

# Cosmological perturbations through the big bang

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Several scenarios have been proposed in which primordial perturbations could originate from quantum vacuum fluctuations in a phase corresponding to a collapse phase (in an Einstein frame) preceding the Big Bang. I briefly review three models which could produce scale-invariant spectra during collapse: (1) curvature perturbations during pressureless collapse, (2) axion field perturbations in a pre big bang scenario, and (3) tachyonic fields during multiple-field ekpyrotic collapse. In the separate universes picture one can derive generalised perturbation equations to describe the evolution of large scale perturbations through a semi-classical bounce, assuming a large-scale limit in which inhomogeneous perturbations can be described by locally homogeneous patches. For adiabatic perturbations there exists a conserved curvature perturbation on large scales, but isocurvature perturbations can change the curvature perturbation through the non-adiabatic pressure perturbation on large scales. Different models for the origin of large scale structure lead to different observational predictions, including gravitational waves and non-Gaussianity.

*to appear in Advanced Science Letters, special issue on Quantum Gravity, Cosmology and Black Holes*

## I. INTRODUCTION

How did the universe begin? The standard Hot Big Bang model, based on four-dimension Friedmann-Robertson-Walker (FRW) cosmology, starts with an initial singularity where our notion of spacetime described by Einstein's general theory of relativity breaks down. But we do not expect general relativity, or any classical theory of spacetime, to hold right up to a Big Bang singularity. Quantum fluctuations about a simple FRW metric in general relativity, including first-order inhomogeneities in the geometry in a semi-classical description, become large as the energy density, and thus the cosmological expansion rate  $H$ , becomes comparable to the Planck scale,  $10^{19}$  GeV. In alternative models, such as models with large extra dimensions, the classical four-dimensional effective theory may break down at much lower energies.

Cosmology can be studied without worrying about what came before the Big Bang as long as we have some prescription for the initial conditions, at whatever time we choose to apply the rules of general relativity, or some model of four-dimensional semi-classical gravity. In the homogeneous and isotropic FRW cosmology it may be sufficient to specify an initial thermal temperature and evolve this forward to the present day. But the standard Hot Big Bang model does not give a unique prescription for the initial distribution of inhomogeneities - spatial variations in the matter and geometry across the initial spatial hypersurface. Indeed there is no reason that they should necessarily be small perturbations, but observations (notably of the cosmic microwave background) suggest they are.

There are two logical possibilities for the origin of primordial perturbations. Either they are produced after the Big Bang, or they originate before the Big Bang.

There is a simple model to generate an initial spectrum of primordial perturbations due to vacuum fluctuations during inflation driven by a slowly-rolling, self-interacting scalar field. The accelerated expansion leads to the vacuum fluctuations on small scales (much smaller than the Hubble length,  $H^{-1}$ ) being swept up to large (super-Hubble) scales where they become "frozen-in" by the cosmological expansion. The amplitude of the fluctuations at the Hubble scale is proportional to  $H$  and the slowly varying expansion rate leads to an almost scale invariant spectrum. The weak interactions required for a slowly-rolling field naturally lead to an almost Gaussian distribution for the primordial perturbations on large scales. After almost 30 years of theoretical development this inflationary picture of the early universe has become the standard model for the origin of structure<sup>1</sup>. There is no single agreed model for which fundamental field is responsible for driving inflation and/or generating structure, but there are numerous possibilities based on extensions beyond the standard model of particle physics<sup>2</sup>.

But it is also possible that the large scale structure of our Universe is inherited from vacuum fluctuations during an earlier non-inflationary phase, *before* the Big Bang<sup>3</sup>. It is this possibility that I will discuss in this paper. There are many similarities with the inflationary model for the origin of structure in that one can calculate a spectrum of perturbations on large, super-Hubble scales in the Hot Big Bang model assuming only vacuum fluctuations on small, sub-Hubble scales in a preceding phase, only one now assumes that the preceding phase was one of accelerated contraction (in the Einstein frame where general relativity applies). This requires an intermediate bounce from contraction to expansion and one of the unresolved problems is whether such a bounce really

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occurs in this manner and if so whether the perturbations spectrum calculated in the collapse phase can be related to perturbations in the standard Hot Big Bang. I will argue that under fairly general conditions the spectrum on large scales of interest can be expected to be preserved through a bounce, while acknowledging that there is as yet no entirely satisfactory physical model for the bounce.

The existence of a cosmological phase before the Big Bang leads to a radically different view regarding the initial conditions for the universe<sup>4,5</sup>, and such models have been criticised<sup>6,7</sup> for requiring a very large universe (relative to the Planck scale) before the big bang. Indeed, as I will discuss, in some cases the generation of a scale-invariant spectrum of primordial perturbations requires the unstable growth of perturbations during collapse. However given the uncertainty in what constitutes a “natural” initial state for the cosmos, I will consider the possible observational consequences of a collapse era.

## II. HOMOGENEOUS COLLAPSE

In this paper I will consider a four-dimensional background cosmological model that is spatially homogeneous and isotropic and therefore described by the Friedmann-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j. \quad (1)$$

where  $\gamma_{ij}$  is the metric on a maximally symmetric 3-space with uniform curvature  $K$ . The Hubble expansion rate (or collapse rate) is  $H \equiv \dot{a}/a$ .

Local energy conservation gives the continuity equation for matter

$$\dot{\rho} = -3H(\rho + P), \quad (2)$$

where  $\rho$  is the energy density and  $P$  the isotropic pressure. For a linear barotropic equation of state  $P = w\rho$  this can be integrated to give  $\rho \propto a^{-3(1+w)}$ .

In a collapsing universe,  $\dot{a} < 0$ , in the presence of matter with  $\rho + 3P > 0$  (or  $w > -1/3$ ) the energy density grows faster than the spatial curvature  $K/a^2$ . Note that in an expanding universe one requires  $\rho + 3P < 0$  (or  $w < -1/3$ ) for the energy density to grow relative to the spatial density, and this is the usual condition for inflation. For simplicity I will assume in the following that spatial curvature is negligible, so that  $\gamma_{ij} = \delta_{ij}$ . More problematic in a collapsing universe is anisotropic shear<sup>8</sup>. In the simplest case of a Bianchi I universe the shear is proportional to  $a^{-6}$  (where in this case we can still think of  $a^3$  as the volume factor). Thus the anisotropic shear grows relative to matter in a collapsing universe for any matter with  $P < \rho$ .

