurrence coefficients here mentioned will be simply called field variables for both lattices. We will study the discrete equations they satisfy considering several flows in the Toda and Pfaff hierarchy of the form (2.1). In the continuum limit at the leading order we will find hierarchies expressed in terms of the continuum version of the field variables. For a suitable reduction of Toda, we will find a scalar hierarchy, expressed in terms of one type of field only. Whereas, for a specific reduction of Pfaff, we will observe a hydrodynamic chain hierarchy, given in terms of infinitely many field variables. In both cases, we will deal with systems of hydrodynamic type, that we will present in the next chapter.

Chapter 3

Hydrodynamic type systems

In this chapter, following [50], we will introduce the Hamiltonian formalism for the description of hydrodynamic systems. We will start considering systems with finitely many components, in section 3.1. We will define the Poisson brackets and describe the manifold spanned by the solutions to the system via the Riemann invariants and the associated characteristic speeds. We will introduce the generalised hodograph method [113] and the related semi-Hamiltonian property, encoding the integrability of this type of systems.

Section 3.2 is dedicated to the study hydrodynamic chains, i.e. a class of hydrodynamic systems composed of infinitely many components. We will follow the approach established in [60] concerning the integrability of hydrodynamic chains via the properties of the Nijenhuis and Haantjes tensors. The latter are involved in the definition of diagonalisability and integrability in the sense of an infinite number of hydrodynamic reductions of the system. Finally, we will introduce the Gibbons–Tsarev system, encoding the information about the integrable chain in a system of equations in terms of characteristic speeds, Riemann invariants, and the seed of the chain.

3.1 Hydrodynamic systems with finitely many components

In this section, we will briefly review the Hamiltonian theory for systems of hydrodynamic type with a finite number of components

$$u_t^i = v_j^i(u)\, u_x^j, \qquad i = 1, \dots, N, \tag{3.1}$$

as described in [50]. In particular, in section 3.1.1 we will give the structure of the Poisson bracket of hydrodynamic type on the manifold $\mathcal{M}$ with local coordinates $u^1, \dots, u^N$, and in section 3.1.2 we will describe the generalised hodograph method, in the context of integrability of systems of kind (3.1).

We start by recalling the main features of the finite-dimensional Poisson bracket. Let $\mathcal{M}$ be a N-dimensional manifold, called the phase space. A Poisson bracket $\{\cdot, \cdot\}$ is defined as an operation on the space of smooth functions on $\mathcal{M}$ manifesting the properties

(*a*) bilinearity

$$\begin{aligned} \{\lambda f + \mu g, h\} &= \lambda \{f, h\} + \mu \{g, h\} \\ \{f, \lambda g + \mu h\} &= \lambda \{f, g\} + \mu \{f, h\}, \end{aligned} \qquad \lambda, \mu = \text{const} \tag{3.2}$$

(*b*) skew-symmetry

$$\{f, g\} = -\{g, f\}, \tag{3.3}$$

(*c*) Jacobi identity

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0, \tag{3.4}$$

(*d*) Leibniz identity

$$\{fg, h\} = f\{g, h\} + g\{f, h\}. \tag{3.5}$$

Considering local coordinates $y^1, \dots, y^N$ on the manifold $\mathcal{M}$, a Poisson bracket is defined by a skew-symmetric $(2, 0)$ tensor

$$h^{ij}(y) = \{y^i, y^j\}, \qquad i, j = 1, \dots, N. \tag{3.6}$$

Hydrodynamic systems with finitely many components

For the Leibniz property (c), the Poisson bracket can also be defined as

$$\{f,g\} = h^{ij}(y) \frac{\partial f(y)}{\partial y^i} \frac{\partial g(y)}{\partial y^j}, \tag{3.7}$$

and the Jacobi identity (d) implies for the tensor h^{ij} to satisfy the relation

$$\frac{\partial h^{ij}}{\partial y^l} h^{lk} + \frac{\partial h^{ki}}{\partial y^l} h^{lj} + \frac{\partial h^{jk}}{\partial y^l} h^{li} = 0. \tag{3.8}$$

If $\det\left(h^{ij}\right) \neq 0$, the constraint (3.8) is equivalent to endowing $\mathcal{M}$ with a symplectic structure, since the inverse matrix $h_{ij} = \left(h^{ij}\right)^{-1}$ contributes to define the 2-form $\Omega = h_{ij} \, dy^i \wedge dy^j$, non-degenerate and closed $d\Omega = 0$. The manifold $\mathcal{M}$ with a non-degenerate Poisson bracket is then called symplectic.

The existence of a Poisson bracket leads to write the Hamiltonian equations as

$$\frac{\partial y^i}{\partial t} = \{y^i, H(y)\}, \tag{3.9}$$

where $H(y)$ is the Hamiltonian of the system (3.1). Any integral F of the system satisfies the property

$$\{F(y), H(y)\} = 0. \tag{3.10}$$

In terms of the field variables $u^i(x,t)$ appearing in (3.1), we now introduce the so called local Poisson bracket, that is defined for a class of functionals on $u^i(x,t)$ with $x = \left(x^1, \dots, x^d\right)$. In particular, they are defined for functionals of local fields and of their derivatives (when they exist) at a point. The Poisson bracket takes the form

$$\{u^i(x), u^j(y)\} = h^{ij}(x,y), \qquad i,j = 1, \dots, N, \tag{3.11}$$

where the tensor $h^{ij}(x,y)$ is now characterised not only by the integer indices i,j, but also by two continuous indices x,y. For functionals $I[u]$, $J[u]$, we have

$$\{I,J\} = \int \frac{\delta I}{\delta u^i(x)} \frac{\delta J}{\delta u^j(y)} h^{ij}(x,y) \, d^d x \, d^d y, \tag{3.12}$$

with variational derivatives given by

$$I[u + \delta u] - I[u] = \int \frac{\delta I}{\delta u^i(x)} \delta u^i(x)\, d^d x + o(\delta u). \tag{3.13}$$

We consider local field functionals

$$I[u] = \int P\left(x, u(x), u^{(1)}(x), \dots, u^{(k)}(x)\right) d^d x, \tag{3.14}$$

where P is a polynomial (or more in general an analytic function) in terms of the variables $\left(u, u^{(1)}, \dots, u^{(k)}\right)$, called the density of the functional. A natural class of local field theoretic brackets is introduced as

$$\{u^i(x), u^j(y)\} = \sum_{|k| \leq K} B_k^{ij}\left(x, u(x), u^{(1)}(x), \dots, u^{(n_k)}(x)\right) \partial_x^k \, \delta(x - y), \qquad i, j = 1, \dots, N, \tag{3.15}$$

where $k = (k_1, \dots, k_d)$, $|k| = k_1 + \cdots + k_d$, $\partial_x^k = \left(\frac{\partial}{\partial x_1}\right)^{k_1} \dots \left(\frac{\partial}{\partial x_d}\right)^{k_d}$ and K the order of the bracket. The derivatives of the Dirac delta function $\delta(x - y)$ are formal symbols defined as

$$\int f(y)\, \delta^{(k)}(x - y)\, d^d y = \partial_x^k f(x). \tag{3.16}$$

Introducing the operator

$$A^{ij} = \sum_{|k| \leq K} B_k^{ij}(x, \dots) \partial_x^k, \tag{3.17}$$

we have for (3.12)

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} A^{ij} \frac{\delta J}{\delta u^j(x)}\, d^d x\, d^d y, \tag{3.18}$$

and the Hamiltonian equations take the form

$$u_t^i(x) = \{u^i(x), H\} = A^{ij} \frac{\delta H}{\delta u^j(x)}, \qquad i = 1, \dots, N, \tag{3.19}$$

where $H = H[u]$ is a local functional of the kind (3.14).

3.1.1 Hamiltonian formalism and Riemannian geometry

A system of hydrodynamic type is represented by an equation of the form [50]

$$u_t^i = v_j^{i\,\alpha}(u)\, u_\alpha^j\,, \qquad i = 1, \ldots, N\,, \qquad \alpha = 1, \ldots, d\,, \tag{3.20}$$

with $u_\alpha^j = \partial u^j / \partial x^\alpha$ for a $d+1$ system with N fields u^i. Considering an invertible smooth change of field variables

$$u^i = u^i\left(R^1, \ldots, R^N\right)\,, \qquad i = 1, \ldots, N\,, \tag{3.21}$$

the coefficients $v_j^{i\,\alpha}$ for each α transform as a $(1,1)$-tensor

$$v_j^{j\,\alpha}(u) \mapsto v_q^{p\,\alpha}(u) = \frac{\partial r^p}{\partial u^i}\, v_j^{i\,\alpha}\left(u(R)\right) \frac{\partial u^j}{\partial r^q}\,. \tag{3.22}$$

Let us introduce the manifold $\mathcal{M}^N$ where the fields $u^1(x,t), \ldots, u^N(x,t)$ take values for each x, t. With this in mind, (3.21) can be interpreted as a change of coordinates in $\mathcal{M}^N$.

For simplicity, let us restrict to the $1+1$ dimensional case. Moreover, let us assume the system described in (3.20) is strictly hyperbolic, i.e. all the eigenvalues $v^1 = \lambda^1, \ldots, v^N = \lambda^N$ of the matrix $\left(v_j^i\right)$ are real and distinct. If it is possible to reduce the system (3.20), via the change of coordinates (3.21), to the diagonal form

$$R_t^i = v^i(R)\, R_x^i\,, \qquad i = 1, \ldots, N\,, \qquad R = \left(R^1, \ldots, R^N\right)\,, \tag{3.23}$$

the variables $R^1, \ldots, R^N$ are called the Riemann invariants for (3.20), while the coefficients $v^1(r), \ldots, v^N(r)$ are the corresponding characteristic speeds. For $N = 2$ it is always possible to obtain the diagonal form in terms of Riemann invariants, while for $N \geq 3$ this is not true in general. The same considerations can be done in the case of complex eigenvalues, involving complex changes of coordinates (3.21).

The study of Hamiltonian systems involves a rich geometry, as it was first pointed out by Dubrovin and Novikov in [52].

For a system of hydrodynamic type

(*a*) the Poisson bracket is defined as

$$\left\{u^i(x), u^j(y)\right\} = g^{ij\alpha}(u)\,\delta'_\alpha(x-y) + b_k^{ij\alpha}(u))\,u^k_\alpha\,\delta(x-y), \qquad (3.24)$$

where $g^{ij\alpha}(u)$, $b_k^{ij\alpha}(u)$ are certain functions, $i,j,k = 1,\dots,N$, $\alpha = 1,\dots,d$ and $\delta'(x)$ is given by (3.16) with $k = 1$;

(*b*) functionals are defined as

$$H[u] = \int h(u)\,d^d x,, \qquad (3.25)$$

where the density $h(u)$ is independent of derivatives u_α, $u_{\alpha\beta}$, $\dots$;

(*c*) if Hamiltonian, it takes the form

$$u^i_t(x) = \left\{u^i(x), H\right\} = \left(g^{ij\alpha}(u)\,\frac{\partial^2 h(u)}{\partial u^j\,\partial u^k} + b_k^{ij\alpha}(u)\,\frac{\partial h(u)}{\partial u^j}\right)u^k_\alpha, \qquad i = 1,\dots,N, \quad (3.26)$$

with $\{\cdot, \cdot\}$ a Poisson bracket of hydrodynamic type (3.24).

In the case of a system of $1+1$ dimensions, omitting the index α, it can be shown [50] the following

(*a*) the class (3.24) of Poisson brackets of hydrodynamic type is invariant under changes of field variables of the form (3.21) $u^i \mapsto v^i(u)$;

(*b*) under these changes of variables, the coefficients $g^{ij}(u)$ transform as tensors of type $(2, 0)$

$$g^{pq}(u) = \frac{\partial v^p}{\partial u^i}\,\frac{\partial v^q}{\partial u^j}\,g^{ij}(u), \qquad p,q = 1,\dots,N; \qquad (3.27)$$

(*c*) assuming that the metric $\left(g^{ij}(u)\right)$ is non-degenerate and defining Γ^k_{ij} from

$$b_k^{ij}(u) = -g^{il}(u)\Gamma^j_{lk}(u), \qquad i,j,k = 1,\dots,N, \qquad (3.28)$$

then under the change of variables (3.21) it transforms as a differential-geometric connection

$$\Gamma^p_{qr}(u) = \frac{\partial v^p}{\partial u^i}\,\frac{\partial u^j}{\partial v^q}\,\frac{\partial u^k}{\partial v^r}\,\Gamma^i_{jk}(u) + \frac{\partial v^p}{\partial u^i}\,\frac{\partial^2 u^i}{\partial v^q\,\partial v^r}. \qquad (3.29)$$

Hydrodynamic systems with finitely many components

If the metric is non-degenerate then $\det\left(g^{ij}\right) \neq 0$ and the corresponding Poisson brackets are called non-degenerate. The latter property is invariant under transformations (3.21).

Theorem 3.1.1 *In the non-degenerate case* $\det\left(g^{ij}\right) \neq 0$, *the expression* (3.24) *defines a Poisson bracket if and only if the tensor* g^{ij} *is symmetric, i.e. it defines a pseudo-Riemannian metric on the manifold* $\mathcal{M}^N$. *The connection* Γ^i_{jk} *of the form* (3.28) *is compatible with the metric* g^{ij} *and has zero curvature and torsion. Therefore, there exist local coordinates* $v^i = v^i\left(u^1, \ldots, u^N\right)$, $i = 1, \ldots, N$ *such that* $g^{ij} = \mathrm{const}$ *and* $b^{ij}_k = 0$. *In these coordinates the Poisson bracket* (3.24) *is constant*

$$\left\{v^i(x), v^j(y)\right\} = g^{ij}_0 \, \delta'(x-y), \qquad g^{ij}_0 = g^{ji}_0 = \mathrm{const}. \tag{3.30}$$

To consider the conditions for which a hydrodynamic system is Hamiltonian in a more explicit form, we start from the observation that the system $u^i_t(x) = \left\{u^i(x), H\right\}$, with Hamiltonian (3.25) ($d = 1$) and Poisson brackets (3.24), can be formulated as

$$u^i_t(x) = v^i_j(u) \, u^j_x, \qquad v^i_j(u) = \nabla^i \nabla_j h(u), \tag{3.31}$$

where ∇_j is the covariant differentiation operator

$$\nabla_j u^i = \partial_j u^i + \Gamma^i_{jk} u^k, \tag{3.32}$$

with $\partial_j = \partial/\partial u_j$. In addition, the controvariant operator is obtained raising indices $\nabla^i = g^{ik} \nabla_k$ and the operators ∇_i, ∇^j commute because of Theorem 3.1.1.

Proposition 3.1.1 *The system* $u^i_t = v^i_j(u) u^j_x$ *is Hamiltonian if and only if there exists a non-degenerate metric* $g^{ij}(u)$ *of zero curvature, such that*

$$g_{ij} v^k_j = g_{jk} v^k_i \tag{3.33}$$

$$\nabla_i v^k_j = \nabla_j v^k_i, \tag{3.34}$$

where ∇_i *is the covariant differentiation generated by the metric* g^{ij}.

In 1983, Novikov conjectured that for a finite-component system of hydrodynamic type to be integrable needs to be Hamiltonian, i.e. it admits a metric as described in the proposition 3.1.1. This was later demonstrated by Tsarev, who established a less strict condition for hydrodynamic systems to be integrable – the semi-Hamiltonian property – and outlined a prescription to integrate these systems called the generalised hodograph method, that we will analyse in the next section.

3.1.2 The generalised hodograph method

In this section we will briefly review the hodograph method [97], with focus on the approach elaborated by Tsarev [113], leading to a generalization of the procedure for multi-component systems.

For a $1 + 1$ dimensional system of hydrodynamic type $u_t = v(u)\, u_x$, with two components $u = \left(u^1, u^2\right)$, it is possible to define a linearization of it through the hodograph transformation

$$x = x\left(u^1, u^2\right), \qquad t = t\left(u^1, u^2\right). \tag{3.35}$$

In particular, the original system of hydrodynamic type

$$\begin{cases} u_t^1 = v_1^1(u)\, u_x^1 + v_2^1(u)\, u_x^2 \\[2mm] u_t^2 = v_1^2(u)\, u_x^1 + v_2^2(u)\, u_x^2 \end{cases}, \tag{3.36}$$

is transformed into the linear version

$$\begin{cases} x_{u_2} = -v_1^1(u)\, t_{u^2} + v_1^2(u)\, t_{u^1} \\[2mm] x_{u_1} = v_1^2(u)\, t_{u^2} - v_2^2(u)\, t_{u^1} \end{cases}. \tag{3.37}$$

The method proposed by Tsarev for the integration of two-component systems is suitable for generalizations to multi-component systems. We start by analysing a two-component system (3.36) that is strictly hyperbolic in some region of the space of coordinates $\left(u^1, u^2\right)$, i.e. the matrix $v_j^i(u)$ has two distinct real eigenvalues $v_1(u)$ and $v_2(u)$. Hence, it is possible to write the system (3.36) in a diagonal form, under a smooth change of coordinates. For

simplicity, let us consider the case in which the system is already diagonal

$$
\begin{cases}
u_t^1 = v_1(u)\, u_x^1 \\[2mm]
u_t^2 = v_2(u)\, u_x^2
\end{cases}
. \tag{3.38}
$$

We introduce $w_1(u)$, $w_2(u)$, solution to the system

$$
\frac{\partial_2 w_1}{w_2 - w_1} = \frac{\partial_2 v_1}{v_2 - v_1}\,, \qquad \frac{\partial_1 w_2}{w_2 - w_1} = \frac{\partial_1 v_2}{v_2 - v_1}\,. \tag{3.39}
$$

Then we have that

(a) the functions $u^1 = u^1(x, t)$, $u^2 = u^2(x, t)$ defined by

$$
\begin{cases}
w_1\left(u^1, u^2\right) = v_1\left(u^1, u^2\right) t + x \\[2mm]
w_2\left(u^1, u^2\right) = v_2\left(u^1, u^2\right) t + x
\end{cases}
, \tag{3.40}
$$

are solutions to the system (3.38), and every smooth solution to (3.38) can be determined in this way;

(b) the system of hydrodynamic type

$$
\begin{cases}
u_\tau^1 = w_1(u)\, u_x^1 \\[2mm]
u_\tau^2 = w_2(u)\, u_x^2
\end{cases}
, \tag{3.41}
$$

defines a symmetry of the system (3.38) ($u_{t\tau}^i = u_{\tau t}^i$), and all the symmetries of the class of systems of hydrodynamic type can be determined in this way.

To show (a), we consider the hodograph transformation applied to the system (3.38), leading to

$$
\begin{cases}
\partial_2 x + v_1(u)\, \partial_2 t = 0 \\[2mm]
\partial_1 x + v_2(u)\, \partial_1 t = 0
\end{cases}
, \tag{3.42}
$$

that can be reformulated as

$$\begin{cases} \partial_2 \left(v_1 \, t + x \right) = t \, \partial_2 v_1 \\[2mm] \partial_1 \left(v_2 \, t + x \right) = t \, \partial_1 v_2 \end{cases} \qquad . \tag{3.43}$$

With the introduction of fields

$$w_i(u) = v_i(u) \, t + x \,, \qquad i = 1, 2 \,, \tag{3.44}$$

we get for the variable t

$$t = \frac{w_1 - w_2}{v_1 - v_2} \,. \tag{3.45}$$

The substitution of (3.45) into (3.43) yields (3.39). Conversely, if we differentiate the implicit functions $u^1(x,t)$, $u^2(x,t)$ from (3.40) and we use (3.39), we get the system (3.43).

For part (b) let us introduce a symmetry of (3.43)

$$u_\tau^i = w_j^i(u) \, u_x^j \,, \qquad i = 1, 2 \,. \tag{3.46}$$

From the symmetry property $u_{t\tau}^i = u_{\tau t}^i$, it follows that the matrix u_j^i commutes with the diagonal matrix $v_j \, \delta_j^i$, hence $w_j^i = w_j \, \delta_j^i$ is diagonal as well. Moreover, this property implies that w_1, w_2 satisfy (3.39). As mentioned abovove, for a a multi-component system of hydrodynamic type, Novikov conjectured that the combination of the existence of the bracket (3.24) with non-degenerate metric and the diagonalization implies the integrability of the system. Then Tsarev proved the conjecture in [113, 114] and introduced a generalization of the hodograph method to integrate these systems. We will briefly review this approach.

Let us consider a multi-component diagonal Hamiltonian system of hydrodynamic type

$$u_t^i = v_i(u) \, u_x^i \,, \qquad i = 1, \dots, N \,, \tag{3.47}$$

with mutually distinct elements and $g^{ij}(u)$ the corresponding metric (assumed to be non-degenerate) describing the Hamiltonian structure.

Lemma 3.1.2 *Let $u^1, \dots, u^N$ be fields variables of a diagonal Hamiltonian system of hydrodynamic type. Then the corresponding metric $g^{ij}(u)$ is diagonal as well.*

Hydrodynamic systems with finitely many components

This is proved by (3.33). From a differential-geometric point of view, a diagonal metric corresponds to a curvilinear orthogonal system of coordinates in a flat space (Euclidean or pseudo-Euclidean). On the other side, if we choose an arbitrary system of curvilinear orthogonal coordinates, then a family of Hamiltonian systems is associated with it.

Lemma 3.1.3 *Let $u^1, \dots, u^N$ be a system of orthogonal curvilinear coordinates, $g_{ij}(u) = g_i(u)\delta_{ij}$ the associated metric, and $\Gamma^k_{ij}(u)$ the generated connection. Then all diagonal systems of hydrodynamic type*

$$u^i_t = w_i(u)\, u^i_x, \qquad i = 1, \dots, N, \tag{3.48}$$

Hamiltonian with respect to the Poisson bracket

$$\left\{ u^i(x), u^j(y) \right\} = g_i\left(u(x)\right)^{-1} \left[\delta^{ij}\, \delta'(x-y) - \sum_k \Gamma^j_{ik}\, u^k_x(x-y) \right], \tag{3.49}$$

are determined by the relations

$$\partial_i w_k = \Gamma^k_{ki}\,(w_i - w_k), \qquad i \neq k. \tag{3.50}$$

All these systems commute pairwise and they are parametrised locally by functions of one variable.

We consider the condition (3.34) for the system to be Hamiltonian and $u^i_j = w_j\, \delta^i_j$

$$\begin{aligned}
0 = \nabla_i u^k_j - \nabla_j u^k_i &= \partial_i u^k_j - \partial_j u^k_i + \sum_{l=1}^N \left(\Gamma^k_{il}\, u^l_j - \Gamma^l_{ij}\, u^k_l - \Gamma^k_{jl}\, u^l_i + \Gamma^l_{ji}\, u^k_l \right) \\
&= \partial_i w_j\, \delta^k_j - \partial_j w_i\, \delta^k_i + \Gamma^k_{ij}\left(w_i - w_j \right).
\end{aligned} \tag{3.51}$$

This is an identity in the case of i, j, k all distinct, since $\Gamma^k_{ij} = 0$ because we have zero curvature and torsion. The non-trivial relation is given for the case $j = k \neq i$, yielding to (3.50).

For a generic diagonal metric $g_{ij} = g_i\, \delta_{ij}$, we have

$$\Gamma^k_{ki} = \partial_i \ln \sqrt{g_k}, \tag{3.52}$$

and inserting this in (3.50) we obtain the relations

$$\partial_i\left(\frac{\partial_j w_k}{w_j - w_k}\right) = \partial_j\left(\frac{\partial_i w_k}{w_i - w_k}\right), \qquad i \neq k,\, j \neq k, \tag{3.53}$$

leading to the definition of a semi-Hamiltonian system. In particular, a diagonal system of hydrodynamic type $u_t^i = w_i(u)\,u_x^i$, $i = 1, \ldots, N$, is called semi-Hamiltonian if its coefficients satisfy (3.53). For $N = 2$ the relations (3.53) reduce to identities, hence every diagonal system is semi-Hamiltonian. For $N \geq 3$ every Hamiltonian system is semi-Hamiltonian, but the converse is not true. Then it is sufficient for a system of hydrodynamic type to be diagonalizable and semi-Hamiltonian in order to be integrable, as it is stated in the following theorem.

Theorem 3.1.4 *Let*

$$u_t^i = v_i(u)\,u_x^i, \qquad i = 1, \ldots, N, \tag{3.54}$$

be a diagonal semi-Hamiltonian system of hydrodynamic type, and $w_1(u), \ldots, w_N(u)$ arbitrary solutions to the system

$$\partial_i w_k = \Gamma_{ki}^k\,(w_i - w_k), \qquad i \neq k, \tag{3.55}$$

with $\Gamma_{ki}^k = \frac{\partial_i v_k}{v_i - v_k}$ coefficients of a hydrodynamic flow commuting with (3.54).
The functions $u^1(x,t), \ldots, u^N(x,t)$ determined by the system

$$w_i(u) = v_i(u)\,t + x, \qquad i = 1, \ldots, N, \tag{3.56}$$

satisfy (3.54); in addition, every smooth solution can be obtained in this way.

To show this, we differentiate (3.56) with respect to t and x, obtaining

$$\begin{cases} \sum_k (\partial_k w_i - t\,\partial_k v_i)\,u_t^k = v_i \\[2mm] \sum_k (\partial_k w_i - t\,\partial_k v_i)\,u_x^k = 1 \end{cases}. \tag{3.57}$$

Introducing the matrix $M(u)$, with elements

$$M_{ik}(u) = \partial_k w_i - t\,\partial_k v_i, \tag{3.58}$$

by (3.55), they can be formulated as

$$M_{ik}(u) = \frac{\partial_k v_i}{v_k - v_i}\left(w_k - w_i - t\left(v_k - v_i\right)\right), \qquad i \neq k. \tag{3.59}$$

If $u = u(x, t)$ is a solution to (3.56), we have

$$w_k - w_i = t\left(v_k - v_i\right) \quad \Longrightarrow \quad M_{ik} = 0, \quad i \neq k. \tag{3.60}$$

Therefore, the only terms remaining are those for which $i = k$ and (3.57) becomes

$$\begin{cases} M_{ii}(u)\, u_t^i = v_i \\[2mm] M_{ii}(u)\, u_x^i = 1 \end{cases} \quad \Longrightarrow \quad u_t^i = v_i(u)\, u_x^i, \quad i = 1, \dots, N, \tag{3.61}$$

and $u = u(x, t)$ is a solution to the system (3.54) as well. Also, because of $M_{ii}(u)\, u_x^i = 1$, we have that $u_x^i \neq 0$ for any smooth solution to (3.54).

Conversely, let us consider $u = u(x, t)$ a solution to (3.54) such that $u_x^i \neq 0$ in the neighbourhood of the point (x_0, t_0) for $i = 1, \dots, N$. Taking $u_0^i = u^i(x, t_0)$ to be the initial condition of the Cauchy problem for the original system (3.54), we have

$$w_i\left(u_0(x)\right) = v_i\left(u_0(x)\right) t_0 + x, \tag{3.62}$$

on the curve $u_0(x)$. Since by assumption $\left(u_0^i\right)_x(x_0) \neq 0$, there exists a unique solution $w_i(u)$ to (3.55) with initial condition (3.62). We introduce the function

$$\Phi_i(u, x, t) = w_i(u) - v_i(t)\, t - x = 0, \qquad i = 1, \dots, N. \tag{3.63}$$

The Jacobian matrix is non degenerate in (u_0^i, x_0, t_0)

$$\frac{\partial \Phi_i}{\partial u^k} = \partial_k w_i - t_0\, \partial_k v_i = M_{ik} \tag{3.64}$$

$$M_{ik} = \begin{cases} 0 & i \neq k \\[2mm] \partial_i w_i - t_0\, \partial_i v_i \neq 0 & i = k \end{cases}. \tag{3.65}$$

We differentiate Φ_i with respect to x at the point (u_0^i, x_0, t_0)

$$M_{ii}\left(u_0^i\right)_x - 1 = 0 \quad \implies \quad M_{ii} = \frac{\partial \Phi_i}{\partial u^i} \neq 0. \tag{3.66}$$

Due to the theorem of implicit function, there exists a unique smooth solution $\bar{u}^i(x,t)$ in a neighbourhood of (u_0^i, x_0, t_0). By construction $\bar{u}(x,t_0) = u(x,t_0)$ and from the previous part, $u(x,t_0)$ is a solution to (3.54). Hence, $\bar{u}(x,t) = u(x,t)$ in a neighbourhood of (x_0,t_0) by the uniqueness of the solution to the Cauchy problem.

Therefore, thanks to the theorem 3.1.4, the integration of a system of hydrodynamic type (3.54) is reduced to that of the linear system (3.55) with the functions $u^1, \dots, u^N$ implicitly determined by (3.56). In this sense, it is evident that this consists in a generalisation of the hodograph method, thus called generalised hodograph method. Integrability for a multi-component system of hydrodynamic type with a finite number of components follows from the diagonalizability of the system and its semi-Hamiltonian property. In the next section, we will investigate the case of systems of hydrodynamic type with an infinite number of components.

3.2 Hydrodynamic chains

Let us now move to systems of hydrodynamic type with an infinite number of components. They are called hydrodynamic chains [60, 101, 102] and are formulated as quasi-linear partial differential equations

$$u_t^i = v_j^i(u)\, u_x^j, \qquad i = 1, 2, \dots, \tag{3.67}$$

with $u = \left(u^1, u^2, \dots\right)^\top$ an infinite vector and $v(u) = \left\{v_j^i(u)\right\}_{i,j=1}^{\infty}$ a $\infty \times \infty$ matrix. The prototypical example of a hydrodynamic chain is given by the Benney's moments' equation

$$u_t^n = u_x^{n+1} + (n-1)\, u^{n-1}\, u_x^1, \qquad n = 1, 2, \dots, \tag{3.68}$$

introduced in [25] to study long waves in shallow fluid with free surface in a gravitational field.

Hydrodynamic chains

In general terms, a hydrodynamic chain takes the form [101]

$$u_t^n = \varphi_1^n u_x^1 + \cdots + \varphi_{n+1}^n u_x^{n+1}, \qquad n = 1, 2, \ldots, \qquad \varphi_{n+1}^n \neq 0, \tag{3.69}$$

where $\varphi_j^n = \varphi_j^n\left(u^1, \ldots, u^{n+1}\right)$. The class of conservative hydrodynamic chains of the type

$$u_t^1 = u_x^2, \qquad u_t^2 = f\left(u^1, u^2, u^3\right)_x, \qquad u_t^3 = g\left(u^1, u^2, u^3, u^4\right)_x, \qquad \ldots, \tag{3.70}$$

has been extensively studied (e.g. [105, 102]). In this case, the function $f\left(u^1, u^2, u^3\right)$ determines all the other equations of the chain (3.70) and the related hierarchy.

We will follow the approach established in [60] for the discussion concerning the integrability of hydrodynamic chains, motivated by the theory of finite-component systems of hydrodynamic type that we have treated in section 3.1. As we have seen, for those systems, the requirements of being diagonalizable in terms of the Riemann invariants and semi-Hamiltonian are sufficient for the system to be integrable. The theory established in [60] relates to the criterion of classification of $(2 + 1)$-dimension integrable systems grounded on the existence of infinite hydrodynamic reductions [59]. This is based on the observation that dispersionless limits of integrable systems in $2 + 1$ dimensions possess infinitely many hydrodynamic reductions. Moreover, if the dispersionless system is not linearly degenerate, in [61] it was shown that hydrodynamic reductions of dispersionless limits of integrable systems can be deformed into those of the dispersive counterpart in $2 + 1$ dimensions. In [62], a definition of integrability for $2 + 1$-dimensional systems is given, claiming that a $2 + 1$-dimensional system is integrable if all the hydrodynamic reductions of its dispersionless limit can be deformed into reductions of their dispersive counterparts.

Ferapontov and Marshall [60] introduce a tensorial criterion for diagonalisability, based on the construction of the so called Nijenhuis tensor and the Haantjes tensor and they extend this concept to infinite-component systems of hydrodynamic type. Their idea comes from the results obtained by Nijenhuis [100] and Haantjes [70]. Their research was aimed to find the conditions for which for a field of endomorphisms of the tangent bundle of a manifold, with the assumption of simple eigenvalues, the distri-

butions spanned by pairs of eigenvectors are integrable [86]. Ferapontov and Marshall in [60] formulate the main result in [70] as a theorem, in the field of integrable systems, as we will see in the following. Their statement is that a system of hydrodynamic type, with mutually distinct characteristic speeds, is diagonalisable if and only if the corresponding Haantjes tensor vanishes identically. The connection relies on the fact that, for systems with finitely many components, the solutions form a manifold in the theory of Riemann invariants. Then they extend the result to infinite-component systems.

For a system of hydrodynamic type with finitely many components, we consider the matrix $v_j^i(u)$. The Nijenhuis tensor of the matrix $v_j^i(u)$ is a $(1,2)$ tensor defined as

$$N_{jk}^i = v_j^p(u) \partial_p v_k^i(u) - v_k^p(u) \partial_p v_j^i(u) - v_p^i(u) \left(\partial_j v_k^p(u) - \partial_k v_j^p(u) \right), \qquad (3.71)$$

with $\partial_i = \partial/\partial u^i$. The Haantjes tensor of the matrix $v_j^i(u)$ is a $(1,2)$ tensor that takes the form

$$H_{jk}^i = N_{pq}^i v_j^p(u) v_k^q(u) - N_{jq}^p v_p^i(u) v_k^q(u) - N_{qk}^p v_p^i(u) v_j^q(u) + N_{jk}^p v_q^i(u) v_p^q(u). \qquad (3.72)$$

The diagonalizability condition for strictly hyperbolic systems can be formulated as the following theorem, introduced in [70] and reformulated by Ferapontov and Marshall in the context of hydrodynamic systems.

Theorem 3.2.1 *A diagonalizable system of hydrodynamic type with mutually distinct characteristic speeds is diagonalizable if and only if the corresponding Haantjes tensor* (3.72) *is identically zero.*

It is remarkable that these tensors can be defined in the infinite-component case (3.67) as well, provided that the matrix $v_j^i(u)$ is "sufficiently sparse".

Definition 3.2.1 *An infinite matrix $V(u) = \left\{ v_j^i(u) \right\}_{i,j=1}^{\infty}$ belongs to the class C (chain class) if it satisfies the properties*

   (*i*) *each row of $V(u)$ contains finitely many non-zero elements;*

   (*ii*) *each matrix element of $V(u)$ depends on finitely many field variables $u^i(x,t)$.*

Hydrodynamic chains

For matrices belonging to class C, the sums on repeated indices in (3.71) and (3.72) reduce to a finite number of terms, hence every component of the tensor H^i_{jk} is well defined and can be computed. In particular, for a fixed value of the upper index i, we have only a finite number of components of H^i_{jk} that are non-zero.

Definition 3.2.2 *A hydrodynamic chain* (3.67) *with* $V(u) \in C$ *is diagonalizable if all components of the Haantjes tensor* (3.72) *are zero.*

As we will show in the following section, the vanishing of the Haantjes tensor is a necessary (and in some cases sufficient) condition for a hydrodynamic chain to have an infinite number of finite-component diagonalizable hydrodynamic reductions.

This approach, based on the construction of the Haantjes tensor has the advantage to be "intrinsic", in the sense that it is not required any knowledge of "estrinsic" objects, like the Hamiltonian structure, the Lax pair or the commuting flows for the system. As we will see in the following, the diagonalizability condition is necessary for the system to possess sufficiently many hydrodynamic reductions.

3.2.1 Hydrodynamic reductions and Gibbons–Tsarev system

A hydrodynamic reduction of an infinite hydrodynamic chain is represented by parametric equations in a finite number m of components, as

$$u^1 = u^1\left(R^1, \dots, R^m\right), \qquad u^2 = u^2\left(R^1, \dots, R^m\right), \qquad u^3 = u^3\left(R^1, \dots, R^m\right), \qquad \dots, \quad (3.73)$$

where $R^1, \dots, R^m$ are the Riemann invariants. They solve the diagonal system

$$R^i_t = \lambda^i(R)\, R^i_x, \qquad i = 1, \dots, m, \qquad R = \left(R^1, \dots, R^m\right), \qquad (3.74)$$

and the characteristic speeds $\lambda^i(R)$ satisfy the semi-Hamiltonian property (3.53) that we recall

$$\partial_i\left(\frac{\partial_j \lambda^k}{\lambda^j - \lambda^k}\right) = \partial_j\left(\frac{\partial_i \lambda^k}{\lambda^i - \lambda^k}\right). \qquad (3.75)$$

All the equations of the chain are satisfied modulo (3.74), hence the infinite-component system reduces to a finite-component one. The notion of hydrodynamic reductions underpins the definition of integrability.

Definition 3.2.3 *A hydrodynamic chain of class C (3.67) is integrable if it admits m-phase solutions of the form*

$$u^k = u^k\left(R^1, \ldots, R^m\right),$$ (3.76)

for arbitrary m.

In [65], Gibbons and Tsarev show that the Benney chain possesses infinitely many *m*-component reductions, parametrized by *m* functions of a single variable. Integrability of more generic hydrodynamic chains has been investigated in [105, 101] with the method of hydrodynamic reductions, that we now briefly review.

We consider the method of hydrodynamic reductions as described in [65, 66], applied to the Benney chain (3.68) for illustrative purposes. The first equations of the chain are

$$u_t^1 = u_x^2$$
$$u_t^2 = u_x^3 + u^1\, u_x^1$$
$$u_t^3 = u_x^4 + 2\, u^2\, u_x^1$$ (3.77)
$$u_t^4 = u_x^5 + 3\, u^3\, u_x^1$$
$$\vdots$$

We look for solutions of the form $u^i = u^i\left(R^1, \ldots, R^m\right)$, where $R^1, \ldots, R^m$ are the Riemann invariants, satisfying the diagonal system (3.74) that we recall

$$R_t^i = \lambda^i(R)\, R_x^i.$$

Using this ansatz in the first of equations (3.77), we obtain

$$\partial_i u^1\, R_t^i = \partial_i u^2\, R_x^i$$
$$\partial_i u^1\left(\lambda^i R_x^i\right) = \partial_i u^2\, R_x^i$$ (3.78)
$$R_x^i\left(\lambda^i\, \partial_i u^1 - \partial_i u^2\right) = 0$$
$$\partial_i u^2 = \lambda^i\, \partial_i u^1,$$

for $i = 1, \ldots, m$ and with $\partial_i = \partial/\partial R^i$.

Hydrodynamic chains

Applying the ansatz recursively in the other equations (3.77), we get

$$\partial_i u^2 = \lambda^i \, \partial_i u^1$$
$$\partial_i u^3 = \left(\left(\lambda^i \right)^2 - u \right) \partial_i u^1$$
$$\partial_i u^4 = \left(\left(\lambda^i \right)^3 - u \, \lambda^i - 2 \, u^2 \right) \partial_i u^1 \tag{3.79}$$
$$\partial_i u^5 = \left(\left(\lambda^i \right)^4 - u \left(\lambda^i \right)^2 - 2 \, u^2 \, \lambda^i - 3 \, u^3 \right) \partial_i u^1$$
$$\vdots$$

Imposing the compatibility conditions

$$\partial_i \partial_j u^k = \partial_j \partial_i u^k \,, \tag{3.80}$$

for $k = 2, 3, 4$ in the expressions in (3.79) yields

$$\partial_i \, \partial_j u^1 = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \, \partial_i u^1 + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} \, \partial_j u^1 \,,$$
$$\partial_j \lambda^i \partial_i u^1 + \partial_i \lambda^j \partial_j u^1 = 0 \,, \tag{3.81}$$
$$\lambda^i \, \partial_j \lambda^i \, \partial_i u^1 + \lambda^j \, \partial_i \lambda^j \, \partial_j u^1 + \partial_i u^1 \, \partial_j u^1 = 0 \,.$$

Solving (3.81) in $\partial_j \lambda^i$, we get the so called Gibbons–Tsarev system for the Benney chain

$$\partial_j \lambda^i = \frac{\partial_j u^1}{\lambda^j - \lambda^i} \,,$$
$$\partial_i \partial_j u^1 = 2 \, \frac{\partial_i u^1 \partial_j u^1}{\left(\lambda^i - \lambda^j \right)^2} \,. \tag{3.82}$$

All the compatibility conditions (3.80) for $k > 4$ in (3.79) are satisfied modulo (3.82) and the semi-Hamiltonian property (3.75) is automatically fulfilled. Hence, the Benney chain is integrable and the m-component reductions of the chain are described by (3.82).

This approach can be applied to any hydrodynamic chain of the class C and this has lead Ferapontov and Marshall to formulate the following theorem, stated in [60].

Theorem 3.2.2 *The vanishing of the Haantjes tensor is a necessary condition for the existence of infinitely many hydrodynamic reductions and, thus, for the integrability of a hydrodynamic*

chain.

To prove the theorem, let us consider a conservative hydrodynamic chain (3.70) written in the form

$$u_t^m = v_n^m(u)\, u_x^n, \qquad m, n = 1, 2, \ldots. \tag{3.83}$$

We apply the ansatz described above, so we look for solutions dependent of a finite number of Riemann invariants. We get

$$\partial_i u^m R_t^i = v_n^m \, \partial_i u^n R_x^i,$$
$$\partial_i u^m \lambda^i R_x^i = v_n^m \, \partial_i u^n R_x^i, \tag{3.84}$$

and equating the coefficients of R_x^i, the previous becomes

$$v_n^m \, \partial_i u^n = \lambda^i \, \partial_i u^m, \tag{3.85}$$

or expressed in vector form as

$$v(u)\, \partial_i u = \lambda^i \, \partial_i u. \tag{3.86}$$

Hence, the characteristic speeds λ^i can be considered eigenvalues of the infinite matrix $v(u)$ and $\partial_i u$ the corresponding eigenvectors. To impose the compatibility condition (3.80), we make use of the operator ∂_j with $j \neq i$ acting on (3.85) (with the notation $v_{n,i}^m = \partial_i v_n^m$)

$$v_{n,k}^m \, \partial_j u^k \, \partial_i u^n + v_n^m \, \partial_j \partial_i u^n = \left(\partial_j \lambda^i\right) \partial_i u^m + \lambda^i \, \partial_j \partial_i u^m, \tag{3.87}$$

and we exchange the indices $i \leftrightarrow j$, yielding

$$v_{n,k}^m \, \partial_i u^k \, \partial_j u^n + v_n^m \, \partial_j \partial_i u^n = \left(\partial_i \lambda^j\right) \partial_j u^m + \lambda^j \, \partial_i \partial_j u^m. \tag{3.88}$$

The compatibility condition, then, gives

$$\left(\lambda^i - \lambda^j\right)\partial_i \partial_j u^m = -\left(\partial_j \lambda^i\right)\partial_i u^m + \left(\partial_i \lambda^j\right)\partial_j u^m + \left(v_{n,k}^m - v_{k,n}^m\right)\partial_i u^k \partial_j u^n$$
$$\partial_i \partial_j u^m = -\frac{\partial_j \lambda^i}{\lambda^i - \lambda^j}\, \partial_i u^m + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j}\, \partial_j u^m + \frac{v_{n,k}^m - v_{k,n}^m}{\lambda^i - \lambda^j}\, \partial_i u^k \partial_j u^n. \tag{3.89}$$

Substituting the latter in (3.87), we obtain

$$v_{n,k}^m \partial_j u^k \partial_i u^n + \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} v_n^m \partial_i u^n + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} v_n^m \partial_j u^n + \frac{v_l^m \left(v_{n,k}^l - v_{k,n}^l\right)}{\lambda^i - \lambda^j} \partial_i u^l \partial_j u^k =$$
$$= \partial_j \lambda^i \partial_i u^m + \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \lambda^i \partial_i u^m + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} \lambda^i \partial_i u^m + \frac{v_{n,k}^m - v_{k,n}^m}{\lambda^i - \lambda^j} \lambda^i \partial_i u^n \partial_j u^k . \tag{3.90}$$

Using (3.85) we get

$$\partial_j \lambda^i \partial_i u^m + \partial_i \lambda^j \partial_j u^m = \frac{\lambda^i - \lambda^j}{\lambda^i - \lambda^j} v_{n,k}^m \partial_i u^k \partial_j u^k + \frac{v_l^m \left(v_{n,k}^l - v_{k,n}^l\right)}{\lambda^i - \lambda^j} \partial_i u^l \partial_j u^k$$
$$+ \frac{v_n^l \left(v_{n,k}^m - v_{k,n}^m\right)}{\lambda^i - \lambda^j} \lambda^i \partial_i u^n \partial_j u^k , \tag{3.91}$$

and using the (3.85) twice on the first term of the right hand side we obtain

$$\partial_j \lambda^i \partial_i u^m + \partial_i \lambda^j \partial_j u^m = \frac{1}{\lambda^i - \lambda^j} \left\{ v_n^l v_{k,l}^m - v_k^l v_{n,l}^m + v_l^m (v_{n,k}^l - v_{k,n}^l) \right\} \partial_i u^n \partial_j u^k . \tag{3.92}$$

Since the Nijenhuis tensor is

$$N_{nk}^m = v_n^l v_{k,l}^m - v_k^l v_{n,l}^m + v_l^m \left(v_{n,k}^l - v_{k,n}^l\right) \tag{3.93}$$

we have

$$\partial_j \lambda^i \partial_i v^m + \partial_i \lambda^j \partial_j v^m = \frac{N_{nk}^m}{\lambda^i - \lambda^j} \partial_i u^n \partial_j u^k . \tag{3.94}$$

where, as usual, the sum is on the reiterated indices, except for i, j.