Although a fluid description yields simple linear barotropic equation of state (for matter  $w = 0$ , or radiation  $w = 1/3$ ) I will be interested in microphysical description of the matter where one can use a quantum

vacuum state to set the initial conditions for inhomogeneous perturbations at early times. Thus I will consider canonical scalar fields  $\varphi_I$  with energy density and pressure

$$\rho = V(\varphi_I) + \sum_I \frac{1}{2} \dot{\varphi}_I^2, \quad (3)$$

$$P = -V(\varphi_I) + \sum_I \frac{1}{2} \dot{\varphi}_I^2. \quad (4)$$

where  $V(\varphi_I)$  is the potential energy. In particular a scalar field with an exponential potential,  $V_I(\varphi_I) \propto \exp(-\lambda_I \kappa \varphi_I)$  where  $\kappa^2 = 8\pi G$ , provides a simple model with  $P = w\rho$  where  $1 + w = \lambda^2/3$ . This is the basis of both power-law inflation<sup>9</sup> for  $\lambda^2 < 2$  and ekpyrotic collapse<sup>10</sup> with  $\lambda^2 \gg 2$ .

Canonical scalar fields also have a kinetic-dominated cosmology if the potential energy can be neglected such that  $P = \rho = \sum_I \dot{\varphi}_I^2/2$ , corresponding to a stiff equation of state with  $w = 1$ . Indeed in a collapsing universe where the energy density grows as the universe collapses, the kinetic energy eventually dominates over any finite potential energy.

We can identify three scalar-field collapse scenarios based on the form of the potential<sup>11,12</sup>:

- Non-stiff collapse with  $P < \rho$ : stable with respect to spatial curvature for  $P > -\rho/3$  but unstable with respect to anisotropic shear.
- Pre-Big Bang<sup>3</sup> collapse  $P = \rho$ : stable with respect to spatial curvature and marginally stable with respect to anisotropic shear<sup>8</sup>.
- Ekpyrotic collapse<sup>10</sup> with  $P \gg \rho$ : stable with respect to spatial curvature and anisotropic shear.

The last two are models which have been inspired by ideas from string theory and are intrinsically higher-dimensional models. Nonetheless most of the quantitative results have been developed for effective theories describing scalar fields in four-dimensional spacetime. Both take as their starting point the notion that string theory should be a self-consistent theory without the singularities found in general relativity.

The pre Big Bang model<sup>3,13,14</sup> assumes that our observable Universe began in a low energy, weak coupling state well described by a low energy effective action of string theory. Although sometimes written in terms of an expanding cosmology in the string frame where the dilaton field is non-minimally coupled to the spacetime curvature, this solution can be conformally transformed to a collapsing cosmology described by general relativity in the so-called Einstein frame<sup>15</sup>. As the dynamics is dominated by the kinetic energy it is independent of the form of the potential or even the number of fields. As the energy density becomes large, and the dilaton becomes large, the low-energy and weak-coupling approximations inevitably break down. This offers the possibility that

the general relativistic singularity is resolved, but this goes beyond the low-energy effective description<sup>16,17</sup>.

The ekpyrotic model<sup>10,18,20</sup> was originally motivated by cosmological solutions describing the motion of branes in a higher-dimensional spacetime. But again this is usually described by an effective theory of scalar fields in a four-dimensional spacetime. In contrast to the pre Big Bang, it incorporates a negative effective potential which is unbounded from below, leading to approximately power-law collapse model with  $w \gg 1$ . This ultra-stiff fast-roll collapse driven by a steep, negative potential is in many ways dual to quasi-de Sitter, slow-roll inflation driven by a flat, positive potential. Unlike the pre Big Bang the model approaches the weak coupling during the collapse phase. In the original model the authors appealed to the higher-dimensional picture to resolve the apparent singularity in the four-dimensional effective theory<sup>21</sup>.

### III. LINEAR PERTURBATIONS DURING COLLAPSE

#### A. Free field perturbations in an FRW cosmology

Let us first consider the dynamics of free field perturbations in a FRW background with scale factor  $a$  and Hubble rate,  $H \equiv \dot{a}/a$ , where a dot denotes derivatives with respect to cosmic time  $t$ .

Consider an inhomogeneous perturbation,  $\varphi_I \rightarrow \varphi_I(t) + \delta\varphi_I(t, \mathbf{x})$ , of the Klein-Gordon equation for a scalar field in an unperturbed FRW universe:

$$\delta\ddot{\varphi}_I + 3H\delta\dot{\varphi}_I + (m_I^2 - \nabla^2)\delta\varphi_I = 0, \quad (5)$$

where the effective mass-squared of the field is  $m_I^2 = \partial^2 V / \partial \varphi_I^2$ , and  $\nabla^2$  is the spatial Laplacian. Decomposing an arbitrary field perturbation into eigenmodes of the spatial Laplacian (Fourier modes in flat space)  $\nabla^2 \delta\varphi_I = -(k^2/a^2)\delta\varphi_I$ , where  $k$  is the comoving wavenumber, we find that small-scale fluctuations undergo underdamped oscillations on sub-Hubble scales (with comoving wavenumber  $k > aH$ ), but on large super-Hubble scales,  $k < aH$ , the modes are overdamped (or ‘‘frozen-in’’).

This is most clearly seen in terms of the rescaled field and conformal time

$$v_I = a\delta\varphi_I, \quad ad\eta = dt. \quad (6)$$

The Klein-Gordon equation (5) becomes

$$v_I'' + \left(k^2 + a^2 m_I^2 - \frac{a''}{a}\right) v_I = 0. \quad (7)$$

We have a simple harmonic oscillator with time-dependent effective mass

$$\mu^2 = m_I^2 a^2 - \frac{a''}{a} = \left(m^2 - \frac{1-3w}{2} H^2\right) a^2. \quad (8)$$

For  $k^2/a^2 \gg |1 - 3w|H^2$  and  $k^2/a^2 \gg |m^2|$  we can neglect the effective mass and we have essentially free oscillations. Normalising the initial amplitude of these small-scale fluctuations to the zero-point fluctuations of a free field in flat spacetime we have<sup>22</sup>

$$\delta\varphi_I \simeq \frac{e^{-ikt/a}}{a\sqrt{2k}}. \quad (9)$$

During an accelerated expansion or collapse  $|\dot{a}| = |aH|$  increases and modes that start on sub-Hubble scales ( $k^2 > a^2 H^2$ ) are stretched up to super-Hubble scales ( $k^2 < a^2 H^2$ ). For  $k^2 \ll |1 - 3w|a^2 H^2$  we can neglect the spatial gradients in Eq. (5). Perturbations in light fields (with mass-squared  $m^2 \ll |1 - 3w|H^2$ ) become over-damped (or ‘‘frozen-in’’) and Eq. (9) evaluated when  $k \simeq aH$  gives the power spectrum for scalar field fluctuations at ‘‘Hubble-exit’’

$$\mathcal{P}_{\delta\varphi_I}|_{k=aH} \equiv \frac{4\pi k^3}{(2\pi)^3} |\delta\varphi_I^2|_{k=aH} \simeq \left(\frac{H}{2\pi}\right)^2. \quad (10)$$

Heavy fields with  $m^2 \gg |1 - 3w|H^2/4$  remain underdamped and have essentially no perturbations on super-Hubble scales. But light fields become over-damped and can be treated as essentially classical perturbations with a Gaussian distribution on super-Hubble scales. Then on large scales we have

$$\delta\varphi_I \simeq C + D \int \frac{dt}{a^3(t)}. \quad (11)$$

where  $C$  and  $D$  are constants of integration. In an expanding universe with  $w < 1$  the integral converges and the field fluctuations become frozen-in on large scales. However in a collapsing universe with  $w < 1$ , there is a growing mode at late times. This is due to the instability with respect to the kinetic energy of the field which (like anisotropic shear) grows as  $\dot{\varphi}^2 \propto a^{-6}$  in a collapsing universe.

Vacuum fluctuations in massless fields during quasi-de Sitter expansion ( $|\dot{H}| \ll H^2$ ) produces approximately constant amplitude of scalar field fluctuations at Hubble-exit which then remain approximately constant on super-Hubble scales, thus producing an approximately scale-invariant spectrum. During an accelerated collapse  $H^2$  grows rapidly ( $\dot{H} = -3(1+w)H^2/2$ ) and thus the typical amplitude of fluctuations at Hubble-exit grows rapidly with time. During pre-big bang or ekpyrotic collapse with  $w \geq 1$  these perturbations are frozen-in and hence minimally coupled, massless fields acquire a steep blue spectrum. On the other hand if  $w < 1$  the instability causes perturbations to grow on super-Hubble scales and in the particular case of a pressureless collapse ( $w = 0$ ) the super-Hubble growth exactly matches the growth of perturbations at Hubble-exit leading to a scale-invariant spectrum of perturbations on super-Hubble scales<sup>23,24,25</sup>.

It is interesting to note that it is the presence of an instability that enables the collapse phase with growing  $H^2$

to produce a scale-invariant spectrum<sup>26</sup>. It is also possible to produce a scale invariant spectrum if the field has a tachyonic mass,  $m^2 < 0$ , which again leads to super-Hubble modes that grow at precisely the same rate as the fluctuations at Hubble-exit<sup>27</sup>. Another means to produce a scale-invariant spectrum is due to a non-minimal coupling, as in the case of pseudo-scalar axion fields in the pre big bang scenario<sup>28</sup>.

Thus far we have neglected the interactions of the field perturbations, including the gravitational coupling, which is valid only for isocurvature field perturbations, whose energy-momentum is negligible. In the next section we will include the effect of linear metric perturbations.

### B. Scalar field and metric perturbations, with interactions

To track the evolution of more general perturbations we need to include interactions between fields and, even in the absence of explicit interactions, we need to include gravitational coupling via metric perturbations.

For an inhomogeneous matter distribution the Einstein equations imply that we must also consider inhomogeneous metric perturbations about the spatially flat FRW metric. The perturbed FRW spacetime is described by the line element<sup>29</sup>

$$ds^2 = -(1 + 2A)dt^2 + 2a\partial_i B dx^i dt + a^2 [(1 - 2\psi)\delta_{ij} + 2\partial_{ij}E + h_{ij}] dx^i dx^j, \quad (12)$$

where  $\partial_i$  denotes the spatial partial derivative  $\partial/\partial x^i$ . We will use lower case latin indices to run over the 3 spatial coordinates.

The metric perturbations have been split into scalar and tensor parts according to their transformation properties on the spatial hypersurfaces. The field equations for the scalar and tensor parts then decouple to linear order. Vector metric perturbations are related to the divergence-free part of the momentum, which vanishes identically at linear order for minimally coupled scalar fields. However vector perturbations have been studied, for example, in the pre big bang model as possible source of primordial magnetic fields due to the non-minimal coupling of the dilaton field<sup>30</sup>.

The tensor perturbations,  $h_{ij}$ , are transverse ( $\partial^i h_{ij} = 0$ ) and trace-free ( $\delta^{ij} h_{ij} = 0$ ). They are automatically independent of coordinate gauge transformations. These describe gravitational waves as they are the free part of the gravitational field and evolve independently of linear matter perturbations.

We can decompose arbitrary tensor perturbations into eigenmodes of the spatial Laplacian,  $\nabla^2 e_{ij} = -(k^2/a^2)e_{ij}^{(+,\times)}$ , with two possible polarisation states, + and  $\times$ , comoving wavenumber  $k$ , and scalar amplitude  $h(t)$ :

$$h_{ij} = h(t)e_{ij}^{(+,\times)}(x). \quad (13)$$

The Einstein equations yield a wave equation for the amplitude of the tensor metric perturbations

$$\ddot{h} + 3H\dot{h} + \frac{k^2}{a^2}h = 0, \quad (14)$$

This is the same as the wave equation (5) for a massless scalar field in an unperturbed FRW metric. Thus initial vacuum fluctuations on sub-Hubble scales give rise to a power spectrum for tensor metric fluctuations at Hubble-exit<sup>31</sup> proportional to that given in Eq. (10)

$$\mathcal{P}_h|_{k=ah} = 64\pi G \left(\frac{H}{2\pi}\right)^2. \quad (15)$$

Note that as the Hubble rate approaches the Planck scale,  $H^2 \rightarrow G^{-1}$ , the power in metric perturbations becomes of order unity, signalling the expected breakdown of the semi-classical description.