We now determine the Gibbons–Tsarev system for the hydrodynamic chain. To do so, we let the matrix $v_j^i(u)$ act on both sides of (3.94) in all the possible ways and using (3.85), we obtain the system[1]

$$\lambda^i \partial_j \lambda^i \partial_i u^m + \lambda^j \partial_i \lambda^j \partial_j u^m = \frac{v_p^m N_{nk}^p \partial_i u^n \partial_j u^k}{\lambda^i - \lambda^j} \tag{3.95}$$

$$\lambda^i \partial_j \lambda^i \partial_i u^m + \lambda^i \partial_i \lambda^j \partial_j u^m = \frac{v_n^p N_{pk}^m \partial_i u^n \partial_j u^k}{\lambda^i - \lambda^j} \tag{3.96}$$

$$\lambda^j \partial_j \lambda^i \partial_i u^m + \lambda^j \partial_i \lambda^j \partial_j u^m = \frac{v_k^p N_{np}^m \partial_i u^n \partial_j u^k}{\lambda^i - \lambda^j} . \tag{3.97}$$

---

[1] Here we compute the expressions for the components of tensors explicitly rather than showing them in a vector form, as reported in [60].

Considering the subtractions side by side of (3.95) – (3.96) and (3.95) – (3.97) we have

$$\partial_i \lambda^j \, \partial_j u^m = \frac{N^m_{pk} v^p_n - N^p_{nk} v^m_p}{(\lambda^i - \lambda^j)^2} \, \partial_i u^n \, \partial_j u^k \tag{3.98}$$

$$\partial_j \lambda^i \, \partial_i u^m = \frac{N^m_{pn} v^p_k - N^p_{kn} v^m_p}{(\lambda^i - \lambda^j)^2} \, \partial_i u^n \, \partial_j u^k . \tag{3.99}$$

As we can easily observe, (3.99) can be obtained from (3.98) exchanging the indices $i \leftrightarrow j$.

Finally, we show that the Hantjees tensor is zero, starting from the equation (3.94). Let us consider twice the action of the matrix $v^i_j(u)$ on both sides of the equation in all the four possible ways

$$\left(\partial_j \lambda^i \right) \left(\lambda^i \right)^2 \partial_i u^m + \left(\partial_i \lambda^j \right) \left(\lambda^j \right)^2 \partial_j u^m = \frac{v^m_q v^q_p N^p_{nk} \partial_i u^n \, \partial_j u^k}{\lambda^i - \lambda^j} \tag{3.100}$$

$$\left(\partial_j \lambda^i \right) \left(\lambda^i \right)^2 \partial_i u^m + \left(\partial_i \lambda^j \right) \lambda^i \, \lambda^j \, \partial_j u^m = \frac{v^m_p v^q_n N^p_{qk} \partial_i u^n \, \partial_j u^k}{\lambda^i - \lambda^j} \tag{3.101}$$

$$\left(\partial_j \lambda^i \right) \lambda^i \, \lambda^j \, \partial_i u^m + \left(\partial_i \lambda^j \right) \left(\lambda^j \right)^2 \partial_j u^m = \frac{v^m_q v^q_k N^p_{nq} \partial_i u^n \, \partial_j u^k}{\lambda^i - \lambda^j} \tag{3.102}$$

$$\left(\partial_j \lambda^i \right) \lambda^i \, \lambda^j \, \partial_i u^m + \left(\partial_i \lambda^j \right) \lambda^i \, \lambda^j \, \partial_j u^m = \frac{v^p_n v^q_k N^m_{pq} . \partial_i u^n \, \partial_j u^k}{\lambda^i - \lambda^j} \tag{3.103}$$

Now, considering the expression given by (3.100) – (3.101) – (3.102) + (3.103), we have

$$0 = \left(v^m_q v^q_p N^p_{nk} - v^m_p v^q_n N^p_{qk} - v^m_q v^q_k N^p_{nq} + v^p_n v^q_k N^m_{pq} \right) \frac{\partial_i u^n \, \partial_j u^k}{\lambda^i - \lambda^j} . \tag{3.104}$$

Given the form of the Hantjees tensor (3.72), that we recall,

$$H^m_{nk} = N^m_{pq} v^p_n v^q_k - N^p_{nq} v^m_p v^q_k - N^p_{qk} v^m_p v^q_n + N^p_{nk} v^m_q v^q_p , \tag{3.105}$$

we have $H^m_{nk} = 0$ and the demonstration of the theorem 3.2.2 is completed.

The Gibbons–Tsarev system is then formulated in terms of explicit indices as

$$\partial_i \partial_j u^m = -\frac{\partial_j \lambda^i}{\lambda^i - \lambda^j} \partial_i u^m + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} \partial_j u^m + \frac{v^m_{n,k} - v^m_{k,n}}{\lambda^i - \lambda^j} \partial_i u^n \partial_j u^k \qquad (3.106)$$

$$\partial_i \lambda^j \partial_j u^m = \frac{N^m_{pk} v^p_n - N^p_{nk} v^m_p}{(\lambda^i - \lambda^j)^2} \partial_i u^n \partial_j u^k, \qquad (3.107)$$

$$\partial_j \lambda^i \partial_i u^m = \frac{N^m_{pn} v^p_k - N^p_{kn} v^m_p}{(\lambda^i - \lambda^j)^2} \partial_i u^n \partial_j u^k, \qquad (3.108)$$

and the semi-Hamiltonian property for the characteristic speeds (3.75) is satisfied.

It is worth noticing that it is possible to build diagonalizable systems that are not semi-Hamiltonian. An explicit example is given in [60], with the hydrodynamic chain

$$u^n_t = u^{n+1}_x + p(u^1) u^n_x + u^{n-1} u^1_x. \qquad (3.109)$$

The Haantjes tensor H^i_{jk} is zero in all its components, hence the system is diagonalizable and it is possible to construct infinitely many hydrodynamic reductions, governed by the same equations valid for the Benney chain. The difference with Benney is that the Riemann invariants satisfy, in this case, the system

$$R^i_t = \left(\lambda^i(R) + p\left(u^1\right)\right) R^i_x, \qquad (3.110)$$

and the characteristic speeds do not fulfill the semi-Hamiltonian property.

Finally, we emphasize that the vanishing of the Haantjes tensor is also a sufficient condition for the integrability of the hydrodynamic chain if the spectrum of the matrix $v(u)$ is simple in the characteristic speeds, as stated in the following theorem.

Theorem 3.2.3 *The vanishing of the Haantjes tensor of a hydrodynamic chain is a necessary and sufficient condition for the existence of two-component reductions parametrized by two arbitrary functions of a single variable in the simple spectrum case.*

In the study of chains, it is worth mentioning that there exists an equivalence between chains and multi-dimensional dispersionless systems. As an example, the Benney chain is related to the dispersionless version of KP (dKP) [125, 82]. The KP hierarchy can be

written as

$$\frac{\partial z}{\partial t_n} = \{(z^n)_+ , z\} , \tag{3.111}$$

where $z = z(p,t)$ is a complex function depending on the complex variable p and the infinite set of complex parameters $t = \{t_1, t_2, \dots\}$. It is assumed to have a Laurent expansion

$$z = p + \sum_{n \geq 1} \frac{a_n(t)}{p^n} , \tag{3.112}$$

in $p \to \infty$. The $(z^n)_+$ is the polynomial part of the expansion in powers of p and the Poisson bracket is

$$\{f , g\} = \partial_p f \, \partial_x g - \partial_x f \, \partial_p g , \qquad x = t_1 . \tag{3.113}$$

The compatibility for equation (3.111) is given imposing the zero curvature condition

$$\frac{\partial (z^m)_+}{\partial t_n} - \frac{\partial (z^n)_+}{\partial t_m} + \{(z^m)_+ , (z^n)_+\} = 0 , \qquad m \neq n . \tag{3.114}$$

From (3.111) we can obtain the Benney equation for $n = 2$

$$(a_{n+1})_t + (a_{n+2})_x + n\, a_1 = 0 , \qquad t = -2\, t_2 . \tag{3.115}$$

For $n = 3$, we obtain the dKP equation

$$\left(u_t - \frac{3}{2} u\, u_x\right)_x = \frac{3}{4} u_{yy} , \qquad u = 2\, a_1 , t = t_3 , y = t_2 . \tag{3.116}$$

In section 7.4, we will use the definitions and the approaches here described to discuss the diagonalizability and the integrability of the new hydrodynamic chain emerging from the study of the Pfaff lattice in the context of the ensemble of random symmetric matrices.

Chapter 4

Nonlinear breaking of critical phenomena

One of the main aspects of the dynamics of nonlinear systems is the emergence of singularities dynamically developed as a result of a gradient catastrophe. In this chapter, we will consider the occurrence of such a phenomenon in section 4.1 and its regularisation via higher order corrections. In particular, we will address two different types of regularisation that will give rise to very different behaviours: the viscous regularisation in section 4.2 and the dispersive regularisation in section 4.3. In the context of hydrodynamic systems, viscous corrections lead to the breaking of the local Hamiltonian structure, while this is not the case for the dispersive ones. We will deal with two equations that we have already encountered in section 1.1, i.e. the Burgers' equation (1.3) and the KdV equation (1.2). Finally, in section 4.4, we will briefly discuss the approach of integrable perturbations to quasi-linear hydrodynamic systems and the universal behaviour of solutions close to critical points.

4.1   Gradient catastrophe

The prototypical nonlinear PDE is the Hopf equation

$$u_t + u\,u_x = 0\,, \tag{4.1}$$

also defined to be quasi-linear since the coefficient of the highest order derivative of the function u (D^k with $k = 1$) depends at most on u itself (D^{k-1}).

The solution to the equation (4.1) is obtained via the characteristic method. Expression (4.1) can be seen as a total derivative of $u(x,t)$ along a line with slope

$$\frac{dx}{dt} = u(x,t), \qquad \frac{du}{dt} = 0, \tag{4.2}$$

at each point of the plane (x,t). We consider the initial condition for the Cauchy problem

$$u(x,0) = f(x), \qquad x \in \mathbb{R}, \tag{4.3}$$

and hence the solution can be written as

$$x + u\,t = f^{-1}(u), \tag{4.4}$$

where $f^{-1}(u)$ is the inverse function of the initial datum $f(x)$. On the axis $(x,0)$ for $x = \xi(0)$ we have $u(x,0) = f(\xi(0))$, where $\xi(t)$ parametrises a point on the characteristic line. We denote by $F(\xi)$ the slope of the characteristic curve intersecting the axis at the point ξ, so that the solution is

$$x = \xi + F(\xi)t. \tag{4.5}$$

In this context, a characteristic curve in the space (x,t) describes the point-like propagation of the initial datum with velocity $u(x,t)$. The solution at a generic time t is given by moving each point on the initial curve $u = f(x)$ at a distance $F(\xi)t$ to the right.

Where the propagation velocity is a decreasing function, as in the case represented in figure 4.1 (a), the profile of $u(x,t)$ undergoes a steepening process, and eventually it breaks giving a multi-valued solution. The breaking occurs when the profile of the solution develops an infinite slope, i.e. $u_x \to \infty$, a so called gradient catastrophe. At the time

$$t = -\frac{1}{F'(\xi)}, \tag{4.6}$$

the breaking of the profile emerges on the characteristic where $F'(\xi) < 0$ and $|F'(\xi)|$ is a
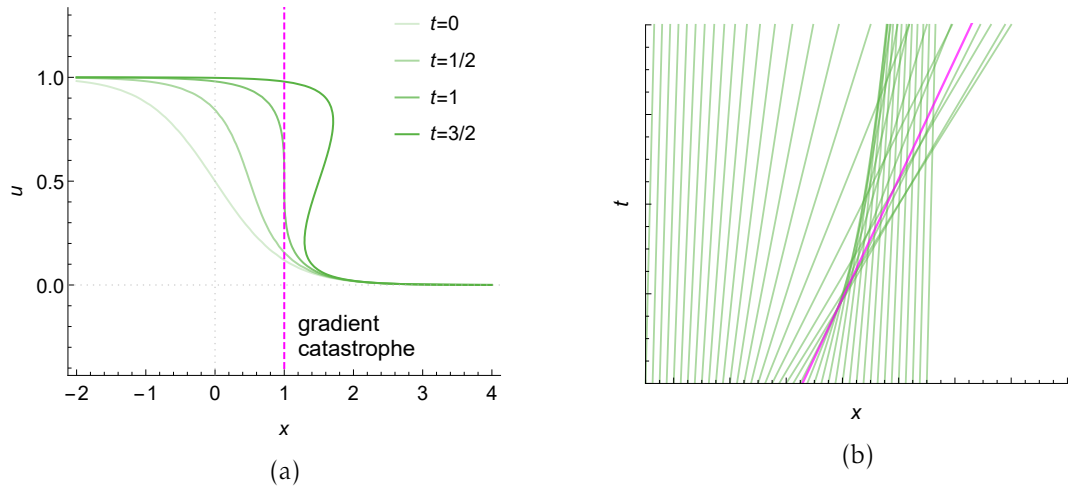
Figure 4.1: (a) The solution to the Hopf equation (4.1) is shown for the initial condition $u(x,0) = (1 - \tanh(x))/2$ evaluated at different values of time in the plane (x, u). (b) Characteristic curves for the Hopf equation (4.1) with initial condition $u(x,0) = (1 + x^2)^{-1}$ in the plane (x, t). The characteristic line corresponding to the occurrence of the profile breaking is drawn in magenta.

maximum, for $\xi = \xi_b$. In figure 4.1 (b), the family of characteristics (8.42) in the parameter ξ is shown for a specific initial condition. The region corresponding to $F'(\xi) < 0$ is the one where the characteristics converge. In the presence of an increasing initial condition the characteristics diverge after the breaking point and the emerging phenomenon is called a rarefaction wave [120].

In the following we will analyse the shock waves arising from two possible mechanisms of regularisation, i.e. viscous and dispersive. We will see how the differences in their structure and evolution reflects the necessity of a different mathematical description of the two phenomena. As pointed out in [55], their modelling represents the essence of their distinction:

▷ the viscous shock wave (figure 4.2 (a)) is described by a travelling wave solution to an ODE,

▷ the dispersive shock wave (figure 4.2 (b)) is represented by a modulated periodic train wave.
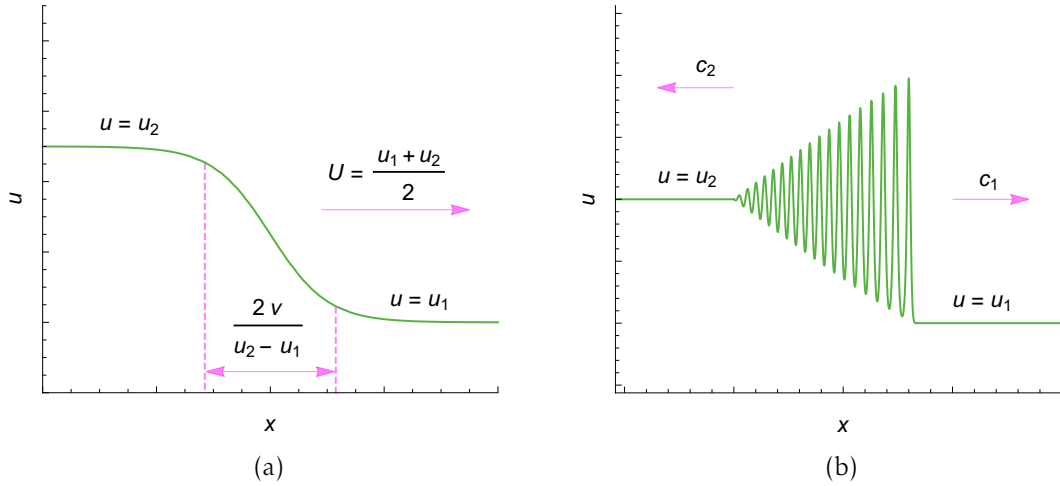
Figure 4.2: (a) The structure of a viscous shock: a smooth steady transition propagating with shock speed U and width proportional to the viscous parameter v. (b) The structure of a dispersive shock: an unsteady nonlinear wavetrain confined to an expanding region.

4.2   Viscous shock wave

The viscous shock wave is the phenomenon emerging from a dissipative regularisation of the gradient catastrophe previously described. It consists of a travelling wave solution, whose evolution is characterised by a fixed width and a single speed. The width depends on the viscous parameter v, whereas the speed is given by a balance of physical integral of motion across the shock and it is independent of the details of the shock internal structure [120].

The Burgers equation is the archetype of a viscous nonlinear integrable PDE

$$u_t + u\,u_x = v\,u_{xx}\,, \tag{4.7}$$

where $v > 0$ is the viscosity[1] parameter. It provides the viscous small amplitude approximation of the Hopf equation[2] (4.1).

As we mentioned in section 1.1, the Cole-Hopf transformation

$$u = -2v\,\partial_x \log \varphi\,, \tag{4.8}$$

---

[1] The terms viscosity, diffusion and dissipation are all used in literature to name this kind of corrections.
[2] The Hopf equation (4.1) is also known as inviscid Burgers' equation.

yields the heat equation in the new field variable φ

$$\varphi_t = \nu\,\varphi_{xx}.\tag{4.9}$$

We consider a decreasing initial condition for the Cauchy problem

$$u(x,0) = f(x).\tag{4.10}$$

The heat equation (4.9) is then solved via the Poisson formula in φ. Recalling the Cole-Hopf transformation we recover the expression for the solution to the Burgers' equation (4.7)

$$u(x,t) = \frac{1}{\int_{-\infty}^{\infty} e^{-G(\eta)/2\nu}\,d\eta} \int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-G(\eta)/2\nu}\,d\eta.\tag{4.11}$$

In the previous expression

$$G(\eta;x,t) = \frac{(x-\eta)^2}{2t} + \int_0^{\eta} F(\eta')\,d\eta',\tag{4.12}$$

where $F(\xi)$ is the function appearing in (8.42). With the assumption that there exists one solution to the equation

$$\left.\frac{\partial G}{\partial \eta}\right|_{\eta=\xi} = F(\xi) - \frac{x-\xi}{t} = 0\,,\tag{4.13}$$

the leading order for the solution ($\nu \to 0$) is obtain by the Laplace transform. Writing the solution as

$$u(x,t) = u^*(x,t) + \mathcal{O}(\nu),\tag{4.14}$$

the leading order $u^*(x,t)$ satisfies the following

$$u^*(x,t) = \frac{x-\xi(x,t)}{t} = F(\xi(x,t))\,,\qquad \tilde{F} = F^{-1}.\tag{4.15}$$

Assuming that $G(\eta)$ has a local minimum at u^* and that the function $F(\xi(x,t))$ is invertible, at least locally

$$x - u^* t = \tilde{F}(u^*).\tag{4.16}$$

Hence, in the inviscid limit $\nu \to 0$ the leading order of the solution to the Burgers' equa-

tion is given by the Hopf equation (4.1). The latter is then a good approximation of the evolution of the solution before the critical time, when the hodograph equation admits one solution.

The viscous shock wave emerges when the equation (4.13) admits multiple solutions. In the inviscid limit, the dominant behaviour is given by the value $\xi_m(x,t)$ for which $G(\xi_m)$ takes the lowest value. We thus have locally

$$u_m^*(x,t) = F(\xi_m(x,t)).\tag{4.17}$$

There exist subsets of the (x,t) plane where the equation

$$G(\xi_l(x,t)) = G(\xi_r(x,t))\tag{4.18}$$

has solution for different indices[3] l and r. Equation (4.18) represents the viscous shock trajectory, the curve representing the jump of the solution form the value that on the left is $u_l^* = F(\xi_l(x,t))$ and on the right $u_r^* = F(\xi_r(x,t))$.

Recalling (4.12), we recover the equal areas rule for F

$$\int_{\xi_l}^{\xi_r} F(\eta)\,d\eta = \frac{1}{2}\left(F(\xi_l) + F(\xi_r)\right)(\xi_r - \xi_l).\tag{4.19}$$

In particular, the viscous shock position is given by placing a discontinuity cutting the solution to the Hopf equation into two lobes of equal areas (as in figure 4.3). This is evident mapping the solution back to $t = 0$ following the characteristics [120].

It is worth noting that the viscous shock wave in the $\nu \to 0$ limit, induces the conservation of the quantity

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \text{const},\tag{4.20}$$

that remains constant also for finite values of ν. Since in this case the expression for the flux is quadratic, the so called shock condition is such that the shock velocity U is given by

$$U = \frac{1}{2}\left(F(\xi_l) + F(\xi_r)\right).\tag{4.21}$$

---

[3]Left and right are intended with respect to the position of the gradient catastrophe.
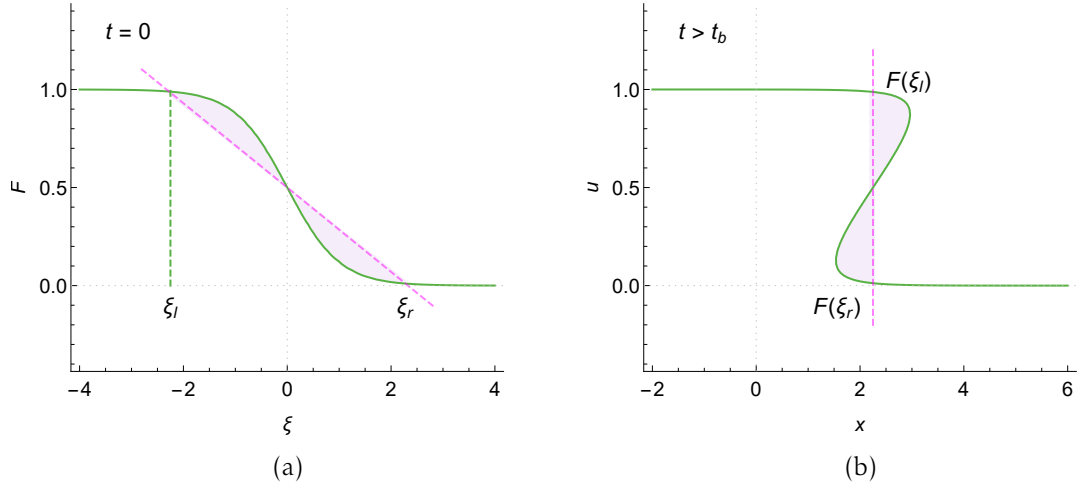
Viscous shock wave

Figure 4.3: Equal area realisation. In (a) at initial time $t = 0$ in the plane (ξ, F), in (b) after the breaking of the profile in the plane (x, u).

The dynamics of a viscous shock wave is described by the propagation of the discontinuity front with velocity given by the shock condition. The solution at the time t is given starting from the initial profile $F(\xi)$ and then translating this of a distance $F(\xi)t$ to the right, as shown in figure 4.3. The shock cuts out the parts $\xi_l < \xi < \xi_r$. The shock is entirely described by the function $F(\xi)$ considering all the chords constructed via the equal area property. In particular, the pairs $\xi = \xi_l$, $\xi = \xi_r$ corresponds to those characteristics that meet on the shock.

The problem is then described by

$$
\begin{aligned}
u_t + q(u)_x = 0, && \text{conservation law} \\
-U\,[u] + [q(u)] = 0, && \text{shock condition.}
\end{aligned}
\tag{4.22}
$$

The flux $q(u)$ is $q(u) = 1/2u^2$ for Hopf. The second expression refers to the compact notation of

$$
q(s^-, t) - q(s^+, t) = \left\{ u(s^-, t) - u(s^+, t) \right\} \dot{s}
\tag{4.23}
$$

where $s(t)$ represents the position of the shock evolving in time and s^+ and s^- represents the limits $x_l \to s^-$ and $x_r \to s^+$. The symbol $[\cdot]$ in (4.22) denotes the jump across the discontinuity and $U(t) = \dot{s}(t)$. The shock solution is a weak solution of the conservation law.

This construction based on the introduction of viscous perturbations allows us to de-

fine suitable solutions to equations of hydrodynamic type, to regularise the discontinuity due to nonlinearity. A solution $u(x,t)$ is considered admissible if there exists a sequence of solutions $u(u,t;v)$ to the Burgers' equation (4.7) such that [51]

$$u(x,t;v) \xrightarrow[v \to 0]{} u(x,t). \tag{4.24}$$

We will encounter viscous shocks in the context of the mean fields described with the formalism of the nonlinear PDEs in chapter 5.

4.3   Dispersive shock wave

Dispersive shock waves appear as a dispersive regularisation mechanism [55] of the gradient catastrophe emerging from nonlinearity. The KdV equation constitutes the paradigmatic example of a dispersion nonlinear integrable PDE

$$u_t + u\,u_x = \varepsilon^2\,u_{xxx}, \tag{4.25}$$

and it serves as the small dispersion approximation to the Hopf equation (4.1). After the occurrence of the wave breaking that we have discussed in the previous section, the solution to the equation (4.25) takes the form of a modulated locally periodic wave φ (see figure 4.2 (b)), whose form will be

$$\varphi(\vartheta) = a + b\,\mathrm{dn}^2(\vartheta), \tag{4.26}$$

where dn is a Jacobi elliptic function and ϑ will encode the modulation. In particular, at the leading edge it exhibits a solitary wave, while close to the trailing edge it transforms into linear wave packet of vanishing amplitude. The unsteady nature of the dispersive shock is manifested by the fact that it expands in time.

The modulation of the dispersive shock is obtained invoking Whitham modulation theory [119] and matched asymptotic analysis. Without going into too much detail, we briefly describe the modulation procedure, following [55]. Starting from the conservation laws associated with the original dispersive equations, the slow modulations of periodic

Dispersive shock wave

nonlinear waves are determined by averaging those conservation laws over a family of periodic travelling wave solutions. Given a n-th order nonlinear evolution equation

$$q_t = K\left(q, q_x, \ldots, q^{(n)}\right), \tag{4.27}$$

for the Whitham method, there exists a n-parameter family of periodic travelling wave solutions

$$q(x, t) = \varphi(\vartheta; \underline{u}), \qquad \text{with } \begin{cases} \underline{u} = (u_1, \ldots, u_n) \\ \vartheta = k(\underline{u}) x - \omega(\underline{u}) t. \end{cases} \tag{4.28}$$

The vector $\underline{u}$ of n components represents the parameters, φ is the phase, $k(\underline{u})$ and $\omega(\underline{u})$ are the wave number and the frequency respectively. Imposing a fixed period of 2π on φ the spatial and temporal periods are determined as

$$L(\underline{u}) = \frac{2\pi}{k(\underline{u})}, \qquad T(\underline{u}) = \frac{2\pi}{\omega(\underline{u})}. \tag{4.29}$$

The assumption for the Whitham method to be applied is the existence of at least $n-1$ conservation densities $\mathcal{P}_i[q]$ and corresponding fluxes $\mathcal{Q}_i[q]$, constituting the conservation laws

$$(\mathcal{P}_i)_t + (\mathcal{Q}_i)_x = 0, \qquad i = 1, \ldots, n-1. \tag{4.30}$$

The modulation equations are derived with the assumption of slow evolution of the parameters $\underline{u} = \underline{u}(x, t)$ both in space and time

$$|\underline{u}_x| \ll \frac{|\underline{u}|}{L}, \qquad |\underline{u}_t| \ll \frac{|\underline{u}|}{T}. \tag{4.31}$$

With the introduction of (4.28) in (4.30), we obtain the modulation equations

$$\left(\overline{\mathcal{P}_i[\varphi]}\right)_t + \left(\overline{\mathcal{Q}_i[\varphi]}\right)_t = 0, \qquad i = 1, \ldots, n-1, \tag{4.32}$$

where the averaged expressions are given by

$$\overline{\mathcal{F}}[\varphi] = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}[\varphi(\vartheta; \underline{u})] \, d\vartheta. \tag{4.33}$$

To completely reconstruct the modulated wave, we need to consider the modulated wave number $\vartheta_x = k(\underline{u})$ and frequency $\vartheta_t = -\omega(\underline{u})$. The compatibility condition brings the conservation of waves

$$\vartheta_{xt} = \vartheta_{tx} \qquad \Longrightarrow \qquad k(\underline{u})_t + \omega(\underline{u})_x = 0. \tag{4.34}$$

The Whitham equations (4.32) and (4.34) are dispersionless and they can be represented as a system of a hydrodynamic equations

$$\underline{u}_t + A(\underline{u})\,\underline{u}_x = 0, \tag{4.35}$$

where the matrix $A(u)$ encodes the information about the nonlinearity and dispersion of the original system.

In the case of the KdV equation (4.25), the modulation requires the introduction of three parameters, the amplitude a, the wave number k, and the average of the wave $\overline{\varphi}$. Whitham realised that the modulated system for KdV can be written in terms of Riemann invariants $R_1 \leq R_2 \leq R_3$ and relative characteristic speeds, that we have defined in section 3.1. The modulated parameters may be expressed in terms of the Riemann invariants as [55]

$$
\begin{aligned}
a &= 2\,(R_2 - R_1) \\
k &= \frac{\pi\sqrt{R_3 - R_1}}{\sqrt{6}\,K(m)}, \qquad m = \frac{R_2 - R_1}{R_3 - R_1} \\
\overline{\varphi} &= R_1 + R_2 - R_3 + 2\,(R_3 - R_1)\,\frac{E(m)}{K(m)},
\end{aligned}
\tag{4.36}
$$

where $E(m)$ and $K(m)$ are complete elliptic integral of the first and second kind respectively

$$K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - m\sin^2(z)}}\,dz, \qquad E(m) = \int_0^{\pi/2} \sqrt{1 - m^2\sin^2(z)}\,dz. \tag{4.37}$$

The periodic wave can be expressed as

$$\varphi(\vartheta) = R_1 + R_2 - R_3 + 2\left(R_3 - R_1\right) \operatorname{dn}^2\left(\frac{K(m)}{\pi}\,\vartheta; m\right), \qquad (4.38)$$

where dn is the so called delta amplitude, a Jacobi elliptic function, as anticipated above. In the limit $m \to 1$, the wave takes the form of a soliton

$$\varphi = \overline{\varphi} + a_s \operatorname{sech}^2\left(\sqrt{\frac{a_s}{12}}\,(x - v_s\,t - \vartheta_0)\right), \qquad v_s = \overline{\varphi} + \frac{a_s}{3}. \qquad (4.39)$$

In the limit $m \to 0$, the solution becomes a vanishing harmonic wave

$$\varphi = \overline{\varphi} + \frac{a_h}{2}\left(\cos\left(k\,x - \omega_0\,t\right) - 1\right) + \mathcal{O}\left(a_h^2\right), \qquad \omega_0 = \overline{\varphi}\,k - k^3. \qquad (4.40)$$

The Whitham method reduces the complexity of the problem, producing a nonlinear modulation system of quasi-linear hydrodynamic form (4.35) with free boundary for the leading and trailing edges of the dispersive shock. The boundary conditions are given by matching the solution of the mean dispersive shock with the dispersionless external solution along double characteristics to the modulation system. In [69], this approach is followed for KdV and the dispersive shock wave arises as a rarefaction wave for the Whitham system.

In figure 4.4, two dispersive shock waves are shown, arising from different initial conditions. In (a), the "Martini glass" shape is obtained for a Riemann problem for KdV, as it was considered in [69]. In (b), the "Bordeaux glass" is produced for KdV in correspondence of a cubic wave breaking, in this sense the latter can be seen as an universal mechanism of dispersive regularisation [55].

In the following, in section 6.4, we will observe the emerging of a structure similar to the one shown in figure 4.4 (b) in the context of the Hermitian matrix ensemble.

4.4   Universality

The Burgers' and the KdV equations represent the universal asymptotic regularisation mechanisms, for viscous and dispersive corrections respectively. An extension to non-
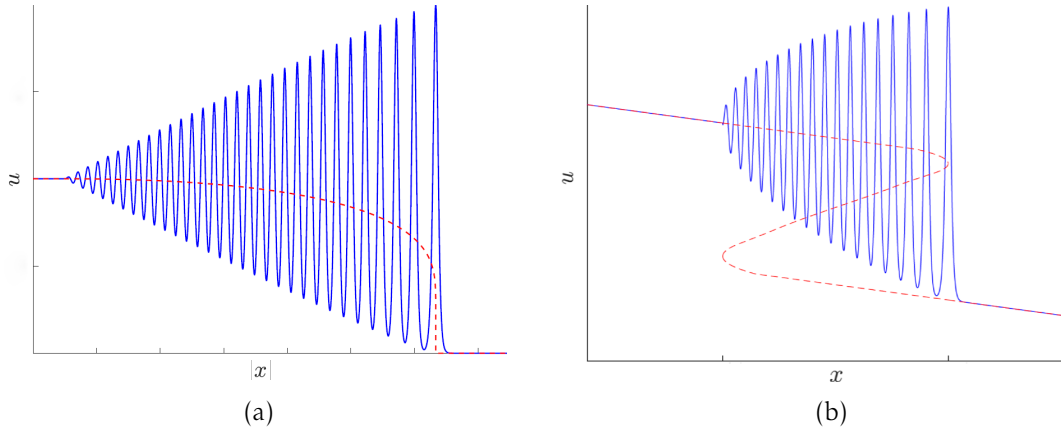
Figure 4.4: (Adapted from [55] with authors' permission.) (a) In blue it is shown the dispersive shock wave for KdV-type equations in a Riemann problem. The dashed red line represents the mean value of the wave. (b) In blue the dispersive shock wave emerging after the cubic wave breaking for KdV-type equations. The dashed red line represents the modulation solution in terms of the three Riemann invariants.

integrable systems introducing the concept of approximate integrability up to a finite order in the perturbation modelled via a small parameter was introduced in [48].

In $1+1$ dimension, the perturbation system is given by

$$u_t + a(u)\, u_x + \varepsilon\Big[b_1(u)\, u_{xx} + b_2(u)\, u_x^2\Big] + \varepsilon^2\Big[b_3(u)\, u_{xxx} + b_4(u)\, u_x\, u_{xx} + b_5(u)\, u_x^3\Big] + \cdots = 0, \quad (4.41)$$

where the unperturbed system is the nonlinear hyperbolic system

$$u_t + a(u)\, u_x = 0. \tag{4.42}$$

This system admits a Hamiltonian description as

$$u_t + \{u(x), H_0\} = u_t + \partial_x \frac{\delta H_0}{\delta u(x)} = 0, \tag{4.43}$$

with the Poisson brackets

$$\{u(x), u(y)\} = \delta'(x - y). \tag{4.44}$$

The solutions to the perturbed equations in (4.41) are considered up to the Miura transformation

$$u \mapsto u + \sum_{k \geq 1} \varepsilon^k F_k\Big(u; u_x, \ldots, u^{(k)}\Big), \tag{4.45}$$

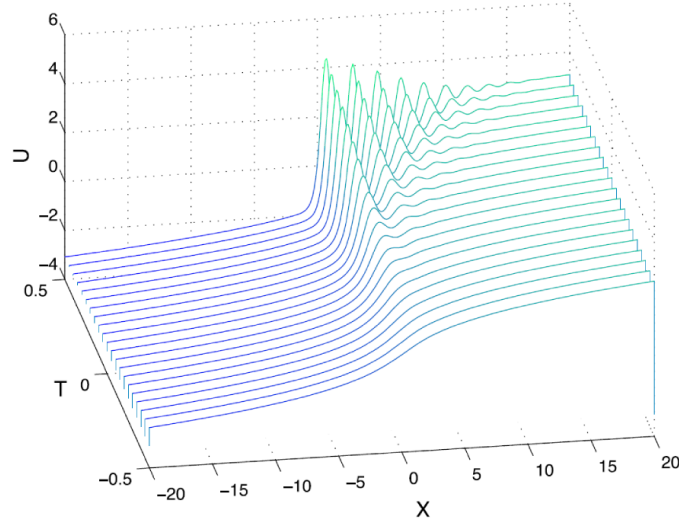Figure 4.5: In a $(2+1)$-dimensional hyperbolic system, the asymptotic universal behaviour of the function $U(X,T)$ is shown. The function specifies the asymptotics of the Riemann invariants and it is a special solution to P_I^2 (figure taken from [54]).

with $F_k\left(u; u_x, \ldots, u^{(k)}\right)$ polynomial in the derivatives of u of degree k. Any Hamiltonian perturbation of the equation (4.42) can be reduced to the form

$$u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0, \qquad H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots. \tag{4.46}$$

We report part of the conjecture formulated in [54]. The main idea underlying the conjecture is the universality of the asymptotic approximation at the leading order.

Conjecture 4.4.1 *The solution to the generic system* (4.43) *with generic ε-independent smooth initial data near a point of cusp catastrophe of the unperturbed hyperbolic system* (4.42) *is described in the limit $\varepsilon \to 0$ by a particular solution to the P_I^2 equation.*

Hence, it is conjectured that the critical behaviour close to the gradient catastrophe is independent of the choice of the initial data and the exact form of the Hamiltonian perturbation. In particular, the solution for (4.41) with a generic ε-independent smooth initial data near a point of cusp catastrophe of the unperturbed hyperbolic system (4.42) is described in the limit $\varepsilon \to 0$ by a particular solution to the P_I^2 equation (Painlevé) [54]. In figure 4.5, it is shown the profile for the solution $U(X,T)$ entering in the asymptotics

for the Riemann invariants and being the solution to the P_I^2, that is

$$X = U\,T - \left[\frac{1}{6}U^3 + \frac{1}{24}\left(U_X^2 + 2U\,U_{XX}\right) + \frac{1}{240}U_{XXXX}\right]. \tag{4.47}$$

The function U satisfies also the KdV equation, in the form

$$U_T + U\,U_X + \frac{1}{12}U_{XXX} = 0. \tag{4.48}$$

The critical behaviour for dispersive hydrodynamic systems has been studied in [53] for scalar hyperbolic systems and generalised in [54] for $(2+1)$-dimensional hyperbolic and elliptic systems. Similar results concerning the universal behaviour of solutions close to critical points hold in generalised viscous systems. These have been explored in [73] and expanded in [51] and [18].

Part II

Case studies

Chapter 5

Mean-field models

In this chapter, we describe the method of differential identities [41, 22] and its first applications to problems in the realm of Statistical Mechanics. The main underlying idea is that the phase transitions typically emerging in thermodynamic systems can be described in terms of nonlinear waves. In particular, we will see how the nonlinear PDEs formalism provides the natural framework to obtain and describe the equations of state. In section 5.1, after some preliminary observations, we will present the connection established in [96] between the main features of thermodynamics and those of nonlinear PDEs theory. Then the method of differential identities will be explicitly applied to treat the Curie-Weiss model in section 5.2, studying the critical behaviour of the order parameter with the formalism developed in chapter 4. We will see how the model is intrinsically related to the Burgers' equation and how viscous shock waves emerge and can be treated in this context.

The approach has been successfully applied to model mean-field theories in a broad class of system [15, 67, 40, 90, 28], as we will see in section 5.3.

5.1   Differential identities and Statistical Mechanics

A novel approach to solve problems historically of competence of Statistical Mechanics relies on the theory of nonlinear PDEs via the method of differential identities [96]. A general class of thermodynamic systems can be effectively described by the theory of nonlinear integrable conservation laws. This approach leads to the description of first or-
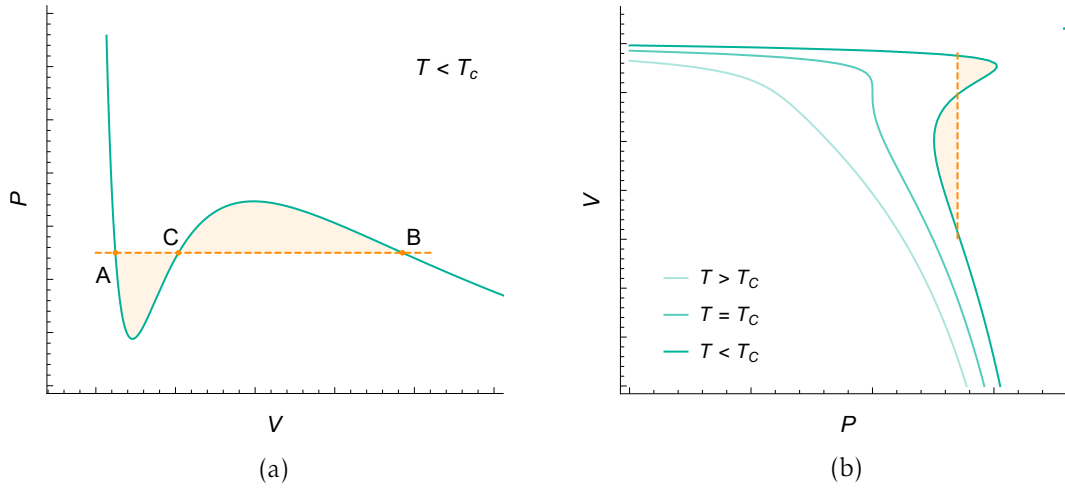
Figure 5.1: (a) The real gas isotherm is shown for a temperature $T < T_c$ with the solid line. The points A and B define the range in the volume V where the phase transition occurs, for a constant value of the pressure P. The dashed line represents the metastable state predicted by the van der Waals model. (b) Van der Waals curves as nonlinear wave solution to a hyperbolic PDE. Beyond the critical temperature the solution is multi-valued and the shock wave emerges.

der phase transitions in terms of shock waves, interpreted as solutions of nonlinear PDEs encoding the whole information on the evolution of the system with respect to some appropriate tunable parameters. In general, it establishes a correspondence between phase transition phenomenology and shock wave dynamics.

The first observation in this direction is reported in [41], where a new perspective is suggested to interpret the occurrence of phase transitions in the van der Waals model. The latter represents the simplest mathematical model providing the description of a phase transitions in a thermodynamic system. For a thermodynamic system in equilibrium, the energy balance equation takes the form

$$dE = T\,dS - P\,dV\,,\tag{5.1}$$

with E the total energy, T the temperature, P the pressure, V the volume and S the entropy, determining the state of the system. The equation of state of the system is given by [88]

$$P + \frac{\partial F}{\partial V}(V,T) = 0\,,\qquad \text{with } F = E(S,V) - T\,S\,.\tag{5.2}$$

The equation of state can be interpreted as a stationary point of the Gibbs potential as a

function of V

$$\Phi = F + PV. \tag{5.3}$$

The system is in a state of stable equilibrium when the Gibbs potential has a minimum for both P and T constant. The existence of points at which the second derivative of Φ with respect to V vanish identifies the phase transition.

The classical example of a phase transition is the change of state of matter from gas to liquid. In figure 5.1 (a), an isothermal curve below the critical temperature is shown. The phase transition takes place between the points A and B. The shape of the solid curve between A and B is given by the van der Waals model and it corresponds to a non-observable metastable state. The correct behaviour of the isotherm is recovered via the Maxwell principle or the equal areas rule. The constant value of the pressure at a phase transition is that for which the area of the lobe AC is the same of that for the lobe CB, in figure 5.1 (a).

The description provided by the equal areas rule is similar to what we have seen in section 4.2 in the context of the viscous shock wave. This is even more evident if we consider the isothermal curves displayed in figure 5.1 (b), after an interchange of the variables P and V and a reflection. At the critical temperature T_c, the gradient catastrophe occurs, then the solution becomes multi-valued. Hence, the behaviour of the isotherm provided by the van der Waals model for V as a function of P can be interpreted as the solution to a hyperbolic PDE [41]

$$\frac{\partial V}{\partial T} = \varphi(V) \frac{\partial V}{\partial P}. \tag{5.4}$$

In section 4.1, we have studied the solution to this equation for $\varphi(V) = -V$, i.e. the Hopf equation. The behaviour of the solution is such that after the gradient catastrophe develops a discontinuity and then it exists in a weak sense only. The position of the shock is obtained via the fitting procedure described in section 4.2, for which the chord cuts off two lobes of equal area. In particular, in [41], it is shown that the function $V(T,P)$ is solution to an equation of the form (5.4) under the assumption that the entropy is a separable function, i.e. it can be decomposed into the sum of a function of V and a function of T. Then the hyperbolic equation (5.4) is equivalent to the balance equation (5.1).

Thermodynamics		Nonlinear conservation laws
Isothermal / isobaric curves	⟷	Nonlinear waves
Critical point	⟷	Gradient catastrophe
Phase transition	⟷	Shock
Maxwell principle	⟷	Equal areas rule
Clapeyron equation	⟷	Shock condition
Triple point	⟷	Shock confluence
Universality	⟷	Universality

Table 5.1: Correspondence between the main features in the framework of thermodynamics and nonlinear conservation laws [96].

The method of characteristics provides an implicit solution for the equation (5.4) and corresponds to the equation of state of the thermodynamic system.

In [96], this framework is expanded and a precise correspondence between phase transition phenomenology and shock waves dynamics is given (see table 5.1). In particular, it is emphasised the connection between universality in the context of the critical behaviour of wave breaking (see section 4.4) and the notion of universality in thermodynamics. The method of differential identities leads to determine the equation of state via a direct integration of the Maxwell's relations with the above mentioned assumption on the entropy, rather than using the ansatz on the asymptotic expansion of the Gibbs potential or its scaling properties.

In [22], it is shown how the approach here described leads to construct the partition function for a finite size system of n interacting particles. Starting from a suitable equation of state defined outside the critical region, the associated partition function is well defined in the whole space of thermodynamic variables and conceals the equal areas rule. The model consists in a fluid of n particles of mass m

$$H_n = \sum_{i=1}^{n} \frac{\vec{p}_i^{\,2}}{2m} - \frac{1}{2} \sum_{i,j=1}^{n} \psi(\vec{r}_i, \vec{r}_j) + P\,v(\vec{r}_1, \dots, \vec{r}_n), \tag{5.5}$$

where $\vec{p}_i$ is the momentum of the i-th particle, $\psi(\vec{r}_i, \vec{r}_j)$ a potential shaping the two-body interaction, $P > 0$ a mean-field coupling constant, and $v(\vec{r}_1, \dots, \vec{r}_n)$ the minimum volume for a configuration $\{\vec{r}_1, \dots, \vec{r}_n\}$. The partition function for the canonical ensemble can be

written as

$$Z = \int e^{-\beta H_n}\, d^n\, \vec{p_i}\, d^n\, \vec{r_i} = \int_b^\infty e^{n(xv+t\,a/v+\ln(v-b))}\, dv\,,\tag{5.6}$$

with $t = \beta/n$, $x = -P\beta/n$, given that the expectation value of the volume $\langle v \rangle$ satisfies the van der Waals equation

$$\left(P + \frac{a}{\langle v \rangle^2}\right)\left(\langle v \rangle - b\right) = n\,R\,T\,,\tag{5.7}$$

outside the critical region. Then the partition function will be solution to the Klein–Gordon equation

$$\frac{\partial^2 Z}{\partial x\, \partial t} = n^2\, a\, Z\,,\tag{5.8}$$

and hence $\langle v \rangle$ satisfies the nonlinear viscous conservation law

$$\frac{\partial \langle v \rangle}{\partial t} = \frac{\partial}{\partial x}\left(\frac{a}{\langle v \rangle} + \frac{1}{n}\frac{\partial \ln\langle v \rangle}{\partial t}\right).\tag{5.9}$$

Here the underlying assumption is that, for any point in the space of parameters (x, t), different configurations occupying the same volume v appear with the same probability density and that the logarithm of the probability density is linear in x and t.