The four scalar metric perturbations  $A$ ,  $\partial_i B$ ,  $\psi\delta_{ij}$  and  $\partial_{ij}E$  are constructed from 3-scalars, their derivatives, and the background spatial metric. In particular the intrinsic Ricci scalar curvature of constant time hypersurfaces is given by

$${}^{(3)}R = \frac{4}{a^2}\nabla^2\psi. \quad (16)$$

First-order perturbations of a canonical scalar field in a first-order perturbed FRW universe obey the wave equation<sup>31</sup>

$$\begin{aligned} \ddot{\delta\varphi}_I + 3H\dot{\delta\varphi}_I + \frac{k^2}{a^2}\delta\varphi_I + \sum_J V_{IJ}\delta\varphi_J \\ = -2V_I A + \dot{\varphi}_I \left[ \dot{A} + 3\dot{\psi} + \frac{k^2}{a^2}(a^2\dot{E} - aB) \right]. \end{aligned} \quad (17)$$

where the mass-matrix  $V_{IJ} \equiv \partial^2 V/\partial\varphi_I\partial\varphi_J$ . The Einstein equations relate the scalar metric perturbations to matter perturbations via the energy and momentum constraints<sup>29</sup>

$$\begin{aligned} -4\pi G\delta\rho &= 3H(\dot{\psi} + HA) \\ &\quad + \frac{k^2}{a^2}[\psi + H(a^2\dot{E} - aB)], \end{aligned} \quad (18)$$

$$-4\pi G\delta q = \dot{\psi} + HA, \quad (19)$$

where the energy and pressure perturbations and momentum for  $n$  scalar fields are given by<sup>31</sup>

$$\delta\rho = \sum_I \left[ \dot{\varphi}_I (\dot{\delta\varphi}_I - \dot{\varphi}_I A) + V_I \delta\varphi_I \right], \quad (20)$$

$$\delta P = \sum_I \left[ \dot{\varphi}_I (\dot{\delta\varphi}_I - \dot{\varphi}_I A) - V_I \delta\varphi_I \right], \quad (21)$$

$$\delta q_{,i} = -\sum_I \dot{\varphi}_I \delta\varphi_{I,i}, \quad (22)$$

where  $V_I \equiv \partial V/\partial\varphi_I$ .

We can construct a variety of gauge-invariant combinations of the scalar metric perturbations. The longitudinal gauge corresponds to a specific gauge-transformation to a (zero-shear) frame such that  $E = B = 0$ , leaving the gauge-invariant variables

$$\Phi \equiv A - \frac{d}{dt} \left[ a^2 (\dot{E} - B/a) \right], \quad (23)$$

$$\Psi \equiv \psi + a^2 H (\dot{E} - B/a). \quad (24)$$

Another variable commonly used to describe scalar perturbations during inflation is the field perturbation in the spatially flat gauge (where  $\psi = 0$ ). This has the gauge-invariant definition<sup>32,33</sup>:

$$\delta\varphi_{I\psi} \equiv \delta\varphi_I + \frac{\dot{\varphi}}{H} \psi. \quad (25)$$

It is possible to use the Einstein equations to eliminate the metric perturbations from the perturbed Klein-Gordon equation (17), and write a wave equation solely in terms of the field perturbations in the spatially flat gauge<sup>34</sup>

$$\begin{aligned} & \delta\ddot{\varphi}_{I\psi} + 3H\delta\dot{\varphi}_{I\psi} + \frac{k^2}{a^2} \delta\varphi_{I\psi} \\ & + \sum_J \left[ V_{IJ} - \frac{8\pi G}{a^3} \frac{d}{dt} \left( \frac{a^3}{H} \dot{\varphi}_I \dot{\varphi}_J \right) \right] \delta\varphi_{J\psi} = 0. \end{aligned} \quad (26)$$

This generalises the single free-field Klein-Gordon equation (5) to multiple, interacting fields.

For any light fields (whose masses are small compared to the Hubble scale) the amplitude of perturbations at Hubble-exit is approximately given by Eq. (10), however the evolution on large, super-Hubble scales now depends on the fields interactions.

It is often useful to identify the ‘‘adiabatic’’ field perturbation which is a perturbation forwards or backwards along the background trajectory in field space<sup>35</sup> (see van Tent<sup>36</sup> for the generalisation to non-canonical fields)

$$\delta\sigma = \sum_I \frac{\dot{\varphi}_I \delta\varphi_I}{\dot{\sigma}^2}, \quad (27)$$

where

$$\dot{\sigma}^2 \equiv \sum_I \dot{\varphi}_I^2. \quad (28)$$

The total energy, pressure and momentum perturbations for multiple fields in Eqs. (20–22) can be written as<sup>31</sup>

$$\delta\rho = \dot{\sigma} (\dot{\delta\sigma} - \dot{\sigma} A) + V_\sigma \delta\sigma + V_s \delta s, \quad (29)$$

$$\delta P = \dot{\sigma} (\dot{\delta\sigma} - \dot{\sigma} A) - V_\sigma \delta\sigma, \quad (30)$$

$$\delta q_{,i} = -\dot{\sigma} \delta\sigma_{,i}, \quad (31)$$

where  $V_\sigma \equiv (\partial V / \partial \varphi_I) \dot{\varphi}_I / \dot{\sigma}$ . The only effect of isocurvature field perturbations, orthogonal to the background

trajectory, is through a non-adiabatic pressure perturbation

$$\begin{aligned} P_{\text{nad},s} &= -2\delta_s V \\ &= 2 \left( V_\sigma \delta\sigma - \sum_I V_I \delta\varphi_I \right). \end{aligned} \quad (32)$$

If the potential gradients vanish orthogonal to the background trajectory,  $\delta_s V = 0$  (for isocurvature fields at a local extremum of their potential), then the adiabatic and isocurvature field perturbations decouple. The isocurvature perturbations obey the Klein-Gordon equation (5) while the adiabatic field perturbations (on spatially flat hypersurfaces) obey the Klein-Gordon equation for a single field in a perturbed FRW cosmology

$$\delta\ddot{\sigma}_\psi + 3H\delta\dot{\sigma}_\psi + \left[ \frac{k^2}{a^2} + V_{\sigma\sigma} - \frac{8\pi G}{a^3} \frac{d}{dt} \left( \frac{a^3}{H} \dot{\sigma}^2 \right) \right] \delta\sigma_\psi = 0. \quad (33)$$

where the final term on the left-hand-side describes the gravitational back-reaction due to metric perturbations. This is compactly written in terms of conformal time and  $v \equiv a\delta\sigma_\psi$  and  $z \equiv a\dot{\sigma}/H$  to give

$$v'' + \left( k^2 - \frac{z''}{z} \right) v = 0. \quad (34)$$

Analogous to Eq. (7) this leads to oscillating solutions on small scales, while on large scales where we neglect the spatial gradients we obtain

$$\delta\sigma \simeq \frac{C\dot{\sigma}}{H} + \frac{D\dot{\sigma}}{H} \int \frac{H^2 dt}{a^3 \dot{\sigma}^2}. \quad (35)$$

In particular we see that the comoving curvature perturbation

$$\mathcal{R} \equiv \psi + \frac{H}{\dot{\sigma}} \delta\sigma = \frac{v}{z}, \quad (36)$$

has a constant mode on large scales. During slow-roll inflation the second mode decays rapidly and is usually neglected, but it may become a growing mode during an accelerated collapse.

#### IV. THREE WAYS TO SCALE-INVARIANT SPECTRA

##### A. Collapse with $P \ll \rho$

If we consider a power-law collapse,  $a \propto (-t)^p$  where  $p = 2/3(1+w)$ , driven by a scalar field with exponential potential then we have  $\dot{H} = -4\pi G\dot{\sigma}^2 = -(1/p)H^2$  and hence  $z''/z = a''/a$  and Eq. (34) for the adiabatic field reduces to Eq. (7) for the isocurvature fields with

$$\frac{z''}{z} = \frac{a''}{a} = \frac{\nu^2 - (1/4)}{\eta^2}. \quad (37)$$

where

$$\nu = \frac{3}{2} + \frac{1}{p-1}. \quad (38)$$

The general solution is given in terms of Hankel functions of order  $|\nu|$

$$v = \sqrt{|k\eta|} \left[ V_+ H_{|\nu|}^{(1)}(|k\eta|) + V_- H_{|\nu|}^{(2)}(|k\eta|) \right]. \quad (39)$$

Normalising to the quantum vacuum on sub-Hubble scales at  $\eta \rightarrow -\infty$  gives a spectrum of field perturbations on super-Hubble scales<sup>25</sup> as  $\eta \rightarrow 0$

$$\mathcal{P}_{\delta\sigma} = \left( \frac{2^{|\nu|} \Gamma(|\nu|)}{(\nu - 1/2) 2^{3/2} \Gamma(3/2)} \right)^2 \left( \frac{H}{2\pi} \right)^2 |k\eta|^{3-2|\nu|}. \quad (40)$$

Thus a power-law collapse gives rise to a power-law spectrum for field fluctuations on super-Hubble scales with spectral tilt

$$\Delta n_{\delta\sigma} \equiv \frac{d \ln \mathcal{P}_{\delta\sigma}}{d \ln k} = 3 - 2|\nu|. \quad (41)$$

I have written these expressions in a way that makes clear that the spectral tilt is invariant under a change of sign of  $\nu \rightarrow -\nu$ , or equivalently<sup>25</sup>

$$p \rightarrow \frac{1-2p}{2-3p}. \quad (42)$$

In particular we see that a scale invariant spectrum of fluctuations in the adiabatic field ( $\Delta n_{\delta\sigma} = 0$ ) may be produced either from slow-roll inflation ( $w = -1$  and  $\nu = 3/2$ ) or a pressureless collapse<sup>23,25</sup> ( $w = 0$  and  $\nu = -3/2$ ). This is because the general solution contains two modes and the transformation (42) swaps the growing and decaying modes at late times. In slow-roll inflation it is the constant mode outside the Hubble-scale which acquires a scale-invariant spectrum whereas in  $w = 0$  collapse it is the time-dependent mode which grows rapidly  $\mathcal{P}_{\delta\sigma} \propto H^2$  outside the Hubble-scale. This is evidence of an instability of the background solution describing pressureless collapse with  $\rho \propto a^{-3}$  which is unstable to the growth of scalar field kinetic energy with  $\dot{\sigma}^2 \propto a^{-6}$ . This raises questions about how fine-tuned the initial conditions would need to be to have a long-lasting, pressureless collapse phase. But if there is such a phase, even if it is short-lived, then it can generate a scale-invariant spectrum of perturbations over some range of scales.

Note that the spectrum of adiabatic field fluctuations, massless isocurvature field fluctuations, and gravitational waves<sup>24</sup> all share the same scale-dependence in a power-law collapse. There is a simple relation between the power of tensor to scalar metric perturbations during power-law collapse

$$r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_\mathcal{R}} = \frac{64\pi G \dot{\sigma}^2}{H^2} = \frac{16}{p}. \quad (43)$$

This is small for slow-roll inflation, but is dangerously large ( $r = 8$ ) during pressureless collapse. Current observational bounds<sup>37</sup> require  $r < 0.3$  at the time of last-scattering of the CMB. However whereas the tensor and scalar amplitudes are constant on large scales during conventional slow-roll inflation, both quantities are rapidly growing during pressureless collapse. Thus the final value for the tensor-to-scalar ratio will be model-dependent. In a simple bounce model<sup>38</sup> it was found that the tensor-scalar ratio was small in only a small corner of parameter space.