The general character of the assumptions considered above makes the approach so developed applicable to a broad class of mean-field theories.

5.2 Differential identities for Curie-Weiss model

We apply the method of differential identities to one of the classical example of mean-field theory, the Curie-Weiss model, as presented in [97]. We will see how the latter is connected to with the Burgers' equation[1], that we have studied in section 4.2.

We will start considering the interaction that models the physical system with a finite number n of components and we will identify the order parameters in the thermodynamic limit, in the limit $n \to \infty$. Then we will introduce suitable differential identities satisfied by the order parameters and valid for finite n. We will define a reasonable initial datum and provide finite n solutions. Taking the thermodynamic limit of the equations $n \to \infty$, we will derive conservation laws in form of hyperbolic systems for the order pa-

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[1]In [34], the interpretation of the mean-field theory in terms of the the Burgers' equation was explicitly given.

rameters. The solutions to this hyperbolic system will represent the equations of state for the model. As we have mentioned above, the shock trajectories will identify the phase diagrams for the order parameters of the system.

The Curie-Weiss model is introduced as the mean-field theory for the Ising model. We consider the Hamiltonian for a system of n spins $\sigma_i = \pm 1$, with $i \in \{1,\ldots,n\}$ interacting with external scalar field $h \in \mathbb{R}$ and with a uniform coupling matrix $J_{ij} = J > 0$

$$H_n(\{\sigma\};J,h) = -\frac{J}{n}\sum_{i<j}\sigma_i\sigma_j - h\sum_i\sigma_i = -\frac{J}{2n}\sum_{i,j}\sigma_i\sigma_j - h\sum_i\sigma_i. \qquad (5.10)$$

The sum refers to a given spin configuration $\{\sigma\}$ and the corresponding partition function is obtained considering the Gibbs distribution

$$Z_n(\beta,J,h) = \sum_{\{\sigma\}}e^{-\beta H_n(\{\sigma\};J,h)}, \qquad (5.11)$$

with $\beta = 1/T$ and T the temperature. From the partition function, the free energy is

$$f_n(\beta,J,h) = -\frac{1}{\beta}\alpha_n(\beta,J,h), \qquad \alpha_n(\beta,J,h) = \frac{1}{n}\ln Z_n(\beta,J,h). \qquad (5.12)$$

In the following we will call α_n the free energy of the system, even though the physical one is f_n. The order parameter of the theory is the magnetisation of the system $m(\sigma)$, determined in the thermodynamic limit $n \to \infty$ for a specific configuration

$$m(\sigma) = \lim_{n\to\infty} m_n(\sigma) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sigma_i. \qquad (5.13)$$

The expected value is defined in terms of the partition function as

$$\langle m \rangle = \frac{1}{Z_n}\sum_{\{\sigma\}}m(\sigma)e^{-\beta H_n}. \qquad (5.14)$$

We now look for the differential identities for finite n [97].

We introduce the variables

$$t = J\beta, \qquad x = h\beta, \tag{5.15}$$

and the partition function written in terms of $m_n(\sigma)$ takes the form

$$Z_n(x,t) = \sum_{\{\sigma\}} e^{n\left(x m_n + \frac{t}{2} m_n^2\right)}. \tag{5.16}$$

Therefore, the partition function satisfies the differential identity

$$\frac{\partial Z_n}{\partial t} = \frac{1}{2n} \frac{\partial^2 Z_n}{\partial x^2}, \tag{5.17}$$

i.e. the partition function for the Curie-Weiss model is a solution to the heat equation.

The initial condition is constructed by considering $t = 0$, hence turning off the two-body interaction

$$Z_n(x,0) = \sum_{\{\sigma\}} e^{n x m_n} = (2 \cosh(x))^n. \tag{5.18}$$

Given the expression (5.17), the free energy α_n satisfies the following equation

$$\frac{\partial \alpha_n}{\partial t} = \frac{1}{2} \left(\frac{\partial \alpha_n}{\partial x} \right)^2 + \nu \frac{\partial^2 \alpha_n}{\partial x^2}, \qquad \nu = \frac{1}{2n}. \tag{5.19}$$

that resembles the Burgers' equation (4.7). The corresponding initial datum is given by $Z_n(x,0)$

$$\alpha_n(x,0) = \ln 2 + \ln \cosh(x). \tag{5.20}$$

The derivatives of the free energy are related to the statistical moments of the order parameters. In particular, we have

$$\begin{aligned} \frac{\partial \alpha_n}{\partial x} &= \langle m_n \rangle \\ \frac{\partial^2 \alpha_n}{\partial x^2} &= n\left(\langle m_n^2 \rangle - \langle m_n \rangle^2 \right) = \text{var}\,(m_n)\,. \end{aligned} \tag{5.21}$$

Differentiating with respect to x the equation (5.19), we get

$$\frac{\partial \langle m_n \rangle}{\partial t} = \langle m_n \rangle \frac{\partial \langle m_n \rangle}{\partial x} + \nu \frac{\partial^2 \langle m_n \rangle}{\partial x^2}, \tag{5.22}$$

hence, $\langle m_n \rangle$ satisfies the Burgers' equation. The corresponding initial datum is

$$\langle m_n(x, 0) \rangle = \tanh(x). \tag{5.23}$$

Then, in the thermodynamic limit, we have for any point where

$$\lim_{n \to \infty} \frac{\partial \langle m_n \rangle}{\partial x} < \infty \qquad \implies \qquad \lim_{n \to \infty} \text{var}(m_n) = 0. \tag{5.24}$$

Therefore, there exists a suitable region in the plane (x, t) for which the viscous term can be neglected in the thermodynamic limit and the order parameter satisfies the Hopf equation

$$\frac{\partial \langle m(\sigma) \rangle}{\partial t} = \langle m(\sigma) \rangle \frac{\partial \langle m(\sigma) \rangle}{\partial x}. \tag{5.25}$$

Following the approach developed in section 4.1, we can consider the method of the characteristics and the equation of state takes the form

$$x + \langle m \rangle t = \text{arctanh}(\langle m \rangle). \tag{5.26}$$

As we have already seen, the solution to the Hopf equation, due to nonlinearity and in presence of a decreasing initial datum, develops a gradient catastrophe at a finite time[2], as in equation (4.6). We have

$$\frac{\partial \langle m \rangle}{\partial x} = \frac{1 - \langle m \rangle^2}{1 + t(\langle m \rangle^2 - 1)} = \infty \qquad \implies \qquad t = \frac{1}{1 - \langle m \rangle^2}. \tag{5.27}$$

The minimum time for which the gradient catastrophe arises gives the critical value for the order parameter

$$\langle m \rangle_{\text{c}} = 0. \tag{5.28}$$

The critical point, where the phase transition occurs, is identified by the coordinates

$$t_{\text{c}} = 1, \qquad x_{\text{c}} = 0, \qquad \langle m \rangle_{\text{c}} = 0. \tag{5.29}$$

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[2]The variable that mimics the time in the hyperbolic equation is related to the temperature T, for a fixed value for the coupling constant J. Thus, the gradient catastrophe occurs for a finite value of the temperature.
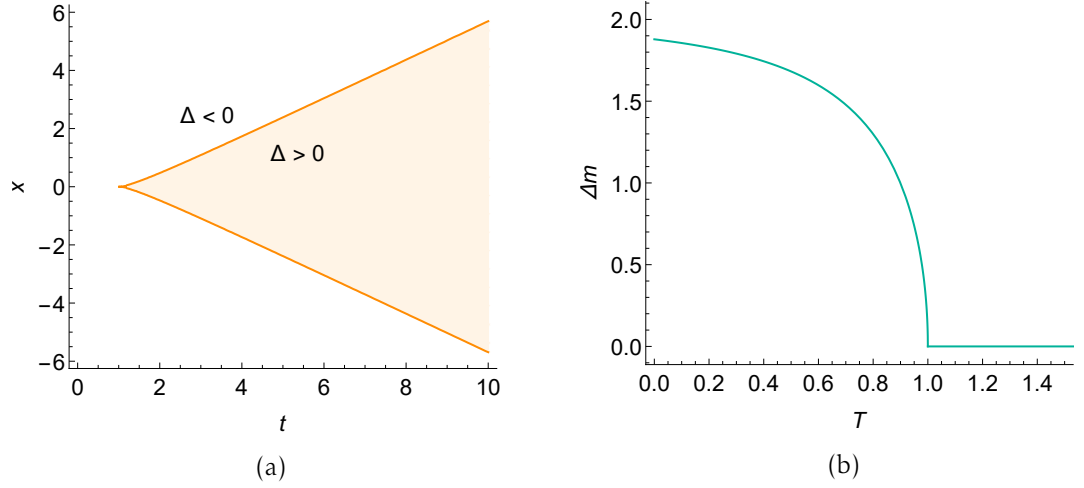
(a)

(b)

Figure 5.2: (a) The solid lines represent the set $\Delta = 0$ sector, splitting the parameter space in two regions. The $\Delta > 0$ region (in orange) corresponds to a multi-valued solution for the magnetisation. In the $\Delta < 0$ region, the solution is single-valued. (b) The shock jump for the magnetisation is depicted for $T \simeq J = 1$.

We will now study the shock structure in the proximity to the critical point, as in section 4.2. We introduce the function

$$G(x,t;\eta) = -\ln \cosh(\eta) + \frac{(x-\eta)^2}{2t}. \tag{5.30}$$

The equation of state (5.26) is recovered for

$$\left.\frac{\partial G}{\partial \eta}\right|_{\eta=\text{arctanh}(x)} = 0. \tag{5.31}$$

The expansion $\eta \to 0$ in the (5.31) gives[3]

$$\frac{1}{3} t \eta^3 - (t-1)\eta - x = 0. \tag{5.32}$$

The discriminant Δ of the expression (5.32) identifies the regions in the space of parameters for which the solution is either multi-valued ($\Delta > 0$) or single-valued ($\Delta < 0$), as shown in figure 5.2 (a). The confluence of the two lines with equation $\Delta = 0$ corresponds to the critical point.

We call the solutions in the multi-valued region $\xi_1(x,t)$, $\xi_2(x,t)$, $\xi_3(x,t)$. The order

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[3]We consider the first term in (5.30) $f(\eta) = -\ln \cosh(\eta)$. Its derivative is $f'(\eta) = -\tanh(\eta)$ and its expansion for $\eta \to 0$ gives $f'(\eta) = -\eta + \eta^3/3 + \mathcal{O}(\eta^5)$.

parameter is given by

$$\langle m \rangle = -\frac{x - \xi^*(x,t)}{t}, \tag{5.33}$$

where ξ^* is the zero such that the function $G(x,t,\xi^*)$ reaches the minimum value. The position of the shock is determined by the equal area condition (4.18)

$$G(x,t,\xi_i(x,t)) = G(x,t,\xi_j(x,t)), \qquad \xi_i \neq \xi_j, \qquad i,j \in \{1,2,3\}. \tag{5.34}$$

This is explicitly represented by the relation

$$R_{ij} = t(\xi_i + \xi_j)\left(\xi_i^2 + \xi_j^2 - 6\right) + 6(2x - \xi_i - \xi_j) = 0. \tag{5.35}$$

Close to critical point, $R_{12} = 0$ is satisfied for $x,t \in \mathbb{R}$ and defines the trajectory of the shock.

Finally, using the roots ξ_1, ξ_2, ξ_3 it is possible to evaluate the jump Δm developed by the order parameter and shown in figure 5.2 (b)

$$\Delta m = \begin{cases} 2 \tanh \sqrt{3\left(1 - \frac{T}{J}\right)} & 0 \leq T < J \\ 0 & T \geq J, \end{cases} \tag{5.36}$$

going back to the original coupling constants.

The method of differential identities represents then a suitable tool to describe the main features emerging in the Curie-Weiss model, taken as an example of a mean-field theory.

5.3 Viscous regularisation in mean-field theories

The approach described above has been successfully applied in the study of different statistical systems that admit a description via a mean-field theory. Starting from the observation in [41], the theory of the van der Waals model with the method of differential identities has been extended in [22], where the possibility to produce solutions at finite size is emphasised. In [67], a multi-parameter extension of the van der Waals theory is given, introducing two more deformation parameters.

The procedure is effective in tackling problems in different fields. In [15], the model of information processing in biochemical reaction is studied. Here, the approach is invoked to provide explicit finite-size solutions in the context of the biochemical reactions. The system is modelled considering the underlying similarities among the collective behaviours in chemical kinetics (biology), spin models (statistical mechanics), and operational amplifiers (cybernetics).

The extension to systems of higher dimension in terms of order parameters is proposed in [40] with the study of liquid crystals, a liquid substance possessing a microstructure that is the result of their molecular anisotropy. Here, phase transitions are defined in terms of the spatial orientation of the crystals. The order parameters are the so called molecular directors. The emergence of a uniaxial phase (the directors point on average in the same direction) and a biaxial phase (simultatneous orientation along two orthogonal axis) is studied. The shock wave corresponding to phase transitions is a shock wave in $(2 + 1)$-dimensions in the space of parameters.

In [90], the Potts model is considered, i.e. the extension of the Curie-Weiss model for $q > 2$ admissible values for spins. The model with $q = 3$ is analysed with the introduction of two order parameters, whose behaviour is studied via the method of differential identities.

More recently, in [28] the procedure has been applied to the theory of exponential random networks. In particular, the so called p-star model is considered, for which the partition function satisfies the heat hierarchy. The order parameter of the theory is defined to be the connectance, obtained as a solution to a nonlinear viscous PDE.

All these examples refer to mean-field theories, that can be modelled via the method of differential identities. At the leading order of the order parameter of the theory we always find the Hopf equation, whose solution after a suitable choice of the initial datum develops a gradient catastrophe. The emerging shock wave undergoes then a viscous regularisation and its trajectory in the space of parameters models the critical behaviour of the system.

In the following, we will illustrate the Hermitian matrix ensemble and the symmetric matrix ensemble. In the case of the Hermitian ensemble, the Hopf equation will emerge again but with a regularisation mechanism consisting in the formation of a dispersive

shock wave.

Chapter 6

Hermitian Matrix Ensemble and dispersive shocks

In this chapter, we analyse the Hermitian matrix ensemble $\mathcal{H}_n$, following the description given by Adler and van Moerbeke in [6, 8, 117] using the tools described in section 2.1. In section 6.1, we will construct the associated discrete integrable structure, the Toda lattice, and observe the emergence of the KP hierarchy for the Toda τ-function, proportional to the partition function of the ensemble $\mathcal{H}_n$. Then we will focus on the study of a particular reduction of the Toda lattice, following [23], called Volterra lattice. This reduction represents the structure arising by selecting the even coupling constants only from the Toda lattice, as we will see in section 6.2. In section 6.3, we will consider the the thermodynamic limit and we will study how the evolution in different even times of the lattice fields takes the form of the Hopf hierarchy. In section 6.4, we will restrict our analysis to the M^6 model, for which the order parameter of the theory expressed in terms of τ-functions undergoes a phase transition in the space of coupling constants near the critical point. Finally, we will see how the singularity can be resolved in terms of a multi-dimensional dispersive shock of the order parameter, leading to the emergence of the already observed chaotic behaviours [78, 109].

6.1 Hermitian Matrix Ensemble

6.1.1 $\mathcal{H}_n$ as a tangent space and partition function

According to the scheme described in section 2.2, we consider the non-compact symmetric space $\mathcal{M} = G/K$, with $G = SL(n,\mathbb{C})$ and the involution map defined on K as

$$\sigma(g) = \left(\bar{g}^\top\right)^{-1}, \tag{6.1}$$

so that the subgroup K is given by

$$K = \{g \in SL(n,\mathbb{C}) \mid \sigma(g) = g\} = \{g \in SL(n,\mathbb{C}) \mid g^{-1} = \bar{g}^\top\} = SU(n). \tag{6.2}$$

The symmetric space $\mathcal{M}$ can be expressed as

$$SL(n,\mathbb{C})/SU(n) \cong \{g\,\bar{g}^\top \mid g \in SL(n,\mathbb{C})\}$$
$$= \{\text{positive definite matrices with det} = 1\}. \tag{6.3}$$

The involution map σ induces the map σ_* in the corresponding algebra, for which $\sigma_*(A) = -\bar{A}^\top$ and the subalgebra $\mathfrak{t}$ is hence $\mathfrak{su}(n)$, consisting of $n \times n$ traceless skew-Hermitian matrices. The vector space $\mathfrak{p}$ tangent to $\mathcal{M}$ at the identity is given by the space of $n \times n$ Hermitian matrices $\mathcal{H}_n$, where $\sigma_*(A) = \bar{A}^\top$. The algebra decomposition is then given by

$$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{t} \oplus \mathfrak{p} = \mathfrak{su}(n) \oplus \mathcal{H}_n. \tag{6.4}$$

For any matrix $M \in \mathcal{H}_n$, the diagonal real elements M_{ii} and the real and imaginary part of non-diagonal elements $\operatorname{Re} M_{ij}$, $\operatorname{Im} M_{ij}$ with $1 \le i < j \le n$ are free variables. The Haar measure on $M \in \mathcal{H}_n$ reads

$$dM := \prod_{i=1}^n dM_{ii} \prod_{1 \le i < j \le n} d\operatorname{Re} M_{ij}\, d\operatorname{Im} M_{ij}. \tag{6.5}$$

A maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p} = \mathcal{H}_n$ is given by the subset of diagonal matrices $z = \operatorname{diag}(z_1, z_2, \ldots, z_n)$, where z_i with $1 \le i \le n$ are eigenvalues. In particular, since the matrices are Hermitian, they can be diagonalised through the action of a unitary operator

and any $M \in \mathcal{H}_n$ can be expressed as

$$M = U z U^{-1}, \qquad U \in K = SU(n). \tag{6.6}$$

The unitary operator can be expressed via the exponential map as $U = e^A$, with $A \in \mathfrak{t} = \mathfrak{su}(n)$, a traceless skew-Hermitian matrix ($\bar{A}^\top = -A$). So, A takes the form

$$A = \sum_{1 \leq k < l \leq n} \left(a_{kl} \left(e_{kl} - e_{lk} \right) + i \, b_{kl} \left(e_{kl} + e_{lk} \right) \right), \tag{6.7}$$

with e_{kl} is the $n \times n$ sparse matrix with $(k, l) = 1$ the only non-zero element and $a_{kl}, b_{kl} \in \mathbb{R}$. From (6.6) and considering a small A, we have

$$dM = d \left(e^A z e^{-A} \right) = d \left(z + [A, z] + \dots \right) \tag{6.8}$$

To evaluate the commutator $[A, z]$, we first consider that both the matrices $e_{kl} - e_{lk}$ and $i \left(e_{kl} + e_{lk} \right)$ are elements of $\mathfrak{t} = \mathfrak{su}(n)$. In addition, with $z \in \mathfrak{p}$ and recalling the Lie bracket (2.12), we have

$$[e_{kl} - e_{lk}, z] = (z_l - z_k)(e_{kl} + e_{lk}) \in \mathfrak{p} = \mathcal{H}_n$$
$$[i \left(e_{kl} + e_{lk} \right), z] = i (z_l - z_k)(e_{kl} - e_{lk}) \in \mathfrak{p} = \mathcal{H}_n. \tag{6.9}$$

The commutator $[A, z]$ is thus given by

$$[A, z] = (z_l - z_k) \sum_{1 \leq k < l \leq n} a_{kl} (e_{kl} + e_{lk}) + i \, b_{kl} (e_{kl} - e_{lk}) \in \mathfrak{p} = \mathcal{H}_n. \tag{6.10}$$

Including this result in (6.8) and referring to (6.5), we have

$$dM = \prod_{i=1}^n dz_i \prod_{1 \leq k < l \leq n} d \left((z_l - z_k) a_{kl} \right) d \left((z_l - z_k) b_{kl} \right)$$
$$= \Delta_n^2(z) \prod_{i=1}^n dz_i \prod_{1 \leq k < l \leq n} da_{kl} \, db_{kl}, \tag{6.11}$$

where $\Delta_n(z)$ is the Vandermonde determinant, defined as

$$\Delta_n(z) = \prod_{1 \le j < k \le n} \left(z_k - z_j\right). \tag{6.12}$$

The $\Delta_n(z)^2$ in (6.11) can be seen as the Jacobian determinant of the map $M \to (z, U)$ and dM can be written in polar coordinates as

$$dM = \Delta_n^2(z) \, dz_1 \, dz_2 \ldots dz_n \, dU, \qquad U \in SU(n). \tag{6.13}$$

As we have seen in section 2.1, $\mathcal{H}_n$ is associated with the probability (2.5)

$$P(M \in dM) = P(M \in dM) = c_n \, \mathrm{e}^{-\mathrm{tr}\, V(M)} dM, \tag{6.14}$$

where the trace can be expressed as a function of the eigenvalues only

$$\mathrm{tr}\, V(M) = \sum_{i=1}^{n} V(z_i), \tag{6.15}$$

and we introduce the weight

$$\rho(dz) = \mathrm{e}^{-V(z)} dz. \tag{6.16}$$

Since (6.13) depends on the trace, the angular part of the polar coordinates dU can be integrated out. Considering an interval $E \subset \mathbb{R}$, we define

$$\mathcal{H}_n(E) = \{M \in \mathcal{H}_n \text{ with spectral points } \in E\} \subset \mathcal{H}_n, \tag{6.17}$$

and the probability associated with the ensemble is obtained by the following matrix integral

$$P(M \in \mathcal{H}_n(E)) = \int_{\mathcal{H}_n(E)} c_n \, \mathrm{e}^{-\mathrm{tr}\, V(M)} dM = \frac{\int_{E^n} \Delta^2(z) \prod_{k=1}^{n} \rho(dz_k)}{\int_{\mathbb{R}^n} \Delta^2(z) \prod_{k=1}^{n} \rho(dz_k)}, \tag{6.18}$$

where c_n is the contribution of the integration of the angular part.

The free theory partition function for the ensemble is then given by [1]

$$Z_n^{(2)}(0) = c_n \int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{k=1}^n \rho(dz_k) = c_n \int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{k=1}^n e^{-V(z_k)} dz_k. \tag{6.19}$$

The weights appearing in the previous expressions are suitable functions

$$\rho(z)\, dz = e^{-V(z)} dz \tag{6.20}$$

defined on an interval $F = [A, B] \in \mathbb{R}$ for which the logarithmic derivative is given by a rational polynomial function

$$-\frac{1}{\rho(z)}\partial_z \rho(z) = \partial_z V(z) = \frac{g(z)}{f(z)}, \tag{6.21}$$

and with boundary conditions

$$\lim_{z \to A, B} f(z)\rho(z) z^k = 0 \qquad \text{for all } k \geq 0. \tag{6.22}$$

Considering the $t-$deformation of the integral in (6.19), we have

$$Z_n^{(2)}(t) = c_n \int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{k=1}^n \rho_t(dz_k) = c_n \int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{k=1}^n e^{-V(z_k) + \sum_{i=1}^\infty t_i z_k^i} dz_k, \tag{6.23}$$

where the elements in $t = (t_1, t_2, \ldots)$ play the role of coupling constants in the formal series.

6.1.2   Toda lattice

Following the approach presented in section 2.3, the Toda lattice arises from a suitable decomposition of the algebra of invertible matrices [10]

$$\mathfrak{gl}(\infty) = \mathfrak{g}_+ \oplus \mathfrak{g}_- = \mathfrak{b} \oplus \mathfrak{s}, \tag{6.24}$$

---

[1]The labeling of the partition function refers to the fact that the considered integral is a $\beta-$integral for $\beta = 2$. The parameter β is associated with the power of the Vandermonde determinant in the integral.

where the subalgebras $\mathfrak{s}$ and $\mathfrak{b}$ are

$$\mathfrak{b} = \{ \text{ lower triangular matrices with diagonal } \}$$
$$\mathfrak{s} = \{ \text{ skew-symmetric matrices } \},$$

(6.25)

and the inner product $\langle A, B \rangle = \mathrm{tr}\,(AB)$. Recalling (2.21) we have $\mathfrak{g}_{\mp}^* \cong \mathfrak{g}_{\pm}^{\perp}$, that in this case are given by

$$\mathfrak{g}_+^{\perp} = \mathfrak{b}^{\perp} = \{ \text{ strictly lower triangular matrices } \}$$
$$\mathfrak{g}_-^{\perp} = \mathfrak{s}^{\perp} = \{ \text{ symmetric matrices } \}.$$

(6.26)

The induced Hamiltonian structure (2.22) on $\mathfrak{s}^{\perp}$ is represented by the equations for the Hamiltonian vector fields

$$\chi_H(L) = \hat{P}_- \left[\nabla_+ H(L), L \right], \qquad \nabla_+ H(L) \in \mathfrak{b},$$

(6.27)

reminding that $\hat{P}_-$ is the projection onto $\mathfrak{g}_+^{\perp}$ along $\mathfrak{g}_-^{\perp}$ (see section 2.3). Setting

$$H_0^{(k)} = -\frac{1}{2} \frac{\mathrm{tr}\, L^{k+1}}{k+1}, \qquad L \in \mathfrak{s}^{\perp},$$

(6.28)

the equation (2.27) for the AKS theorem reads

$$\frac{\partial L}{\partial t_k} = \left[\frac{1}{2} \left(L^k \right)_{\mathfrak{s}}, L \right] = - \left[\frac{1}{2} \left(L^k \right)_{\mathfrak{b}}, L \right].$$

(6.29)

The matrix L is built from the dressing of the shift operator $\Lambda = \{ \delta_{i,j-1} \}_{1 \le i,j < \infty}$ as

$$L(t) = S(t) \Lambda\, S(t)^{-1},$$

(6.30)

with S belonging to the group G_+ associated with the algebra $\mathfrak{g}_+$, as stated in the end of the section 2.3, being a lower triangular matrix with non zero diagonal.

Hermitian Matrix Ensemble

As anticipated in the end of section 2.4, we follow the approach described in [6, 117] to determine the matrix L in terms of suitable τ-functions, that are proportional to the partition function (6.23) given in terms of the weight $\rho_t(z)$

$$Z_n^{(2)}(t) = c_n \int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{k=1}^n \rho_t(dz_k). \tag{6.31}$$

The weight $\rho_t(z)$ is considered to define the following inner product in $\mathbb{R}$

$$(f, g)_t = \int_{\mathbb{R}} f(z) g(z) \rho_t(z) \, dz \qquad \text{with } \rho_t(z) = e^{\sum_{i=1}^\infty t_i z^i} \rho(z) = e^{-V(z) + \sum_{i=1}^\infty t_i z^i}. \tag{6.32}$$

The corresponding moments matrix is thus given by

$$m_n(t) = \left(\mu_{ij}(t) \right)_{0 \le i, j < n} = \left(\left(z^i, z^j \right)_t \right)_{0 \le i, j < n}, \tag{6.33}$$

that is symmetric being a Hänkel matrix, since μ_{ij} depends on $i + j$. Because of the form of the weight, it is easy to see that the moments $\mu_{ij}(t)$ satisfy

$$\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k, j}, \tag{6.34}$$

leading to the following, for the corresponding semi-infinite moments matrix $m_\infty(t)$

$$\frac{\partial m_\infty(t)}{\partial t_k} = \Lambda^k m_\infty(t), \tag{6.35}$$

where Λ is the shift matrix previously recalled. The moments matrix so constructed admits a Borel decomposition in a lower and upper triangular matrices, as

$$m_\infty(t) = S(t)^{-1} S(t)^{\top -1}, \tag{6.36}$$

with $S(t)$ a lower triangular matrix with non zero diagonal.

In the following, we will state the theorem due to Adler and van Moerbeke concerning the τ-function for the Hermitian ensemble.

Theorem 6.1.1 *The τ-functions defined as determinants of the moments matrix*

$$\tau_n(t) := \det m_n(t) = \frac{1}{n!} \int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{k=1}^{n} \rho_t(z_k) \, dz_k \propto Z_n^{(2)}(t),$$ (6.37)

(*i*) *satisfy the equation in the KP hierarchy*

$$\left(s_{k+4}(\tilde{\partial}) - \frac{1}{2} \partial_{t_1} \partial_{t_{k+3}} \right) \tau_n(t) \circ \tau_n(t) = 0, \qquad k = 0,1,2,\dots;$$ (6.38)

(*ii*) *constitute the elements of the Toda lattice $L(t) = S(t) \Lambda S(t)^{-1}$*

$$L(t) = \begin{pmatrix} \partial_{t_1} \log \frac{\tau_1}{\tau_0} & \left(\frac{\tau_0 \tau_2}{\tau_1^2} \right)^{1/2} & 0 & 0 & 0 & \cdots \\ \left(\frac{\tau_0 \tau_2}{\tau_1^2} \right)^{1/2} & \partial_{t_1} \log \frac{\tau_2}{\tau_1} & \left(\frac{\tau_1 \tau_3}{\tau_2^2} \right)^{1/2} & 0 & 0 & \\ 0 & \left(\frac{\tau_1 \tau_3}{\tau_2^2} \right)^{1/2} & \partial_{t_1} \log \frac{\tau_3}{\tau_2} & \left(\frac{\tau_2 \tau_4}{\tau_3^2} \right)^{1/2} & 0 & \\ 0 & 0 & \left(\frac{\tau_2 \tau_4}{\tau_3^2} \right)^{1/2} & \partial_{t_1} \log \frac{\tau_4}{\tau_3} & \left(\frac{\tau_3 \tau_5}{\tau_4^2} \right)^{1/2} & \\ \vdots & & & \ddots & \ddots & \ddots \end{pmatrix}$$ (6.39)

for which the commuting equations in different flows are (6.29)

$$\frac{\partial L}{\partial t_k} = \left[\frac{1}{2} \left(L^k \right)_{\mathfrak{s}}, L \right] = -\left[\frac{1}{2} \left(L^k \right)_{\mathfrak{b}}, L \right];$$ (6.40)

(*iii*) *enter in the definition of the two classes of eigenvectors of L*

▷ $p(t,z) = (p_n(t,z))_{n \geq 0}$, *satisfying*

$$L(t)p(t,z) = z \, p(t,z),$$ (6.41)

where $p_n(t,z)$ are the n-th degree polynomials in z, orthonormal with respect to

the t-dependent inner product

$$(p_k(t,z), p_l(t,z))_t = \delta_{kl}, \tag{6.42}$$

which admit the representation, with $\chi(z) = \left(1, z, z^2, \dots\right)^\top$

$$p_n(t,z) = (S(t)\chi(z))_n = z^n h_n^{-1/2} \frac{\tau_n\left(t - [z^{-1}]\right)}{\tau_n(t)}, \qquad \text{with } h_n = \frac{\tau_{n+1}(t)}{\tau_n(t)}; \tag{6.43}$$

▷ $q(t,z) = (q_n(t,z))_{n\geq 0}$, *satisfying*

$$L(t)q(t,z) = z q(t,z), \tag{6.44}$$

where $q_n(t,z)$ *are defined as the Cauchy transform of* $p_n(t,z)$

$$q_n(t,z) = z \int_{\mathbb{R}^n} \frac{p_n(t,z)}{z-u} \rho_t(u)\,du, \tag{6.45}$$

and admit the following representation

$$q_n(t,z) = \left(S(t)^{\top-1}\chi\left(z^{-1}\right)\right)_n = z^{-n} h_n^{-1/2} \frac{\tau_n\left(t + [z^{-1}]\right)}{\tau_n(t)}. \tag{6.46}$$

To show (6.37) we write the Vandermonde determinant as

$$\Delta_n(z) = \prod_{1 \leq i,j \leq n} \left(z_i - z_j\right) = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_n \\ \vdots & \vdots & & \vdots \\ z_1^{n-1} & z_2^{n-1} & & z_n^{n-1} \end{pmatrix} = \det\left(z_j^{i-1}\right)_{1 \leq i,j \leq n}. \tag{6.47}$$

Using the definition of the determinant and the property $(\det(A))^2 = \det(A)\det(A^\top) = \det(AA)$

$$\begin{aligned} \Delta_n^2(z) &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{k=1}^n \left(z_{\sigma(k)}^{k-1}\right) \sum_{\sigma' \in S_n} (-1)^{\sigma'} \prod_{l=1}^n \left(z_l^{\sigma'(l)-1}\right) \\ &= \sum_{\sigma \in S_n} \det\left(z_{\sigma(k)}^{l+k-2}\right)_{1 \leq k,l \leq n}, \end{aligned} \tag{6.48}$$

with σ, σ' permutations belonging to the symmetric group S_n with $n!$ elements. In the partition function $Z_n^{(2)}(t)$ (6.31), the Vandermonde determinant appears in the n-fold integral, that can be written as

$$
\int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{k=1}^n \rho_t(z_k)\, dz_k = \sum_{\sigma \in S_n} \int_{\mathbb{R}^n} \det\left(z_{\sigma(k)}^{l+k-2}\right)_{1 \le k,l \le n} \rho_t\left(z_{\sigma(k)}\right) dz_{\sigma(k)}
$$
$$
= \sum_{\sigma \in S_n} \det\left(\int_{\mathbb{R}^n} z_{\sigma(k)}^{l+k-2} \rho_t\left(z_{\sigma(k)}\right) dz_{\sigma(k)}\right)_{1 \le k,l \le n} \tag{6.49}
$$
$$
= n! \det m_n(t) = n!\, \tau_n(t).
$$

We now focus on property (i). The τ-functions satisfy the bilinear Hirota identity (6.38), coming from the relation

$$
\mathrm{Res}_{z=\infty}\left\{\tau_n\left(t - [z^{-1}]\right) \tau_n\left(t' + [z^{-1}]\right) e^{\xi(t-t',z)}\right\} = 0, \qquad \forall t, t' \in \mathbb{C}, \tag{6.50}
$$
$$
\xi(t,z) = \sum_{n=1}^{\infty} t_n z^n, \tag{6.51}
$$

consisting of all the evolution equations of the KP hierarchy. The functions $\tau_n\left(t \pm [z^{-1}]\right)$ are written in terms of the Schur polynomials

$$
\sum_{j=0}^{\infty} s_j(t) z^j = e^{\sum_{n=1}^{\infty} t_n z^n}, \tag{6.52}
$$

and $\tilde{\partial} = \left(\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \dots\right)$ as

$$
\tau_n\left(t \pm [z^{-1}]\right) = \sum_{n=0}^{\infty} s_n(\pm\tilde{\partial})\, \tau_n(t) z^{-n}. \tag{6.53}
$$

We consider the change of variables $(t, t') \rightarrow (x, y)$

$$
\begin{cases} t = x - y \\ t' = x + y, \end{cases} \tag{6.54}
$$

the Hirota derivation

$$s\left(\partial_t\right) f \circ g = s\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots\right) f\left(t+y\right) g\left(t-y\right)\Bigg|_{y=0}, \tag{6.55}$$

the Schur polynomials (6.52) and the expressions (6.53) to evaluate the residue (6.50)

$$
\begin{aligned}
0 &= \mathrm{Res}_{z=\infty}\left\{e^{\xi(-2y,z)}\, \tau_n\left(x-y-[z^{-1}]\right) \tau_n\left(x+y+[z^{-1}]\right)\right\} \\
&= \frac{1}{2\pi i}\oint_C dz \left(\sum_{i=0}^{\infty} z^i\, s_i(-2y)\right)\left(\sum_{j=0}^{\infty} z^{-j}\, s_j\left(\tilde{\partial}\right)\right) e^{\sum_k y_k\, \partial_k}\, \tau_n(x) \circ \tau_n(x) \\
&= \frac{1}{2\pi i}\oint_C dz \left(\sum_{i,j=0}^{\infty} z^{i-j}\, s_i(-2y)\, s_j\left(\tilde{\partial}\right)\right) e^{\sum_k y_k\, \partial_k}\, \tau_n(x) \circ \tau_n(x) \\
&= \left(\sum_{j=0}^{\infty} s_j(-2y)\, s_{j+1}\left(\tilde{\partial}\right)\right) e^{\sum_k y_k\, \partial_k}\, \tau_n(x) \circ \tau_n(x).
\end{aligned}
\tag{6.56}
$$

Considering $s_0(-2y)\, s_1\left(\tilde{\partial}\right) = \partial_{x_1}$ and a Taylor expansion in $y = (t-t')/2$, we have

$$
\begin{aligned}
&\left(\partial_{x_1} + \sum_{j=1}^{\infty} s_{j+1}\left(\tilde{\partial}\right)\left(-2y_j + \mathcal{O}\left(y^2\right)\right)\right)\left(1 + \sum_{k=1}^{\infty} y_k\, \partial_{x_k} + \mathcal{O}\left(y^2\right)\right)\tau_n(x) \circ \tau_n(x) = 0 \\
&\left(\partial_{x_1} + \sum_{k=1}^{\infty} y_k\left(\partial_{x_k}\, \partial_{x_1} - 2\, s_{k+1}\left(\tilde{\partial}\right)\right)\right)\tau_n(x) \circ \tau_n(x) + \mathcal{O}\left(y^2\right) = 0.
\end{aligned}
\tag{6.57}
$$

Since $\partial_{x_1}\tau_n(x) \circ \tau_n(x) = 0$ and the coefficient of y_k is trivial for $k = 1, 2$, with $x \to t$ we obtain the Hirota bilinear representation of the KP hierarchy (6.38).

The first part of property (ii), i.e. the explicit form of $L(t)$ in terms of τ-functions follows from its definition via the decomposition (6.30). The second part of property (ii) is a consequence of (6.30). Indeed we have

$$
\begin{aligned}
L(t) &= S(t)\Lambda\, S(t)^{-1} = S(t)\Lambda\, S(t)^{-1}\, S(t)^{\top-1}\, S(t)^{\top} \\
&= S(t)\Lambda\, m_\infty(t)\, S(t)^{\top} = S(t)\, m_\infty(t)\Lambda^{\top}\, S(t)^{\top} \\
&= S(t)\, S(t)^{-1}\, S(t)^{\top-1}\Lambda^{\top}\, S(t)^{\top} = \left(S(t)\Lambda\, S(t)^{-1}\right)^{\top} = L(t)^{\top},
\end{aligned}
\tag{6.58}
$$

hence $L(t)$ is symmetric and thus tridiagonal.

Conjugating (6.35) with $S(t)$, we have

$$
\begin{aligned}
0 &= S(t)\left(\Lambda(t)^k m_\infty(t) - \frac{\partial m_\infty(t)}{\partial t_k}\right) S(t)^\top \\
&= S(t)\Lambda(t)^k S(t)^{-1} - \frac{\partial}{\partial t_k}\left(S(t)^{-1} S(t)^{\top-1}\right) S(t)^\top \\
&= L(t)^k + \frac{\partial S}{\partial t_k} S(t)^{-1} + S(t)^{\top-1} \frac{\partial S(t)^\top}{\partial t_k}.
\end{aligned}
\tag{6.59}
$$

Taking into account the projections $(\ )_\pm$, selecting the upper/lower triangular part of the matrix respectively plus the diagonal and $(\ )_0$ the projection selecting only the diagonal, we can construct the two projections constituting the algebra splitting (6.24)

$$
a = (a)_\flat + (a)_\sharp \qquad
\begin{cases}
(a)_\flat = 2(a)_- - (a)_0 \\[2mm]
(a)_\sharp = (a)_+ - (a)_- .
\end{cases}
\tag{6.60}
$$

From (6.60) and (6.59), we obtain the flows equations for the Toda lattice (6.40).

Finally, we consider the property (iii). The Borel decomposition of $m_\infty(t)$ (6.36) induces the orthonormality of the polinomials $p_n(z,t) = (S(t)\chi(z))_n$

$$
\begin{aligned}
(p_k(z,t), p_k(z,t))_t\big|_{0\le k,l\le n} &= \int_\mathbb{R} S(t)\chi(z)\chi(z)^\top S(t)^\top \rho_t(z)\,dz \\
&= S(t) m_\infty(t) S(t)^\top = S(t) S(t)^{-1} S(t)^{\top-1} S(t)^\top = I.
\end{aligned}
\tag{6.61}
$$

In particular, $\left(p_n(z,t), z^k\right)_t = 0$ for $0 \le k \le n-1$. Introducing $h_n(t) = (\tau_{n+1}(t)/\tau_n(t))^{1/2}$, a classical result [111] is that they admit the integral representation

$$
\begin{aligned}
h_n(t) p_n(z,t) &= \frac{1}{n!\,\tau_n(t)} \int_{\mathbb{R}^n} \Delta_n^2(u) \prod_{k=1}^n (z - u_k)\,\rho_t(u_k)\,du_k \\
&= \frac{z^n}{n!\,\tau_n(t)} \int_{\mathbb{R}^n} \Delta_n^2(u) \prod_{k=1}^n \left(1 - \frac{u_k}{z}\right)\rho_t(u_k)\,du_k \\
&= \frac{z^n}{n!\,\tau_n(t)} \int_{\mathbb{R}^n} \sum_{\sigma\in S_n} \det\left(u_{\sigma(k)}^{k+l-2}\right)_{1\le k,l\le n} \prod_{k=1}^n \left(1 - \frac{u_k}{z}\right)\rho_t(u_k)\,du_k \\
&= \frac{z^n}{n!\,\tau_n(t)} \int_{\mathbb{R}^n} \sum_{\sigma\in S_n} \det\left(u_{\sigma(k)}^{k+l-2} - \frac{u_{\sigma(k)}^{k+l-1}}{z}\right)_{1\le k,l\le n} \rho_t(u_{\sigma(k)})\,du_{\sigma(k)} \\
&= \frac{z^n}{\tau_n(t)} \det\left(\mu_{i,j}(t) - \frac{1}{z}\mu_{i,j+1}(t)\right)_{0\le i,j\le n-1}.
\end{aligned}
\tag{6.62}
$$

The expression in the parenthesis is

$$\mu_{i,j}(t) - \frac{1}{z}\mu_{i,j+1}(t) = \int_{\mathbb{R}} u^{i+j}\left(1 - \frac{u}{z}\right)\rho_t(u)\,du$$

$$= \int_{\mathbb{R}} u^{i+j}e^{\log\left(1-\frac{u}{z}\right)}\rho_t(u)\,du$$

$$= \int_{\mathbb{R}} u^{i+j}e^{\sum_k\left(t_k - \frac{1}{kz^k}\right)u^k}\rho(u)\,du \qquad (6.63)$$

$$= \mu_{i,j}\left(t - [z^{-1}]\right),$$

hence (6.62) becomes

$$h_n(t)p_n(z,t) = z^n\frac{\tau_n\left(t - [z^{-1}]\right)}{\tau_n(t)}. \qquad (6.64)$$

An analogous computation can be done for $q_n(z,t)$ defined in (6.45), giving

$$h_n(t)q_n(z,t) = z^{-n}\frac{\tau_n\left(t + [z^{-1}]\right)}{\tau_n(t)}. \qquad (6.65)$$

Recalling $\chi(z) = \left(1, z, z^2, \dots\right)^{\top}$ and the shift operator Λ

$$\Lambda\,\chi(z) = z\,\chi(z), \qquad \Lambda^{\top}\chi(z^{-1}) = z\,\chi(z^{-1}) - z\,e_1, \qquad (6.66)$$

with $e_1 = (1,0,0,\dots)^{\top}$, the vectors

$$p(z,t) = S(t)\chi(z), \qquad q(z,t) = S(t)^{\top-1}\chi(z^{-1}) \qquad (6.67)$$

are eigenvectors of the Toda lattice

$$L(t)p(z,t) = S(t)\Lambda\,S(t)^{-1}S(t)\chi(z) = S(t)\Lambda\chi(z)$$

$$= z\,S(t)\chi(z) = z\,p(z,t), \qquad (6.68)$$

$$L(t)^{\top}q(z,t) = S(t)^{\top-1}\Lambda\,S(t)^{\top}S(t)^{\top-1}\chi(z^{-1}) = S(t)^{\top-1}\Lambda\chi(z^{-1})$$

$$= z\,S(t)^{\top-1}\chi(z^{-1}) - z\,S(t)^{\top-1}e_1 = z\,q(z,t) - z\,e_1. \qquad (6.69)$$

Since for (6.57) $L(t) = L(t)^\top$ we have

$$((L(t) - zI)p(z,t))_n = 0, \qquad n \geq 0$$
$$((L(t) - zI)q(z,t))_n = 0, \qquad n \geq 1.$$

(6.70)

So far we have reviewed the fundamental features and general aspects of the theory to describe the $\mathcal{H}_n$. In the following, we will consider the description provided in [23], defining a suitable reduction of the Toda lattice, corresponding to the selection of the even times only in the coupling constants in the expression for the partition function.

6.2 From Toda lattice to Volterra lattice

We recall the form of the Toda lattice, associated to the study of the $\mathcal{H}_n$, introduced in section 6.1.2

$$L(t) = \begin{pmatrix} \partial_{t_1} \log \frac{\tau_1}{\tau_0} & \left(\frac{\tau_0 \tau_2}{\tau_1^2}\right)^{1/2} & 0 & 0 & 0 & \cdots \\ \left(\frac{\tau_0 \tau_2}{\tau_1^2}\right)^{1/2} & \partial_{t_1} \log \frac{\tau_2}{\tau_1} & \left(\frac{\tau_1 \tau_3}{\tau_2^2}\right)^{1/2} & 0 & 0 & \\ 0 & \left(\frac{\tau_1 \tau_3}{\tau_2^2}\right)^{1/2} & \partial_{t_1} \log \frac{\tau_3}{\tau_2} & \left(\frac{\tau_2 \tau_4}{\tau_3^2}\right)^{1/2} & 0 & \\ 0 & 0 & \left(\frac{\tau_2 \tau_4}{\tau_3^2}\right)^{1/2} & \partial_{t_1} \log \frac{\tau_4}{\tau_3} & \left(\frac{\tau_3 \tau_5}{\tau_4^2}\right)^{1/2} & \\ \vdots & & & \ddots & \ddots & \ddots \end{pmatrix},$$

(6.71)

where the fields entering the matrix are explicitly reported in terms of τ-functions of the system. The matrix $L(t)$ representing the lattice satisfies the Lax equations (6.40)

$$\frac{\partial L}{\partial t_k} = \left[\frac{1}{2}\left(L^k\right)_{\mathfrak{s}}, L\right].$$

(6.72)

The matrix $L(t)$ is by construction symmetric and the projection $\mathfrak{s}$ acts on a generic symmetric matrix a as

$$(a)_{\mathfrak{s}} = (a)_+ - (a)_-,$$

(6.73)

where the projections $(\ )_\pm$ selects the upper / lower triangular part of the matrix respectively, giving a skew-symmetric matrix.