## B. Pre big bang with $P = \rho$

The pre big bang scenario is based upon bosonic fields in the low-energy string effective action including the dilaton and other moduli fields<sup>3,13,14</sup>. Any finite potential becomes negligible as the energy density grows in a collapsing universe and hence the universe becomes dominated by the kinetic energy of the fields, leading to power-law collapse with  $w = 1$  and  $p = 1/3$ . In this case adiabatic and canonical isocurvature field perturbations have a general solution given by Eq. (39) with Hankel functions of order  $\nu = 0$ . This gives a strong blue spectral tilt  $\Delta n = +3$  in Eq. (41) for both the scalar and tensor metric perturbations during the pre big bang phase<sup>39</sup>, leaving essentially no perturbations on large scales.

Originally it was hoped that if the pre big bang could provide a homogeneous universe on large scales then causal mechanisms such as cosmic strings or other topological defects could source primordial perturbations. However subsequent observations<sup>37</sup> have shown that an approximately Gaussian distribution of adiabatic density perturbations is required on super-Hubble scales by the time of last scattering.

It turns out that some isocurvature fields will have very different perturbation spectra if they are non-minimally coupled to fields such as the dilaton which are rapidly evolving during the pre big bang. In particular the pseudo-scalar axion in the four-dimensional effective action is coupled to the dilaton in an  $SL(2, \mathbb{R})$  invariant Lagrangian<sup>14</sup>

$$\mathcal{L} = -\frac{1}{2}(\nabla\sigma)^2 - \frac{1}{2}e^{2\sigma}(\nabla\chi)^2. \quad (44)$$

The Klein-Gordon equation for isocurvature fluctuations in the axion field is

$$\ddot{\delta\chi} + (3H + 2\dot{\sigma})\dot{\delta\chi} + \frac{k^2}{a^2}\delta\chi = 0. \quad (45)$$

Analogous to Eq. (7) this can be written as

$$u'' + \left( k^2 - \frac{\bar{a}''}{\bar{a}} \right) u = 0, \quad (46)$$

where  $u = \bar{a}\delta\chi$  and  $\bar{a} \equiv e^\sigma a$  is the ‘‘scale factor’’ in the conformal frame in which the axion (rather than the

dilaton) is minimally coupled. As a result the spectral index for axion field perturbations turns out to be given by<sup>28</sup>

$$\Delta n_{\delta\chi} = 3 - 2|\cos\xi|, \quad (47)$$

where  $\xi$  is an angle describing the rate at which the dilaton rolls relative to other moduli fields. The invariance of the spectra under  $\cos\xi \rightarrow -\cos\xi$  corresponds to the previously noted invariance under  $\nu \rightarrow -\nu$  which here coincides with invariance under duality transformations of the string effective action<sup>14</sup>. Perturbations of the coupled dilaton-axion system can be shown to be invariant under  $SL(2, R)$  transformations of the background solutions<sup>28</sup>.

More generally there are many axion-type fields in the four-dimensional effective theories with different couplings to the dilaton and/or other moduli fields. For specific parameters these may acquire scale-invariant, or almost scale-invariant spectra<sup>14</sup>.

These isocurvature perturbations during the pre big bang phase still need to be converted into adiabatic density perturbations in the primordial era. This will happen if the axion field leads to a non-adiabatic pressure perturbation,  $\delta P_{\text{nad}}$ , and hence a perturbation in the local equation of state which changes the large-scale curvature perturbation

$$\dot{\mathcal{R}} \simeq H \frac{\delta P_{\text{nad}}}{\rho + P}. \quad (48)$$

In recent years a number of such mechanisms have been investigated in the context of inflationary cosmology with multiple fields.

In the curvaton scenario<sup>40,41</sup>, the axion survives from the pre big bang into the hot big bang phase, as a massive, weakly coupled field. Although its initial energy density is negligible, once it becomes non-relativistic its energy density grows relative to the radiation and can eventually come to contribute a significant fraction of the total energy density. The curvaton must decay before primordial nucleosynthesis, but when it does so, any perturbation in its energy density is transferred to the radiation density. Curvaton models have distinctive observational signatures including the possibility of residual isocurvature modes<sup>42</sup> or non-Gaussianity in the primordial density perturbation<sup>42,43</sup>.

Note that the pre big bang is only marginally stable with respect to anisotropic shear<sup>8</sup> in a collapsing universe, and anisotropies grow during any collapse with  $P < \rho$ .

### C. Ekpyrotic collapse with $P \gg \rho$

The ekpyrotic scenario involves a collapse phase driven by a steep and negative exponential potential in the four-dimensional effective action. This leads to an ultra-stiff equation of state  $w \gg 1$  and a rapidly increasing Hubble rate while the scale factor only slowly decreases. This

is in many ways the collapse equivalent of slow-roll inflation where the scale factor rapidly increases while the Hubble rate slowly decreases. The ekpyrotic collapse is the stable attractor during collapse with respect to spatial curvature and shear, just as slow-roll inflation is the stable attractor during expansion.

However for  $w \gg 1$  and thus power-law collapse with  $p \ll 1$  the spectral index given in Eqs. (38) and (41) for scalar and tensor metric perturbations produced during collapse is steep and blue<sup>44</sup>  $\Delta n = +2$ . This is in contrast to the original ekpyrotic papers which calculated the spectrum of scalar metric perturbations in the longitudinal gauge, where one finds a scale-invariant spectrum<sup>45</sup>. We will show in the next section that if the collapse phase is connected to the hot big bang expansion by a non-singular bounce then we expect the comoving curvature perturbation, and not the curvature perturbation in the longitudinal gauge to be conserved on large scales.

As in the pre big bang model, one requires instead a spectrum of almost scale-invariant perturbations in an isocurvature field to lead to an almost scale-invariant spectrum of primordial density perturbations. As in the the pre big bang model this could be a pseudo-scalar axion non-minimally coupled to the adiabatic field which evolves during the ekpyrotic phase<sup>46</sup>. However in the ekpyrotic phase the masses of the fields are not negligible compared with the Hubble rate and one can also consider scale invariance due to a tachyonic mass of an isocurvature field.

A simple example is the case of two canonical scalar fields, both with steep negative exponential potentials<sup>27,47,48,49</sup>

$$V(\varphi_1, \varphi_2) = -V_1 \exp(-\lambda_1 \kappa \varphi_1) - V_2 \exp(-\lambda_2 \kappa \varphi_2). \quad (49)$$

In slow-roll inflation it is known that a potential that is a separable sum of exponentials leads to ‘‘assisted’’ inflation<sup>50</sup> which is a power-law expansion with power  $p = \sum_I 2/\lambda_I^2$  which is larger than the power  $p_I = 2/\lambda_I^2$  that would be obtained for any of the fields on their own, ‘‘assisting’’ slow-roll. The same happens in ekpyrotic collapse with the potential (49), although the fact that  $p$  is larger than  $p_I$  for a single field takes it further from the ekpyrotic limit  $p \rightarrow 0$ .

The dynamics with multiple exponential potentials is most easily understood via a fixed rotation in field space<sup>51,52</sup>

$$\phi = \frac{\lambda_2 \varphi_1 + \lambda_1 \varphi_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \quad \chi = \frac{\lambda_1 \varphi_1 - \lambda_1 \varphi_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}. \quad (50)$$

The potential (49) is then given by

$$V(\phi, \chi) = U(\chi) \exp(-\lambda \kappa \phi), \quad (51)$$

where

$$U(\chi) = -U_0 \left[ 1 + \frac{\lambda^2}{2} \kappa^2 (\chi - \chi_0)^2 + \dots \right]. \quad (52)$$

The “assisted” power-law solution corresponds to a solution where  $\phi$  evolves along the extremum,  $\chi = \chi_0 = \text{constant}$ . One can verify that perturbations in  $\phi$  describes adiabatic field perturbations (27) along this trajectory. Thus perturbations in  $\chi$  are isocurvature perturbations described by Eq. (5) with a tachyonic mass

$$m_\chi^2 = \frac{2\kappa^2 V}{p} = -\frac{9}{2}(w^2 - 1)H^2. \quad (53)$$

The time dependent effective mass term in Eq. (7) is then

$$\mu^2 = m_\chi^2 a^2 - \frac{a''}{a} = \frac{(1/4) - \nu^2}{\eta^2}, \quad (54)$$

and the general solution is of the form given in Eq. (39) where the order

$$\nu^2 = \frac{9}{4} - \frac{18(w+1)}{(3w+1)^2}. \quad (55)$$

Thus for  $w \gg 1$  we have  $\nu \simeq (3/2) - (2/3w)$  and the spectral index for isocurvature field perturbations on super-Hubble scales which originate from vacuum fluctuations is given by Eq. (41) with  $\Delta n_{\delta\chi} \rightarrow 0$  as  $w \rightarrow \infty$ . Note that the effective mass of the isocurvature field depends only on the combined  $\lambda^2$  and not on the individual  $\lambda_I$  so we do not require any cancellations between different parameters to obtain a scale-invariant spectrum; this comes out automatically for a sufficiently steep potentials with combined  $\lambda \gg 1$ . On the other hand we have obtained an isocurvature spectrum from vacuum fluctuations only at the expense of perturbing about an unstable solution with a tachyonic direction whose mass grows proportional to the Hubble rate.

Different mechanisms have been studied which could transfer these isocurvature field perturbations to produce an almost scale invariant spectrum of scalar metric perturbations. This could occur at the bounce if the non-adiabatic field perturbations give rise to a significant non-adiabatic pressure perturbation during the bounce, or it could occur during the collapse phase as the field rolls away from the extremum  $\chi = \chi_0$  either due to terms in the potential<sup>47,49</sup> with  $\partial V/\partial\chi \neq 0$ , or simply due to the tachyonic instability itself<sup>53</sup> which naturally causes any small initial deviation from  $\chi = \chi_0$  to grow. The stable late time attractor for the potential (49) is an ekpyrotic collapse driven by a single field,  $\phi_1$  or  $\phi_2$ , as in the original ekpyrotic model. Note however that isocurvature fluctuations in the other field are not then close to scale-invariant.

It is worth noting that the simple model in Eq. (49) cannot produce a red tilt,  $\Delta n < 0$  favoured by current observations<sup>37</sup>. Thus one must introduce additional time-dependent terms either in  $U(\chi)$  or breaking the exact exponential potential for  $\phi$  in Eq. (51). One might also hope to include positive mass terms for  $\chi$  to stabilise the  $\chi_0$  at early times<sup>54</sup>, e.g., a constant mass term which becomes dominant at early times but becomes negligible relative

to the growing tachyonic mass term at late times. And eventually one must consider additional effects which will produce a transition from collapse to expansion.

## V. PERTURBATIONS THROUGH A BOUNCE

The calculations presented thus far are based on the dynamics of scalar fields in general relativity. This is a familiar framework for theoretical cosmology and has been thoroughly explored in the context of inflationary models of the early universe and quintessence models of the late universe. Thus the results are generally uncontroversial, although differences in approach, notably choice of gauge and conformal rescalings of the metric and/or non-minimal coupling of fields to the spacetime curvature lead to differences in the presentation of results.

However any collapse model must be connected to an expanding phase if it is to provide an explanation of initial conditions for the hot big bang cosmology, and in particular the observed spectrum of almost Gaussian, almost scale-invariant, and almost adiabatic density perturbations before the last scattering of the cosmic microwave background<sup>37</sup>. This is not easy. In the context of spatially flat FRW models any bounce ( $H = 0$ ,  $\dot{H} > 0$ ) requires violation of the null energy condition,  $\rho + P < 0$ . This is not possible for any number of scalar fields with a canonical kinetic field for which  $\rho + P = \dot{\sigma}^2 \geq 0$  regardless of the potential energy. Instead one requires ghost fields with negative kinetic energy which generally leads to instabilities<sup>55</sup>. The effective energy-momentum tensor of non-minimally coupled fields may violate the null energy condition<sup>56</sup> but we have calculated our metric perturbations during collapse in the Einstein frame, and wish to set initial conditions in the primordial expanding phase where general relativity is again assumed to be valid. Therefore we will require effective violation of the null energy condition in an Einstein frame. Problems controlling the instabilities - usually by requiring some UV-completion of a low energy effective field theory containing ghosts<sup>19,47,48</sup> - leave models invoking a bounce on much less secure foundation than other realisations of scalar fields in cosmology.