The τ-functions appearing within the elements of the Toda lattice are proportional to the partition function for $n \times n$ Hermitian matrices (6.23)

$$Z_n^{(2)}(t) = c_n \int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{k=1}^{n} \rho_t(dz_k) = c_n \int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{k=1}^{n} e^{-\frac{1}{2}z^2 + \sum_{i=1}^{\infty} t_i z_k^i} \, dz_k, \tag{6.74}$$

as described in section 6.1.1. As we can see, in (6.74) the free theory is given by the Gaussian Unitary Ensemble, obtained setting all the coupling constants t_i to zero.

The τ-function is defined as the determinant of the moments matrix

$$\tau_n(t) = \det\left(m_n(t)\right), \tag{6.75}$$

built via the symmetric inner product (6.32), as

$$\begin{aligned} m_n(t) &= \left(\mu_{ij}(t)\right)_{0,\leq i,j\leq n} = \left(\left(z^i, z^j\right)_t\right)_{0,\leq i,j\leq n} \\ \left(z^i, z^j\right)_t &= \int_{\mathbb{R}} z^i z^j e^{-\frac{1}{2}z^2 + \sum_{i=1}^{\infty} t_i z^i} \, dz. \end{aligned} \tag{6.76}$$

6.2.1 Initial condition with the GUE

We now consider the Lax matrix of the Toda lattice at the initial condition, with $t = 0$. Looking at the structure of its elements and following a notation commonly used in the literature, we distinguish between two different fields, $a_n(t)$ and $b_n(t)$. The fields $a_n(t)$ occupy the positions along the main diagonal and they are defined in terms of τ-functions as

$$a_n(t) = \partial_{t_1} \log \tau_n(t), \tag{6.77}$$

whereas the fields $b_n(t)$ appear in the first above and lower diagonals of $L(t)$ and they are given by

$$b_n(t) = \left(\frac{\tau_{n+1}(t)\,\tau_{n-1}(t)}{\tau_n(t)^2}\right)^{1/2}. \tag{6.78}$$

The expressions in (6.77) and (6.78) can be evaluated for $t = 0$, when the Hermitian matrix ensemble reduces to the GUE. Following [92], the typical integrals appearing in

the representation of τ-functions for Gaussian ensembles at $t = 0$ can be given in terms of a Selberg's integral. In particular,

$$\int_{\mathbb{R}^n} |\Delta(x)|^{2\gamma} \prod_{k=1}^{n} e^{-ax_k^2} dx_k = (2\pi)^{n/2} (2a)^{-n(\gamma(n-1)+1)/2} \prod_{k=1}^{n} \frac{\Gamma(1+k\gamma)}{\Gamma(1+\gamma)}. \tag{6.79}$$

In this, case $\gamma = 1$ and $a = 1/2$

$$\int_{\mathbb{R}^n} \Delta(x)^2 \prod_{k=1}^{n} e^{-\frac{1}{2}x_k^2} dx_k = (2\pi)^{n/2} \prod_{k=1}^{n} \Gamma(1+k). \tag{6.80}$$

Recalling that the τ-function is given in terms of the determinant of the moments matrix, we have

$$\tau_n(0) = (2\pi)^{n/2} \prod_{k=1}^{n} \frac{k!}{n!}, \tag{6.81}$$

and we can compute $b_n(0)$ as

$$b_n(0) = \left(\frac{\tau_{n+1}(0)\tau_{n-1}(0)}{\tau_n(0)^2} \right)^{1/2} = \sqrt{n}. \tag{6.82}$$

To evaluate the fields $a_n(0)$, we observe that if we enable the t_1 interaction in the τ-functions, we have

$$\tau_n(t_1, 0, 0, \dots) = (2\pi)^{n/2} e^{\frac{nt_1^2}{2}} \prod_{k=1}^{n} \frac{k!}{n!}. \tag{6.83}$$

Then we have

$$a_n(0) = \partial_{t_1} \log \tau_n(t_1, 0, 0, \dots)\big|_{t_1=0} = 0. \tag{6.84}$$

The corresponding Lax matrix representation of the Toda lattice at $t = 0$ is entirely described by the fields $b_n(0)$, while the elements of the main diagonal vanish. In the next section, we will see that if we set the fields along the diagonal to zero also after the initial time, we obtain the Volterra lattice.

6.2.2   Discrete equations for the fields in the Volterra lattice

The Volterra lattice emerges from Toda by letting all the elements of the main diagonal be identically zero for $t \neq 0$. It was introduced in [79] and its Lax operator takes the form

$$
L(t) = \begin{pmatrix}
0 & b_1(t) & 0 & 0 & 0 & \cdots \\
b_1(t) & 0 & b_2(t) & 0 & 0 & \\
0 & b_2(t) & 0 & b_3(t) & 0 & \\
0 & 0 & b_3(t) & 0 & b_4(t) & \\
\vdots & & & \ddots & \ddots & \ddots
\end{pmatrix},
\tag{6.85}
$$

with $t = (t_2, t_4, \dots)$. It is worth mentioning that the model involving odd weights differs from the case here studied and it has been analysed in [56].

The Lax equations representing the Volterra lattice are the following

$$
\frac{\partial L}{\partial t_{2k}} = \left[\frac{1}{2} \left(L^{2k} \right)_{\mathfrak{s}}, L \right], \qquad k = 1, 2, \dots.
\tag{6.86}
$$

In terms of the field variables of the lattice the previous equation reads

$$
\frac{\partial b_n}{\partial t_{2k}} = \frac{b_n}{2} \left(b_{n+1} \left(L^{2k-1} \right)_{n+1,n+2} - b_{n-1} \left(L^{2k-1} \right)_{n-1,n} \right).
\tag{6.87}
$$

Multiplying both sides of the equation by $b_n(t)$ and introducing the notation

$$
\begin{cases}
B_n(t) = b_n^2(t) \\[2mm]
V_n^{(2k)}(t) = b_n(t) \left(L^{2k-1}(t) \right)_{n,n+1},
\end{cases}
\tag{6.88}
$$

the equation (6.87) becomes

$$
\frac{\partial B_n}{\partial t_{2k}} = B_n \left(V_{n+1}^{(2k)} - V_{n-1}^{(2k)} \right), \qquad k = 1, 2, \dots.
\tag{6.89}
$$

The field variable $B_n(t)$ is addressed as the order parameter of the system. In particular, the fields $V_n^{(2k)}(t)$ can be obtained as linear combinations of products involving the variable $B_n(t)$. For the first three terms we have

$$V_n^{(2)} = B_n$$
$$V_n^{(4)} = V_n^{(2)}\left(V_{n-1}^{(2)} + V_n^{(2)} + V_{n+1}^{(2)}\right) \tag{6.90}$$
$$V_n^{(6)} = V_n^{(2)}\left(V_{n-1}^{(2)} V_{n+1}^{(2)} + V_{n-1}^{(4)} + V_n^{(4)} + V_{n+1}^{(4)}\right).$$

Considering the theory of the orthogonal polynomials described in section 2.5, we have that the field variables are essentially the coefficients of the recursion relation (2.55). This result is in this context established in [31, 27] and gives rise to the so called string equation. We will refer to its expression given in [30], which in the case of even times only takes the form

$$n = B_n - \sum_{k=1}^{\infty} 2k\, t_{2k}\, V_n^{(2k)}. \tag{6.91}$$

In particular, the form of the operator in the second term of the right hand side of (6.91) is produced imposing the string equation

$$[L, P] = 1. \tag{6.92}$$

In (6.92), the operator P is expressed as

$$P = \frac{1}{2}(L)_{\mathrm{s}} + \sum_{k \geq 1} k\, t_k \left(L^{2k-1}\right)_{\mathrm{s}}. \tag{6.93}$$

In the following we will study the behaviour of the order parameter via the expression (6.91) selecting the model where $t_{2k} = 0$ for $k > 3$, evaluating the thermodynamic limit (for $n \to \infty$). The order parameter will develop a singularity, that is regularised by oscillations, observed in [78, 109] and interpreted as a chaotic behaviour. In [23], this chaotic phase is instead interpreted as the occurrence of a propagating dispersive shock, the regularisation mechanism described in 4.3.

6.3   Thermodynamic limit and scalar hierarchy

We introduce a typical scale of the system N and the rescaled field variables

$$u_n = \frac{B_n}{N}, \qquad T_{2k} = N^{k-1} t_{2k}, \qquad W_n^{2k} = \frac{V_n^{(2k)}}{N^k}. \tag{6.94}$$

Then the expression (6.91) reads

$$\frac{n}{N} = u_n - \sum_{k=1}^{\infty} 2k \, T_{2k} \, W_n^{2k}. \tag{6.95}$$

We define the interpolating function $u(x)$ that will be the continuous field variable

$$\begin{cases} u(x) = u_n \\ \\ u(x \pm \varepsilon) = u_{n \pm 1} \end{cases} \qquad \text{with } x = \frac{n}{N} \text{ and } \varepsilon = \frac{1}{N}. \tag{6.96}$$

As previously mentioned, we will focus on the case for which only the first three terms in the corresponding coupling constants are on. In particular, the coupling constant $T_6 < 0$, so that the convergence of the integral in the partition function (6.74) is ensured. The Taylor expansion for $\varepsilon \to 0$ gives at the leading order

$$x = (1 - T_2) u - 12 \, T_4 \, u^2 - 60 \, T_6 \, u^3. \tag{6.97}$$

We consider the continuum limit evaluated for the Volterra lattice equations (6.88) to better understand the evolution of the solution to the recurrence relation $u(x)$. It is worth noting that $u(x)$ is indeed an order parameter, since in the thermodynamic limit it can be express in terms of the derivative of the "free energy" of the system (recalling that τ is essentially the partition function)

$$u(x) = \partial_x^2 \ln \tau(x). \tag{6.98}$$

Using (6.94) in (6.88), we get the corresponding expression involving the interpolation function $u(x)$ and $u(x \pm \varepsilon)$.

Evaluating the Taylor series for $\varepsilon \to 0$, the hierarchy can be written as

$$u_{T_{2k}} = \sum_{n=0}^{\infty} \varepsilon^n g_n^{(k)}(u; \partial_x u, \ldots, \partial_x^n u), \tag{6.99}$$

with $g_n^{(k)}$ differential polynomials of u. In the thermodynamic limit, with $\varepsilon \to 0$ the previous expression at the leading order gives us the Hopf hierarchy (also known as Burgers-Hopf hierarchy [81])

$$u_{T_{2k}} = c_k u^k u_x, \qquad \text{with } c_k = (-1)^k \frac{(2n+1)!!}{2^n n!}. \tag{6.100}$$

The solution to this equation is implicitly given by (6.97). From the latter, we can determine the condition for extremising the free energy

$$F[u] = \int_0^\beta f_0(u) \, dx, \qquad \beta > 0$$

$$f_0(u) = -xu + \frac{1}{2}(1 - T_2) u^2 - 4 T_4 u^3 - 15 T_6 u^4. \tag{6.101}$$

The number of local minima and maxima of the free energy density depend on the signature of the discriminant Δ of (6.97), such as

$$\begin{cases} \Delta > 0 & \text{two local minima and one local maximum,} \\ \Delta = 0 & \text{boundary of the multi-valued region,} \\ \Delta > 0 & \text{one minimum.} \end{cases} \tag{6.102}$$

The phase transition occurs at the critical point, represented by the cusp point in the figure 6.1 (a). For a given choice of T_2, T_4 in the plane (x, T_6) the colored region represents the condition for which (6.97) has three different solutions, corresponding to the stationary points of the free energy density, displayed in (b) for two different values of T_6 as a function of the field variable u.

All the figures reported in this section are reproductions of those appeared in the work [23].

The behaviour above described was expected from equation (6.100), since a generic solution to the Hopf hierarchy develops a singularity for finite value of the time vari-
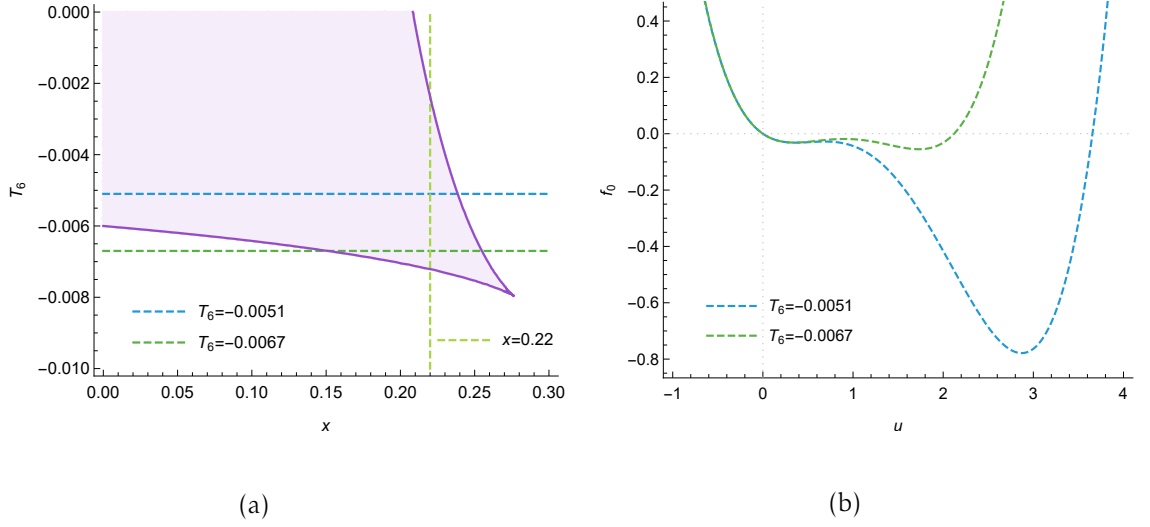
(a) (b)

Figure 6.1: In (a) the critical set $\Delta = 0$ for $T_2 = 0$, $T_4 = 0.1$ in the plane (x, T_6). The filled region corresponds to the case $\Delta > 0$, where (6.97) admits multiple roots. In (b) the free energy density is depicted as a function of the solution u for the points $(0.22, -0.0051)$ and $(0.22, -0.0067)$ identified in (a), in the region $\Delta > 0$.

ables T_{2k}, as we have seen in section 4.1. In particular, we will see how the singularity is associated to the occurrence of a dispersive shock induced by dispersive corrections to the Hopf hierarchy, as discussed in section 4.3. Approaching the gradient catastrophe, the dispersive corrections appearing in (6.99) induce oscillations manifesting the emergence of a dispersive shock, as discussed in the following section.

6.4 Dispersive regularisation and possible scenarios in the multivalued region

In figure 6.2 (a)–(e), it is shown the evolution of the solution $u(x)$ to (6.97) and of u_n evaluated via the recurrence relation (6.95) with $k \in \{1, 2, 3\}$. Recalling that $u_n = \varepsilon B_n$, the recurrence relation becomes

$$n = B_n - 2\, T_2\, V_n^{(2)} - 4\, T_4\, \varepsilon\, V_n^{(4)} - 6\, T_6\, \varepsilon^2\, V_n^{(6)}, \tag{6.103}$$

where $V_n^{(2)}, V_n^{(4)}, V_n^{(6)}$ are given by the relations (6.90).

The initial constraint of the recurrence relation are given by the fields

$$B_0(t_2,t_4,t_6) = 0 \qquad B_i(t_2,t_4,t_6) = \frac{\tau_{i+1}(t_2,t_4,t_6)\,\tau_{i-1}(t_2,t_4,t_6)}{\tau_i(t_2,t_4,t_6)^2}, \qquad i = 1,2,3, \qquad (6.104)$$

with $\tau_1 = 1$ and τ_i as defined in (6.75) with only nonzero times (t_2,t_4,t_6).

In figure 6.2 (a) the behaviour of the two overlapping solutions is represented for values of T_2, T_4, T_6 such that the order parameter is single valued (i.e. in the region $\Delta < 0$ in figure 6.1 (a)).

In figure 6.2 (b), in proximity to the gradient catastrophe, we observe a deviation in the evolution of the two functions and the profile representing the exact solution develops oscillations, that become evident in figures 6.2 (c)–(e). Figure 6.1 (a) illustrates the passage from the region $\Delta < 0$ to $\Delta > 0$.

The micro-oscillatory behaviour reveals the occurrence of a dispersive regularisation. The ostensible chaotic phase, as it was interpreted in [78, 109], is then describable as the onset of this mechanism of regularisation, for the presence of higher order corrections to the leading order in (6.97). In [78], the phase transition is interpreted in terms of the spectral distribution associated to the matrix model, extending to the M^6 theory the approach laid out in [27] for the M^4 theory. In particular, the single valued phase is connected to a spectral distribution with one single cut, whereas the multi-valued phase corresponds to a spectral distribution with three cuts.
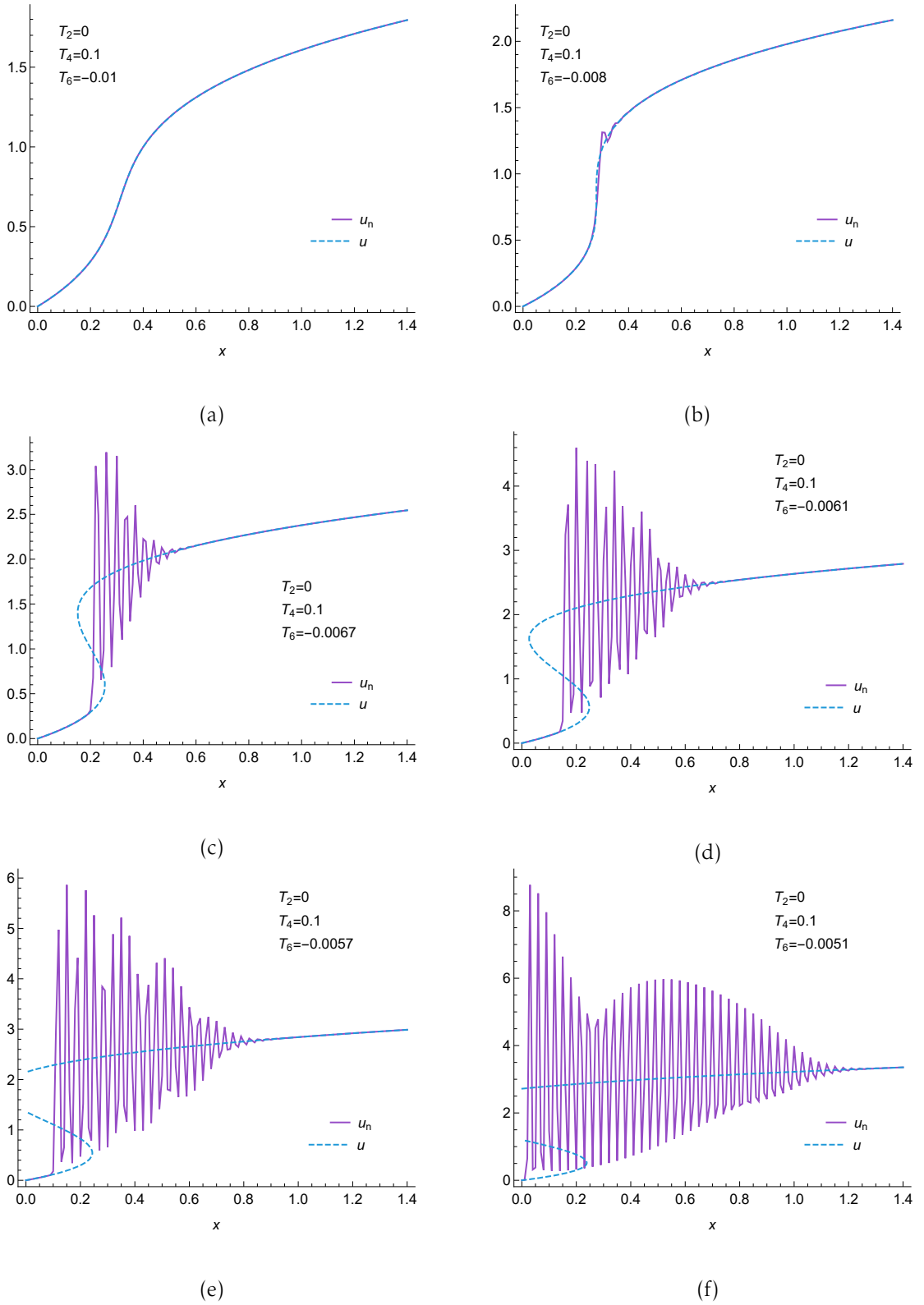
(a)

(b)

(c)

(d)

(e)

(f)

Figure 6.2: The function $u(x)$ and u_n are shown for constant values of T_2, T_4 and $\varepsilon = 0.01$. In (a) the behaviour of the function at a point in the $\Delta < 0$ region is represented. In (b) $\Delta = 0$ in correspondence of the gradient catastrophe, from (c) to (e) it is shown the solutions at the selected points of the region $\Delta > 0$ in figure 6.1.

We now analyse the region $\Delta > 0$, for which (6.97) admits three real and distinct solutions. Here, different scenarios are possible, depending on the ranges associated to the coupling constants T_2, T_4, T_6, obtained studying the sign of the coefficients in (6.97). As it was mentioned above, to ensure the converge of the integral to evaluate the τ-function, the time T_6 must be strictly negative. We identify the possible three cases

$$(1 - 2\, T_2) < 0 \,, T_4 > 0 \qquad \text{scenario 1,}$$

$$(1 - 2\, T_2) > 0 \qquad \text{scenario 2,} \qquad (6.105)$$

$$(1 - 2\, T_2) < 0 \,, T_4 < 0 \qquad \text{scenario 3.}$$

Scenario 1 It is represented in figures 6.2 (c)–(e) and it constitutes the same case analysed in [77, 78]. Since $u(x) \geq 0$, only non negative branches of the field variable represent admissible states of the system. In particular, the three branches of the cubic, corresponding to stationary points of the free energy density, are positive. We can observe a complex structure that qualitatively looks like a dispersive shock wave, but displaying an additional so called beating pattern [35].
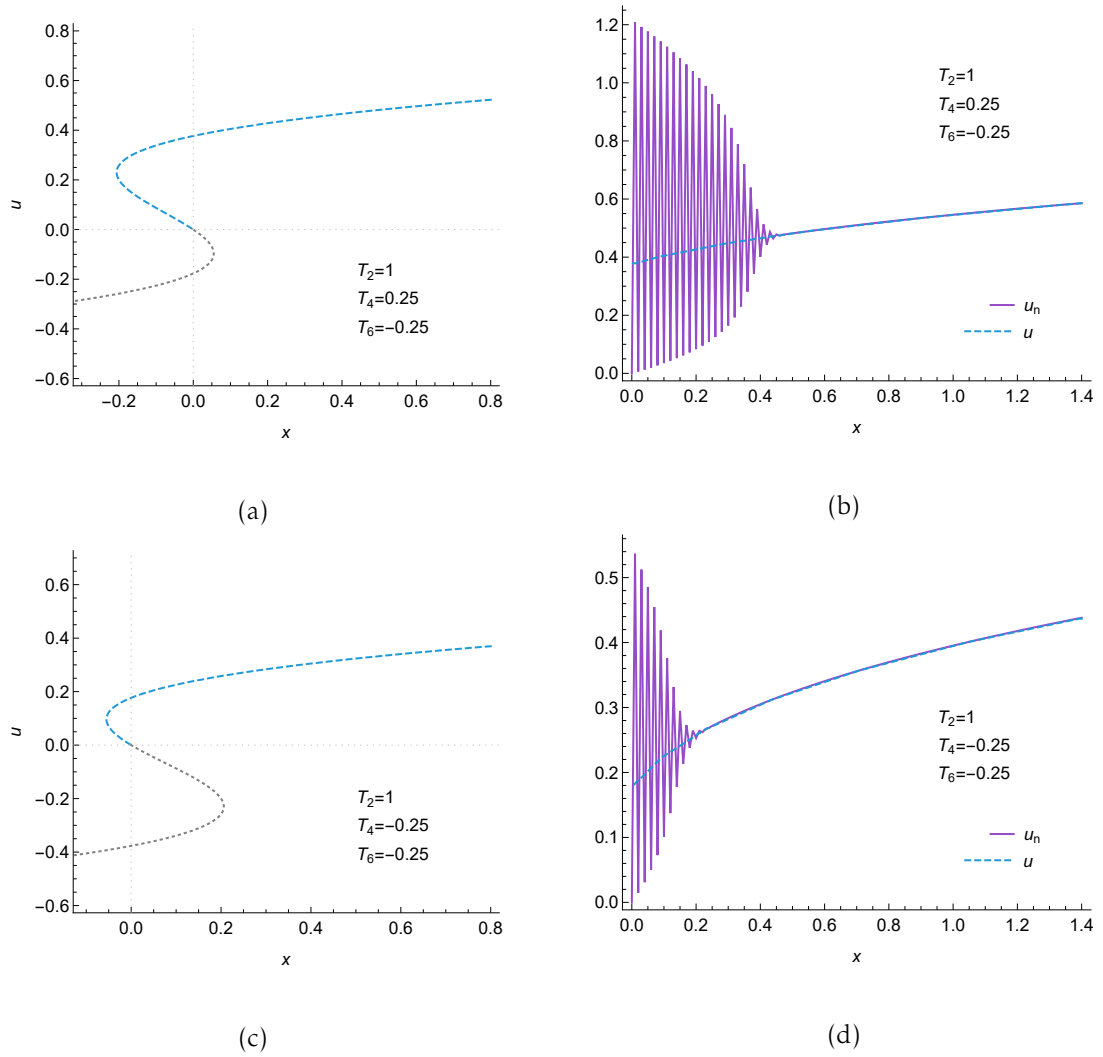
Figure 6.3: In (a), $u(x)$ is reported for $(1 - 2\,T_2) > 0$, $T_4 > 0$. In (b), it is also represented u_n for $\varepsilon = 0.01$, where it is visible the propagation of the dispersive shock originated in the region $x < 0$. The same is shown in (c) and (d) for the sector $(1 - 2\,T_2) > 0$, $T_4 < 0$.

Scenario 2 The solution $u(x)$ to (6.97) is shown in figure 6.3 (a) and (c) for different values of T_4, while the parameters T_2 and T_6 are kept constant. The function is three valued, but one of the roots is negative and it does not lead to a stable state for the system. Nonetheless, the existence of two other possible states, one stable and one unstable, leads to the emergence of the dispersive shock, visible in figure 6.3 (b) and (d). This is the case even if in the region $x > 0$ there exists a non-negative branch of the cubic only: the gradient catastrophe occurs for $x < 0$. In this case, the profile of the regulation mechanism qualitatively resembles the one of the dispersive shock wave appearing in KdV with a cubic wavebreaking [55] (the so called Bordeaux glass profile, see section 4.3).
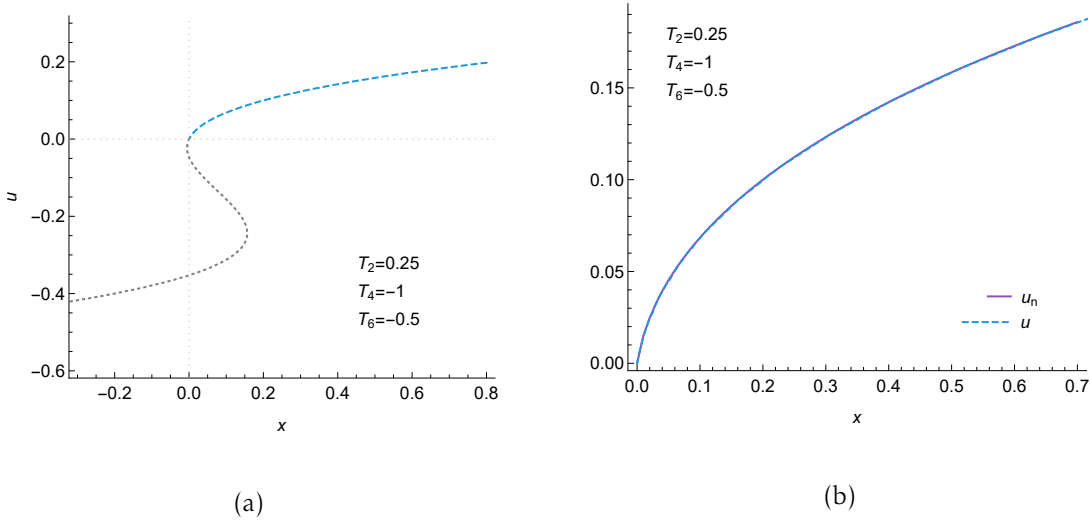
(a)   (b)

Figure 6.4: In (a) $u(x)$ is shown for $(1 - 2\,T_2) > 0$, $T_4 > 0$. In (b) it is also represented u_n for $\varepsilon = 0.01$. There is no dispersive shock in this case and the solution for $u > 0$ is single valued.

Scenario 3   The solution $u(x)$ is multivalued with only one positive branch for $u(x) > 0$, as represented in figure 6.4 (a), hence the system can be in one state only. In this case, the two solutions shown in figure 6.4 (b) overlap and there is no oscillation. Then one can conclude that the regularisation mechanism given by the dispersive shock is associated with the existence of accessible both stable and unstable states.

The emergence of a dispersive shock is a specific feature of matrix ensembles, whereas viscous shocks are a specific feature of classical magnetic and fluid models. In section 5.3 we have observed the onset of a viscous shock in the order parameter for mean-field statistical mechanical models. In this case, the underlying hydrodynamic system at the leading order in the order parameter is given by the Hopf equation (e.g. equation (5.25) for the Curie-Weiss model). In the context of the Hermitian matrix ensemble, at the leading order in the thermodynamic limit, we encounter the Hopf hierarchy (6.100) in even slow times T_{2k}. This may suggest that from this point of view, the two systems are specified by the initial condition for the differential identity.

In the following chapter, we will show how ensembles of symmetric give rise to more complex structures, i.e. hydrodynamic chains, compared to the scalar hierarchy emerging for the Hermitian matrix ensemble.

Part III

Results

Chapter 7

Symmetric Matrix Ensemble and hydrodynamic chains

This chapter is dedicated to the study of the symmetric matrix ensemble $\mathcal{S}_n$ and a new associated hydrodynamic chain [24]. Firstly, in section 7.1, we will use the tools developed in section 2.1 to describe the discrete integrable structure associated with the ensemble, referred to as Pfaff lattice. We follow [12, 13, 117] and we will see how the KP-Pfaff hierarchy emerges for the Pfaffian τ-function, the latter being proportional to the partition function for the ensemble $\mathcal{S}_n$. In section 7.2, we will introduce a suitable notation for the field variables of the Pfaff lattice with the aim of unveiling the existing underlying double-chain structure.

Then, in section 7.3, we will describe the lattice emerging by selecting the even interaction terms only. This Pfaff reduction is realised with the aim of reducing the complexity of the problem, passing from a double-chain structure to a single-chain one. In the thermodynamic limit at the leading order, the resultant system of equations for the evolution in the first even time (i.e. t_2) can be recast in the form of a hydrodynamic chain, as we will discuss in section 7.4. Then we will investigate the integrability of the new infinite hydrodynamic chain, following the approach developed in section 3.2. In section 7.5, we will conjecture that generalised hydrodynamic chains can be found at the leading order of the continuum limit for higher even flows as well, i.e. with respect to times $t_4, t_6, \dots$.

Finally, section 7.6 is devoted to the comparison between the hierarchies emerging at

the leading order of the thermodynamic limit in the case of the Volterra lattice and the corresponding even time reduction of the Pfaff lattice.

7.1 Symmetric Matrix Ensemble

7.1.1 $\mathcal{S}_n$ as a tangent space and partition function

Similarly to the Hermitian case treated in section 6.1.1, we follow the approach presented in section 2.2. We consider the non-compact symmetric space $\mathcal{M} = G/K$, with $G = SL(n, \mathbb{R})$ and the involution map defined in K as

$$\sigma(g) = \left(g^\top\right)^{-1},\tag{7.1}$$

so that the subgroup K is in this case given by

$$K = \{g \in SL(n, \mathbb{R}) \mid \sigma(g) = g\} = \{g \in SL(n, \mathbb{R}) \mid g^{-1} = g^\top\} = SO(n).\tag{7.2}$$

The symmetric space $\mathcal{M}$ can be expressed as

$$SL(n, \mathbb{R})/SO(n) \cong \{g\,g^\top \mid g \in SL(n, \mathbb{R})\}$$
$$= \{\text{positive definite matrices with det} = 1\}.\tag{7.3}$$

The involution map σ induces the map $\sigma_*(A) = -A^\top$, the subalgebra being $\mathfrak{t} = \mathfrak{so}(n)$, of $n \times n$ traceless skew-symmetric matrices. The tangent vector space $\mathfrak{p}$ to $\mathcal{M}$ at the identity is the space of $n \times n$ symmetric matrices $\mathcal{S}_n$, for which $\sigma_*(A) = A^\top$. The algebra decomposition is

$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{t} \oplus \mathfrak{p} = \mathfrak{so}(n) \oplus \mathcal{S}_n.\tag{7.4}$$

The free variables in $M \in \mathcal{S}_n$ are the real entries M_{ij} for $1 \leq i \leq j \leq n$ and the Haar measure on $\mathcal{S}_n$ is

$$dM = \prod_{1 \leq i \leq j \leq n}^{n} dM_{ij}.\tag{7.5}$$

Also in this case, a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p} = \mathcal{S}_n$ is given by the subset of diagonal matrices $z = \text{diag}(z_1, z_2, \ldots, z_n)$, where z_i with $1 \leq i \leq n$ are eigenvalues. Any

symmetric matrix can be diagonalised through an orthogonal operator, so that

$$M = O z O^{-1}, \qquad O \in K = SO(n). \tag{7.6}$$

The orthogonal operator can be expressed via the exponential map as $O = \mathrm{e}^A$, with the matrix A such that $A \in \mathfrak{t} = \mathfrak{so}(n)$, a traceless skew-symmetric matrix ($A^\top = -A$). Then A takes the form

$$A = \sum_{1 \le k < l \le n} (a_{kl}(e_{kl} - e_{lk})), \tag{7.7}$$

with the same formalism used in section 6.1.1. In this case, we have

$$[A, z] = (z_l - z_k) \sum_{1 \le k \le l \le n} a_{kl}(e_{kl} + e_{lk}) \in \mathfrak{p} = \mathcal{S}_n, \tag{7.8}$$

so that the Haar measure on $\mathcal{S}_n$ becomes

$$
\begin{aligned}
dM &= \prod_{i=1}^{n} dz_i \prod_{1 \le k < l \le n} d\left((z_l - z_k)a_{kl}\right) \\
&= |\Delta_n(z)| \prod_{i=1}^{n} dz_i \prod_{1 \le k < l \le n} da_{kl}.
\end{aligned}
\tag{7.9}
$$

Here, $|\Delta_n(z)|$ is the Jacobian determinant of the map $M \to (z, O)$ to the polar coordinates

$$dM = |\Delta_n(z)| \, dz_1 \, dz_2 \dots dz_n \, dO, \qquad O \in SO(n). \tag{7.10}$$

Similarly to the Hermitian case, we have

$$P(M \in \mathcal{S}_n(E)) = \int_{\mathcal{H}_n(E)} c_n \mathrm{e}^{-\mathrm{tr}\, V(M)} dM = \frac{\int_{E^n} |\Delta(z)| \prod_{k=1}^n \rho(dz_k)}{\int_{\mathbb{R}^n} |\Delta(z)| \prod_{k=1}^n \rho(dz_k)}, \tag{7.11}$$

and the free theory partition function is defined as

$$Z_n^{(1)}(0) = c_n \int_{\mathbb{R}^n} |\Delta_n(z)| \prod_{k=1}^{n} \rho(dz_k) = c_n \int_{\mathbb{R}^n} |\Delta_n(z)| \prod_{k=1}^{n} \mathrm{e}^{-V(z_k)} dz_k. \tag{7.12}$$

Deforming the potential we obtain

$$Z_n^{(1)}(t) = c_n \int_{\mathbb{R}^n} |\Delta_n(z)| \prod_{k=1}^n \rho_t(dz_k) = c_n \int_{\mathbb{R}^n} |\Delta_n(z)| \prod_{k=1}^n e^{-V(z_k) + \sum_{i=1}^\infty t_i z_k^i} \, dz_k. \tag{7.13}$$

7.1.2   Pfaff lattice

As in the case of Toda, the Pfaff lattice emerges from a suitable decomposition of the algebra $\mathfrak{go}(\infty)$ of invertible matrices[1](see section 2.3). It is seen as composed of 2×2 blocks [10] and then it admits the natural decomposition

$$\mathfrak{go}(\infty) = \mathfrak{d}_- \oplus \mathfrak{d}_0 \oplus \mathfrak{d}_+ = \mathfrak{d}_- \oplus \mathfrak{d}_0^- \oplus \mathfrak{d}_0^+ \oplus \mathfrak{d}_+, \tag{7.14}$$

where $\mathfrak{d}_0$ has 2×2 blocks along the diagonal and zeros elsewhere, $\mathfrak{d}_\pm$ are the subalgebras of upper/lower triangular matrices with 2×2 zero blocks along the diagonal. In addition, $\mathfrak{d}_0$ can be further decomposed into

$$\mathfrak{d}_0^- = \{ \text{ all } 2 \times 2 \text{ blocks } \in \mathfrak{d}_0 \text{ are proportional to Identity } \}$$

$$\mathfrak{d}_0^+ = \{ \text{ all } 2 \times 2 \text{ blocks } \in \mathfrak{d}_0 \text{ are traceless } \}. \tag{7.15}$$

We introduce the skew-symmetric semi-infinite matrix J

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & -1 & 0 & \\ \vdots & & & & & & \ddots \end{pmatrix}. \tag{7.16}$$

and the associated involution $\sigma \colon \mathfrak{go}(\infty) \to \mathfrak{go}(\infty)$

$$\sigma \colon a \mapsto \sigma(a) = J \, a^\top J. \tag{7.17}$$

---

[1]In [38] $\mathfrak{go}(\infty)$ is addressed as the algebra behind the so called BKP hierarchy for the corresponding τ-functions (here called Pfaff-KP), as well as $\mathfrak{gl}(\infty)$ is the one for the KP hierarchy in the associated τ-functions.

Symmetric Matrix Ensemble

To apply the AKS theorem, we consider the splitting

$$\mathfrak{go}(\infty) = \mathfrak{g}_+ \oplus \mathfrak{g}_- = \mathfrak{t} \oplus \mathfrak{n}, \tag{7.18}$$

where the subalgebras $\mathfrak{t}$ and $\mathfrak{n}$ are

$$\mathfrak{t} = \mathfrak{d}_- \oplus \mathfrak{d}_0^- = \{ \text{ lower triangular matrices with } 2 \times 2 \text{ diagonal blocks } \propto \text{ Id } \} \tag{7.19}$$

$$\mathfrak{n} = \mathfrak{d}_0^+ \oplus \mathfrak{d}_+ = \{ a \in \mathfrak{go}(\infty) \mid a = J\, a^\top J \} = \mathfrak{sp}(\infty),$$

and the inner product $\langle A, B \rangle = \mathrm{tr}\,(AB)$. Setting

$$H_0^{(k)} = -\frac{\mathrm{tr}\, L^{k+1}}{k+1}, \tag{7.20}$$

the equation (2.27) for the AKS theorem reads

$$\frac{\partial L}{\partial t_k} = -\left[\left(L^k \right)_{\mathfrak{t}}, L \right] = \left[\left(L^k \right)_{\mathfrak{n}}, L \right]. \tag{7.21}$$

The matrix L is, in this case, built from the dressing of the shift operator $\Lambda = \{\delta_{i,j-1}\}_{1 \le i,j < \infty}$ as

$$L(t) = Q(t)\, \Lambda\, Q(t)^{-1}, \tag{7.22}$$

with the matrix $Q \in G_+$, belonging to the group associated with the subalgebra $\mathfrak{g}_+$, thus being a lower triangular matrix with the 2×2 blocks along the diagonal proportional to the identity. The projectors entering in (7.21) are explicitly given as follows. Given $a \in \mathfrak{gl}(\infty)$

$$a = (a)_- + (a)_0 + (a)_+$$

$$= (a)_{\mathfrak{t}} + (a)_{\mathfrak{n}} \tag{7.23}$$

$$= \left(\left((a)_- - J\,(a)_+^\top\, J \right) + \frac{1}{2}\left((a)_0 - J\,(a)_0^\top\, J \right) \right) + \left(\left((a)_+ + J\,(a)_+^\top\, J \right) + \frac{1}{2}\left((a)_0 + J\,(a)_0^\top\, J \right) \right).$$

We now will follow the approach shown in [12, 13, 117] to express the Pfaff lattice in terms of the corresponding Pfaffian τ-functions. As for the Toda lattice, the τ-function is proportional to the partition function for the symmetric ensemble determined in (7.13)

$$Z_n^{(1)}(t) = c_n \int_{\mathbb{R}^n} |\Delta_n(z)| \prod_{k=1}^n \rho_t(dz_k). \tag{7.24}$$

Let us introduce the inner product on the skew-symmetric weight $\rho_t(y,z) = -\rho_t(z,y)$

$$\begin{aligned}\langle f, g \rangle_t &= \int\int_{\mathbb{R}^2} f(y)\,g(z)\,\rho_t(y,z)\,dy\,dz \\ &= \int\int_{\mathbb{R}^2} f(y)\,g(z)\,\varepsilon(y-z)\,e^{-V(y)-V(z)+\sum_{i=1}^\infty t_i(y^i+z^i)}\,dy\,dz,\end{aligned} \tag{7.25}$$

where $\varepsilon(x) = \mathrm{sgn}(x)$ and $\varepsilon(0) = 0$. The moments matrix is, in this case, given by

$$m_n(t) = \left(\mu_{ij}(t)\right)_{0\le i,j<n} = \left(\langle y^i, z^j \rangle_t\right)_{0\le i,j<n}, \tag{7.26}$$

that is skew-symmetric. Due to the form of the inner product, for the moments $\mu_{ij}(t)$ we have

$$\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j} + \mu_{i,j+k}. \tag{7.27}$$

For the corresponding semi-infinite moments matrix $m_\infty(t)$ this leads to

$$\frac{\partial m_\infty(t)}{\partial t_k} = \Lambda^k m_\infty(t) + m_\infty(t)\Lambda^{\top k}, \tag{7.28}$$

where Λ is the shift matrix mentioned above. The moments matrix so constructed admits a unique decomposition in terms of the inverse of the matrix of the aforementioned group G_+ and the semi-infinite matrix J, as

$$m_\infty(t) = Q(t)^{-1} J Q(t)^{\top-1}, \tag{7.29}$$

with the matrix $Q(t)$ of the form

$$Q(t) = \begin{pmatrix} \ddots & 0 & 0 & 0 & 0 & \cdots \\ & Q_{2n,2n} & 0 & 0 & 0 & \cdots \\ & 0 & Q_{2n,2n} & 0 & 0 & \cdots \\ * & * & Q_{2n+2,2n+2} & 0 & \cdots \\ * & * & 0 & Q_{2n+2,2n+2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7.30}$$

We can now state the theorem by Adler and van Moerbeke[2].

Theorem 7.1.1 *The τ-functions defined as Pfaffian of the moments matrix*

$$\tau_{2n}(t) := \mathrm{pf}\, m_{2n}(t) = (\det m_{2n}(t))^{1/2} \propto Z^{(1)}_{2n}(t), \tag{7.31}$$

(i) *satisfy the equation in the Pfaff-KP hierarchy*

$$\left(s_{k+4}(\tilde{\partial}) - \frac{1}{2} \partial_{t_1} \partial_{t_{k+3}} \right) \tau_{2n}(t) \circ \tau_{2n}(t) = s_k(\tilde{\partial})\, \tau_{2n+2}(t) \circ \tau_{2n-2}(t), \qquad k = 0, 1, 2, \ldots; \tag{7.32}$$

(ii) *constitute the elements of the Pfaff lattice $L(t) = Q(t) \Lambda\, Q(t)^{-1}$*

$$L(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ * & \partial_{t_1} \log \tau_2 & \left(\frac{\tau_4 \tau_0}{\tau_2^2} \right)^{1/2} & 0 & 0 & 0 & \ddots \\ * & * & -\partial_{t_1} \log \tau_2 & 1 & 0 & 0 & \ddots \\ * & * & * & \partial_{t_1} \log \tau_4 & \left(\frac{\tau_6 \tau_2}{\tau_4^2} \right)^{1/2} & 0 & \ddots \\ * & * & * & * & -\partial_{t_1} \log \tau_4 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{7.33}$$

---

[2]The theorem here presented is in the form reported in [117], where the results of several works are collected (see e.g. [12, 13, 7]).

for which the commuting equations in different flows are (7.21)

$$\frac{\partial L}{\partial t_k} = -\left[\left(L^k\right)_{\text{t}}, L\right] = \left[\left(L^k\right)_{\text{n}}, L\right];$$
(7.34)

(*iii*) *enter in the definition of the class of eigenvectors of L $q(t,z) = (q_n(t,z))_{n\geq 0}$, satisfying*

$$L(t)\,q(t,z) = z\,q(t,z).$$
(7.35)

Here, $q_n(t,z)$ are the n-th degree polynomials in z, skew-orthonormal with respect to the t-dependent inner product

$$\langle q_i(t,z), q_j(t,z)\rangle_t = J_{ij}.$$
(7.36)

They admit the representation $(q_n(z,t))_{n\geq 0} = Q(t)\chi(z)$, with $\chi(z) = \left(1, z, z^2, \dots\right)^{\top}$, and

$$q_{2n}(t,z) = z^{2n}\,h_{2n}^{-1/2}\,\frac{\tau_{2n}\left(t - [z^{-1}]\right)}{\tau_{2n}(t)}, \qquad \text{with } h_{2n} = \frac{\tau_{2n+2}(t)}{\tau_{2n}(t)}$$

$$q_{2n+1}(t,z) = z^{2n}\,h_{2n}^{-1/2}\,\frac{1}{\tau_{2n}(t)}\left(z + \frac{\partial}{\partial t_1}\right)\tau_{2n}\left(t - [z^{-1}]\right).$$
(7.37)

In order to prove the formula (7.31), we consider the integral in the definition of the partition function $Z_n^{(1)}(t)$ in (7.13). It involves the Vandermonde determinant (see (6.47))

$$\int_{\mathbb{R}^{2n}} |\Delta_{2n}(z)| \prod_{k=1}^{2n} \rho_t(z_k)\,dz_k =$$

$$= (2n)! \int_{-\infty < z_1 < z_2 < \dots < z_{2n} < \infty} \det\left(z_{j+1}^i\right)_{0 \leq i,j \leq 2n-1} \prod_{k=1}^{2n} \rho_t(z_k)\,dz_k,$$
(7.38)

where we impose an ordering of the eigenvalues to remove the absolute value of the determinant and, as a consequence of this, a factorial factor appears in front of the integral.

Symmetric Matrix Ensemble

We consider a shift of the indices i and j

$$(2n)! \int_{-\infty < z_1 < z_2 < \ldots < z_n < \infty} \det\left(z_{j+1}^i\right)_{0 \leq i,j \leq 2n-1} \prod_{k=1}^{2n} \rho_t(z_k)\, dz_k =$$

$$= (2n)! \int_{-\infty < z_1 < z_2 < \ldots < z_n < \infty} \det\left(z_j^{i-1}\right)_{1 \leq i,j \leq 2n} \prod_{k=1}^{2n} \rho_t(z_k)\, dz_k \qquad (7.39)$$

$$= (2n)! \int_{-\infty < z_1 < z_2 < \ldots < z_n < \infty} \det\left(z_j^{i-1} \rho_t(z_j)\right)_{1 \leq i,j \leq 2n} \prod_{k=1}^{2n} dz_k\,.$$

Given the ordering of the eigenvalues and the fact that z_1 appears only in the first column of the matrix $\left(z_j^{i-1} \rho_t(z_j)\right)_{1 \leq i,j \leq 2n}$, this can be integrated for each element as

$$F_i(z_2) = \int_{-\infty}^{z_2} \rho_t(z_1) z_1^{i-1}\, dz_1\,, \qquad \forall i = 1, \ldots, 2n \qquad (7.40)$$

and i-th element of the column substituted with $F_i(z_2)$

$$\det\left(z_j^{i-1}\rho_t(z_j)\right)_{1 \leq i,j \leq 2n} = \begin{vmatrix} \rho_t(z_1) & \rho_t(z_2) & \ldots & \rho_t(z_{2n}) \\ z_1\,\rho_t(z_1) & z_2\,\rho_t(z_2) & \ldots & z_{2n}\,\rho_t(z_{2n}) \\ \vdots & \vdots & & \vdots \\ z_1^{2n-1}\,\rho_t(z_1) & z_2^{2n-1}\,\rho_t(z_2) & \ldots & z_{2n}^{2n-1}\,\rho_t(z_{2n}) \end{vmatrix}.$$

$(F_i(z_2))_{1 \leq i \leq 2n}$

$$(7.41)$$

After the substitution, the eigenvalue z_2 appears in the first two columns. We can reiterate the procedure for z_3, substituting the third column with

$$F_i(z_4) = \int_{-\infty}^{z_4} \rho_t(z_3) z_3^{i-1}\, dz_3\,, \qquad \forall i = 1, \ldots, 2n\,, \qquad (7.42)$$

and then subtracting $F_i(z_2)$

$$F_i(z_4) - F_i(z_2) = \int_{z_2}^{z_4} \rho_t(z) z^{i-1}\, dz\,, \qquad \forall i = 1, \ldots, 2n\,. \qquad (7.43)$$

In this way, all the variables $z_1, z_3, z_5, \ldots$ can be integrated out and the expression in the

last row of (7.39) becomes

$$(2n)! \int_{-\infty < z_2 < z_4 < \ldots < z_{2n} < \infty} \prod_{k=1}^{n} \rho_t(z_{2k}) \, dz_{2k}$$

$$\det \left(F_i(z_2), z_2^i, F_i(z_4) - F_i(z_2), \ldots, F_i(z_{2n}) - F_i(z_{2n-2}), z_{2n}^i \right)_{0 \le i \le 2n-1}. \tag{7.44}$$

We introduce the function

$$G_i(z) = F_i'(z) = \frac{\partial}{\partial z} \int_{-\infty}^{z} x^i \rho_t(x) \, dx = z^i \rho_t(z), \tag{7.45}$$

and using the invariance of the determinant with respect to the addition and subtraction of columns, the expression in (7.44) takes the form

$$(2n)! \int_{-\infty < z_2 < z_4 < \ldots < z_{2n} < \infty} \det \left(F_i(z_2), G_i(z_2), \ldots, F_i(z_{2n}), G_i(z_{2n}) \right)_{0 \le i \le 2n-1} \prod_{k=1}^{n} dz_{2k} =$$

$$= \frac{(2n)!}{n!} \int_{\mathbb{R}^n} \det \left(F_i(z_1), G_i(z_1), \ldots, F_i(z_n), G_i(z_n) \right)_{0 \le i \le 2n-1} \prod_{k=1}^{n} dz_k, \tag{7.46}$$

where in the last step the ordering of eigenvalues is removed invoking the symmetry of the expression and this last can be seen as a sum over n terms of the kind α_{ij}

$$\alpha_{ij} = \int_{\mathbb{R}} \left(F_i(z) G_j(z) - F_j(z) G_i(z) \right) dz$$

$$= \int_{-\infty < y < x < \infty} \left(G_i(y) G_j(x) - G_j(y) G_i(x) \right) dx \, dy$$

$$= \frac{1}{2} \int_{\mathbb{R}^2} G_i(x) G_j(y) \, \varepsilon(x-y) \, dx \, dy, \qquad \text{with } \varepsilon(z) = \begin{cases} 1 & z > 0 \\ -1 & z < 0 \\ 0 & z = 0, \end{cases} \tag{7.47}$$

using (7.45).

Thus, the (7.46) brings to

$$\frac{(2n)!}{n!} \sum_{\sigma \in S_{2n}} (-1)^\sigma \prod_{k=0}^{2n-1} \alpha_{\sigma(k)\sigma(k+1)} =$$

$$= \frac{(2n)!}{2^n n!} \sum_{\sigma \in S_{2n}} (-1)^\sigma \prod_{k=0}^{2n-1} \left(\int_{\mathbb{R}^2} x^{\sigma(k)} y^{\sigma(k+1)} e^{\sum_i t_i (x^i + y^i)} \varepsilon(x-y) \rho(x) \rho(y) \, dx \, dy \right) \quad (7.48)$$

$$= (2n)! \, \mathrm{pf} \, m_{2n}(t) = (2n)! \, \tau_{2n}(t),$$

where we recognize the Pfaffian τ-function, that in the case of a matrix with an even number of rows and columns is given by

$$\mathrm{pf} \, m_{2n}(t) = (\det m_{2n}(t))^{1/2} . \quad (7.49)$$

The Pfaffian τ-functions satisfy the bilinear identity [13, 7]

$$\mathrm{Res}_{z=\infty} \left\{ \tau_{2n}\left(t - [z^{-1}]\right) \tau_{2m+2}\left(t' + [z^{-1}]\right) e^{\xi(t-t',z)} z^{2n-2m-2} \right\}$$

$$+ \mathrm{Res}_{z=0} \left\{ \tau_{2n+2}(t - [z]) \tau_{2m}(t' + [z]) e^{\xi(t'-t,z)} z^{2n-2m} \right\} = 0, \qquad \forall t, t' \in \mathbb{C}, \quad (7.50)$$

with $\xi(t,z)$ defined in (6.51). Considering the change of variables $(t,t') \to (x,y)$ as in (6.54), the Schur polynomials (6.52) and the Taylor expansion in $y = (t-t')/2$, we obtain

$$\frac{1}{2\pi i} \oint_{z=\infty} e^{-\sum_k 2y_k z^k} \tau_{2n}\left(x - y - [z^{-1}]\right) \tau_{2m+2}\left(x + y + [z^{-1}]\right) z^{2n-2m-2} \, dz$$

$$+ \frac{1}{2\pi i} \oint_{z=0} e^{\sum_k 2y_k z^k} \tau_{2n+2}(x - y - [z]) \tau_{2m}(x + y + [z]) z^{2n-2m} \, dz =$$

$$= \frac{1}{2\pi i} \oint_{z=\infty} \sum_{j=0}^\infty z^j s_j(-2y) e^{\sum_k -y_k \partial_k} \sum_{k=0}^\infty z^{-k} s_k(-\tilde\partial) \tau_{2n}(x) \circ \tau_{2m+2}(x) z^{2n-2m-2} \, dz$$

$$+ \frac{1}{2\pi i} \oint_{z=0} \sum_{j=0}^\infty z^{-j} s_j(2y) e^{\sum_k -y_k \partial_k} \sum_{k=0}^\infty z^k s_k(\tilde\partial) \tau_{2n+2}(x) \circ \tau_{2m}(x) z^{2n-2m} \, dz \quad (7.51)$$

$$= \sum_{j-k+2n-2m=1} s_j(-2y) e^{\sum_i -y_i \partial_i} s_k(-\tilde\partial) \tau_{2n}(x) \circ \tau_{2m+2}(x)$$

$$+ \sum_{k-j+2n-2m=-1} s_j(2y) e^{\sum_i -y_i \partial_i} s_k(\tilde\partial) \tau_{2n+2}(x) \circ \tau_{2m}(x)$$

$$= \ldots + y_k \left(\left(\frac{1}{2} \partial_{x_1} \partial_{x_k} - s_{k+1}(\tilde\partial) \right) \tau_{2n}(x) \circ \tau_{2n}(x) + s_{k-3}(\tilde\partial) \tau_{2n+2}(x) \circ \tau_{2n-2}(x) \right) + \ldots .$$

With $x \to t$, imposing that the coefficients of y_k are zero, we identify the Pfaff KP hier-

achy (7.32).

The procedure to build the Pfaff lattice from the matrix $Q(t)$ described at the beginning of the section allows us to write the elements in the diagonal and in the above diagonal showed in (7.33) in terms of the sequence of Pfaffian τ-functions explicitly. The lower triangular part is composed of terms involving combinations of Schur polynomials whose form is not specified. In what follows, we will focus on a specific reduction of the Pfaff lattice, obtained selecting the even times only in the weight of the inner product, mimicking the way in which Volterra is obtained from Toda. A study of the equations of the flows obtained for the first even times in the reduction shows that the above mentioned expressions given in [7] for the fields occupying the first lower diagonal are valid only up to a truncated finite lattice for $n = 4$ and cannot be generalised[3].

For the last part of the property (*ii*), we consider the equations for the flows of the moments matrix (7.28) in conjugation with the matrix $Q(t)$

$$
\begin{aligned}
0 &= Q(t)\left(\Lambda^k m_\infty(t) + m_\infty(t)\Lambda^{\top k} - \frac{\partial m_\infty(t)}{\partial t_k}\right)Q(t)^\top \\
&= \left(Q(t)\Lambda^k Q(t)^{-1}\right)J - \left(J\,Q(t)^{\top -1}\Lambda^{\top k}Q(t)^\top J\right)J + \frac{\partial Q(t)}{\partial t_k}Q(t)^{-1}J - \left(J\,Q(t)^{-1\top}\frac{\partial Q(t)^\top}{\partial t_k}J\right)J \\
&= \left(L(t)^k + \frac{\partial Q(t)}{\partial t_k}Q(t)^{-1}\right) - J\left(L(t)^k + \frac{\partial Q(t)}{\partial t_k}Q(t)^{-1}\right)^\top J,
\end{aligned}
$$

$$(7.52)$$

where we use the definition of the Pfaff lattice and the property $J^2 = -I$. Evaluating the projections $(\ )_0$ and $(\ )_\pm$, corresponding to selecting 2×2 blocks along the diagonal and upper/lower triangular part with zero elements in the 2×2 blocks along the diagonal respectively, we obtain the equations for the commuting vector fields (7.34).

Finally, for (*iii*), from the skew-Borel decomposition, we get the skew-orthonormality of the polynomials $q_n(z, t)$

$$
\langle q_k(z,t), q_l(z,t)\rangle_t\big|_{k,l \geq 0} = Q(t)\left(\langle y^k, z^l\rangle_t\right)_{k,l \geq 0}Q(t)^\top = Q(t)\,m_\infty(t)\,Q(t)^\top = J. \qquad (7.53)
$$

---

[3]Whenever we have explored an aspect of the lattices here reported, we have of course considered a truncated version of the lattices, for which the boundary effects have been neglected.

Then, using the first of (6.66) and the definition of the Pfaff lattice, we get

$$
\begin{aligned}
L(t)\,q(z,t) &= Q(t)\,\Lambda\,Q(t)^{-1}\,Q(t)\,\chi(z) = Q(t)\,\Lambda\,\chi(z)\\
&= z\,Q(t)\,\chi(z) = z\,q(z,t).
\end{aligned}
\tag{7.54}
$$

Hence, the skew-orthogonal polynomials are eigenvectors of the Pfaff lattice.

7.2 Lattice equations in the first two flows for the Pfaff hierarchy

In the following, we will introduce a suitable notation for the fields constituting the Pfaff lattice. This is meant to highlight how certain fields evolve similarly, and will be helpful later on in clarifying the underlying double chain structure for the field variables.

We start by recalling the form of the Pfaff lattice introduced in section 7.1.2

$$
L(t) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \dots \\[2mm]
* & \partial_{t_1}\log\tau_2 & \left(\dfrac{\tau_4\,\tau_0}{\tau_2^2}\right)^{1/2} & 0 & 0 & 0 & \ddots \\[2mm]
* & * & -\partial_{t_1}\log\tau_2 & 1 & 0 & 0 & \ddots \\[2mm]
* & * & * & \partial_{t_1}\log\tau_4 & \left(\dfrac{\tau_6\,\tau_2}{\tau_4^2}\right)^{1/2} & 0 & \ddots \\[2mm]
* & * & * & * & -\partial_{t_1}\log\tau_4 & 1 & \ddots \\[2mm]
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},
\tag{7.55}
$$

with the fields of the main diagonal and the upper above diagonal explicitly given in terms of τ-functions of the system. The matrix $L(t)$ satisfies the Lax equations (7.34)

$$
\frac{\partial L}{\partial t_k} = -\left[\left(L^k\right)_{\mathrm{t}},\,L\right].
\tag{7.56}
$$

From (7.23) the action of the projection labelled by t in (7.56) on a 2×2 blocks matrix a

$$
(a)_{\mathrm{t}} = \left((a)_- - J\,(a)_+^\top\,J\right) + \frac{1}{2}\left((a)_0 - J\,(a)_0^\top\,J\right),
\tag{7.57}
$$

131

where J is the skew-symmetric matrix (7.16), $(\ )_{\pm}$ is the projection selecting the upper/lower triangular part without the 2×2 blocks along the main diagonal and $(\ )_0$ selects the 2×2 blocks along the main diagonal.

The τ-functions are proportional to the partition function describing $\mathcal{S}_n$ for $2n \times 2n$ symmetric matrices

$$Z_{2n}^{(1)}(t) = c_{2n} \int_{\mathbb{R}^{2n}} |\Delta_{2n}(z)| \prod_{k=1}^{2n} \rho_t(dz_k) = c_{2n} \int_{\mathbb{R}^{2n}} |\Delta_{2n}(z)| \prod_{k=1}^{2n} e^{-\frac{1}{2}z_k^2 + \sum_{i=1}^{\infty} t_i z_k^i} \, dz_k, \qquad (7.58)$$

as discussed in section 7.1.1, see in particular equation (7.13). Setting to zero the coupling constants t_i in (7.58), we find the free theory given by the Gaussian Orthogonal Ensemble. We recall that the τ-function is defined as the Pfaffian of the moments matrix

$$\tau_{2n}(t) = \mathrm{pf}(m_{2n}(t)) = (\det(m_{2n}(t)))^{1/2}$$

$$\tau_{2n}(t) = \frac{1}{2^n \, n!} \sum_{\sigma \in S_{2n}} (-1)^{\sigma} \prod_{k=0}^{2n-1} \left(\int_{\mathbb{R}^2} x^{\sigma(k)} y^{\sigma(k+1)} e^{-\frac{1}{2}(x^2 + y^2) + \sum_i t_i(x^i + y^i)} \varepsilon(x - y) \, dx \, dy \right). \tag{7.59}$$

We observe that Lax equations for the Pfaff lattice (7.56) can be recast in the form of a two-component infinite chain. We introduce the following notation for the entries of the lattice

$$L(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ w_1^{-1} & v_1^0 & w_1^0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ v_1^{-1} & w_1^1 & -v_1^0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ w_1^{-2} & v_1^1 & w_2^{-1} & v_2^0 & w_2^0 & 0 & 0 & 0 & 0 & \dots \\ v_1^{-2} & w_1^2 & v_2^{-1} & w_2^1 & -v_2^0 & 1 & 0 & 0 & 0 & \dots \\ w_1^{-3} & v_1^2 & w_2^{-2} & v_2^1 & w_3^{-1} & v_3^0 & w_3^0 & 0 & 0 & \dots \\ v_1^{-3} & w_1^3 & v_2^{-2} & w_2^2 & v_3^{-1} & w_3^1 & -v_3^0 & 1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{7.60}$$

We distinguish between the entries in the odd and even diagonals of the lattice, respectively w_n^k and v_n^k, with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. The first upper diagonal is the highest non-zero odd diagonal of the lattice, and the non-constant fields belonging to it are named w_n^0.

Lattice equations in the first two flows for the Pfaff hierarchy

The main diagonal is populated by the fields v_n^0. For the fields appearing in the lower triangular part of the lattice, the upper index in absolute value identifies the diagonal to which the field belongs, and negative and positive values refer to odd and even positions of the diagonal respectively. Hence, the fields w_n^k occupy the odd positions of the $(2|k|-1)$-th below diagonal for $k < 0$ and the even positions for $k > 0$. The same is valid for the fields v_n^k in the $(2|k|)$-th below diagonal. From (7.56), we can investigate the evolution for the fields v_n^k and w_n^k with respect to the different times. With the chosen notation the structure characterising the matrix L in 2×2 blocks is evident

$$
L(t) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
w_1^{-1} & v_1^0 & w_1^0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
v_1^{-1} & w_1^1 & -v_1^0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
w_1^{-2} & v_1^1 & w_2^{-1} & v_2^0 & w_2^0 & 0 & 0 & 0 & 0 & \cdots \\
v_1^{-2} & w_1^2 & v_2^{-1} & w_2^1 & -v_2^0 & 1 & 0 & 0 & 0 & \cdots \\
w_1^{-3} & v_1^2 & w_2^{-2} & v_2^1 & w_3^{-1} & v_3^0 & w_3^0 & 0 & 0 & \cdots \\
v_1^{-3} & w_1^3 & v_2^{-2} & w_2^2 & v_3^{-1} & w_3^1 & -v_3^0 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}.
$$

We can rewrite this in terms of 2×2 blocks b_{ij} as follows

$$
L(t) = \begin{pmatrix}
\ddots & \ddots & 0 & 0 & 0 & 0 & \cdots \\
\ddots & \ddots & 0 & 0 & 0 & \cdots \\
\cdots & b_{1\,j-1} & b_{0\,j-1} & 0 & 0 & \cdots \\
\cdots & b_{ij} & \cdots & b_{1\,j} & b_{0\,j} & 0 & \cdots \\
& & \cdots & b_{1\,j+1} & b_{0\,j+1} & 0 \\
& & & & \ddots & \ddots
\end{pmatrix}, \tag{7.61}
$$

where the blocks are given in terms of the previously introduced fields v_n^k and w_n^k, as

$$
b_{ij} = \begin{pmatrix} w_{i-j+1}^{-i} & v_{i-j+1}^{i-1} \\ v_{i-j+1}^{-i} & w_{i-j+1}^{i} \end{pmatrix}, \qquad b_{0j} = \begin{pmatrix} w_j^0 & 0 \\ v_j^0 & 1 \end{pmatrix}, \qquad \text{with } j \leq i, i \geq 0. \tag{7.62}
$$

133

In the following, we will display the equations for the first two flows (t_1, t_2) in terms of the fields v_n^k and w_n^k. We start by considering the equations for the t_1-flow in the fields v_n^k

$$\partial_{t_1} v_n^k = \frac{1}{2}\left(v_{n-1}^0 + v_n^0 - v_{n-k-1}^0 - v_{-k+n}^0\right)v_n^k + w_n^{k-1} - w_n^0 w_{n+1}^{-(k+1)}$$

$$- w_n^{-1} w_n^{-k} - w_{n-1}^0 w_{n-1}^{-(k-1)}, \qquad k < -1$$

$$\partial_{t_1} v_n^{-1} = \frac{1}{2}\left(v_{n-1}^0 - v_{n+1}^0\right)v_n^{-1} + w_n^{-2} - w_n^0 - w_n^{-1} w_n^1 - w_{n-1}^0 w_{n-1}^2$$

$$\partial_{t_1} v_n^0 = w_n^0 w_n^1$$

$$\partial_{t_1} v_n^1 = \frac{1}{2}\left(v_{n+1}^0 - v_{n-1}^0\right)v_n^1 - w_n^{-2} + w_n^0 + w_{n+1}^{-1} w_n^1 + w_{n+1}^0 w_n^2$$

$$\partial_{t_1} v_n^k = \frac{1}{2}\left(v_{k+n}^0 + v_{k+n-1}^0 - v_n^0 - v_{n-1}^0\right)v_n^k + w_{n+k-1}^0 w_n^{k-1} + w_{n+k}^{-1} w_n^k$$

$$+ w_{n+k}^0 w_n^{k+1} - w_n^{-(k+1)}, \qquad k > 1.$$

The equations for the fields w_n^k in the time t_1 are

$$\partial_{t_1} w_n^k = \frac{1}{2}\left(v_{n-k-1}^0 + v_{n-k-2}^0 + v_n^0 + v_{n-1}^0\right)w_n^k + w_{n-k-2}^0 v_n^{k+2} - w_n^0 v_{n+1}^{-(k+2)}$$

$$+ w_{n-k-1}^{-1} v_n^{k+1} - w_n^{-1} v_n^{-(k+1)} + w_{n-k-1}^0 v_n^k - w_{n-1}^0 v_{n-1}^{-k}, \qquad k < -1$$

$$\partial_{t_1} w_n^{-1} = w_n^0 v_n^{-1} - w_{n-1}^0 v_{n-1}^1$$

$$\partial_{t_1} w_n^0 = \frac{1}{2}\left(v_{n+1}^0 - 2v_n^0 + v_{n-1}^0\right)w_n^0$$

$$\partial_{t_1} w_n^k = -\frac{1}{2}\left(v_{n+k}^0 + v_{n+k-1}^0 + v_n^0 + v_{n-1}^0\right)w_n^k + v_n^k - v_n^{-k}, \qquad k > 0.$$

Lattice equations in the first two flows for the Pfaff hierarchy

For the second flow in the time t_2 we have for the fields v_n^k

$$\partial_{t_2} v_n^k = -\frac{1}{2} v_n^k \left((v_{n-k}^0)^2 - (v_{n-k-1}^0)^2 - (v_n^0)^2 + (v_{n-1}^0)^2 + w_{n-k}^0 w_{n-k}^1 - w_{n-k-1}^0 w_{n-k-1}^1 \right.$$

$$\left. - w_n^0 w_n^1 + w_{n-1}^0 w_{n-1}^1 \right) + w_{n-k}^0 v_n^{k-1} - w_{n-k-1}^0 v_n^{k+1} + w_n^0 v_{n+1}^{k+1} - w_{n-1}^0 v_{n-1}^{k-1}$$

$$+ \left(v_{n-k}^0 - v_{n-k-1}^0 \right) w_n^{k-1} - \left(v_n^0 - v_{n-1}^0 \right) w_n^{-1} w_n^{-k}$$

$$- \left(w_n^0 v_n^{k+1} - w_{n-1}^0 v_{n-1}^{-(k+1)} \right) w_n^{-k}, \qquad k < 0$$

$$\partial_{t_2} v_n^0 = w_n^0 \left(v_n^1 + v_n^{-1} \right)$$

$$\partial_{t_2} v_n^k = \frac{1}{2} v_n^k \left((v_{n+k}^0)^2 - (v_{n+k-1}^0)^2 + (v_n^0)^2 - (v_{n-1}^0)^2 + w_{n+k}^0 w_{n+k}^1 - w_{n+k-1}^0 w_{n+k-1}^1 \right.$$

$$\left. + w_n^0 w_n^1 - w_{n-1}^0 w_{n-1}^1 \right) + w_{n+k}^0 v_n^{k+1} - w_{n+k-1}^0 v_n^{k-1} + w_n^0 v_{n+1}^k - w_{n-1}^0 v_{n-1}^{k+1}$$

$$+ \left(v_{n+k}^0 - v_{n+k-1}^0 \right) w_{n+k}^{-1} w_n^k - \left(v_n^0 - v_{n-1}^0 \right) w_n^{-(k+1)}$$

$$+ \left(w_{n+k}^0 v_{n+k}^{-(k-1)} - w_{n+k-1}^0 v_{n+k-1}^{k-1} \right) w_n^k, \qquad k > 0$$

Finally, the evolution of the fields w_n^k in t_2 is given by the equations

$$\partial_{t_2} w_n^k = \frac{1}{2} w_n^k \left((v_{n-k-1}^0)^2 - (v_{n-k-2}^0)^2 + (v_n^0)^2 - (v_{n-1}^0)^2 \right.$$

$$\left. + w_{n-k-1}^0 w_{n-k-1}^1 - w_{n-k-2}^0 w_{n-k-2}^1 + w_n^0 w_n^1 - w_{n-1}^0 w_{n-1}^1 \right)$$

$$+ w_{n-k-1}^0 w_n^{k-1} - w_{n-k-2}^0 w_n^{k+1} + w_n^0 w_{n+1}^{k+1}$$

$$- w_{n-1}^0 w_{n-1}^{k-1} + \left(v_{n-k-1}^0 - v_{n-k-2}^0 \right) w_{n-k-1}^{-1} v_n^{k+1} - \left(v_n^0 - v_{n-1}^0 \right) w_n^{-1} v_n^{-(k+1)}$$

$$+ \left(w_{n-k-1}^0 v_{n-k-1}^{k+2} + w_{n-k-2}^0 v_{n-k-2}^{-(k+2)} \right) v_n^{k+1}$$

$$- \left(w_n^0 v_n^{k+2} + w_{n-1}^0 v_{n-1}^{-(k+2)} \right) v_n^{-(k+1)}, \qquad k < 0$$

$$\partial_{t_2} w_n^0 = \frac{1}{2} w_n^0 \left((v_{n+1}^0)^2 - (v_{n-1}^0)^2 + w_{n+1}^0 w_{n+1}^1 - w_{n-1}^0 w_{n-1}^1 \right) + w_n^0 \left(w_{n+1}^{-1} - w_{n-1}^{-1} \right)$$

$$\partial_{t_2} w_n^1 = -\frac{1}{2} w_n^1 \left((v_{n+1}^0)^2 - (v_{n-1}^0)^2 + w_{n+1}^0 w_{n+1}^1 - w_{n-1}^0 w_{n-1}^1 \right) + w_{n+1}^0 w_n^2 - (w_n^0)^2$$

$$+ w_n^0 w_{n+1}^0 - w_{n-1}^0 w_{n-1}^2 + \left(v_{n+1}^0 - v_n^0 \right) v_n^1 - \left(v_n^0 - v_{n-1}^0 \right) v_n^{-1}$$

$$\partial_{t_2} w_n^k = -\frac{1}{2} w_n^k \left((v_{n+k}^0)^2 - (v_{n+k-1}^0)^2 + (v_n^0)^2 - (v_{n-1}^0)^2 + w_{n+k}^0 \, w_{n+k}^1 \right.$$

$$\left. - w_{n+k-1}^0 \, w_{n+k-1}^1 + w_n^0 \, w_n^1 - w_{n-1}^0 \, w_{n-1}^1 \right) + w_{n+k}^0 \, w_n^{k+1} - w_{n+k-1}^0 \, w_n^{k-1}$$

$$+ w_n^0 \, w_{n+1}^{k-1} - w_{n-1}^0 \, w_{n-1}^{k+1} + \left(v_{n+k}^0 - v_{n+k-1}^0 \right) v_n^k - \left(v_n^0 - v_{n-1}^0 \right) v_n^{-k}, \qquad k > 1.$$

We can see how the complexity of the equations increases for higher flows, comparing the evolution of the fields in t_2 with those in t_1. Nevertheless, it is worth noticing that the number of fields on which these expressions depend remains finite. In the following we will see how the number of elements of every equation reduces if we consider a particular restriction, inspired by the form of initial condition of the Pfaff lattice related to the symmetric matrix ensemble.

7.3 The even Pfaff lattice

We look for a suitable reduction of the Pfaff lattice with the aim of simplifying the structure, inspired by how the Volterra lattice is determined starting from the Toda lattice. We consider the initial datum $t = 0$, for which the number of field variables entering the lattice is considerably reduced. In particular, only the fields w_n^k survive in the new configuration. Then we look for a suitable selection of the coupling constants such that the expression for the lattice is completely given in terms of w_n^k, the even Pfaff lattice.

7.3.1 Initial condition with the GOE

We now consider the initial condition for the Lax matrix $L(t = 0)$. From (7.33), we have the explicit form in terms of τ-functions for the functions $w_n^0(t)$ and $v_n^0(t)$. Specifically, the component $w_n^0(t)$ can be expressed as follows

$$w_n^0(t) = \left(\frac{\tau_{2n+2}(t) \, \tau_{2n-2}(t)}{\tau_{2n}^2(t)} \right)^{1/2}. \tag{7.63}$$

and the component $v_n^0(t)$ as

$$v_n^0(t) = \partial_{t_1} \log \tau_{2n}(t). \tag{7.64}$$

The even Pfaff lattice

Both expressions can be evaluated for $t = 0$, emphasising that in this case the symmetric matrix ensemble reduces to the GOE. We will consider again the Selberg's integral (6.79), with in this case $\gamma = 1/2$ and $a = 1/2$, giving

$$\int_{\mathbb{R}^n} |\Delta(x)| \prod_{k=1}^{n} e^{-\frac{1}{2}x_k^2} \, dx_k = (2\pi)^{n/2} \prod_{k=1}^{n} \frac{\Gamma(1 + \frac{k}{2})}{\Gamma(1 + \frac{1}{2})}, \tag{7.65}$$

and since $\Gamma(1 + 1/2) = \sqrt{\pi}/2$ we have

$$\int_{\mathbb{R}^n} |\Delta(x)| \prod_{k=1}^{n} e^{-\frac{1}{2}x_k^2} \, dx_k = 2^{(n+2)/2} \pi^{(n-1)/2} \prod_{k=1}^{n} \Gamma\left(1 + \frac{k}{2}\right). \tag{7.66}$$

Recalling that the τ-function is in this case written in terms of the Pfaffian of the moment matrix with a skew-symmetric inner product and using the properties of the Gamma function, we have

$$\tau_{2n}(0) = \pi^{n/2} \prod_{k=0}^{n-1} 2^{-2k}(2k)!. \tag{7.67}$$

Therefore, equations (7.63) and (7.67) imply

$$w_n^0(0) = 2\sqrt{\pi}\sqrt{2n(2n-1)}. \tag{7.68}$$

To evaluate (7.64), we consider only the dependence on t_1 in the expression of the Pfaffian τ-function. Recalling that $t = \{t_1, t_2, t_3, \dots\}$, we have

$$\tau_{2n}(t_1, 0, 0, \dots) = \pi^{n/2} e^{\frac{nt_1^2}{2}} \prod_{k=0}^{n-1} 2^{-2k}(2k)!. \tag{7.69}$$

Hence, the fields constituting the main diagonal of the Pfaff lattice vanish

$$\partial_{t_1} \tau_{2n}(t)\Big|_{t=0} = n\, t_1\, \pi^{n/2} e^{\frac{nt_1^2}{2}} \prod_{k=0}^{n-1} 2^{-2k}(2k)!\Big|_{t=0} = 0 \qquad \Longrightarrow \qquad v_n^0(0) = 0. \tag{7.70}$$

We observe that the fields $v_n^k(0)$ for $k \neq 0$ vanish at $t = 0$ as well. This can be seen by considering that the moment matrix $m_{2n}(0)$ in the case of a symmetric weight is

$$m_{2n}(0) = \left(\mu_{i,j}\right)_{0 \leq i,j \leq n-1} = \begin{pmatrix} 0 & \mu_{0,1} & 0 & \mu_{0,3} & 0 & \mu_{0,5} & \cdots \\ -\mu_{0,1} & 0 & \mu_{1,2} & 0 & \mu_{1,4} & 0 & \cdots \\ 0 & -\mu_{1,2} & 0 & \mu_{2,3} & 0 & \mu_{2,5} & \cdots \\ -\mu_{0,3} & 0 & -\mu_{2,3} & 0 & \mu_{3,4} & 0 & \cdots \\ 0 & -\mu_{1,4} & 0 & -\mu_{3,4} & 0 & \mu_{4,5} & \cdots \\ -\mu_{0,5} & 0 & -\mu_{2,5} & 0 & -\mu_{4,5} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{7.71}$$

because of the skew-symmetry of the inner product.

When considering the decomposition of the moment matrix to obtain the matrix $Q(0)$ that allows to build the Pfaff lattice (see equations (7.29) and (7.30)), we have

$$Q(0) = \begin{pmatrix} Q_{0,0} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & Q_{0,0} & 0 & 0 & 0 & 0 & \cdots \\ Q_{3,1} & 0 & Q_{2,2} & 0 & 0 & 0 & \cdots \\ 0 & Q_{4,2} & 0 & Q_{2,2} & 0 & 0 & \cdots \\ Q_{5,1} & 0 & Q_{5,3} & 0 & Q_{4,4} & 0 & \cdots \\ 0 & Q_{6,2} & 0 & Q_{6,4} & 0 & Q_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{7.72}$$

and the corresponding Pfaff lattice has the form

$$L(0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ w_1^{-1} & 0 & w_1^0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & w_1^1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ w_1^{-2} & 0 & w_2^{-1} & 0 & w_2^0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & w_1^2 & 0 & w_2^1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ w_1^{-3} & 0 & w_2^{-2} & 0 & w_3^{-1} & 0 & w_3^0 & 0 & 0 & \cdots \\ 0 & w_1^3 & 0 & w_2^2 & 0 & w_3^1 & 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{7.73}$$

The even Pfaff lattice

In what follows, we will study the Pfaff lattice in the specific restriction for which the fields that are zero at $t = 0$ remain zero for different times.

7.3.2 The even reduction

We consider the symmetric matrix ensemble S_n with even power interactions specified by the partition function (7.13)

$$Z_{2n}^{(1)}(t) = c_{2n} \int_{\mathbb{R}^{2n}} |\Delta_{2n}(z)| \prod_{k=1}^{2n} e^{-\frac{z_k^2}{2} + \sum_{i=1}^{\infty} t_{2i} z_k^{2i}} \, dz_k. \tag{7.74}$$

We are going to show that it provides a solution to a reduction of the even Pfaff lattice, i.e. the commuting flows (7.56) associated to the even times t_{2k} only. We will see how the notation we have introduced is suitable for the description of this system, involving one type of fields only.

In this case, the equation $\tau_{2n}(t) = \text{pf}(m_{2n}(t))$ still holds with $m_{2n}(t) = (\mu_{ij}(t))_{0 \le i, j \le 2n-1}$ and in the inner product we select only the times labelled by even indices

$$\mu_{ij}(t) = \langle x^i, y^j \rangle_t = \int \int_{\mathbb{R}^2} x^i y^j \, \sigma(x - y) e^{\sum_{k \ge 1} t_{2k} (x^{2k} + y^{2k})} e^{-\frac{1}{2}(x^2 + y^2)} \, dx \, dy. \tag{7.75}$$

Hence, the moments matrix $m_{2n}(t)$ reads as

$$m_{2n}(t) = \left(\mu_{ij} \right)_{0 \le i, j \le 2n-1} = \begin{pmatrix} 0 & \mu_{01} & 0 & \mu_{03} & 0 & \mu_{05} & \cdots \\ -\mu_{01} & 0 & \mu_{12} & 0 & \mu_{14} & 0 & \cdots \\ 0 & -\mu_{12} & 0 & \mu_{23} & 0 & \mu_{25} & \cdots \\ -\mu_{03} & 0 & -\mu_{23} & 0 & \mu_{34} & 0 & \cdots \\ 0 & -\mu_{14} & 0 & -\mu_{34} & 0 & \mu_{45} & \cdots \\ -\mu_{05} & 0 & -\mu_{25} & 0 & -\mu_{45} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{7.76}$$

for all times $t = \{t_2, t_4, t_6, \dots\}$. The moments (7.75) satisfy the evolution equations

$$\frac{\partial \mu_{ij}}{\partial t_{2k}} = \mu_{i+2k,j} + \mu_{i,j+2k} \tag{7.77}$$

which imply for the semi-infinite moment matrix

$$\frac{\partial m_\infty}{\partial t_{2k}} = \Lambda^{2k} m_\infty + m_\infty \Lambda^{2k}. \tag{7.78}$$

We consider the reduction of the Lax equation (7.56) of the form

$$\frac{\partial L}{\partial t_{2k}} = \left[-(L^{2k})_{\rm t}, L \right], \tag{7.79}$$

with

$$L(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ w_1^{-1} & 0 & w_1^0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & w_1^1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ w_1^{-2} & 0 & w_2^{-1} & 0 & w_2^0 & 0 & 0 & 0 & 0 & \dots \\ 0 & w_1^2 & 0 & w_2^1 & 0 & 1 & 0 & 0 & 0 & \dots \\ w_1^{-3} & 0 & w_2^{-2} & 0 & w_3^{-1} & 0 & w_3^0 & 0 & 0 & \dots \\ 0 & w_1^3 & 0 & w_2^2 & 0 & w_3^1 & 0 & 1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{7.80}$$

i.e. the Lax matrix associated with $\mathcal{S}_n$ with even power interactions is obtained from the general one by setting the variables v_n^0, v_n^k identically equal to zero for any t. In other words, the partition function gives a solution to a reduction of the even Pfaff hierarchy which preserves the zeros of the initial Lax matrix $L(0)$ given by the expression (7.73).

The evolution equations for the first non trivial flow t_2 of the fields constituting the even Pfaff lattice read as

$$\partial_{t_2} w_n^k = \frac{1}{2}\left(w_n^k w_n^0 w_n^1 + w_n^k w_{n-k-1}^0 w_{n-k-1}^1 - w_n^k w_{n-1}^0 w_{n-1}^1 - w_n^k w_{n-k-2}^0 w_{n-k-2}^1\right)$$

$$+ w_{n+1}^{k+1} w_n^0 + w_n^{k-1} w_{n-k-1}^0 - w_{n-1}^{k-1} w_{n-1}^0 - w_n^{k+1} w_{n-k-2}^0, \qquad k < -1$$

$$\partial_{t_2} w_n^{-1} = w_n^0\left(w_n^{-1} w_n^1 + w_n^{-2} + w_n^0\right) - w_{n-1}^0\left(w_n^{-1} w_{n-1}^1 + w_{n-1}^{-2}\right) - \left(w_{n-1}^0\right)^2$$

$$\partial_{t_2} w_n^0 = \frac{1}{2}\left(w_{n+1}^0 w_{n+1}^1 - w_{n-1}^0 w_{n-1}^1\right) w_n^0 + \left(w_{n+1}^{-1} - w_n^{-1}\right) w_n^0 \tag{7.81}$$

$$\partial_{t_2} w_n^1 = \frac{1}{2}\left(w_{n-1}^0 w_{n-1}^1 w_n^1 - w_{n+1}^0 w_n^1 w_{n+1}^1\right) + w_{n+1}^0 w_n^2 - w_{n-1}^0 w_{n-1}^2$$

$$\partial_{t_2} w_n^k = \frac{1}{2}\left(w_{n-1}^0 w_{n-1}^1 w_n^k + w_{n+k-1}^0 w_{n+k-1}^1 w_n^k - w_n^0 w_n^1 w_n^k - w_{n+k}^0 w_{n+k}^1 w_n^k\right)$$

$$+ w_n^0 w_{n+1}^{k-1} + w_{n+k}^0 w_n^{k+1} - w_{n-1}^0 w_{n-1}^{k+1} - w_{n+k-1}^0 w_n^{k-1}, \qquad k > 1.$$

7.4 Thermodynamic limit and integrable hydrodynamic chain

We now consider the continuum limit of the equations for the Pfaff lattice, exploring the asymptotic properties of the symmetric matrix ensemble $\mathcal{S}_n$ for large n, with a focus on the case of even power interactions, where the Pfaff lattice is given by (7.80), satisfying the Lax equations (7.79).

As mentioned above, the lattice equations for the reduced even Pfaff hierarchy (7.81) constitute an infinite chain for the variables w_n^k, where $k \in \mathbb{Z}$ identifies the components of the chain and $n \in \mathbb{N}$ labels points on the lattice. As $n \to \infty$, for the variables w_n^k we have

$$w_{n+1}^k - w_n^k = O(\varepsilon), \qquad \varepsilon \to 0, \tag{7.82}$$

with ε such that $x = \varepsilon n$ remains finite. In the following, we derive the continuum limit equations for the chain and study the integrability at the leading order with respect to the ε expansion. We illustrate the result for the first equation of the hierarchy given by the t_2-flows. Our considerations extend to the flows in t_4 and t_6 as well, and we conjecture they hold for any equation of the hierarchy.

We introduce the interpolation function $w^k(x/\varepsilon)$ with $x = \varepsilon n$ so that $w^k(n) = w_n^k$, and

define

$$u^k(x) := w^k\left(\frac{x}{\varepsilon}\right)$$

with $u^k(x \pm \varepsilon) = w^k_{n\pm1}$. Substituting $u^k(x)$ into the equations (7.81), expanding in Taylor series for $\varepsilon \to 0$ and setting $t = \varepsilon\, t_2$, at the leading order $O(\varepsilon^0)$ we get the following system of PDEs

$$u^k_t = \left((k+2)u^{k+1} - ku^{k-1} + u^1u^k\right)u^0_x + u^0u^ku^1_x + u^0u^{k-1}_x + u^0u^{k+1}_x, \qquad k < 0$$

$$u^0_t = u^0u^1u^0_x + \left(u^0\right)^2 u^1_x + u^0u^{-1}_x$$

$$u^1_t = \left(2u^2 - \left(u^1\right)^2\right)u^0_x - u^0u^1u^1_x + u^0u^2_x$$

$$u^k_t = \left((k+1)u^{k+1} - (k-1)u^{k-1} - u^1u^k\right)u^0_x - u^0u^ku^1_x + u^0u^{k-1}_x + u^0u^{k+1}_x, \qquad k > 1$$

$$(7.83)$$

with the notation $f_t = \partial_t f$, $f_x = \partial_x f$. In particular, we note that the system (7.83) is an infinite chain of quasilinear PDEs of hydrodynamic type. In fact, the equations of the chain are of the form

$$u^k_t = a^k_0\, u^0_x + a^k_1\, u^1_x + a^k_{k-1}\, u^{k-1}_x + a^k_{k+1}\, u^{k+1}_x, \tag{7.84}$$

or equivalently in vector form

$$u_t = A(u)\,u_x, \qquad u = \left(\dots, u^{-1}, u^0, u^1, \dots\right)^\top, \tag{7.85}$$

where $A(u) = \left\{a^k_j\right\}^{+\infty}_{j,k=-\infty}$ is an infinite matrix such that $a^k_j = 0$ if $j \notin \{0, 1, k-1, k+1\}$ and

$$a^k_0 = \begin{cases} (k+2)u^{k+1} - ku^{k-1} + u^1u^k & \text{if } k < 0 \\ u^0u^1 & \text{if } k = 0 \\ (k+1)u^{k+1} - (k-1)u^{k-1} - u^1u^k & \text{if } k \geq 1 \end{cases} \qquad a^k_1 = \begin{cases} u^0u^k & \text{if } k \leq 0 \\ -u^0u^k & \text{if } k \geq 1 \end{cases}$$

$$a^k_{k-1} = \begin{cases} u^0 & \text{if } k \neq 1 \\ \left(2u^2 - (u^1)^2\right) & \text{if } k = 1 \end{cases} \qquad a^k_{k+1} = \begin{cases} u^0 & \text{if } k \neq 0 \\ (u^0)^2 & \text{if } k = 0 \end{cases}$$

Thermodynamic limit and integrable hydrodynamic chain

By applying the same procedure, one can construct a hierarchy of infinitely many commuting flows, each of them in the form of a hydrodynamic chain from the thermodynamic limit of the higher flows of the hierarchy (7.79).

The hydrodynamic chain (7.83) is integrable as it possesses an infinite hierarchy of commuting flows. In the following, we show that the hydrodynamic chain (7.83) is diagonalisable and integrable according to the criterion introduced and discussed in section 3.2, namely the existence of integrable hydrodynamic reductions in an arbitrary number of components.

Referring to definition 3.2.1 describing the chain class, and bearing in mind the form of the matrix $A(u)$ as specified in (7.4), we have the following

Proposition 7.4.1 *Given the chain* (7.83), *the associated matrix $A(u)$ in* (7.85) *belongs to the chain class.*

Now, we can construct the Nijenhuis and Haantjes tensors of the infinite (sufficiently sparse) matrix $A(u)$, to study the diagonalisability of the chain, as described in section 3.2. We state the following proposition.

Proposition 7.4.2 *Given the chain* (7.83), *the Haantjes tensor of the associated matrix $A(u)$ vanishes.*

The proof proceeds by direct inspection. We recall the form of the Nijenhuis tensor in (3.71)

$$N^i_{jk} = a^p_j(u)\,\partial_p a^i_k(u) - a^p_k(u)\,\partial_p a^i_j(u) - a^i_p(u)\left(\partial_j a^p_k(u) - \partial_k a^p_j(u)\right).$$

Observing that, by definition, N^i_{jk} is antisymmetric under the exchange of j and k, a direct calculation shows that $N^0_{jk} = 0$ for any j and k. Similarly, for $i \neq 0$ the only nonzero elements of N^i_{jk} are

$$N^i_{0\pm1},\, N^i_{0\,i},\, N^i_{0\,i\pm1},\, N^i_{1\,i\pm1}\, N^i_{-1\,i\pm1},\, N^i_{-1\,+1} \tag{7.86}$$

and their counterparts with the lower indices exchanged. These components can be computed for a generic value of i, yielding

▷ for $|i| > 2$

$$N_{01}^i = \begin{cases} u^0\left((i-1)u^{i-1} - (i+1)u^{i+1}\right) & \text{if } i > 2 \\ u^0\left(iu^{i-1} - (i+2)u^{i+1}\right) & \text{if } i < -2 \end{cases}$$

$$N_{0-1}^i = \begin{cases} (i-1)u^{i-1} + u^1 u^i - (i+1)u^{i+1} & \text{if } i > 2 \\ iu^{i-1} - u^i u^1 - (i+2)u^{i+1} & \text{if } i < -2 \end{cases}$$

$$N_{-11}^i = -\text{sgn}(i)u^0 u^i$$

$$N_{0i}^i = -4u^0$$

$$N_{1\,i+1}^i = N_{1\,i-1}^i = (u^0)^2$$

$$N_{0\,i+1}^i = N_{0\,i-1}^i = u^0 u^1$$

$$N_{-1\,i+1}^i = N_{-1\,i-1}^i = u^0$$

▷ for $|i| \leq 2$

$$N_{01}^2 = u^0(2u^1 - 3u^3) \qquad\qquad N_{01}^{-2} = -2u^{-3}u^0$$

$$N_{0-1}^2 = u^1(1 + u^2) - 3u^3 \qquad\qquad N_{0-1}^{-2} = -2u^{-3} + (-u^{-2} + u^0)u^1$$

$$N_{-11}^2 = -u^0(-1 + u^2) \qquad\qquad N_{-11}^{-2} = (u^{-2} - u^0)u^0$$

$$N_{02}^2 = -4u^0 \qquad\qquad N_{0-2}^{-2} = -4u^0$$

$$N_{03}^2 = u^0 u^1 \qquad\qquad N_{0-3}^{-2} = u^0 u^1$$

$$N_{13}^2 = (u^0)^2 \qquad\qquad N_{1-3}^{-2} = (u^0)^2$$

$$N_{-13}^2 = u^0 \qquad\qquad N_{-1-3}^{-2} = u^0$$

$$N_{01}^1 = -2u^0(2 + u^2) \qquad\qquad N_{01}^{-1} = -u^0(u^{-2} + 2u^0)$$

$$N_{02}^1 = u^0 u^1 \qquad\qquad N_{0-1}^{-1} = -u^{-2} - 6u^0 - u^{-1}u^1$$

$$N_{12}^1 = (u^0)^2 \qquad\qquad N_{0-2}^{-1} = u^0 u^1$$

$$N_{-10}^1 = -(u^1)^2 + 2u^2 \qquad\qquad N_{1-2}^{-1} = (u^0)^2$$

$$N_{-11}^1 = -u^0 u^1 \qquad\qquad N_{-1-2}^{-1} = u^0$$

$$N_{-12}^1 = u^0 \qquad\qquad N_{-11}^{-1} = u^0 u^{-1}$$

Thermodynamic limit and integrable hydrodynamic chain

We recall the expression of the Haantjes tensor given in (3.72)

$$H^i_{jk} = N^i_{pq}\, a^p_j(u)\, a^q_k(u) - N^p_{jq}\, a^i_p(u)\, a^q_k(u) - N^p_{qk}\, a^i_p(u)\, a^q_j(u) + N^p_{jk}\, a^i_q(u)\, a^q_p(u).$$

The structure of N^i_{jk} and $A(u)$ induces constraints on the range of values the indices p and q can take in the expression of the Haantjies tensor (3.72), and consequently on potential nonzero elements. Indeed, the form of N^i_{jk}, specified by the elements (7.86), implies that

$$H^i_{jk} \neq 0, \qquad i \in \mathbb{Z}, \quad j,k \in \{0, \pm 1, \pm 2, 3, i, i \pm 1, i \pm 2, i \pm 3\}. \tag{7.87}$$

Given the explicit expressions for a^k_j in (7.4) and N^i_{jk} above, a direct calculation demonstrates that $H^i_{jk} = 0$ for the listed values of the lower indices. This proves the statement.

We now study the integrability of the chain (7.85) by following the approach based on the method of hydrodynamic reductions applied to the system (7.83) and reported in section 3.2.1. We look for solutions of the form

$$u^k = u^k\left(R^1, R^2, \ldots, R^N\right) \tag{7.88}$$

for an arbitrary number N of components $R^i = R^i(x,t)$. The functions $\left\{R^i\right\}_{i=1}^N$ are the Riemann invariants and satisfy by definition the diagonal system

$$R^i_t = \lambda^i\left(R^1, \ldots, R^N\right) R^i_x \tag{7.89}$$

where the characteristic speeds λ^i are such that the system (7.89) possesses the semi-Hamiltonian property, that is

$$\partial_k\left(\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}\right) = \partial_j\left(\frac{\partial_k \lambda^i}{\lambda^k - \lambda^i}\right), \tag{7.90}$$

with the notation $\partial_i = \partial_{R^i}$. The diagonal form of the system (7.89) and the condition (7.90) guarantee that equations (7.89) constitute a system of conservation laws [110] which is integrable via the generalised hodograph method, described in section 3.1.2. Substituting the assumption (7.88) into the system (7.85) and using (7.89), we obtain the

equations of the form

$$\lambda^i \partial_i u = A(u) \partial_i u, \qquad i = 1, 2, \dots N \tag{7.91}$$

where we used the fact that R_x^i for $i = 1, \dots, N$ are independent. We observe that, due to the specific sparse structure of the matrix $A(u)$, the components of the eigenvectors $\partial_i u$ can be parametrised in terms of the components $\partial_i u^0$ and $\partial_i u^1$.

Let us consider, for example, the equations for $\partial_i u^{-2}$, $\partial_i u^{-1}$, $\partial_i u^2$ and $\partial_i u^3$:

$$\partial_i u^{-2} = \frac{1}{(u^0)^2} \left((\lambda^i)^2 - u^0 u^1 \lambda^i - u^0 (2u^0 + u^{-2} + u^{-1} u^1) \right) \partial_i u^0 - \left(\lambda^i + u^{-1} \right) \partial_i u^1$$

$$\partial_i u^{-1} = \left(\frac{\lambda^i}{u^0} - u^1 \right) \partial_i u^0 - u^0 \partial_i u^1$$

$$\partial_i u^2 = \frac{1}{u^0} \left((u^1)^2 - 2u^2 \right) \partial_i u^0 + \frac{1}{u^0} \left(\lambda^i + u^0 u^1 \right) \partial_i u^1 \tag{7.92}$$

$$\partial_i u^3 = \frac{1}{(u^0)^2} \left(\left((u^1)^2 - 2u^2 \right) \lambda^i + u^0 \left(u^1 (1 + u^2) - 3u^3 \right) \right) \partial_i u^0$$

$$+ \frac{1}{(u^0)^2} \left((\lambda^i)^2 + u^0 u^1 \lambda^i + (u^0)^2 (u^2 - 1) \right) \partial_i u^1, \qquad i = 1, \dots, N.$$

The compatibility conditions

$$\partial_j \partial_i u^{-2} = \partial_i \partial_j u^{-2} \qquad \partial_j \partial_i u^{-1} = \partial_i \partial_j u^{-1} \qquad \partial_j \partial_i u^2 = \partial_i \partial_j u^2 \qquad \partial_j \partial_i u^3 = \partial_i \partial_j u^3$$

lead to the associated Gibbons–Tsarev system. For our chain, this takes the form

$$\partial_j \lambda^i = \frac{4(u^0)^2 - \lambda^i \lambda^j}{u^0 (\lambda^i - \lambda^j)} \partial_j u^0$$

$$\partial_i \lambda^j = \frac{4(u^0)^2 - \lambda^i \lambda^j}{u^0 (\lambda^j - \lambda^i)} \partial_i u^0$$

$$\partial_i \partial_j u^0 = \frac{(\lambda^i)^2 + (\lambda^j)^2 - 8(u^0)^2}{u^0 (\lambda^i - \lambda^j)^2} \partial_i u^0 \partial_j u^0 \tag{7.93}$$

$$\partial_i \partial_j u^1 = -\frac{(\lambda^j - 2\lambda^i)\lambda^j + 4(u^0)^2}{u^0 (\lambda^i - \lambda^j)^2} \partial_i u^0 \partial_j u^1 - \frac{(\lambda^i - 2\lambda^j)\lambda^i + 4(u^0)^2}{u^0 (\lambda^i - \lambda^j)^2} \partial_j u^0 \partial_i u^1.$$

A direct calculation shows that the system of equations (7.93) is in involution, i.e. compatibility conditions of the form

$$\partial_k \partial_j \lambda^i = \partial_j \partial_k \lambda^i \qquad \partial_k \partial_i \partial_j u^0 = \partial_i \partial_k \partial_j u^0 \qquad \partial_k \partial_i \partial_j u^1 = \partial_i \partial_k \partial_j u^1$$

are satisfied modulo the equations (7.93) for all permutation of the derivatives with respect to R^i, R^j, R^k. A first classification of Gibbons–Tsarev systems has been provided by Odesskii and Sokolov [101, 102]. We note that, at the best of our knowledge, the system (7.93) has not appeared before in the literature and it is not included in the class considered in the above mentioned works.

The compatibility of the Gibbons–Tsarev system (7.93) guarantees, that for any solution to the Riemann invariants system (7.83), it is possible to construct a solution to the hydrodynamic chain, as reported in section 3.2.1.

Therefore, the above calculations prove the following

Theorem 7.4.1 *The hydrodynamic chain* (7.83) *is integrable in the sense of the hydrodynamic reductions.*

We will then explore the structure of the chains associated with higher terms in the Pfaff hierarchy.

7.5 Extension to higher flows and generalisation of the chain

As mentioned in section 3.2, a hydrodynamic chain takes the form

$$u_t^n = \varphi_1^n u_x^1 + \cdots + \varphi_{n+1}^n u_x^{n+1}, \qquad n \in \mathbb{N}, \qquad \varphi_{n+1}^n \neq 0, \tag{7.94}$$

where $\varphi_j^n = \varphi_j^n \left(u^1, \ldots, u^{n+1} \right)$ and the integrability of the chain is studied by analysing the corresponding Gibbons–Tsarev system, as in [101, 102]. The most known example is given by the Benney chain

$$u_t^n = u_x^{n+1} + (n-1) u^{n-1} u_x^1, \qquad n = 1, 2, \ldots, \tag{7.95}$$

whose Gibbons–Tsarev system is given by (3.82), that we recall

$$\partial_j \lambda^i = \frac{\partial_j u^1}{\lambda^j - \lambda^i}$$

$$\partial_i \lambda^j = \frac{\partial_i u^1}{\lambda^i - \lambda^j} \tag{7.96}$$

$$\partial_i \partial_j u^1 = 2 \frac{\partial_i u^1 \partial_j u^1}{\left(\lambda^i - \lambda^j\right)^2},$$

in terms of the characteristic speeds λ^j, the derivative with respect to the Riemann invariants R^i, and the field u^1, that we call the *seed* of the chain.

The hydrodynamic chain associated to the continuum limit at the leading order of the first flow for the even Pfaff lattice is given by

$$u_t^k = a_0^k u_x^0 + a_1^k u_x^1 + a_{k-1}^k u_x^{k-1} + a_{k+1}^k u_x^{k+1}, \qquad k \in \mathbb{Z}, \tag{7.97}$$

and the associated Gibbons–Tsarev system is reported in (7.93). The new chain associated with the symmetric ensemble is double infinite, unlike the Benney chain and other known examples of integrable chains. In addition, the new chain is initialised by the two central elements or seeds (the fields u^0 and u^1) rather than one, as in the Benney chain.

From (7.97), we notice that the evolution of the the field u^k with respect to the first even slow time variable t depends on the spatial derivatives of the seeds of the chain u^0 and u^1 and of its nearest neighbours u^{k-1} and u^{k+1}. In the following, we will see that it is possible to associate a hydrodynamic chain for the evolution of the fields in t_4 and t_6 as well, observing a nominal proliferation of the seeds of the chain and a dependence of the dynamics on an increasing number of nearest neighbours.

We study the continuum limit for the evolution equations of the fields entering the discrete Pfaff lattice for higher even flows, t_4 and t_6. We recall the Lax equations for the even flows

$$\frac{\partial L}{\partial \tilde{t}_{2q}} = \left[-\left(L^{2q}\right)_t, L \right], \qquad q = \{1, 2, \ldots\} \tag{7.98}$$

in terms of the semi-infinite Lax matrix L, whose elements are given by the discrete variables w_n^k, with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

We have verified that for t_4 and t_6 the leading order of the continuum limit can be recast as

$$u_{t_{2q}} = A_q(u) u_x, \qquad u = \left(\dots, u^{-1}, u^0, u^1, \dots \right)^\top, \qquad (7.99)$$

where we consider slow time variables $t_{2q} = \varepsilon \, \tilde{t}_{2q}$ and the interpolating functions $u^k(x) = w^k(x/\varepsilon) = w^k(n) = w_n^k$, where $\varepsilon = 1/N$, in the limit $N \to \infty$. The matrix $A_q(u)$ is an infinite matrix of the chain class, for $q = 1, 2, 3$. Since the flows of the hierarchy commute pairwise, we have

$$\left[A_q, A_p \right] = 0. \qquad (7.100)$$

We have verified by direct inspection (7.100) for the matrices for $q, p \in \{1, 2, 3\}$. Therefore, the continuum limit at the leading order can be written in terms of a hydrodynamic chain for $q = 1, 2, 3$ and this has led us to conjecture a generalisation for $q > 3$. In the following, we will show the form of the hydrodynamic chains associated to the higher flows and the possible generalisation. As we will see, the number of seeds of the chain and the number of nearest neighbours in the interaction term increases with q. The explicit form of the entries of the matrices is reported in appendix C.

- $q = 1$, $a_j^i \in A_1$

$$u_{t_2}^k = a_0^k u_x^0 + a_1^k u_x^1 + a_{k-1}^k u_x^{k-1} + a_{k+1}^k u_x^{k+1} \qquad (7.101)$$

  the seeds of the chain are u^0, u^1 and the interaction is with first nearest neighbours;

- $q = 2$, $a_j^i \in A_2$

$$\begin{aligned}
u_{t_4}^k = a_{-1}^k u_x^{-1} + a_0^k u_x^0 + a_1^k u_x^1 + a_2^k u_x^2 \\
+ a_{k-2}^k u_x^{k-2} + a_{k-1}^k u_x^{k-1} + a_{k+1}^k u_x^{k+1} + a_{k+2}^k u_x^{k+2}
\end{aligned} \qquad (7.102)$$

  the seeds are u^{-1}, u^0, u^1, u^2 and the interaction is with second nearest neighbours;

- $q = 3$, $a_j^i \in A_3$

$$u_{t_6}^k = a_{-2}^k u_x^{-2} + a_{-1}^k u_x^{-1} + a_0^k u_x^0 + a_1^k u_x^1 + a_2^k u_x^2 + a_3^k u_x^3$$
$$+ a_{k-3}^k u_x^{k-3} + a_{k-2}^k u_x^{k-2} + a_{k-1}^k u_x^{k-1} + a_{k+1}^k u_x^{k+1} + a_{k+2}^k u_x^{k+2} + a_{k+3}^k u_x^{k+3} \qquad (7.103)$$

the seeds are $u^{-2}, u^{-1}, u^0, u^1, u^2, u^3$ and the interaction is with third nearest neighbours.

We can generalise the form of the hydrodynamic chain for the generic q–th flow

$$u_{t_{2q}}^k = \sum_{p=-(q-1)}^{q} a_p^k u_x^p + \sum_{p=1}^{q} \left(a_{k-p}^k u_x^{k-p} + a_{k+p}^k u_x^{k+p} \right) \qquad (7.104)$$

with $a_j^i \in A_q$. The generic hydrodynamic chain for the q-th flow of the even Pfaff lattice is then characterised by $2q$ seeds and interaction with q-th nearest neighbours.

7.6   Leading order of even times hierarchy for Toda and Pfaff

We have studied the hydrodynamic system of PDEs emerging from the study of the symmetric matrix ensemble, consisting of infinitely many components in terms of the field variables. The system is described by an infinite number of order parameters. The order parameters of the model with even order interactions satisfy a reduction of the even Pfaff hierarchy.

It is worth to emphasise that only the model with even rescaled interactions leads to an integrable hydrodynamic chain hierarchy, given by (7.104)

$$u_{t_{2q}}^k = \sum_{p=-(q-1)}^{q} a_p^k u_x^p + \sum_{p=1}^{q} \left(a_{k-p}^k u_x^{k-p} + a_{k+p}^k u_x^{k+p} \right), \qquad k \in \mathbb{Z}, q \in \mathbb{N}.$$

This represents the main difference with the case of the Hermitian matrix ensemble, where the leading order in even slow times is described by a scalar hydrodynamic system, i.e. the Volterra lattice. The collection of the equations representing the evolution at

different times gives the Hopf hierarchy (6.100)

$$u_{t_{2k}} = c_k \, u^k \, u_x, \qquad k \in \mathbb{N}.$$

In the context of the perspective offered by the approach of differential identities, we have seen in section 6.4 the emergence of a dispersive shock. The latter characterises a phase transition where asymptotic stable states are connected by an intermediate state, where the dispersive nature of finite size corrections induce fast oscillations in the order parameter. In this case, the description is possible in terms of one order parameter only.

Part IV

Explorative studies

Chapter 8

Towards networks

In this chapter we will consider a particular case of symmetric matrices, the adjacency matrices arising in the context of graph theory. Section 8.1 is dedicated to a brief introduction of the subject. In section 8.2 we will address the problem of the so called two-star model as presented in [103]. The same problem will be treated with the tools developed in chapter 5 in section 8.3, invoking the method of differential identities. We then will consider the one-dimensional Ising model, in section 8.4, and write a suitable partition function constructed from the corresponding adjacency matrices. Finally, we will give an insight on the automorphisms of different configurations in section 8.5 and we will discuss the form of the partition function for exponential random graphs in section 8.6.

8.1 Graphs

A network is described in mathematical term by the graph, a collection of vertices connected by edges [99]. Those primary elements acquire different nomenclature with respect to the field we are considering (nodes and links, sites and bonds, actors and ties).

We will focus on the study of the simple graph in figure 8.1, with n vertices and m edges, without neither multiedges (more than one edge connects a pair of vertices) nor self-edges (connecting a vertex with itself).

One of the possible way to represent a network is via its edge list, not so useful in terms of a mathematical analysis. Another more proper way is to consider the adjacency matrix
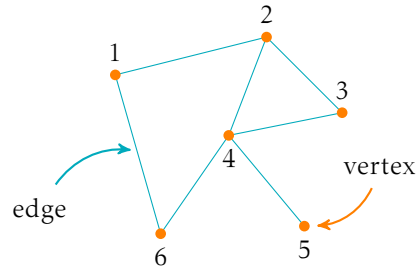
Figure 8.1: Simple graph with $n = 6$ vertices

of the graph representing the network, defined by

$$A_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases} \tag{8.1}$$

The adjacency matrix describing the graph shown in figure 8.1 is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{8.2}$$

with every element of the diagonal is zero, since there are no self-edge and it is symmetric, due to the fact that if there is a connection between i and j, the same is for the connection between j and i ($A = A^{\top}$).

In order to represent self-edges of specific vertices, once they are addressed with a label, the related element in the diagonal will be twice the multiplicity of the edge: if there is a simple loop connecting vertex i with itself, we have two legs on that vertex.

Sometimes it may be useful to consider edges with a certain weight. In that case, the element corresponding to an edge will account for the weight of that connection. One can pass from this kind of description to a multi-edge one, in which the weight is rearranged in term of multiplicity in units of the minimum weighted edge in the network.

We now introduce the degree k_i of a vertex i as the number of edges adjacent to it.

The adjacency matrix depends on the enumeration of vertices, but of course different enumerations are conjugates of each other via a permutation matrix. Also, the fact that the matrix is symmetric implies that there exist an orthonormal basis of eigenvectors $u_1, u_2, \ldots, u_n \in \mathbb{R}^n$ with real eigenvalues

$$k_{\max} \geq \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \geq -k_{\max} \qquad \text{with } k_{\max} = \max_{i \in V} k_i \qquad (8.3)$$

The multiplicities are taken into account, so that multiplicity 3 means that $\kappa_j = \kappa_{j+1} = \kappa_{j+2}$. The κ_i are the zeros of the characteristic polynomial

$$p(\lambda) = \det(\lambda \, 1_n - A) \qquad (8.4)$$

and the eigenvalues $\kappa_1, \kappa_2, \ldots, \kappa_n$ form the adjacency spectrum of the graph G. We will see that just by studying the adjacency spectrum we can have different information about the characteristics of the graph.

A graph is named directed if its edges have a direction, they are represented by arrows connecting the vertices and the adjacency matrix describing is

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases} \qquad (8.5)$$

corresponding to an asymmetric matrix. The undirected network can be seen as a directed one where the undirected edges may be seen as two directed edges going in opposite directions.

We will study the problem of the so called two-star model, that allow a description of the Curie-Weiss model. We will review the problem both with the mean-field approach and via the method of differential identities developed in chapter 5 and in particular following the development of section 5.2.

8.2   The two-star model in the mean-field approach

The two-star model appears in the context of statistical mechanics of networks [104, 103, 16]. With this approach it is possible to study different configurations of a system by considering an ensemble of networks, where every possible configuration is represented by a graph G. We can associate a Hamiltonian $H(G)$ to any configuration and then define a partition function Z for the ensemble, being this the starting point for a statistical approach. For every graph the probability is given by

$$P(G) = \frac{e^{-H(G)}}{Z}, \qquad Z = \sum_{\{G\}} e^{-H(G)}, \qquad (8.6)$$

with the partition function defined as the sum over all the possible graphs. The Hamiltonian is written in terms of the adjacency matrix elements taking into account the number of edges connecting vertices m and the number of two-stars m_{2s} for each configuration.

The two-star is an elemental structure that can be found in a graph, denoting a vertex shared by two different edges. By counting these kind of subgraphs, it is possible to have information about the way in which the edges are distributed in the entire graph, either they tend to appear in clusters or they are randomly spread.

Given the form of A_G, one can define m and m_{2s} as

$$m = \sum_{i<j} a_{ij} = \frac{1}{2} \sum_{i \neq j} a_{ij}, \qquad (8.7)$$

$$m_{2s} = \sum_{i} \sum_{j \neq i} \sum_{k \neq i,j} a_{ij} a_{ik} = \frac{1}{2} \sum_{i \neq j} a_{ij} \sum_{k \neq i,j} (a_{ik} + a_{jk}). \qquad (8.8)$$

The hamiltonian for a configuration G can be written in terms of the Lagrange multipliers β, γ by using the definitions in (8.7) and (8.8)

$$H(G) = -\beta \frac{1}{2} \sum_{i \neq j} a_{ij} - \gamma \frac{1}{2} \sum_{i \neq j} a_{ij} \sum_{k \neq i,j} (a_{ik} + a_{jk}). \qquad (8.9)$$

In order to evaluate the previous expression, we adopt the mean-field approach, replacing all the quantities with their mean values on the graph. The mean value for the term multiplying β is given by (8.7) and we have to determine the mean value for the re-

maining term, multiplied by γ. The latter counts for all the possible pairs of vertices in the graph connected by an edge. In the mean-field approach, every pair of vertices can be connected by an edge with a given probability p, so that what is actually fixed in the graph is the number of vertices N and the probability p. We can now evaluate the expectation value of the two-stars, as

$$\left\langle \sum_{k \neq i,j} a_{ik} \right\rangle = \sum_{k \neq i,j} \langle a_{ik} \rangle = \sum_{k \neq i,j} p = (N-2)p \sim Np, \tag{8.10}$$

for large values of N.

The probability associated to any graph in the ensemble with m edges and N vertices, in terms of the probability that an edge connects two vertices p, is given by

$$P(G) = p^m (1-p)^{\frac{N(N-1)}{2}-m} = p^m (1-p)^{\binom{N}{2}-m}. \tag{8.11}$$

It is possible to determine the expectation value of the number of edges $\langle m \rangle$, by considering that the number of distinct configurations with N vertices and m edges is equal to the number of ways we can pick the position of edges among $N(N-1)/2$ distinct pairs. Every graph enters in the ensemble with the same probability $P(G)$, so that the probability distribution for the number of edges m is

$$P(m) = \binom{\binom{N}{2}}{m} p^m (1-p)^{\binom{N}{2}-m}, \tag{8.12}$$

hence the binomial distribution, with expectation value

$$\langle m \rangle = \binom{N}{2} p. \tag{8.13}$$

Substituting the results (8.10) and (8.13) in (8.9), we obtain the mean-field hamiltonian H_N

$$H_N(G) = -(\beta + 2\gamma Np)m = -\vartheta(N,p)m \tag{8.14}$$

with the introduction of the function $\vartheta(N,p)$ in order to simplify the next calculations. The partition function associated to the networks' ensemble in the mean-field approach

is given by

$$Z_N = \sum_{\{G\}} e^{-H_N(G)} = \sum_{\{a_{ij}\}} e^{\vartheta \sum_{i<j} a_{ij}} = \sum_{\{a_{ij}\}} \prod_{i<j} e^{\vartheta a_{ij}} = \prod_{i<j} \left(1 + e^{\vartheta}\right) = \left(1 + e^{\vartheta}\right)^{\binom{N}{2}}. \qquad (8.15)$$

In order to obtain a consistence relation involving the probability p, we can use the definition of the expected value of the number of edges $\langle m \rangle$ via the partition function as

$$\langle m \rangle = \frac{\sum_G m \, e^{-H_N(G)}}{Z_N(G)} = \frac{1}{Z_N} \frac{\partial Z_N}{\partial \vartheta} = \binom{N}{2} \frac{e^{\vartheta}}{1 + e^{\vartheta}} = \binom{N}{2} \frac{1}{1 + e^{-\vartheta}}. \qquad (8.16)$$

By considering the equivalence between (8.13) and (8.16), we get

$$\binom{N}{2} p = \binom{N}{2} \frac{1}{1 + e^{-\vartheta}} \qquad \Longrightarrow \qquad p = \frac{1}{1 + e^{-\vartheta}}. \qquad (8.17)$$

We can write the previous in terms of the fixed variables, N and p, as

$$p = \frac{1}{2} \left[1 + \tanh\left(\frac{\beta}{2} + \gamma N p\right)\right]. \qquad (8.18)$$

By rescaling the parameters in the previous equation, with $b = \beta/2$ and $c = \gamma N/2$, so that the equation is

$$p = \frac{1}{2} \left[1 + \tanh\left(b + 2cp\right)\right], \qquad (8.19)$$

that will be evaluated in terms of the parameters b and c.

In figure 8.2 the behaviour of $p(b)$ as a function of b is shown, with different values for the parameter c. We can observe that in correspondence of $c = 1$ a gradient catastrophe occurs, the function becomes multivalued for $c > 1$, beyond the critical point. The spontaneous symmetry breaking produces two different possible configuration for the present graph for values of the parameter $c > 1$. In particular, the network can be either dense, with a high number of edges connecting vertices, or sparse.

By differentiating the expression in (8.19) with respect to p on both sides we get

$$1 = c \operatorname{sech}^2(b + 2cp)$$
$$\frac{1}{c} = 1 - \tanh^2(b + 2cp), \qquad (8.20)$$
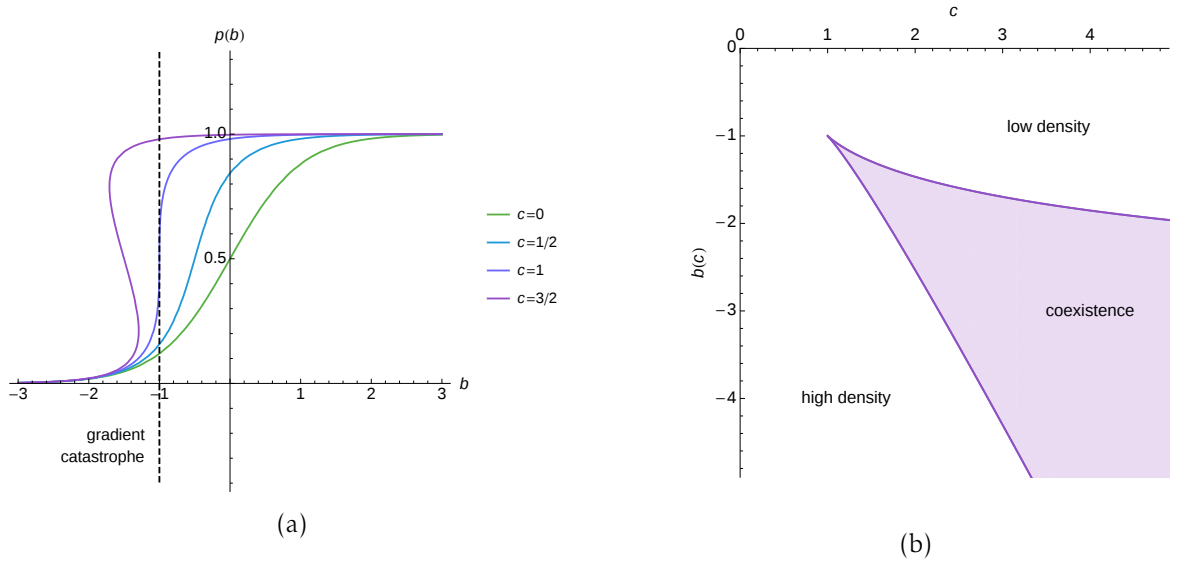
Figure 8.2: (a) The probability that an edge connects two vertices p is shown as a function of the parameter b, for different values of the parameter c. The dashed line represents the gradient catastrophe, beyond which the profile of the function (for $c > 1$) is not more single-valued in b. (b) The phase diagram in terms of the parameters b and c.

using the fact that $\operatorname{sech}^2(x) = 1 - \tanh^2(x)$. From (8.19) we can write (8.20) as

$$
\begin{aligned}
1 - (2p - 1)^2 &= \frac{1}{c} \\
p^2 - p + \frac{1}{4c} &= 0.
\end{aligned}
$$

$$(8.21)$$

The roots of the equation are easily found as

$$
p_{1,2} = \frac{1}{2} \pm \sqrt{1 - \frac{1}{c}}.
$$

$$(8.22)$$

In order to produce the diagram for the phase transition observed, we write b from (8.19) as a function of p and c

$$
\begin{aligned}
b &= \operatorname{arctanh}(2p - 1) - 2cp \\
b &= \pm \operatorname{arctanh}\sqrt{1 - \frac{1}{c}} - c\left(1 \pm \sqrt{1 - \frac{1}{c}}\right).
\end{aligned}
$$

$$(8.23)$$

These curves are the boundaries for the coexistence region shown in the right of figure 8.2, where the regions of the space of parameters are represented. The critical point is

localised by the coordinates

$$b_c = 1, \qquad c_c = 1. \tag{8.24}$$

In the next section, we will tackle the same problem by considering the approach established in chapter 5.

8.3 Differential identities for the two-star model

The same problem can be treated with the formalism of non-linear PDEs by starting from the analogy with the Curie-Weiss system, mean-field approach for the Ising model. The two-star model can be seen as an Ising model, if we look at the Hamiltonian (8.9) previously discussed, we can identify a term referring to en external field and a term related to the interaction between pairs. The elements of the adjacency matrix play the role of spins, represented by the edges. In contrast with the original Ising model, where spins can assume values in $\{+1, -1\}$, here the set of possible values is $\{0, 1\}$. With the Curie-Weiss approach the two-body interaction term is substituted by a mean-field term, as if every spin interacts with all the others. With this prescriptions, the mean-field Hamiltonian can be written as

$$H_N(G) = -\frac{J}{N(N-1)} \sum_{ij} a_{ij} \sum_{kl} a_{kl} - h \sum_{ij} a_{ij}, \tag{8.25}$$

where the first term describes the mean-field interaction, with the coupling constant J, and the second refers to the interaction with the external field h. The term referring to the interaction between "spins" is long ranged and weak, of order $1/N^2$ (since here we are considering pairs of vertices). Since the indices are not correlated, the first term of (8.25) can be written as

$$\sum_{ij} a_{ij} \sum_{kl} a_{kl} = \left(\sum_{ij} a_{ij} \right)^2. \tag{8.26}$$

Differential identities for the two-star model

The order parameter is given in this context by the number of edges appearing in the graph, normalized to the maximum number of possible pairs of vertices, as

$$m = \frac{2}{N(N-1)} \sum_{i \neq j} a_{ij} \, . \tag{8.27}$$

The aim is to determine a partial differential equation in terms of the expectation value of the order parameter $\langle m \rangle$, in order to discuss the phase diagram of the system.

By using (8.27) the Hamiltonian becomes

$$H_N(G) = -J \frac{N(N-1)}{2} m^2 - h \frac{N(N-1)}{2} m = -\binom{N}{2}\left(Jm^2 + hm\right). \tag{8.28}$$

The partition function for the system is defined, with $\beta = 1/T$, as

$$Z_N(G) = \sum_G e^{-\beta H_N(G)} = \sum_G e^{\binom{N}{2}\left(\frac{1}{2}tm^2 + xm\right)}, \tag{8.29}$$

having rescaled the coupling constants J and h as

$$\begin{aligned} t &= J\beta \\ x &= h\beta \, . \end{aligned} \tag{8.30}$$

By taking the derivative of $Z_N(x,t)$ with respect to x and t, we can write a differential identity for Z_N

$$\begin{aligned} \frac{\partial Z_N}{\partial t} &= \frac{N(N-1)}{4} m^2 Z_N \\ \frac{\partial^2 Z_N}{\partial x^2} &= \frac{(N(N-1))^2}{4} m^2 Z_N \end{aligned} \qquad \Longrightarrow \qquad \frac{\partial Z_N}{\partial t} = \frac{1}{N(N-1)} \frac{\partial^2 Z_N}{\partial x^2} \tag{8.31}$$

hence the partition function satisfies the heat equation. We have to involve an initial condition, given by $t = 0$. The evaluation is the same as in (8.15), so that we have

$$Z_N(x,0) = (1 + e^x)^{\binom{N}{2}} \, . \tag{8.32}$$

The free energy of the system is f_N

$$f_N(x,t) = -\frac{1}{\beta} \alpha_N(x,t) \qquad \text{with} \quad \alpha_N = \frac{2}{N(N-1)} \ln Z_N. \tag{8.33}$$

By imposing the result of (8.31), we obtain the PDE satisfied by α_N

$$\partial_t \alpha_N = \frac{1}{2} (\partial_x \alpha_N)^2 + \frac{1}{N(N-1)} \partial_x^2 \alpha_N$$

$$\partial_t \alpha_N = \frac{1}{2} (\partial_x \alpha_N)^2 + \nu \partial_x^2 \alpha_N, \tag{8.34}$$

with $\nu = 1/N(N-1)$, multiplying a dispersion term in α_N. The initial condition is given by (8.32), as

$$\alpha_N(x,0) = \frac{2}{N(N-1)} \ln Z_N(x,0) = \ln(1 + e^x). \tag{8.35}$$

By writing explicitly the first and second derivatives of α_N with respect to x we can identify the expectation value of the order parameter $\langle m \rangle$ and the variance $\text{var}\langle m \rangle$ respectively

$$\partial_x \alpha_N = \frac{1}{Z_N} \sum_G m \, e^{\binom{N}{2}\left(xm + \frac{t}{2}m^2\right)} = \langle m \rangle. \tag{8.36}$$

$$\partial_x^2 \alpha_N = \frac{1}{Z_N^2} \frac{N(N-1)}{2} \left\{ \sum_G m^2 \, e^{\binom{N}{2}\left(xm + \frac{t}{2}m^2\right)} - \left[\sum_G m \, e^{\binom{N}{2}\left(xm + \frac{t}{2}m^2\right)} \right]^2 \right\} = \text{var}\langle m \rangle. \tag{8.37}$$

We can now take the derivative with respect to x of the PDE (8.34), using the fact that the order of derivatives with respect to x and t is not important, obtaining

$$\partial_t(\partial_x \alpha_N) = (\partial_x \alpha_N)\partial_x(\partial_x \alpha_N) + \nu \partial_x^2(\partial_x \alpha_N) \tag{8.38}$$

hence, we have that the order parameter satisfies the Burgers equation

$$\partial_t \langle m \rangle = \langle m \rangle \partial_x \langle m \rangle + \nu \, \partial_x^2 \langle m \rangle. \tag{8.39}$$

The initial condition is given by

$$\langle m(x,0) \rangle = \partial_x \alpha_N(x,0) = \frac{e^x}{1 + e^x} = \frac{1}{2}\left(1 + \tanh \frac{x}{2}\right). \tag{8.40}$$

Differential identities for the two-star model

The initial datum does not depend on the number of vertices N of the graph. In the thermodynamic limit, in a neighbourhood of the critical point, we can neglect the viscous term, proportional to $\mathrm{var}\langle m \rangle$, so that the Burgers equation reduces to the Hopf equation

$$\partial_t \langle m \rangle - \langle m \rangle \, \partial_x \langle m \rangle = 0. \tag{8.41}$$

The solution to this equation is implicitly given via the method of characteristics

$$x + \langle m \rangle \, t = f(\langle m \rangle), \tag{8.42}$$

with $\langle m \rangle$ representing the characteristic speed. The form of the function $f(\langle m \rangle)$ is given by inverting the function for the initial datum

$$\langle m(x,0) \rangle = f^{-1}(x) = \frac{1}{2}\left(1 + \tanh \frac{x}{2}\right) \quad \implies \quad x = 2\,\mathrm{arctanh}\,(2\langle m(x,0)\rangle - 1). \tag{8.43}$$

With this prescription for the initial profile of the function, since the value of $\langle m \rangle$ is
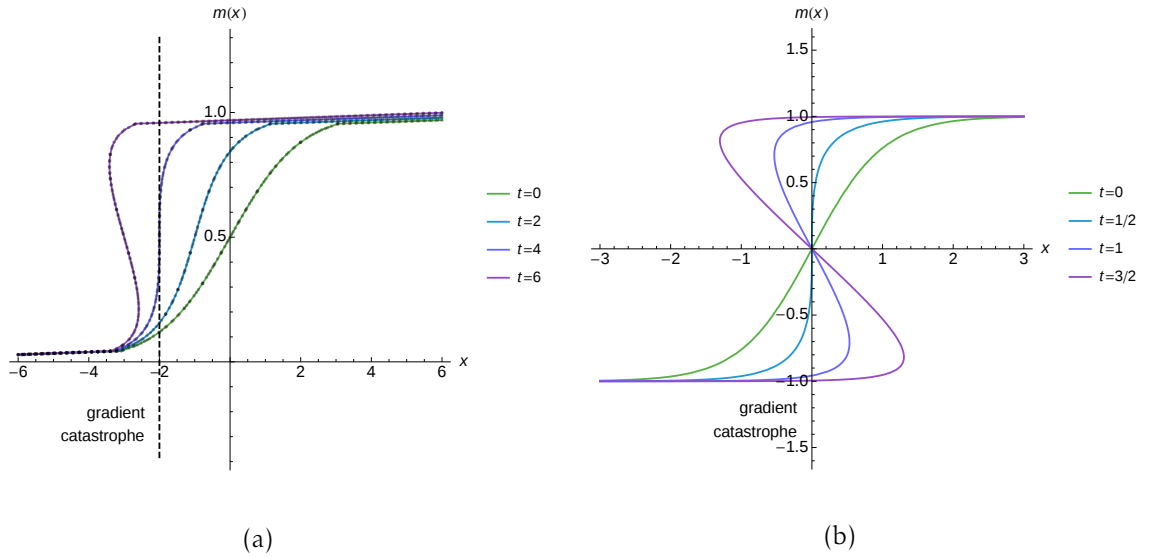
Figure 8.3: (a) Behaviour of the order parameter m as a function of the coupling x for different value of the coupling t. (b) Magnetization in the space of parameters for spins with values in $\{+1, -1\}$.

constant along the characteristic curve, the equation is

$$x + \langle m \rangle t = 2\,\mathrm{arctanh}\,(2\langle m \rangle - 1). \tag{8.44}$$

In order to determine the coordinates for the critical point in terms of the parameters x and t, we look for the gradient catastrophe, expected since the non-linearity of the equation (8.41). We take the derivative with respect to x

$$1 + \partial_x \langle m \rangle t = 4 \operatorname{sech}^2 (2 \langle m \rangle - 1) \partial_x \langle m \rangle, \tag{8.45}$$

so that we get

$$\partial_x \langle m \rangle = -\frac{1}{t - 4 \operatorname{sech}^2 (1 - 2 \langle m \rangle)} = -\frac{1}{t - 4 + 4 \tanh^2 (1 - 2 \langle m \rangle)}. \tag{8.46}$$

The critical time t_c is defined as the value of t for which the derivative $\partial_x \langle m \rangle \to \infty$. This is if the denominator of (8.46) is zero. Hence

$$t_c = 4 - 4 \tanh^2 (1 - 2 \langle m \rangle). \tag{8.47}$$

The degeneracy condition corresponds to the half of the possible edges connecting vertices, the most disordered phase. Since we are considering the normalized order parameter, this is given by $\langle m \rangle_c = 1/2$. The critical point is identified by the coordinates

$$t_c = 4, \qquad x_c = -2. \tag{8.48}$$

Finally, we can compare the result obtained with this description with that of the classical Curie-Weiss problem, for which the spin values in $\{+1, -1\}$. In the first case (Figure 8.3(a)), we have that a shock wave is formed propagating backwards, towards negative values of x. In the second case (Figure 8.3(b)), we observe a fixed point in the origin, since the reference frame corresponds to the characteristic curve.

8.4   Ising model in one dimension

In our search for differential identities, following the Curie-Weiss model, we will try to build the partition function for the Ising model in one dimension, starting from suitably defined adjacency matrices.

We start by considering the Ising model in one dimension. The Ising model describe a system composed of N interacting spins σ_i, where σ_i assumes values in $\{+1, -1\}$. In one dimension the system is represented by a chain, with first neighbours interaction between spins. The Hamiltonian describing the model is

$$H_N = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i \,, \tag{8.49}$$

where the first term describes the interaction between spins, with coupling constant J, and the second term refers to the interaction with an external field, with coupling constant h. In order to neglect border effects, we introduce the periodical boundary condition, for which $\sigma_{N+1} = \sigma_1$, the structure of the system modifying as in Figure 8.4.
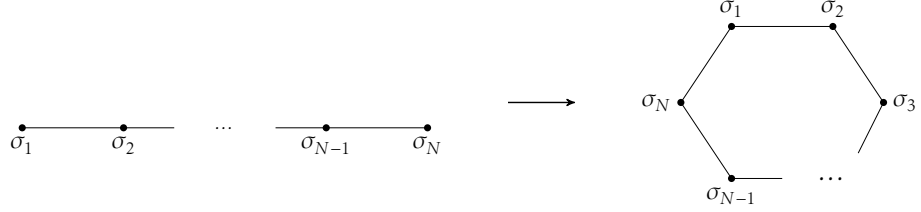
Figure 8.4: Chain of spins σ_i (left) with periodical boundary conditions (right).

By taking into account the constraint, the Hamiltonian reads as

$$H_N(J,h) = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - \frac{h}{2} \sum_{i=1}^{N} (\sigma_i + \sigma_{i+1}). \tag{8.50}$$

The partition function Z_N is given in terms of all the possible spin configurations, for which the Hamiltonian is defined as

$$Z_N(J,h) = \sum_{\{\sigma\}} e^{-\beta H_N(J,h)} \tag{8.51}$$

with $\{\sigma\}$ representing the set of configurations and $\beta = 1/(k_B T)$ is related to the inverse of temperature via the Boltzmann constant.

The standard procedure to define the partition function associated to the Ising model in one dimension involves a 2×2 transfer matrix and the solution is given in terms of its the two eigenvalues $\lambda_\pm$ as

$$Z_N = \lambda_+^N + \lambda_-^N \,, \tag{8.52}$$

167

where the eigenvalues are written in terms of the coupling constants J and h as

$$\lambda_\pm = e^J \cosh h \pm \sqrt{e^{2J} \cosh^2 h - 2 \sinh 2J}. \tag{8.53}$$

The idea is to define the partition function in terms of the adjacency matrix associated to a properly defined graph. In order to draw a graph describing the structure of the system, we consider a change of variables $\sigma_i \rightarrow c_i$, defined as

$$\sigma_i = 2c_i - 1 \qquad c_i = \frac{\sigma_i + 1}{2} \qquad c_i \in \{0, 1\}. \tag{8.54}$$

In terms of the variables c_i the transformed Hamiltonian is written as

$$
\begin{aligned}
H_N &= -J \sum_{i=1}^{N} (2c_i - 1)(2c_{i+1} - 1) - \frac{h}{2} \sum_{i=1}^{N} (2c_i - 1 + 2c_{i+1} - 1) \\
&= -4J \sum_{i=1}^{N} c_i c_{i+1} - (h - 2J) \sum_{i=1}^{N} (c_i + c_{i+1}) - (J - h) N \\
&= -\alpha_1 \sum_{i=1}^{N} c_i c_{i+1} - \alpha_2 \sum_{i=1}^{N} (c_i + c_{i+1}) + \left(\alpha_2 + \frac{\alpha_1}{4} \right) N,
\end{aligned}
\tag{8.55}
$$

where in the last expression a scaling of the coupling constants is taken into account

$$
\begin{cases}
\alpha_1 = 4J \\
\alpha_2 = h - 2J
\end{cases}
\tag{8.56}
$$

Since the values of the variables c_i are in $\{0, 1\}$ we can define the adjacency matrix associated to a graph $G = (V, E)$ representing the system, characterised by V vertices and E edges. In this case, the graph is a simple undirected cycle C_N, where each vertex is connected to two other vertices via two edges, being a 2-regular graph. The graph is represented by its adjacency matrix, the $N \times N$ matrix A with elements

$$
A_{ij} = \begin{cases} 1 & \text{if } c_i \sim c_j \\ 0 & \text{otherwise} \end{cases}
\tag{8.57}
$$

Since the graph is simple and undirected, the corresponding matrix A has zeros as di-

agonal elements and it is symmetric. Every vertex is labelled by the variable c_i and in order to involve these variables in the adjacency matrix, we consider the dual-edge graph, where for each edge in the original graph, a vertex is drawn in the dual and in the latter two vertices are connected by an edge if the corresponding edges in the original one share a vertex.
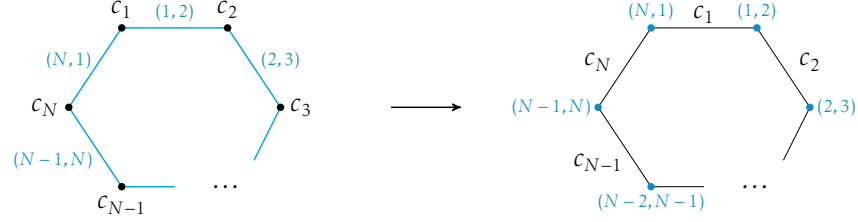
Figure 8.5: Graph representing the spin chain with periodical boundary (left) with vertices c_i and edges (i, j) and its edge-dual (right).

In this way, the variables c_i label the edges mapping the spins. In particular, the original spin +1 corresponds to an existing edge between two vertices, a spin -1 to the situation in which two vertices are not connected. The adjacency matrix associated to the edge-dual graph $\hat{A}$ for N spins is given by

$$\hat{A} = \begin{pmatrix} 0 & c_2 & 0 & 0 & \dots & 0 & c_1 \\ c_2 & 0 & c_3 & 0 & \dots & 0 & 0 \\ 0 & c_3 & 0 & c_4 & \dots & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & c_N \\ c_1 & 0 & 0 & 0 & \dots & c_N & 0 \end{pmatrix} \qquad \text{with } c_i \in \{0, 1\} \ \forall i = 1, \dots, N. \tag{8.58}$$

The generic element of the previous matrix is given by

$$(\hat{A})_{i,j} = c_{i+1}\, \delta_{i,j-1} + c_i\, \delta_{i,j+1} \qquad \text{with } c_{N+1} = c_1 \tag{8.59}$$

where δ is the Kronecker delta and we have stressed the significant constraint given by the periodical conditions. It is possible to reproduce the structures involved in the Hamiltonian (8.55) by considering traces of powers of $\hat{A}$. In particular, we seek expressions with the purpose of reproducing the interaction with the external field (one-body) and the interaction between first neighbours (two-body).

We start by computing odd powers of the adjacency matrix, for which we have

$$\operatorname{tr}\hat{A}^{2n+1} = 2(2n+1)\sum_{i=1}^{N} c_i c_{i+1}\dots c_{i+2n+1}\delta_{2n+1,N} \tag{8.60}$$

where we assume $n \geq 1$. As we can see, for the Kronecker delta involving the number of spins, the contribution of the expression is non zero only for the configurations in which the number of spins coincides with the power of the adjacency matrix and when they have values $c_i = 1$ for each i. Since the first non-zero term in (8.60) appears for $n = 1$ and describes a three-body interaction, we can exclude the presence of such terms in the specific problem we are studying. For the quadratic term we get

$$
\begin{aligned}
\operatorname{tr}\hat{A}^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{A}_{ij}\hat{A}_{ji} = \sum_{i=1}^{N}\sum_{j=1}^{N}\Big(c_{i+1}\,\delta_{i,j-1}+c_i\,\delta_{i,j+1}\Big)\Big(c_{j+1}\,\delta_{j,i-1}+c_j\,\delta_{j,i+1}\Big) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\Big(c_{i+1}c_{j+1}\,\delta_{i,j-1}\,\delta_{j,i-1}+c_{i+1}c_j\,\delta_{i,j-1}\,\delta_{j,i+1}+c_i c_{j+1}\,\delta_{i,j+1}\,\delta_{j,i-1}+c_i c_j\,\delta_{i,j+1}\,\delta_{j,i+1}\Big) \\
&= \sum_{i=1}^{N}\Big(c_{i+1}c_i\,\delta_{2,N}+c_{i+1}^2+c_i^2+c_i c_{i-1}\,\delta_{2,N}\Big) \\
&= \sum_{i=1}^{N}\Big(2\,c_i^2+2\,c_i c_{i+1}\,\delta_{2,N}\Big) \\
&= \sum_{i=1}^{N}\big(2\,c_i+2\,c_i c_{i+1}\,\delta_{2,N}\big)
\end{aligned}
\tag{8.61}
$$

where we have used the fact that $c_i \in \{0,1\}$. The term multiplied by $\delta_{2,N}$ is derived by considering that the highlighted terms appearing in the second row give in general a zero contribution, except for the case in which the index j can be at the same time equal to $i-1$ and $i+1$. By involving the periodical condition $c_{N+1} = c_1$, this occurrence is verified only for $N = 2$ for each $i = 1, 2$ and the term is non zero only for $c_i = 1$.

Ising model in one dimension

We will now consider the trace of the quartic power of the adjacency matrix, as follows

$$\operatorname{tr}\hat{A}^4 = \sum_{i=1}^{N} \sum_{j,k,l=1}^{N} \hat{A}_{ij}\hat{A}_{jk}\hat{A}_{kl}\hat{A}_{li}$$

$$= \sum_{i=1}^{N} \sum_{j,k,l=1}^{N} \left(c_{i+1}\,\delta_{i,j-1} + c_i\,\delta_{i,j+1}\right)\left(c_{j+1}\,\delta_{j,k-1} + c_j\,\delta_{j,k+1}\right)$$

$$\times \left(c_{k+1}\,\delta_{k,l-1} + c_k\,\delta_{k,l+1}\right)\left(c_{l+1}\,\delta_{l,i-1} + c_l\,\delta_{l,i+1}\right) \tag{8.62}$$

for which we can evaluate the terms giving a non zero contribution, obtaining

$$\operatorname{tr}\hat{A}^4 = \sum_{i=1}^{N} \left(2\,c_i^4 + 4\,c_i^2 c_{i+1}^2 + 2\,c_i c_{i+1} c_{i+2} c_{i+3}\delta_{4,N}\right)$$

$$= \sum_{i=1}^{N} \left(2\,c_i + 4\,c_i c_{i+1} + 2\,c_i c_{i+1} c_{i+2} c_{i+3}\delta_{4,N}\right), \tag{8.63}$$

where the term involving $\delta_{4,N}$ is derived in an analogous way to the term $\delta_{2,N}$ in (8.61), by imposing the periodical condition. Summarizing we get

$$\operatorname{tr}\hat{A}^2 = \sum_{i=1}^{N} \left(c_i + c_{i+1}\right) + 2\,\delta_{2,N} \sum_{i=1}^{N} c_i c_{i+1} \tag{8.64}$$

$$\operatorname{tr}\hat{A}^4 = \sum_{i=1}^{N} \left(c_i + c_{i+1}\right) + 4 \sum_{i=1}^{N} c_i c_{i+1} + 2\,\delta_{4,N} \sum_{i=1}^{N} c_i c_{i+1} c_{i+2} c_{i+3} \tag{8.65}$$

We can rearrange the terms appearing in (8.55) in order to write the Hamiltonian involving the traces of the matrices, giving that

$$\sum_{i=1}^{N} c_i c_{i+1} = \frac{1}{4}\left(\operatorname{tr}\hat{A}^4 - 2\,\delta_{4,N} \sum_{i=1}^{N} c_i c_{i+1} c_{i+2} c_{i+3} - \operatorname{tr}\hat{A}^2 + 2\,\delta_{2,N} \sum_{i=1}^{N} c_i c_{i+1}\right) \tag{8.66}$$

$$\sum_{i=1}^{N} \left(c_i + c_{i+1}\right) = \operatorname{tr}\hat{A}^2 - 2\,\delta_{2,N} \sum_{i=1}^{N} c_i c_{i+1}. \tag{8.67}$$

Hence, the Hamiltonian of the system becomes

$$H_N(\kappa_2,\kappa_4) = -\kappa_2\left(\operatorname{tr}\hat{A}^2 - 2\,\delta_{2,N} \sum_{i=1}^{N} c_i c_{i+1}\right) - \kappa_4\left(\operatorname{tr}\hat{A}^4 - 2\,\delta_{4,N} \sum_{i=1}^{N} c_i c_{i+1} c_{i+2} c_{i+3}\right) + \kappa N, \tag{8.68}$$

where a redefinition of the coupling constants $(\alpha_1, \alpha_2) \to (\kappa_2, \kappa_4)$ is considered

$$
\begin{cases}
\kappa_2 = \alpha_1 - \dfrac{\alpha_2}{4} \\[2ex]
\kappa_4 = \dfrac{\alpha_1}{4}
\end{cases}
\tag{8.69}
$$

and κ is written as a function of κ_2 and κ_4 as

$$
\kappa(\kappa_2, \kappa_4) = \alpha_2 + \frac{\alpha_1}{4} = \kappa_2 + 2\kappa_4.
\tag{8.70}
$$

By assuming a Boltzmann distribution for the adjacency matrices of the form (8.58), representing the set of all possible configurations of the system $\{\hat{A}\}$, the partition function for the system can be written as

$$
Z_N(\kappa_2, \kappa_4) = \sum_{\{\hat{A}\}} e^{\beta H_N\left(\{\hat{A}\}; \kappa_2, \kappa_4\right)},
\tag{8.71}
$$

and it can easily checked that it corresponds to (8.52). The partition function can be rewritten in terms of eigenvalues. Since the matrices belonging to the set $\{\hat{A}\}$ are symmetric they are diagonalizable, with eigenvalues $\lambda_i \in \mathbb{R}$. By using the cyclic property of the trace, we have

$$
\operatorname{tr} \hat{A}^m = \operatorname{tr}\left(O\hat{D}O^{-1}\right)^m = \operatorname{tr}\left(O^{-1}O\hat{D}\right)^m = \operatorname{tr}\left(\hat{D}\right)^m = \sum_{i=1}^{N} \lambda_i^m \qquad \text{for } m = 2, 4.
\tag{8.72}
$$

After a suitable redefinition of the coupling constants, the partition function takes the form

$$
Z_N(x_2, x_4) = \sum_{\{\alpha\}} c_N(\alpha) \prod_{i=1}^{N} e^{-x_2 \lambda_i^2 - x_4 \lambda_i^4},
\tag{8.73}
$$

where α represents a configuration. As we can see in (8.73), it is not possible express the partition function in terms of the spectrum only. This is because to express the original adjacency matrix we start with N degrees of freedom and we have $N - 1$ degrees of freedom when we consider the eigenvalues. The invariance of the trace imposes a constraint on the total sum of eigenvalues. Then new parameters are needed to restore the proper number of degrees of freedom and this may be related to the degree sequence for the

graph (see section 8.1). The coefficients $c_N(\alpha)$ encode the information about the symmetry of the specific configuration, related to the symmetry group underpinning the graph structure and its automorphisms, as we will see in section 8.5.

It seems that with our description, we are in some way modelling the interaction term for the spins. Indeed, if we consider the mean-field version of the Ising model, the interaction will be represented by a star
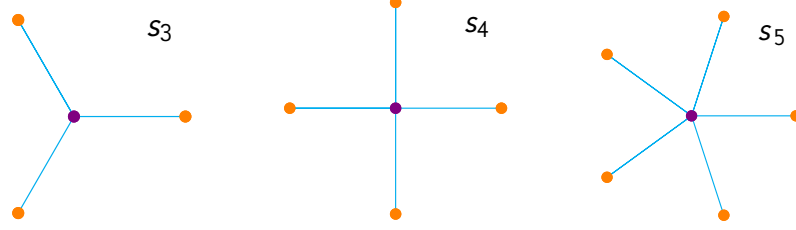
Figure 8.6: Star graphs representing the mean-field interaction term for a system composed of 3, 4 and 5 spins respectively.

The spectrum σ of a star graph is given by

$$\sigma(S_{N-1}) = \{-\sqrt{N-1}, \sqrt{N-1}, 0, \ldots, 0\}, \tag{8.74}$$

therefore in this case we have

$$\mathrm{tr}A_{\mathrm{star}}^4 = \frac{1}{2}\left(\mathrm{tr}A_{\mathrm{star}}^2\right)^2, \tag{8.75}$$

and we recover the typical form of the partition function for the two-star version of the Curie-Weiss model.

8.5 Automorphisms of graphs

In order to give a proper expression for the coefficients $c(\alpha)$ in equation (8.73), we consider the following definitions in the context of graph theory [19].

Definition 8.5.1 Isomorphisms *of graphs are bijections of the vertex sets preserving adjacency as well as non-adjacency.*

Definition 8.5.2 Automorphisms *of the graph $X = (V,E)$ are $X \to X$ isomorphisms, they form the subgroup $Aut(X)$ of the symmetric group $Sym(V)$.*

Definition 8.5.3 Homomorphisms *of graphs are defined as adjacency preserving maps. A map $f : V_1 \to V_2$ is a homomorpism of the graph $X_1 = (V_1, E_1)$ to the graph $X_2 = (V_2, E_2)$ if $(f(x), f(y)) \in E_2$ whenever a $(x, y) \in E_1$. Non-adjacency is not preserved in a homomorphism, so a bijective homomorphism is not necessarily an isomorphism. The chromatic number of the graph X is the smallest cardinal number m such that the set $\mathrm{Hom}(X, K_m)$ of $X \to K_m$ homomorphisms is nonempty.*

A graph and its complement have the same automorphisms. The automorphism group of the complete graph K_n and the empty graph $\bar{K}_n$ is the symmetric group S_n (of order $n!$). The automorphism group of the cycle of length n is the dihedral group D_n (of order $2n$). A star has S_n as automorphism group (of order $n!$). A path of length ≥ 1 has 2 automorphisms.

The automorphism group of a graph is determined by the automorphism groups and the isomorphisms of its connected components.

In the case of the Ising model the coefficients $c(\alpha)$ in (8.73) are given in terms of automorphisms of a configuration, as

$$c_N(\alpha) = \frac{|D_N|}{|\mathrm{Aut}(\alpha)|} = \frac{2N}{|\mathrm{Aut}(\alpha)|}, \tag{8.76}$$

since we are essentially considering cycles (elements of the dihedral group D_n).

The relevance of the symmetry factor becomes evident when we consider more complex structures than cycles, as we will see in the nex section.

8.6   Exponential random graphs

A random graph is defined to be $G(n, p)$, where p is the probability associated to an edge between a pair of vertices [58, 104, 16]. We consider the adjacency matrix of an undi-
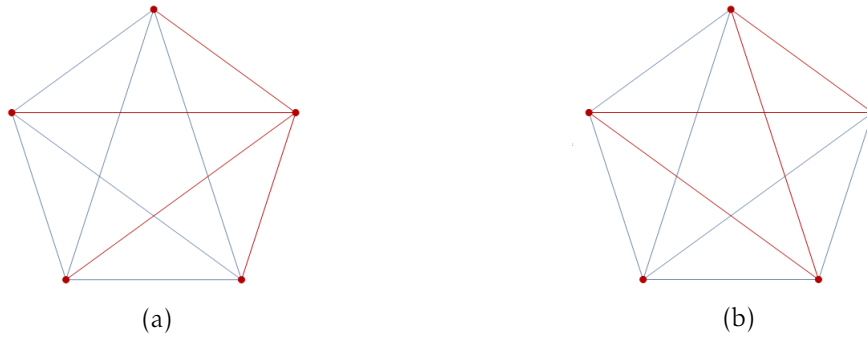
Figure 8.7: Example of two different configurations sharing the same spectrum $\sigma(a) = \sigma(b) = \{-2, 2, 0, 0, 0\}$ and possessing different symmetry factors $c_5(a) = 5$, $c_5(b) = 15$.

rected random graph

$$A = \begin{pmatrix} 0 & a_{12} & \dots & a_{1N} \\ a_{12} & 0 & \dots & a_{2N} \\ \vdots & & \ddots & \\ a_{1N} & & & 0 \end{pmatrix} \qquad a_{ij} = \begin{cases} 1 & i \sim j \text{ with probability } p \\ 0 & \text{otherwise} \end{cases}$$

We consider the occurrence of an edge with probability $p \leq 1/2$ and every entry in the matrix $a_{ij} \in \{0, 1\}$. The probability distribution for every entry is then the Bernoulli distribution. Since the graph is undirected, the adjacency matrix is symmetric, i.e. it is invariant under orthogonal transformations

$$A \rightarrow O^\top A O, \qquad \text{with } OO^\top = 1. \tag{8.77}$$

With the assumption that the entries are independent, we have a symmetric matrix with entries independent and identically distributed. As we have seen in section 2.1, this leads to a Gaussian weight in the partition function. The partition function is

$$Z_N(a) = \sum_{\{\alpha\}} c_N(\alpha) \prod_{i=1}^{N} e^{-a \lambda_i^2}. \tag{8.78}$$

Here, it is crucial distinguishing between configurations having the same spectrum and different symmetry factor, related to the number of the associated automorphisms. In figure 8.7 it is shown an example of two isospectral configurations with different number

of automorphisms. The symmetry factor is given by

$$c_N(\alpha) = \frac{|S_N|}{|\mathrm{Aut}(\alpha)|} = \frac{N!}{|\mathrm{Aut}(\alpha)|}, \tag{8.79}$$

with S_N the symmetric group. In the jargon of graphs what is different between the two configurations reported in figure 8.7 (a) and (b) is the degree sequence

$$\begin{aligned} d(a) &= \{1,1,1,1,4\}, \\ d(b) &= \{2,2,0,2,2\}, \end{aligned} \tag{8.80}$$

whereas the sum of the degrees is the same, this given by the fact that the number of edges for the two graphs is the same.

Giving the similarity of the forms of the partition functions constructed in this chapter with those encountered in the theory of random matrix ensembles, we expect that some results can be applied to real networks.

Conclusions

In this thesis we have investigated several integrable systems in the framework offered by the approach of differential identities. We have seen how mean-field theories can be suitably described via the introduction of nonlinear equations of hydrodynamic type satisfied by the order parameters of the theory. The breaking of the solutions induced by the effects of nonlinearity is regularised via a viscous shock solution. We have explicitly applied the method to the Curie-Weiss model, where we have found that the order parameter satisfies a Hopf equation.

We have then studied the Volterra reduction of the Toda lattice, connected with the Hermitian matrix ensemble with even interactions only. At the leading order in the continuum limit of the field variable, we have obtained the Hopf hierarchy. We have analysed the specific case of all but the first three times set to zero. With this assumption, we have studied the dynamics of the solution and we have observed the emergence of a structure characterised by fast oscillations after the breaking. This feature resembles the structure of a dispersive shock and occurs in different scenarios in the space of parameters.

Within the perspective of the corresponding hydrodynamic systems, it seems that the magnetisation in the Curie-Weiss model and the continuum limit of the order parameter in the Volterra lattice belong to the same class of solutions, both being solutions of the Hopf hierarchy. What distinguish the two systems is the initial datum and the regularisation mechanism.

Particular emphasis has been given to the study of the symmetric matrix ensemble and its underpinning integrable structure, the Pfaff lattice. We have introduced a suitable notation of the field variables constituting the elements of the lattice, making the double-chain structure shared by the field variables manifest. We have considered the

GOE as the free theory (or initial datum) for the Pfaff lattice and we have introduced a suitable reduction, by selecting the even times only, in analogy with the construction of the Volterra lattice from the Toda lattice. We have studied the behaviour at the leading order in the continuum limit of the field variables in the first flow, where the equations can be recast in form of a new hydrodynamic chain.

The introduced hydrodynamic chain constitutes an interesting object per se, given that it differs from the standard integrable hydrodynamic chains studied in literature, for the presence of an additional seed. We have addressed the question of integrability of the chain, analysing the geometric structure behind it via the evaluation of the Nijenhuis and Haantjes tensors and we have obtained the corresponding Gibbons–Tsarev system. We have extended the study to the next two flows, finding a hydrodynamic chain-like structure as well. Also, we have observed a nominal proliferation of seeds in the hydrodynamic chains associated to higher flows and a dependence on an increasing number of nearest neighbours in the dynamics. We have then conjectured the existence of a hydrodynamic chain hierarchy. From these observations, it seems that the symmetric matrix ensemble is a system characterised by a sort of intrinsic multi-dimensionality. This is something that is evident starting from the more complex structure of the underlying Pfaff lattice compared to the Toda lattice. Therefore, we expect a broader and richer patterns of possible behaviours in the context of the symmetric matrix ensemble compared to those observed for the Hermitian matrix ensemble.

In the last part of the work we have applied the above mentioned method of differential identities to the two-star model, in the context of graph theory, reproducing the classical result of the mean-field underlying theory. Finally, we have determined the partition function for the one-dimensional Ising model in terms of the elements of the adjacency matrix associated to cycles.

Appendix A

Exploring the Pfaff lattice

A.1 Observations on the structure of the Pfaff lattice

Let us consider, for $N = 8$, the matrix L of the form

$$
L(t) = \begin{pmatrix}
L_{1,1}(t) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
L_{2,1}(t) & L_{2,2}(t) & L_{2,3}(t) & 0 & 0 & 0 & 0 & 0 \\
L_{3,1}(t) & L_{3,2}(t) & L_{3,3}(t) & 1 & 0 & 0 & 0 & 0 \\
L_{4,1}(t) & L_{4,2}(t) & L_{4,3}(t) & L_{4,4}(t) & L_{4,5}(t) & 0 & 0 & 0 \\
L_{5,1}(t) & L_{5,2}(t) & L_{5,3}(t) & L_{5,4}(t) & L_{5,5}(t) & 1 & 0 & 0 \\
L_{6,1}(t) & L_{6,2}(t) & L_{6,3}(t) & L_{6,4}(t) & L_{6,5}(t) & L_{6,6}(t) & L_{6,7}(t) & 0 \\
L_{7,1}(t) & L_{7,2}(t) & L_{7,3}(t) & L_{7,4}(t) & L_{7,5}(t) & L_{7,6}(t) & L_{7,7}(t) & 1 \\
L_{8,1}(t) & L_{8,2}(t) & L_{8,3}(t) & L_{8,4}(t) & L_{8,5}(t) & L_{8,6}(t) & L_{8,7}(t) & L_{8,8}(t)
\end{pmatrix}
$$

where $L_{i,j}$ are functions of $t = \{t_1, t_2, \ldots\}$ and $L_{2n,2n+1}$ are related to the Pfaffian τ−functions

$$
L_{2n,2n+1} = \left(\frac{h_{2n}}{h_{2n-2}} \right)^{1/2} \tag{A.1}
$$

The Hamiltonian commuting equations are

$$
\frac{\partial L}{\partial t_k} = \left[-\left(L^k \right)_t, L \right]. \tag{A.2}
$$

In order to explore the structure of the different elements of the matrix L, we will consider the equations (A.2) for different t_k and solve the corresponding system of equations in terms of elements $L_{i,j}$.

To determine other constraints on the elements of the matrix, we will consider that L is introduced as a matrix given by dressing the shift matrix Λ with the matrix Q, decomposition of the moments matrix

$$L(t) = Q(t)\Lambda Q(t)^{-1}. \tag{A.3}$$

A.2 Equations for derivatives w.r.t. t_1

We now consider the equation

$$\frac{\partial L}{\partial t_1} = [-(L)_t, L] \tag{A.4}$$

explicitly for $N = 4, 6$.

With $N = 4$, the matrix L is

$$L(t) = \begin{pmatrix} L_{1,1}(t) & 1 & & \\ L_{2,1}(t) & L_{2,2}(t) & L_{2,3}(t) & 0 \\ L_{3,1}(t) & L_{3,2}(t) & L_{3,3}(t) & 1 \\ L_{4,1}(t) & L_{4,2}(t) & L_{4,3}(t) & L_{4,4}(t) \end{pmatrix} \tag{A.5}$$

The system of equations that we can write by considering every non zero element of the matrix L is the following ($\partial/\partial t_1 = \partial_1$)

$$\partial_1 L_{1,1} = 0$$

$$\partial_1 L_{1,2} = 0$$

$$\partial_1 L_{2,1} = L_{2,3} L_{3,1} \tag{A.6}$$

$$\partial_1 L_{2,2} = L_{2,3} L_{3,2}$$

$$\partial_1 L_{2,3} = \frac{1}{2} L_{2,3} \left(-L_{1,1} - L_{2,2} + L_{3,3} + L_{4,4}\right)$$

Equations for derivatives w.r.t. t_1

$$\partial_1 L_{3,1} = -L_{2,3} - L_{2,1} L_{3,2} + L_{4,1} + \frac{1}{2} L_{3,1} \left(-L_{1,1} + L_{2,2} + L_{3,3} - L_{4,4}\right)$$

$$\partial_1 L_{3,2} = L_{3,1} + L_{4,2} + \frac{1}{2} L_{3,2} \left(L_{1,1} - L_{2,2} + L_{3,3} - L_{4,4}\right)$$

$$\partial_1 L_{3,3} = -L_{2,3} L_{3,2}$$

$$\partial_1 L_{3,4} = 0$$

$$\partial_1 L_{4,1} = -L_{2,1} L_{4,2} + L_{3,1} L_{4,3} + L_{2,3} \left(L_{1,1} - L_{4,4}\right) + \frac{1}{2} L_{4,1} \left(-L_{1,1} + L_{2,2} - L_{3,3} + L_{4,4}\right)$$

$$\partial_1 L_{4,2} = L_{2,3} - L_{4,1} + L_{3,2} L_{4,3} + \frac{1}{2} L_{4,2} \left(L_{1,1} - L_{2,2} - L_{3,3} + L_{4,4}\right)$$

$$\partial_1 L_{4,3} = -L_{2,3} L_{4,2}$$

$$\partial_1 L_{4,4} = 0$$

(A.7)

Inserting the constraints related to the form of Q, it is possible to solve the system. The matrix L for $N = 4$ have the structure in terms of Q elements given by

$$L = Q \Lambda Q^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\[1em] -\dfrac{q_{3,1}}{q_{4,4}} & -\dfrac{q_{3,2}}{q_{4,4}} & \dfrac{q_{2,2}}{q_{4,4}} & 0 \\[1em] -\dfrac{q_{3,1} q_{3,2} + q_{4,1} q_{4,4}}{q_{2,2} q_{4,4}} & \dfrac{-q_{3,2}^2 + (q_{3,1} - q_{4,2}) q_{4,4}}{q_{2,2} q_{4,4}} & \dfrac{q_{3,2}}{q_{4,4}} & 1 \\[1em] \dfrac{q_{3,1} q_{4,2}}{q_{2,2} q_{4,4}} & \dfrac{-q_{3,2} q_{4,2} + q_{4,1} q_{4,4}}{q_{2,2} q_{4,4}} & \dfrac{q_{4,2}}{q_{4,4}} & 0 \end{pmatrix}$$

(A.8)

where Q has the following structure

$$Q = \begin{pmatrix} q_{2,2} & 0 & & \\ 0 & q_{2,2} & & \\ q_{3,1} & q_{3,2} & q_{4,4} & 0 \\ q_{4,1} & q_{4,2} & 0 & q_{4,4} \end{pmatrix}$$

(A.9)

The added constraints are, then,

$$L_{1,1} = L_{4,4} = 0$$

$$L_{2,2} = -L_{3,3} \tag{A.10}$$

$$L_{1,2} = L_{3,4} = 1$$

With these prescriptions, the system is

$$\partial_1 L_{2,1} = L_{3,1} L_{2,3}$$

$$\partial_1 L_{2,2} = L_{3,2} L_{2,3}$$

$$\partial_1 L_{2,3} = -L_{2,2} L_{2,3}$$

$$\partial_1 L_{3,1} = -L_{2,1} L_{3,2} + L_{4,1} - L_{2,3}$$

$$\partial_1 L_{3,2} = L_{3,1} + L_{4,2} - L_{3,2} L_{2,2} \tag{A.11}$$

$$\partial_1 L_{4,1} = -L_{2,1} L_{4,2} + L_{3,1} L_{4,3} + L_{4,1} L_{2,2}$$

$$\partial_1 L_{4,2} = L_{3,2} L_{4,3} - L_{4,1} + L_{2,3}$$

$$\partial_1 L_{4,3} = -L_{4,2} L_{2,3},$$

reducing the number of equations to 8, with variables $\vec{L} = \{L_{2,1}, L_{2,2}, L_{2,3}, L_{3,1}, L_{3,2}, L_{4,1}, L_{4,2}, L_{4,3}\}$. The latter elements are effected by more constraints than those previously considered, thus yielding to the fact that the variables $L_{i,j}$ are not the best option to treat the system. If we consider as new variables the entries of the matrix Q, we will produce a system with 5 distinct equations and 6 variables $\{q_{2,2}, q_{3,1}, q_{3,2}, q_{4,1}, q_{4,2}, q_{4,4}\}$. By analysing the form of the L matrix in terms of Q entries, we recognise a suitable change of independent variables, for which it is possible to write a closed system of 5 equations.

The number of independent variables per N is

$$N_{\text{var}} = \sum_{i=1}^{(N-2)/2} i + \frac{N-2}{2}. \tag{A.12}$$

Equations for derivatives w.r.t. t_1

We introduce the variables a_i with $i = 0, 1, \ldots, 4$, written in terms of Q entries

$$a_0 = \frac{q_{2,2}}{q_{4,4}}$$

$$a_1 = \frac{q_{3,1}}{q_{4,4}}$$

$$a_2 = \frac{q_{3,2}}{q_{4,4}}$$

$$a_3 = \frac{q_{4,2}}{q_{4,4}}$$

$$a_4 = \frac{q_{4,1}}{q_{4,4}}$$

$$\begin{pmatrix} q_{2,2} & 0 & 0 & 0 \\ 0 & q_{2,2} & 0 & 0 \\ q_{3,1} & q_{3,2} & q_{4,4} & 0 \\ q_{4,1} & q_{4,2} & 0 & q_{4,4} \end{pmatrix}$$

The form of the matrix L in terms of this set of variables is

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -a_1 & -a_2 & a_0 & 0 \\ -\dfrac{a_1 a_2 + a_4}{a_0} & \dfrac{a_1 - a_2^2 - a_3}{a_0} & a_2 & 1 \\ -\dfrac{a_1 a_3}{a_0} & \dfrac{-a_2 a_3 + a_4}{a_0} & a_3 & 0 \end{pmatrix} \tag{A.13}$$

Starting from the system (A.11), we obtain the following in terms of a_i variables

$$\partial_1 a_0 = a_0 a_2$$

$$\partial_1 a_1 = a_1 a_2 + a_4$$

$$\partial_1 a_2 = -a_1 + a_2^2 + a_3 \tag{A.14}$$

$$\partial_1 a_3 = a_2 a_3 - a_4$$

$$\partial_1 a_4 = a_0^2 + a_1 a_3$$

The elements of the matrix L show a precise structure in terms of a_i.

The "skeleton" of the matrix in terms of the independent variables is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-a_1 & -a_2 & a_0 & 0 \\
-a_4 & -a_3 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

First, we notice that the independent fields appear as shown in the following.

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-a_1 & -a_2 & a_0 & 0 \\
-a_4 & a_1 - a_3 & a_2 & 1 \\
0 & a_4 & a_3 & 0
\end{pmatrix}
$$

Then the additional terms in the lower part of the matrix are given by multiplying the entries appearing in the more external frame

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-a_1 & -a_2 & a_0 & 0 \\
-a_4 - a_1 a_2 & a_1 - a_3 - a_2^2 & a_2 & 1 \\
-a_1 a_3 & a_4 - a_2 a_3 & a_3 & 0
\end{pmatrix}
$$

Finally, every element is rescaled by the entry a_0 and we reproduce the form of L for $N = 4$.

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-a_1 & -a_2 & a_0 & 0 \\
\dfrac{-a_4 - a_1 a_2}{a_0} & \dfrac{a_1 - a_3 - a_2^2}{a_0} & a_2 & 1 \\
\dfrac{-a_1 a_3}{a_0} & \dfrac{a_4 - a_2 a_3}{a_0} & a_3 & 0
\end{pmatrix}
$$

Equations for derivatives w.r.t. t_1

It is possible to identify the system (A.14) by analysing the second step of the construction for the L matrix.

We can identify the equations for the system in a_i variables by comparing the skeleton matrix and the matrix obtained in the steps previously shown

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
\partial_1 \\
-a_1 & \partial_1 \\
-a_2 & a_0 & 0 \\
\partial_1 & \partial_1 \\
-a_4 & -a_3 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\qquad
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-a_1 & -a_2 & a_0 & 0 \\
-a_4 - a_1 a_2 & a_1 - a_2^2 - a_3 & a_2 \; a_0 & 1 \\
-a_1 a_3 - a_0^2 & a_4 - a_2 a_3 & a_3 & 0
\end{pmatrix}
$$

We reproduce the system

$$\partial_1 a_1 = a_1 a_2 + a_4$$

$$\partial_1 a_2 = -a_1 + a_2^2 + a_3$$

$$\partial_1 a_3 = a_2 a_3 - a_4$$

$$\partial_1 a_4 = a_0^2 + a_1 a_3$$

$$\partial_1 a_0 = a_0 a_2$$

For $N = 6$ the matrix L is

$$L(t) = \begin{pmatrix} L_{1,1}(t) & 1 & 0 & 0 & 0 & 0 \\ L_{2,1}(t) & L_{2,2}(t) & L_{2,3}(t) & 0 & & \\ L_{3,1}(t) & L_{3,2}(t) & L_{3,3}(t) & 1 & 0 & 0 \\ L_{4,1}(t) & L_{4,2}(t) & L_{4,3}(t) & L_{4,4}(t) & L_{4,5}(t) & 0 \\ L_{5,1}(t) & L_{5,2}(t) & L_{5,3}(t) & L_{5,4}(t) & L_{5,5}(t) & 1 \\ L_{6,1}(t) & L_{6,2}(t) & L_{6,3}(t) & L_{6,4}(t) & L_{6,5}(t) & L_{6,6}(t) \end{pmatrix} \qquad (A.15)$$

Investigating the form of L written in terms of the decomposition matrix Q, we obtain a pretty complicate form, still useful to deduce some constraints on L elements

$$L_{1,1} = L_{6,6} = 0$$
$$L_{3,3} = -L_{2,2} \qquad (A.16)$$
$$L_{5,5} = -L_{4,4}$$

The form of the Q matrix for $N = 6$ is

$$Q = \begin{pmatrix} q_{2,2} & 0 & 0 & 0 & & \\ 0 & q_{2,2} & 0 & 0 & 0 & 0 \\ q_{3,1} & q_{3,2} & q_{4,4} & 0 & 0 & 0 \\ q_{4,1} & q_{4,2} & 0 & q_{4,4} & 0 & 0 \\ q_{5,1} & q_{5,2} & q_{5,3} & q_{5,4} & q_{6,6} & 0 \\ q_{6,1} & q_{6,2} & q_{6,3} & q_{6,4} & 0 & q_{6,6} \end{pmatrix} \qquad (A.17)$$

As in the previous case, we introduce a new set of variables, related to ratios of Q ele-

ments

$$a_0 = \frac{q_{2,2}}{q_{4,4}} \qquad a_1 = \frac{q_{3,1}}{q_{4,4}}$$

$$a_2 = \frac{q_{3,2}}{q_{4,4}} \qquad a_3 = \frac{q_{4,2}}{q_{4,4}}$$

$$a_4 = \frac{q_{4,1}}{q_{4,4}} \qquad b_0 = \frac{q_{4,4}}{q_{6,6}}$$

$$b_1 = \frac{q_{5,3}}{q_{6,6}} \qquad b_2 = \frac{q_{5,4}}{q_{6,6}}$$

$$b_3 = \frac{q_{6,4}}{q_{6,6}} \qquad b_4 = \frac{q_{6,3}}{q_{6,6}}$$

$$b_5 = \frac{q_{5,1}}{q_{6,6}} \qquad b_6 = \frac{q_{5,2}}{q_{6,6}}$$

$$b_7 = \frac{q_{6,2}}{q_{6,6}} \qquad b_8 = \frac{q_{6,1}}{q_{6,6}}$$
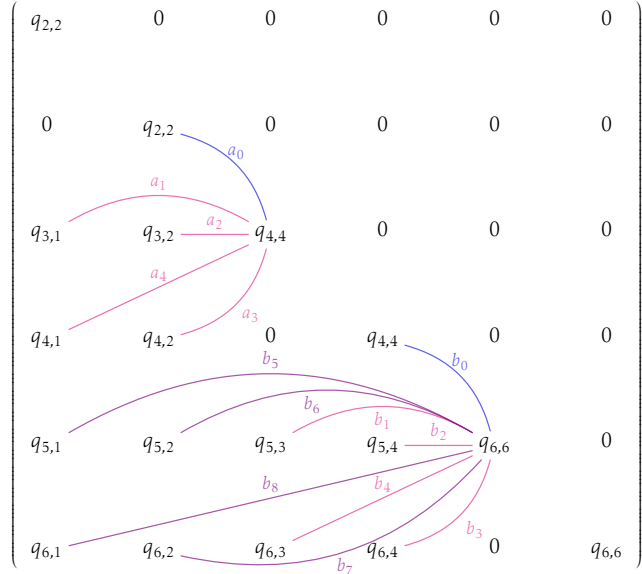
With this set of variables, we produce a closed system of 14 differential equations.

$$\partial_1 a_0 = a_0\, a_2 - \frac{1}{2} a_0\, b_2 \qquad\qquad \partial_1 b_0 = b_0\, b_2 - \frac{1}{2} b_0\, a_2$$

$$\partial_1 a_1 = a_1\, a_2 + a_4 \qquad\qquad \partial_1 b_1 = b_1\, b_2 + b_4 - b_6$$

$$\partial_1 a_2 = -a_1 + a_2^2 + a_3 \qquad\qquad \partial_1 b_2 = -b_1 + b_2^2 + b_3$$

$$\partial_1 a_3 = a_2(a_3 - b_1) - a_3\, b_2 - (a_4 - b_6) \qquad\qquad \partial_1 b_3 = b_2\, b_3 - b_4$$

$$\partial_1 a_4 = a_0^2 + a_1(a_3 - b_1) - a_4\, b_2 + b_5 \qquad\qquad \partial_1 b_4 = b_0^2 + b_1\, b_3 + b_7$$

$$\partial_1 b_5 = b_2\, b_5 + b_8$$

$$\partial_1 b_6 = b_5 + b_2\, b_6 + b_7$$

$$\partial_1 b_7 = a_2\, b_0^2 + b_3\, b_6 - b_8$$

$$\partial_1 b_8 = a_1\, b_0^2 + b_3\, b_5$$

The matrix L has the following form expressed with variables a_i and b_i

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\[2mm] -a_1 & -a_2 & a_0 & 0 & 0 & 0 \\[2mm] \dfrac{-a_1 a_2 - a_4}{a_0} & \dfrac{a_1 - a_2^2 - a_3}{a_0} & a_2 & 1 & 0 & 0 \\[2mm] \dfrac{-a_1(a_3-b_1)+a_4 b_2 - b_5}{a_0} & \dfrac{-a_2(a_3-b_1)+a_4+a_3 b_2 - b_6}{a_0} & a_3 - b_1 & -b_2 & b_0 & 0 \\[2mm] \dfrac{-a_4(b_1-b_2^2-b_3)+a_1(b_1 b_2+b_4-b_6)-b_2 b_5 - b_8}{a_0 b_0} & \dfrac{-a_3(b_1-b_2^2-b_3)+a_2(b_1 b_2+b_4-b_6)+b_5-b_2 b_6-b_7}{a_0 b_0} & \dfrac{-b_1 b_2 - b_4+b_6}{b_0} & \dfrac{b_1-b_2^2-b_3}{b_0} & b_2 & 1 \\[2mm] \dfrac{-a_4(-b_2 b_3+b_4)-a_1(-b_1 b_3+b_7)-b_3 b_5-b_7}{a_0 b_0} & \dfrac{-a_3(-b_2 b_3-b_4)+a_2(b_1 b_3-b_7)-b_3 b_6+b_8}{a_0 b_0} & \dfrac{-b_1 b_3+b_7}{b_0} & \dfrac{-b_2 b_3+b_4}{b_0} & b_3 & 0 \end{pmatrix}$$

We now analyse the structure of the matrix L, starting from the independent variables a_i and b_i

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_1 & -a_2 & a_0 & 0 & 0 & 0 \\ -a_4 & -a_3 & 0 & 1 & 0 & 0 \\ -b_5 & -b_6 & -b_1 & -b_2 & b_0 & 0 \\ -b_8 & -b_7 & -b_4 & -b_3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We consider the diagonal shifting of every independent variable

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_1 & -a_2 & a_0 & 0 & 0 & 0 \\ -a_4 & a_1 - a_3 & a_2 & 1 & 0 & 0 \\ -b_5 & a_4 - b_6 & a_3 - b_1 & -b_2 & b_0 & 0 \\ -b_8 & b_5 - b_7 & b_6 - b_4 & b_1 - b_3 & b_2 & 1 \\ & b_8 & b_7 & b_4 & b_3 & 0 \end{pmatrix}$$

Equations for derivatives w.r.t. t_1

We obtain the additional terms by multiplying the terms appearing in the external rows of the blocks progressively. First, we consider separately the action of the a_i and of b_i

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_1 & -a_2 & a_0 & 0 & 0 & 0 \\ -a_4 & -a_3 & a_2 & 1 & 0 & 0 \\ -b_5 & a_4 - b_6 & a_3 - b_1 & -b_2 & b_0 & 0 \\ -b_2 b_5 - b_8 & -b_2 b_6 + b_5 - b_7 & -b_1 b_2 + b_6 - b_4 & b_1 - b_2^2 - b_3 & b_2 & 1 \\ -b_3 b_5 & -b_3 b_6 + b_8 & -b_1 b_3 + b_7 & -b_2 b_3 + b_4 & b_3 & 0 \end{pmatrix}$$

Then we consider the mixed action of the two classes of variables in two steps. The first considering the second row and the third column

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_1 & -a_2 & a_0 & 0 & 0 & 0 \\ -a_1 a_2 - a_4 & a_1 - a_2^2 - a_3 & a_2 & 1 & 0 & 0 \\ -a_1(a_3 - b_1) - b_5 & -a_2(a_3 - b_1) + a_4 - b_6 & a_3 - b_1 & -b_2 & b_0 & 0 \\ -a_1(-b_1 b_2 + b_6 - b_4) - b_2 b_5 - b_8 & -a_2(-b_1 b_2 + b_6 - b_4) - b_2 b_6 + b_5 - b_7 & -b_1 b_2 + b_6 - b_4 & b_1 - b_2^2 - b_3 & b_2 & 1 \\ -a_1(-b_1 b_3 + b_7) - b_3 b_5 & -a_2(-b_1 b_3 + b_7) - b_3 b_6 + b_8 & -b_1 b_3 + b_7 & -b_2 b_3 + b_4 & b_3 & 0 \end{pmatrix}$$

and the second step considering the independent variables of the third row ($-a_4$ for the first element and $-a_3$ for the second) and the forth column

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_1 & -a_2 & a_0 & 0 & 0 & 0 \\ -a_1 a_2 - a_4 & a_1 - a_2^2 - a_3 & a_2 & 1 & 0 & 0 \\ -a_1(a_3 - b_1) + a_4 b_2 - b_5 & -a_2(a_3 - b_1) + a_3 b_2 + a_4 - b_6 & a_3 - b_1 & -b_2 & b_0 & 0 \\ -a_1(-b_1 b_2 + b_6 - b_4) - a_4(b_1 - b_2^2 - b_3) - b_2 b_5 - b_8 & -a_2(-b_1 b_2 + b_6 - b_4) - a_3(b_1 - b_2^2 - b_3) - b_2 b_6 + b_5 - b_7 & -b_1 b_2 + b_6 - b_4 & b_1 - b_2^2 - b_3 & b_2 & 1 \\ -a_1(-b_1 b_3 + b_7) - a_4(-b_2 b_3 + b_4) - b_3 b_5 & -a_2(-b_1 b_3 + b_7) - a_3(-b_2 b_3 + b_4) - b_3 b_6 + b_8 & -b_1 b_3 + b_7 & -b_2 b_3 + b_4 & b_3 & 0 \end{pmatrix}$$

Finally, we rescale the entries in the bulk blocks with the variables a_0 and b_0

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-a_1 & -a_2 & \boxed{a_0} & 0 & 0 & 0 \\
\frac{-a_1 a_2 - a_4}{a_0} & \frac{a_1 - a_2^2 - a_3}{a_0} & a_2 & 1 & 0 & 0 \\
\frac{-a_1(a_3-b_1)+a_4 b_2 - b_5}{a_0} & \frac{-a_2(a_3-b_1)+a_3 b_2 + a_4 - b_6}{a_0} & a_3 - b_1 & -b_2 & b_0 & 0 \\
\frac{-a_1(-b_1 b_2 + b_6 - b_4) - a_4(b_1 - b_2^2 - b_3) - b_2 b_5 - b_8}{a_0} & \frac{-a_2(-b_1 b_2 + b_6 - b_4) - a_3(b_1 - b_2^2 - b_3) - b_2 b_6 + b_5 - b_7}{a_0} & -b_1 b_2 + b_6 - b_4 & b_1 - b_2^2 - b_3 & b_2 & 1 \\
\frac{-a_1(-b_1 b_3 + b_7) - a_4(-b_2 b_3 + b_4) - b_3 b_5}{a_0} & \frac{-a_2(-b_1 b_3 + b_7) - a_3(-b_2 b_3 + b_4) - b_3 b_6 + b_8}{a_0} & -b_1 b_3 + b_7 & -b_2 b_3 + b_4 & b_3 & 0
\end{pmatrix}$$

and we can reproduce entirely the complete form of the matrix L for $N = 6$

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-a_1 & -a_2 & a_0 & 0 & 0 & 0 \\
\frac{-a_1 a_2 - a_4}{a_0} & \frac{a_1 - a_2^2 - a_3}{a_0} & a_2 & 1 & 0 & 0 \\
\frac{-a_1(a_3-b_1)+a_4 b_2 - b_5}{a_0} & \frac{-a_2(a_3-b_1)+a_3 b_2 + a_4 - b_6}{a_0} & a_3 - b_1 & -b_2 & \boxed{b_0} & 0 \\
\frac{-a_1(-b_1 b_2 + b_6 - b_4) - a_4(b_1 - b_2^2 - b_3) - b_2 b_5 - b_8}{a_0 b_0} & \frac{-a_2(-b_1 b_2 + b_6 - b_4) - a_3(b_1 - b_2^2 - b_3) - b_2 b_6 + b_5 - b_7}{a_0 b_0} & \frac{-b_1 b_2 + b_6 - b_4}{b_0} & \frac{b_1 - b_2^2 - b_3}{b_0} & b_2 & 1 \\
\frac{-a_1(-b_1 b_3 + b_7) - a_4(-b_2 b_3 + b_4) - b_3 b_5}{a_0 b_0} & \frac{-a_2(-b_1 b_3 + b_7) - a_3(-b_2 b_3 + b_4) - b_3 b_6 + b_8}{a_0 b_0} & \frac{-b_1 b_3 + b_7}{b_0} & \frac{-b_2 b_3 + b_4}{b_0} & b_3 & 0
\end{pmatrix}$$

Analogously to what we did for the case $N = 4$, we will analyse the structure of the matrix L in terms of the variables a_i and b_i in order to easily produce the equations composing the system for $N = 6$. We start considering the lower part of the matrix, involving equations in b_i variables for $i = 1, \ldots, 8$

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-a_1 & -a_2 & a_0 & 0 & 0 & 0 \\
-a_4 & -a_3 & 0 & 1 & 0 & 0 \\
-b_5 & -b_6 & -b_1 & -b_2 & b_0 & 0 \\
-b_8 & -b_7 & -b_4 & -b_3 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-a_1 & -a_2 & a_0 & 0 & 0 & 0 \\
-a_1 a_2 - a_4 & a_1 - a_2^2 - a_3 & a_2 & 1 & 0 & 0 \\
-b_5 & a_4 - b_6 & a_3 - b_1 & -b_2 & b_0 & 0 \\
-b_2 b_5 - b_8 & -b_2 b_6 + b_5 - b_7 & -b_1 b_2 + b_6 - b_4 & b_1 - b_2^2 - b_3 & b_2 & 1 \\
-a_1 b_0^2 - b_3 b_5 & -a_2 b_0^2 - b_3 b_6 + b_8 & -b_0^2 - b_1 b_3 + b_7 & -b_2 b_3 + b_4 & b_3 & 0
\end{pmatrix}$$

Equations for derivatives w.r.t. t_1

We reproduce the following equations

$$\partial_1 b_1 = b_1 b_2 + b_4 - b_6 \qquad\qquad \partial_1 b_5 = b_2 b_5 + b_8$$

$$\partial_1 b_2 = -b_1 + b_2^2 + b_3 \qquad\qquad \partial_1 b_6 = b_5 + b_2 b_6 + b_7$$

$$\partial_1 b_3 = b_2 b_3 - b_4 \qquad\qquad \partial_1 b_7 = a_2 b_0^2 + b_3 b_6 - b_8$$

$$\partial_1 b_4 = b_0^2 + b_1 b_3 + b_7 \qquad\qquad \partial_1 b_8 = a_1 b_0^2 + b_3 b_5$$

We now consider the equations with a_i variables by comparing the skeleton matrix with independent fields and the step of the matrix involving all the products in a_i variables

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-a_1 & -a_2 & a_0 & 0 & 0 & 0 \\
-a_4 & -a_3 & 0 & 1 & 0 & 0 \\
-b_5 & -b_6 & -b_1 & -b_2 & b_0 & 0 \\
-b_8 & -b_7 & -b_4 & -b_3 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-a_1 & -a_2 & a_0 & 0 & 0 & 0 \\
-a_1 a_2 - a_4 & a_1 - a_2^2 - a_3 & a_2 & 1 & 0 & 0 \\
-a_0^2 - a_1(a_3 - b_1) + a_4 b_2 - b_5 & -a_2(a_3 - b_1) + a_3 b_2 + a_4 - b_6 & a_3 - b_1 & -b_2 & b_0 & 0 \\
-a_1(-b_1 b_2 + b_6 - b_4) - a_4(b_1 - b_2^2 - b_3) - b_2 b_5 - b_8 & -a_2(-b_1 b_2 + b_6 - b_4) - a_3(b_1 - b_2^2 - b_3) - b_2 b_6 + b_5 - b_7 & -b_1 b_2 + b_6 - b_4 & b_1 - b_2^2 - b_3 & b_2 & 1 \\
-a_1(-b_1 b_3 + b_7) - a_4(-b_2 b_3 + b_4) - b_3 b_5 & -a_2(-b_1 b_3 + b_7) - a_3(-b_2 b_3 + b_4) - b_3 b_6 + b_8 & -b_1 b_3 + b_7 & -b_2 b_3 + b_4 & b_3 & 0
\end{pmatrix}
$$

We have the equations

$$\partial_1 a_1 = a_1 b_2 + a_4$$

$$\partial_1 a_2 = -a_1 + a_2^2 + a_3$$

$$\partial_1 a_3 = a_2(a_3 - b_1) - a_3 b_2 - (a_4 - b_6)$$

$$\partial_1 a_4 = a_0^2 + a_1(a_3 - b_1) - a_4 b_2 + b_5$$

Finally, for the equations involving the variables a_0 and b_0 with derivatives in the skeleton matrix, they are connected respectively to a_2 and b_2 and we obtain the last two equations in the system

$$\partial_1 a_0 = a_0 a_2 - \frac{1}{2} a_0 b_2 \qquad \partial_1 b_0 = b_0 b_2 - \frac{1}{2} b_0 a_2.$$

Appendix B

Continuous limit of the even Pfaff lattice

B.1 Equations for t_2-flow and higher order corrections

We provide, for the reduced even Pfaff Hierarchy, the corrections to the leading order of equation (7.83) up to $O(\varepsilon^3)$:

$$u_t^k = \left(\left((k+2)u^{k+1} - ku^{k-1} - u^1 u_x^0 u^k\right)u_x^0 - u^0 u_x^1 u^k + u^0 u_x^{k-1} + u^0 u_x^{k+1}\right) + \frac{1}{2}\left(k^2 u_{xx}^0(-u^{k-1})\right.$$

$$+(k^2 + 2k)u_{xx}^0 u^{k+1} - ku^k\left(2u_x^0 u_x^1 + u^1 u_{xx}^0 + u^0 u_{xx}^1\right) - 2u_x^0 u_x^{k+1} + u^0\left(u_{xx}^{k-1} - u_{xx}^{k+1}\right)\left.\right)\varepsilon$$

$$+ \frac{1}{12}\left(2\left(u_{xxx}^0\left(((k+1)^3 + 1)u^{k+1} - k^3 u^{k-1}\right) + 3u_{xx}^0 u_x^{k+1} + 3u_x^0 u_{xx}^{k+1} + u^0\left(u_{xxx}^{k-1} + u_{xxx}^{k+1}\right)\right)\right.$$

$$\left.-u^k\left(3k^2 + 3k + 2\right)\left(3u_x^1 u_{xx}^0 + 3u_x^0 u_{xx}^1 + u^1 u_{xxx}^0 + u^0 u_{xxx}^1\right)\right)\varepsilon^2 + O\left(\varepsilon^3\right), \qquad\qquad k < 0$$

$$u_t^0 = u^0\left(u_x^{-1} + u^1 u_x^0 + u^0 u_x^1\right) + \frac{1}{2}\left(u^0 u_{xx}^{-1}\right)\varepsilon + \frac{1}{6}u^0\left(3u_x^1 u_{xx}^0 + 3u_x^0 u_{xx}^1 + u_{xxx}^{-1} + u^1 u_{xxx}^0\right.$$

$$\left.+u^0 u_{xxx}^1\right)\varepsilon^2 + O\left(\varepsilon^3\right)$$

$$u_t^1 = \left(2u^2 u_x^0 - u^1\left(u^1 u_x^0 + u^0 u_x^1\right) + u^0 u_x^2\right) + \left(-u_x^0 u_x^2 - \frac{1}{2}u^0 u_{xx}^2\right)\varepsilon + \frac{1}{6}\left(-u_{xxx}^0(u^1)^2\right.$$

$$\left.-\left(3u_x^1 u_{xx}^0 + 3u_x^0 u_{xx}^1 + u^0 u_{xxx}^1\right)u^1 + 3u_x^2 u_{xx}^0 + 3u_x^0 u_{xx}^2 + 2u^2 u_{xxx}^0 + u^0 u_{xxx}^2\right)\varepsilon^2 + O\left(\varepsilon^3\right)$$

$$u_t^k = \left(\left((k+1)u^{k+1} - (k-1)u^{k-1} + u^1 u^k \right) u_x^0 + u^0 u_x^1 u^k + u^0 u_x^1 u^k + u^0 u_x^{k-1} + u^0 u_x^{k+1} \right)$$

$$+ \frac{1}{2} \left(u_{xx}^0 \left((k^2-1)u^{k+1} - (k^2-2k+1)u^{k-1} \right) - 2u_x^0 u_x^{k+1} + (k-1)\left(2u_x^0 u_x^1 + u^1 u_{xx}^0 + u^0 u_{xx}^1 \right) u^k \right.$$

$$+ u^0 u_{xx}^{k-1} - u^0 u_{xx}^{k+1} \Big) \varepsilon + \frac{1}{12} \left(2 \left(u_{xxx}^0 \left((k^3+1)u^{k+1} - (k-1)^3 u^{k-1} \right) + 3u_{xx}^0 u_x^{k+1} + 3u_x^0 u_{xx}^{k+1} \right. \right.$$

$$+ u^0 \left(u_{xxx}^{k-1} + u_{xxx}^{k+1} \right) \Big) + \left(3k^2 - 3k + 2 \right) \left(3u_x^1 u_{xx}^0 + 3u_x^0 u_{xx}^1 + u^1 u_{xxx}^0 + u^0 u_{xxx}^1 \right) u^k \Big) \varepsilon^2$$

$$+ O\left(\varepsilon^3 \right), \qquad k > 1.$$

Appendix C

Hydrodynamic chains for flow t_4

Here, we list the discrete equations in t_4 and the leading order of the continuum limit, recasting the expressions in terms of the corresponding hydrodynamic chains reported in (7.102)

$$u^k_{t_4} = a^k_{-1} u^{-1}_x + a^k_0 u^0_x + a^k_1 u^1_x + a^k_2 u^2_x + a^k_{k-2} u^{k-2}_x + a^k_{k-1} u^{k-1}_x + a^k_{k+1} u^{k+1}_x + a^k_{k+2} u^{k+2}_x \qquad \text{(C.1)}$$

C.1 Discrete equations in t_4

- field u^{-1}_n

$$
\begin{aligned}
\partial_{t_4} u^{-1}_n = {} & -u^1_{n-1}\left(u^0_{n-1}\right)^3 - u^{-1}_n\left(u^1_{n-1}\right)^2\left(u^0_{n-1}\right)^2 - u^{-1}_{n-1}\left(u^0_{n-1}\right)^2 - u^{-1}_n\left(u^0_{n-1}\right)^2 \\
& - u^0_{n-2}u^1_{n-2}\left(u^0_{n-1}\right)^2 - u^{-3}_{n-2}u^0_{n-2}u^0_{n-1} - \left(u^{-1}_n\right)^2 u^1_{n-1}u^0_{n-1} - u^{-1}_{n-1}u^{-1}_n u^1_{n-1}u^0_{n-1} \\
& - u^{-2}_{n-1}\left(u^{-1}_{n-1} + u^{-1}_n + u^0_{n-2}u^1_{n-2} + u^0_{n-1}u^1_{n-1}\right)u^0_{n-1} - u^{-1}_n u^0_{n-2}u^2_{n-2}u^0_{n-1} \\
& + u^{-1}_n\left(u^0_n\right)^2 + u^{-1}_{n+1}\left(u^0_n\right)^2 + u^{-1}_n\left(u^0_n\right)^2\left(u^1_n\right)^2 + u^{-2}_n u^{-1}_n u^0_n + u^{-2}_n u^{-1}_{n+1}u^0_n \\
& + u^{-3}_n u^0_n u^0_{n+1} + \left(u^0_n\right)^3 u^1_n + u^{-2}_n\left(u^0_n\right)^2 u^1_n + \left(u^{-1}_n\right)^2 u^0_n u^1_n + u^{-1}_n u^{-1}_{n+1}u^0_n u^1_n \\
& + \left(u^0_n\right)^2 u^0_{n+1}u^1_{n+1} + u^{-2}_n u^0_n u^0_{n+1}u^1_{n+1} + u^{-1}_n u^0_n u^0_{n+1}u^2_n,
\end{aligned}
$$

$$\text{(C.2)}$$

- field u_n^0

$$\partial_{t_4} u_n^0 = \frac{1}{2} u_n^0 \Big(-2\left(u_n^{-1}\right)^2 - \left(3u_{n-1}^0 u_{n-1}^1 + 2u_n^0 u_n^1\right) u_n^{-1} + 2\left(u_{n+1}^{-1}\right)^2 - \left(u_{n-1}^0\right)^2 \left(u_{n-1}^1\right)^2$$

$$+ \left(u_{n+1}^0\right)^2 \left(u_{n+1}^1\right)^2 - 2u_{n-1}^{-2} u_{n-1}^0 + 2u_{n+1}^{-2} u_{n+1}^0 - u_{n-1}^{-1} u_{n-1}^0 u_{n-1}^1 + u_{n+2}^{-1} u_{n+1}^0 u_{n+1}^1$$

$$+ u_{n+1}^{-1} \left(2u_n^0 u_n^1 + 3u_{n+1}^0 u_{n+1}^1\right) - u_{n-2}^0 u_{n-1}^0 u_{n-2}^2 - u_{n-1}^0 u_n^0 u_{n-1}^2 + u_n^0 u_{n+1}^0 u_n^2$$

$$+ u_{n+1}^0 u_{n+2}^0 u_{n+1}^2 \Big),$$

$$\tag{C.3}$$

- field u_n^1

$$\partial_{t_4} u_n^1 = \frac{1}{2} \Big(u_{n-1}^1 \left(u_{n-1}^1 u_n^1 - 2u_{n-1}^2\right) \left(u_{n-1}^0\right)^2 + \left(u_n^{-1} u_{n-1}^1 u_n^1 + u_{n-2}^0 u_{n-2}^2 u_n^1 - 2u_n^{-1}\right.$$

$$u_{n-1}^2 + u_{n-1}^{-1} \left(u_{n-1}^1 u_n^1 - 2u_{n-1}^2\right) + u_n^0 (2u_{n-1}^1 - u_n^1 u_{n-1}^2) - 2u_{n-2}^0 u_{n-2}^3 \Big) u_{n-1}^0$$

$$- u_{n+1}^0 \Big(u_{n+1}^0 u_n^1 \left(u_{n+1}^1\right)^2 + u_{n+2}^{-1} u_n^1 u_{n+1}^1 - 2u_{n+1}^0 u_n^2 u_{n+1}^1 - 2u_{n+2}^{-1} u_n^2 + u_{n+1}^{-1}$$

$$\left(u_n^1 u_{n+1}^1 - 2u_n^2\right) + u_n^0 (2u_{n+1}^1 - u_n^1 u_n^2) + u_{n+2}^0 u_n^1 u_{n+1}^2 - 2u_{n+2}^0 u_n^3 \Big) \Big),$$

$$\tag{C.4}$$

- field u_n^2

$$\partial_{t_4} u_n^2 = \frac{1}{2} \Big(\left(u_{n-1}^1\right)^2 u_n^2 \left(u_{n-1}^0\right)^2 - 2u_{n-1}^1 u_{n-1}^3 \left(u_{n-1}^0\right)^2 + 2u_n^0 u_{n-1}^1 u_{n+1}^1 u_{n-1}^0$$

$$+ u_{n-1}^{-1} u_{n-1}^1 u_n^2 u_{n-1}^0 + u_n^{-1} u_{n-1}^1 u_n^2 u_{n-1}^0 + u_{n-2}^0 u_{n-2}^2 u_n^2 u_{n-1}^0 - 2u_{n-1}^{-1} u_{n-1}^3 u_{n-1}^0$$

$$- 2u_n^{-1} u_{n-1}^3 u_{n-1}^0 - 2u_n^0 u_n^1 u_{n-1}^3 u_{n-1}^0 - 2u_{n-2}^0 u_{n-2}^4 u_{n-1}^0 + 2u_n^{-1} u_n^0 u_{n+1}^1$$

$$+ 2\left(u_n^0\right)^2 u_n^1 u_{n+1}^1 - 2\left(u_{n+1}^0\right)^2 u_n^1 u_{n+1}^1 - 2u_{n+1}^0 u_{n+2}^0 u_n^1 u_{n+2}^1$$

$$- \left(u_n^0\right)^2 \left(u_n^1\right)^2 u_n^2 + \left(u_{n+1}^0\right)^2 \left(u_{n+1}^1\right)^2 u_n^2 - \left(u_{n+2}^0\right)^2 \left(u_{n+2}^1\right)^2 u_n^2 - u_n^{-1} u_n^0 u_n^1 u_n^2$$

$$- u_{n+3}^{-1} u_{n+2}^0 u_{n+2}^1 u_n^2 + u_{n+1}^{-1} \left(u_{n+1}^0 \left(u_{n+1}^1 u_n^2 - 2u_n^1\right) + u_n^0 \left(2u_{n+1}^1 - u_n^1 u_n^2\right)\right)$$

$$- u_{n+2}^0 u_{n+3}^0 u_n^2 u_{n+2}^2 + 2u_{n+3}^{-1} u_{n+2}^0 u_n^3 + 2u_{n+1}^0 u_{n+2}^0 u_{n+1}^1 u_n^3 + 2\left(u_{n+2}^0\right)^2 u_{n+2}^1 u_n^3$$

$$+ u_{n+2}^{-1} (u_{n+1}^0 \left(u_{n+1}^1 u_n^2 - 2u_n^1\right) + u_{n+2}^0 (2u_n^3 - u_{n+2}^1 u_n^2)) + 2u_{n+2}^0 u_{n+3}^0 u_n^4 \Big),$$

$$\tag{C.5}$$

Discrete equations in t_4

- field u_n^k for $k > 2$

$$\partial_{t_4} u_n^k = \frac{1}{2}\Bigg(\left(u_{n-1}^1\right)^2 \left(u_{n-1}^0\right)^2 u_n^k - 2u_{n-1}^1 \left(u_{n-1}^0\right)^2 u_{n-1}^{k+1} + u_{n-1}^{-1} u_{n-1}^1 u_{n-1}^0 u_n^k$$

$$+ u_n^{-1} u_{n-1}^1 u_{n-1}^0 u_n^k + u_{n-2}^0 u_{n-2}^2 u_{n-1}^0 u_n^k$$

$$- 2u_{n-1}^{-1} u_{n-1}^0 u_{n-1}^{k+1} - 2u_n^{-1} u_{n-1}^0 u_{n-1}^{k+1} - 2u_{n-2}^0 u_{n-1}^0 u_{n-2}^{k+2} - 2u_{k+n-1}^{-1} u_{k+n-1}^0 u_n^{k-1}$$

$$- 2u_{k+n}^{-1} u_{k+n-1}^0 u_n^{k-1} - 2\left(u_{k+n-1}^0\right)^2 u_{k+n-1}^1 u_n^{k-1} - 2u_{k+n-1}^0 u_{k+n}^0 u_{k+n}^1 u_n^{k-1}$$

$$+ \left(u_{k+n-1}^0\right)^2 \left(u_{k+n-1}^1\right)^2 u_n^k - \left(u_{k+n}^0\right)^2 \left(u_{k+n}^1\right)^2 u_n^k + u_{k+n-1}^{-1} u_{k+n-1}^0 u_{k+n-1}^1 u_n^k$$

$$+ u_{k+n}^{-1} u_{k+n-1}^0 u_{k+n-1}^1 u_n^k - u_{k+n}^{-1} u_{k+n}^0 u_{k+n}^1 u_n^k - u_{k+n+1}^{-1} u_{k+n}^0 u_{k+n}^1 u_n^k$$

$$- u_{k+n}^0 u_{k+n+1}^0 u_{k+n}^2 u_n^k + u_{k+n-2}^0 u_{k+n-1}^0 (u_{k+n-2}^2 u_n^k - 2u_n^{k-2}) + \left(u_n^0\right)^2 u_n^1 (2u_{n+1}^{k-1} - u_n^1 u_n^k)$$

$$+ u_n^0 (2u_{n+1}^{-1} u_{n+1}^{k-1} + 2u_{n-1}^0 u_{n-1}^1 u_{n+1}^{k-1} - u_{n+1}^{-1} u_n^1 u_n^k + u_n^{-1}(2u_{n+1}^{k-1} - u_n^1 u_n^k)$$

$$+ u_{n+1}^0 (2u_{n+2}^{k-2} - u_n^2 u_n^k) - 2u_{n-1}^0 u_n^1 u_{n-1}^{k+1}) + 2u_{k+n}^{-1} u_{k+n}^0 u_n^{k+1} + 2u_{k+n+1}^{-1} u_{k+n}^0 u_n^{k+1}$$

$$+ 2u_{k+n-1}^0 u_{k+n}^0 u_{k+n-1}^1 u_n^{k+1} + 2\left(u_{k+n}^0\right)^2 u_{k+n}^1 u_n^{k+1} + 2u_{k+n}^0 u_{k+n+1}^0 u_n^{k+2} \Bigg),$$

$$\text{(C.6)}$$

- field u_n^{-k} for $k > 1$

$$\partial_{t_4} u_n^{-k} = \frac{1}{2}\Big(-2u_{n-1}^1 \left(u_{n-1}^0\right)^2 u_{n-1}^{1-k} - \left(u_{n-1}^1\right)^2 \left(u_{n-1}^0\right)^2 u_n^{-k} - 2u_{n-1}^{-1} u_{n-1}^0 u_{n-1}^{1-k}$$

$$- 2u_n^{-1} u_{n-1}^0 u_{n-1}^{1-k} - 2u_{n-2}^0 u_{n-2}^1 u_{n-1}^0 u_{n-1}^{1-k} - 2u_{n-2}^0 u_{n-1}^0 u_{n-2}^{2-k} - u_{n-1}^{-1} u_{n-1}^1 u_{n-1}^0 u_n^{-k}$$

$$- u_n^{-1} u_{n-1}^1 u_{n-1}^0 u_n^{-k} - u_{n-2}^0 u_{n-2}^2 u_{n-1}^0 u_n^{-k} - 2u_{k+n-2}^{-1} u_{k+n-2}^0 u_n^{-k-1}$$

$$- 2u_{k+n-1}^{-1} u_{k+n-2}^0 u_n^{-k-1} - 2\left(u_{k+n-2}^0\right)^2 u_{k+n-2}^1 u_n^{-k-1} + 2u_{k+n-1}^{-1} u_{k+n-1}^0 u_n^{1-k}$$

$$+ 2u_{k+n}^{-1} u_{k+n-1}^0 u_n^{1-k} + 2\left(u_{k+n-1}^0\right)^2 u_{k+n-1}^1 u_n^{1-k} + 2u_{k+n-1}^0 u_{k+n}^0 u_{k+n}^1 u_n^{1-k}$$

$$+ 2u_{k+n-1}^0 u_{k+n}^0 u_n^{2-k} - \left(u_{k+n-2}^0\right)^2 \left(u_{k+n-2}^1\right)^2 u_n^{-k} + \left(u_{k+n-1}^0\right)^2 \left(u_{k+n-1}^1\right)^2 u_n^{-k}$$

$$- u_{k+n-2}^{-1} u_{k+n-2}^0 u_{k+n-2}^1 u_n^{-k} - u_{k+n-1}^{-1} u_{k+n-2}^0 u_{k+n-2}^1 u_n^{-k} + u_{k+n-1}^{-1} u_{k+n-1}^0 u_{k+n-1}^1 u_n^{-k}$$

$$+ u_{k+n}^{-1} u_{k+n-1}^0 u_{k+n-1}^1 u_n^{-k} + u_{k+n-1}^0 u_{k+n}^0 u_{k+n-1}^2 u_n^{-k} + \left(u_n^0\right)^2 u_n^1 (2u_{n+1}^{-k-1} + u_n^1 u_n^{-k})$$

$$- u_{k+n-3}^0 u_{k+n-2}^0 (2u_n^{-k-2} + 2u_{k+n-3}^1 u_n^{-k-1} + u_{k+n-3}^2 u_n^{-k})$$

$$+ u_n^0 ((u_n^{-1} + u_{n+1}^{-1})(2u_{n+1}^{-k-1} + u_n^1 u_n^{-k}) + u_{n+1}^0 (2u_{n+2}^{-k-2} + 2u_{n+1}^1 u_{n+1}^{-k-1} + u_n^2 u_n^{-k})) \Big),$$

$$\tag{C.7}$$

C.2   Continuum limit in t_4

The leading order of the thermodynamic limit with the time rescaled as $t = \varepsilon t_4$ can be recast in the hydrodynamic chain

$$u_t^k = a_{-1}^k u_x^{-1} + a_0^k u_x^0 + a_1^k u_x^1 + a_2^k u_x^2 + a_{k-2}^k u_x^{k-2} + a_{k-1}^k u_x^{k-1} + a_{k+1}^k u_x^{k+1} + a_{k+2}^k u_x^{k+2},$$

(C.8)

whose coefficients are listed below.

$$a_{k\pm1}^k = \begin{cases} -2(-u_x^{-1} u_x^0 - (u_x^0)^2 u_x^1) & k > 2 \\ \\ 0 & k \le 2 \end{cases}$$

(C.9)

$$a_{k\pm2}^k = \begin{cases} 2(u_x^0)^2 & k > 2 \\ \\ 0 & k \le 2 \end{cases}$$

(C.10)

$$a_2^k = \begin{cases} -2(u_x^0)^2 u_x^k & k > 2 \\ \\ -2(u_x^0)^2 u_x^2 & k = 2 \\ \\ 2u_x^{-1} u_x^0 & k = 1 \\ \\ 2(u_x^0)^3 & k = 0 \\ \\ 2u_x^{-1}(u_x^0)^2 & k = -1 \\ \\ 2(u_x^0)^2 u_x^k & k < -1 \end{cases}$$

(C.11)

$$a_1^k = \begin{cases} -2\left(k(u_x^0)^2 u_x^{-1+k} + u_x^{-1} u_x^0 u_x^k + (u_x^0)^2 u_x^1 u_x^k - k(u_x^0)^2 u_x^{1+k}\right) & k > 2 \\[2ex] -2\left(-u_x^{-1} u_x^0 + (u_x^0)^2 u_x^1 + u_x^{-1} u_x^0 u_x^2 + (u_x^0)^2 u_x^1 u_x^2 - 2(u_x^0)^2 u_x^3\right) & k = 2 \\[2ex] -2\left((u_x^0)^2 + u_x^{-1} u_x^0 u_x^1 + (u_x^0)^2 (u_x^1)^2 - (u_x^0)^2 u_x^2\right) & k = 1 \\[2ex] 2u_x^0 \left(2u_x^{-1} u_x^0 + (u_x^0)^2 u_x^1\right) & k = 0 \\[2ex] 2\left((u_x^{-1})^2 u_x^0 + 2u_x^{-2}(u_x^0)^2 + 2(u_x^0)^3 + u_x^{-1}(u_x^0)^2 u_x^1\right) & k = -1 \\[2ex] 2\left(3(u_x^0)^2 u_x^{-1+k} + k(u_x^0)^2 u_x^{-1+k} + (u_x^0)^2 u_x^{1+k} + k(u_x^0)^2 u_x^{1+k}\right. \\[1ex] \left. + u_x^{-1} u_x^0 u_x^k + (u_x^0)^2 u_x^1 u_x^k\right) & k < -1 \end{cases}$$

$$(C.12)$$

$$a_{-1}^k = \begin{cases} -2\left(-u_x^0 u_x^{k-1} + k u_x^0 u_x^{k-1} + u_x^0 u_x^1 u_x^k - k u_x^0 u_x^{k+1} - u_x^0 u_x^{k+1}\right) & k > 2 \\[2ex] -2\left(u_x^0 u_x^1 + u_x^0 u_x^2 u_x^1 - 3u_x^0 u_x^3\right) & k = 2 \\[2ex] -2\left(u_x^0 \left(u_x^1\right)^2 - 2u_x^0 u_x^2\right) & k = 1 \\[2ex] 2u_x^0 \left(u_x^{-1} + 2u_x^0 u_x^1\right) & k = 0 \\[2ex] 2\left(\left(u_x^0\right)^2 + u_x^{-2} u_x^0 + u_x^{-1} u_x^1 u_x^0\right) & k = -1 \\[2ex] 2\left(-k u_x^0 u_x^{-k-1} + 2u_x^0 u_x^{-k-1} + k u_x^0 u_x^{1-k} + u_x^0 u_x^1 u_x^{-k}\right) & k < -1 \end{cases}$$

$$(C.13)$$

$$a_0^k = \begin{cases} -2\Big(-2u_x^0 u_x^{-2+k} + k u_x^0 u_x^{-2+k} - u_x^{-1} u_x^{-1+k} + k u_x^{-1} u_x^{-1+k} - u_x^0 u_x^1 u_x^{-1+k} + 2k u_x^0 u_x^1 u_x^{-1+k} \\ \quad\quad + u_x^{-1} u_x^1 u_x^k + u_x^0 (u_x^1)^2 u_x^k + 2u_x^0 u_x^2 u_x^k - u_x^{-1} u_x^{1+k} - k u_x^{-1} u_x^{1+k} - u_x^0 u_x^1 u_x^{1+k} \\ \quad\quad - 2k u_x^0 u_x^1 u_x^{1+k} - 2u_x^0 u_x^{2+k} - k u_x^0 u_x^{2+k} \Big) \quad\quad k > 2 \\[2ex]
-2\Big(u_x^{-1} u_x^1 + 3u_x^0 (u_x^1)^2 + u_x^{-1} u_x^1 u_x^2 + u_x^0 (u_x^1)^2 u_x^2 + 2u_x^0 (u_x^2)^2 \\ \quad\quad -3u_x^{-1} u_x^3 - 5u_x^0 u_x^1 u_x^3 - 4u_x^0 u_x^4 \Big) \quad\quad k = 2 \\[2ex]
-2\Big(u_x^0 u_x^1 + u_x^{-1} (u_x^1)^2 + u_x^0 (u_x^1)^3 - 2u_x^{-1} u_x^2 - u_x^0 u_x^1 u_x^2 - 3u_x^0 u_x^3 \Big) \quad\quad k = 1 \\[2ex]
2u_x^0 \Big(u_x^{-2} + 2u_x^{-1} u_x^1 + u_x^0 (u_x^1)^2 + 2u_x^0 u_x^2 \Big) \quad\quad k = 0 \\[2ex]
2\Big(u_x^{-2} u_x^{-1} + 2u_x^{-3} u_x^0 + 2u_x^{-1} u_x^0 + (u_x^{-1})^2 u_x^1 + 3u_x^{-2} u_x^0 u_x^1 \\ \quad\quad + 4(u_x^0)^2 u_x^1 + u_x^{-1} u_x^0 (u_x^1)^2 + 2u_x^{-1} u_x^0 u_x^2 \Big) \quad\quad k = -1 \\[2ex]
2\Big(3u_x^0 u_x^{-2+k} + k u_x^0 u_x^{-2+k} + 2u_x^{-1} u_x^{-1+k} + k u_x^{-1} u_x^{-1+k} + 5u_x^0 u_x^1 u_x^{-1+k} \\ \quad\quad - 2k u_x^0 u_x^1 u_x^{-1+k} + k u_x^{-1} u_x^{1+k} + u_x^0 u_x^1 u_x^{1+k} + 2k u_x^0 u_x^1 u_x^{1+k} + u_x^0 u_x^{2+k} \\ \quad\quad + k u_x^0 u_x^{2+k} + u_x^{-1} u_x^1 u_x^k + u_x^0 (u_x^1)^2 u_x^k + 2u_x^0 u_x^2 u_x^k \Big) \quad\quad k < -1 \end{cases}$$

$$\text{(C.14)}$$

Appendix D

Permutations for one-dimensional Ising model

The partition function for the Ising model in one dimension is written in terms of the adjacency matrix $\hat{A}$ as

$$Z_N(x_2, x_4) = \sum_{\{\hat{A}\}} \left(e^{x_2 \operatorname{tr} \hat{A}^2 + x_4 \operatorname{tr} \hat{A}^4} \right) e^{-N(x_2 + 2x_4)}. \tag{D.1}$$

Since $\hat{A}$ is symmetric it can be diagonalised

$$\hat{A} = O\hat{D}O^T, \tag{D.2}$$

with $\hat{D}$ is the diagonal matrix of the eigenvalues of $\hat{A}$ and the matrix O is given by the corresponding eigenvectors.

D.1 Transformation of adjacency matrix for different configurations

We observe that, given the adjacency matrix $\hat{A}_N$ for a specific configuration, it is possible to identify a set of transformations P_i acting on the associated eigenvectors orthogonal

matrix by permuting its rows, that leave invariant the structure of $\hat{A}_N$ itself

$$\hat{A}_N = O\hat{D}O^T \qquad \tilde{O} = P_i O$$

$$\hat{A}_N = \tilde{O}\hat{D}\tilde{O}^T = P_i O\hat{D}O^T P_i^T \qquad \text{(D.3)}$$

$$\hat{A}_N = P_i \hat{A}_N P_i^{-1}.$$

where the last expression is given by the fact that a permutation matrix is orthogonal. The effect of the transformation P_i on the adjacency matrix is to permute rows (acting from the left) and columns (acting from the right via the inverse). Hence, we have

$$\hat{A}_N P_i = P_i \hat{A}_N \implies \left[\hat{A}_N, P_i\right] = 0 \qquad \text{(D.4)}$$

that means that the permutations that leave invariant the adjacency matrix are those that commute with $\hat{A}_N$.

The permutations of n objects form a group, called the symmetric group S_n of order $n!$. In terms of matrices every element of the group is given by a permutation of the eigenvectors of the identity matrix.

The permutation π of n elements $\pi: \{1,\ldots,n\} \to \{1,\ldots,n\}$ can be represented in the following two-line form as

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix} \qquad \text{(D.5)}$$

and there are two natural ways to represent a permutation with a permutation matrix, starting from the $n \times n$ identity matrix I_n. The first way is to consider a permutation of columns of I_n, the second a permutation of rows. We will consider the matrix $P_\pi = p_{ij}$ associated to the permutation of rows of I_n, as

$$p_{ij} = \begin{cases} 1 & \text{if } i = \pi(j) \\ 0 & \text{otherwise} \end{cases} \qquad \text{(D.6)}$$

The entries in the i−th column are all 0 except for 1 in correspondence of the row $\pi(j)$, we can write

$$P_\pi = \begin{bmatrix} \mathbf{e}_{\pi(1)} & \mathbf{e}_{\pi(2)} & \cdots & \mathbf{e}_{\pi(n)} \end{bmatrix} \qquad \text{(D.7)}$$

where $\mathbf{e}_j$ is a standard basis vector, a column of length n with 1 in the j−th position and 0 in every other position.

Let us consider for example the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix} \tag{D.8}$$

the corresponding permutation matrix P_π is given by

$$P_\pi = \begin{bmatrix} \mathbf{e}_{\pi(1)} & \mathbf{e}_{\pi(2)} & \mathbf{e}_{\pi(3)} & \mathbf{e}_{\pi(4)} & \mathbf{e}_{\pi(5)} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_4 & \mathbf{e}_2 & \mathbf{e}_5 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{D.9}$$

where the second row in I_5 occupies the forth one in P_π, the third row occupies the second one and so forth, following the prescription (D.8).

The $n \times n$ permutation matrices that can be constructed starting from the I_n identity matrix form a group under matrix multiplication with identity matrix as identity element. Any permutation may be written as a product of *transpositions*, a cycle composed of two elements. In general a cycle of degree m is a permutation interchanging m objects cyclically.

The group S_n of permutations of n objects $\{1,\dots,n\}$ can be generated by the $n-1$ fundamental transpositions $(1 \leftrightarrow 2)$, $(1 \leftrightarrow 3)$, $\dots$, $(1 \leftrightarrow n)$.

Case N=3

We consider the group of row permutations for 3×3 matrices. It is composed of 6 elements, each of them can be expressed in terms of the 2 fundamental elements

$$p_1 = (1 \leftrightarrow 2) \qquad p_2 = (1 \leftrightarrow 3) \tag{D.10}$$

The symmetric group S_3 is then

$$S_3 = \{1, p_1, p_2, p_1 p_2, p_2 p_1, p_1 p_2 p_1\} \tag{D.11}$$

where 1 corresponds to the identical permutation, $p_1^2 = p_2^2 = 1$ and $p_1 p_2 p_1 = p_2 p_1 p_2$. It can also represented formally as

$$S_3 = \langle p_1, p_2 \mid p_1^2 = p_2^2 = (p_1 p_2)^3 = 1 \rangle \tag{D.12}$$

The operators associated to the elements are identified by matrices $\mathcal{M}_{3\times3}$ as follows

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P_1 P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad P_2 P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad P_1 P_2 P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

By considering the possible configurations represented by the adjacency matrices for $N = 3$ we have the following relations between the specific adjacency matrix and the various permutation matrices.

The adjacency associated to the configuration with no link and labelled as $\hat{A}_3(0)$ trivially commutes with all the elements of the group S_3

$$\hat{A}_3(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \sigma(\hat{A}_3(0)) = \{0, 0, 0\}$$

$$\tag{D.13}$$

$$\left[\hat{A}_3(0), I_3\right] = \left[\hat{A}_3(0), P_1\right] = \left[\hat{A}_3(0), P_2\right] = 0$$

$$\left[\hat{A}_3(0), P_1 P_2\right] = \left[\hat{A}_3(0), P_2 P_1\right] = \left[\hat{A}_3(0), P_1 P_2 P_1\right] = 0$$

The same is verified by the complementary configuration, where all the links are on being

represented by the adjacency matrix $\hat{A}_3(3)$

$$\hat{A}_3(3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \triangle \qquad \sigma(\hat{A}_3(3)) = \{2, -1, -1\}$$

(D.14)

$$\left[\hat{A}_3(3), I_3\right] = \left[\hat{A}_3(3), P_1\right] = \left[\hat{A}_3(3), P_2\right] = 0$$

$$\left[\hat{A}_3(3), P_1 P_2\right] = \left[\hat{A}_3(3), P_2 P_1\right] = \left[\hat{A}_3(3), P_1 P_2 P_1\right] = 0$$

We notice that with equal commutation relations with the elements of S_3 what allows us to distinguish $\hat{A}_3(0)$ from $\hat{A}_3(3)$ is the spectrum of eigenvalues related to the matrices.

The situation is different for the intermediate configurations, where each adjacency matrix commutes with just one of the elements of S_3, other than with I_3, this being trivial. The other elements of the group, fixed the adjacency matrix, allow us to pass from a configuration to another representing the same structure (having the same spectrum).

The possible configurations represented by the adjacency matrix with spectrum $\sigma(\hat{A}_3(1))$ are shown in the following. Each adjacency matrix is accompanied by the corresponding graph, the permutation matrix with which it commutes (the trivial I_3 is not shown) and the transformations taking it to the equivalent configurations are represented.

$\sigma(\hat{A}_3(1)) = \{-1, 1, 0\}$

$$\hat{A}_3(1,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left[\hat{A}_3(1,2), P_1\right] = 0$$

$P_2 P_1, P_1 P_2 P_1$

$P_1 P_2, P_1 P_2 P_1$

$P_2, P_1 P_2$

$P_2, P_2 P_1$

$$\hat{A}_3(1,1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$P_1, P_1 P_2$

$$\hat{A}_3(1,3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$P_1, P_2 P_1$

$$\left[\hat{A}_3(1,1), P_2\right] = 0$$

$$\left[\hat{A}_3(1,3), P_1 P_2 P_2\right] = 0$$

The same is given for the configurations related to the spectrum $\sigma(\hat{A}_3(2))$
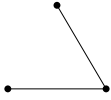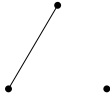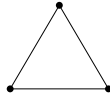
$\sigma(\hat{A}_3(2)) = \{-\sqrt{2}, \sqrt{2}, 0\}$

$$\hat{A}_3(2,3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$\left[\hat{A}_3(2,3), P_1\right] = 0$

$P_2 P_1, \ P_1 P_2 P_1$

$P_1 P_2, \ P_1 P_2 P_1$

$P_2, \ P_1 P_2$

$P_2, \ P_2 P_1$

$$\hat{A}_3(2,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\left[\hat{A}_3(2,2), P_2\right] = 0$

$P_1, \ P_1 P_2$

$P_1, \ P_2 P_1$

$$\hat{A}_3(2,1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

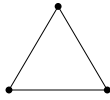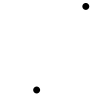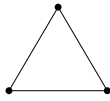$\left[\hat{A}_3(2,1), P_1 P_2 P_2\right] = 0$
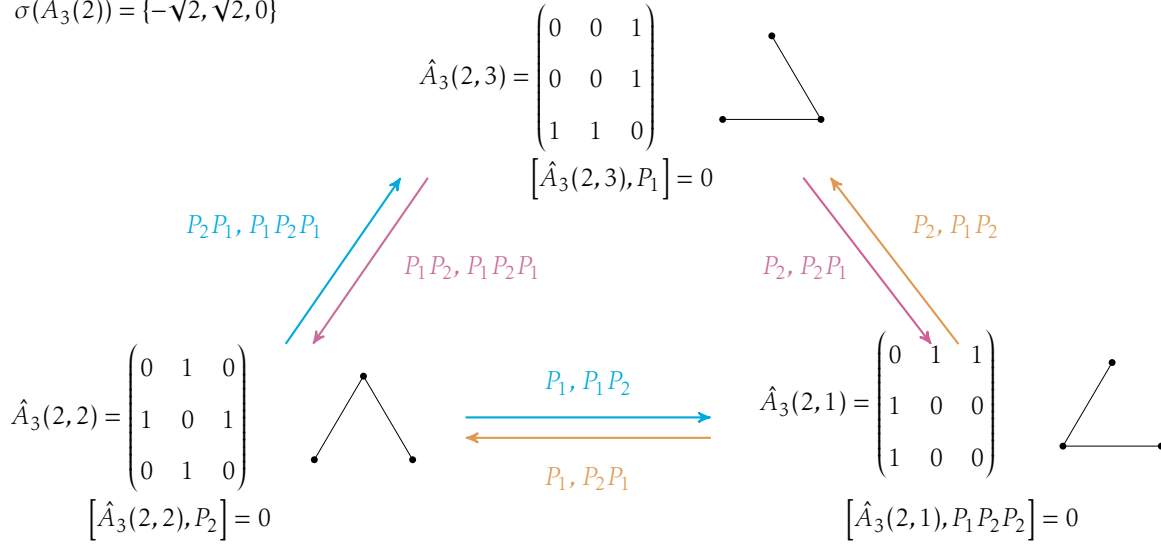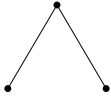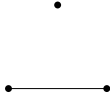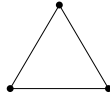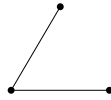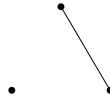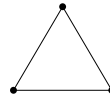
Now we consider the general form of the adjacency matrix for $N = 3$ and look for the constraints on the values of the entries, by imposing the relations of commutation with the different permutation matrices, representations of the group S_3.

Transformation of adjacency matrix for different configurations

The general form for $\hat{A}_3$ is

$$\hat{A}_3 = \begin{pmatrix} 0 & c_2 & c_1 \\ c_2 & 0 & c_3 \\ c_1 & c_3 & 0 \end{pmatrix}$$

(D.15)

Imposing that the commutator respectively with P_1, P_2, $P_1P_2P_1$, P_1P_2 and P_2P_1 is zero we get

	$\sigma(\hat{A}_3(2))$	$\sigma(\hat{A}_3(1))$	$\sigma(\hat{A}_3(3))$	$\sigma(\hat{A}_3(0))$
	$\{-\sqrt{2}, \sqrt{2}, 0\}$	$\{-1, 1, 0\}$	$\{2, -1, 1\}$	$\{0, 0, 0\}$

$\left[\hat{A}_3, P_1\right] = 0 \quad \rightarrow \quad c_3 = c_1$

$\left[\hat{A}_3, P_2\right] = 0 \quad \rightarrow \quad c_3 = c_2$

$\left[\hat{A}_3, P_1P_2P_1\right] = 0 \quad \rightarrow \quad c_2 = c_1$

$\left[\hat{A}_3, P_1P_2\right] = 0 \quad \rightarrow \quad c_3 = c_2 = c_1$

$\left[\hat{A}_3, P_2P_1\right] = 0 \quad \rightarrow \quad c_3 = c_2 = c_1$

209

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