One might fear that no useful predictions can be made in the absence of a detailed physical model for the bounce. However we can make some statements about the primordial perturbations inherited from a preceding collapse phase if we make apparently reasonable assumptions about the bounce. A key assumption is that causality which limits the physical scale over which a sudden bounce can alter the scale dependence of perturbations.

We will consider first the case of isocurvature field perturbations which depend solely on the background evolution before considering the behaviour of scalar metric perturbations.



### A. Isocurvature field perturbations

Let us consider the simplest case of a non-interacting scalar field perturbation obeying the Klein-Gordon equation (5) in an FRW cosmology.

The effective mass,  $\mu^2$  in Eq. (8), contains terms from both the physical mass,  $m^2$  and the expansion,  $a''/a$ . If the mass is bounded from below then all Fourier modes with wavenumber  $k \ll |am|_{\min}$  will follow the same evolution  $\delta\varphi_I/\delta\varphi_I(t_0) = f(t)$  independent of wavenumber  $k$ . In this case the scale dependence of the spectrum of perturbations (41) will not be changed. But if the spatial gradients grow larger than the physical mass, then we need to compare the gradients with the expansion. During collapse we related  $a''/a$  to the comoving Hubble scale,  $a''/a = -(3w-1)a^2H^2$ . Clearly the Hubble length,  $H^{-1}$ , diverges at a bounce, but  $a''$  is finite and non-zero at a simple bounce. In fact  $a''/a$  will go through zero at some point before a smooth bounce (where  $a(\eta)$  is analytic) as it is negative during the collapse phase and positive at the bounce, and then negative again during a radiation dominated expansion after the bounce.

Nonetheless if the bounce has a finite duration then there is always a finite scale over which the perturbations evolution is significantly affected by the spatial gradients, and thus a long-wavelength regime in which the scale dependence is conserved.

In the long-wavelength limit we can model the bounce by a junction condition obtained by integrating the Klein-Gordon equation while imposing continuity of the field and scale factor

$$\left[\frac{v'_I}{v_I}\right]_+ = \left[\frac{a'}{a}\right]_- - a^2 M_I^2, \quad (56)$$

where we can allow for a divergent mass-term through the bounce

$$M_I^2 = \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{+\epsilon} m_I^2 d\eta. \quad (57)$$

which could lead to a scale-independent change in the perturbations.

### B. Adiabatic perturbations

We can derive a generalised equation for adiabatic perturbations by requiring that there exists a long-wavelength limit in which the evolution of the perturbed universe is the same as that of the FRW background<sup>57</sup>. The notion that long-wavelength perturbations can be modelled as piecewise homogeneous universes is known as the ‘‘separate universes’’ picture<sup>58,59,60</sup>. This is also sometimes called the ultra-local approximation<sup>61</sup>. Once the background solution is specified, the evolution of adiabatic perturbations on large scales is also implicitly specified since adiabatic perturbations are simply local

perturbations forwards or backwards along this background solution. Consistency then requires that even if a bounce solution invokes new physics, the same new physics applies locally to long-wavelength perturbations as applies to homogeneous FRW cosmology.

We consider a gravitational theory where homogeneous and isotropic spacetimes obey a Friedmann-type constraint equation, determining the expansion rate of comoving worldlines,  $\theta$ , and an equation for its evolution with respect to the proper time,  $\tau$ , along these worldlines,

$$\theta^2 = 3f, \quad (58)$$

$$\frac{d}{d\tau}\theta = -\frac{3}{2}g. \quad (59)$$

For example, in loop quantum cosmology a modified effective Friedmann equation (58) can be derived<sup>62</sup> where  $f = f(\rho)$  that leads to a cosmological bounce, and Cardassian models<sup>63</sup> where  $f(\rho) \propto \rho + C\rho^n$  have been investigated. In both these examples local energy conservation along comoving worldlines then fixes the form of  $g(\rho, p)$ .

In general relativity we have  $f = 8\pi G\rho$  and  $g = 8\pi G(\rho + P)$ , where  $G$  is Newton’s constant. More generally, one can always define an effective energy-momentum tensor such that the Einstein tensor  $G_{\mu\nu} = 8\pi GT_{\mu\nu}^{\text{eff}}$ . From Eqs. (58) and (59) we can identify an effective density and pressure:

$$\rho^{\text{eff}} \equiv \frac{f}{8\pi G}, \quad p^{\text{eff}} \equiv \frac{g-f}{8\pi G}. \quad (60)$$

Conservation of the Einstein tensor,  $\nabla^\mu G_{\mu\nu} = 0$ , then requires conservation of the effective energy-momentum tensor, which implies

$$\frac{d}{d\tau}\rho^{\text{eff}} = -\theta(\rho^{\text{eff}} + p^{\text{eff}}), \quad (61)$$

or equivalently, from Eqs. (58) and (59),

$$\frac{d}{d\tau}f = -\theta g. \quad (62)$$

However, in the following we will allow  $f$  and  $g$  to be arbitrary functions of energy, pressure or other variables.

In the linearly perturbed FRW cosmology (12) there is a unit time-like vector field orthogonal to constant- $\eta$  spatial hypersurfaces<sup>64</sup>,

$$N^\mu = \frac{1}{a}(1 - A, -B^i), \quad (63)$$

whose expansion rate is given by

$$\theta = 3\frac{a'}{a^2}(1 - A) - \frac{3}{a}\psi' + \frac{1}{a}\nabla^2\sigma, \quad (64)$$

where a prime denotes a derivative with respect to the conformal time  $\eta$ , and the anisotropic shear is

$$\nabla^2\sigma = \nabla^2(E' - B). \quad (65)$$

At zeroth-order the shear vanishes and the background expansion rate is  $\theta_0 = 3\mathcal{H}/a$ , where  $\mathcal{H} \equiv a'/a$  is the conformal Hubble parameter.

For the zeroth-order homogeneous (FRW) background the equations (58) and (59) can be written as

$$\mathcal{H}^2 = \frac{a^2}{3}f_0, \quad (66)$$

$$\mathcal{H}^2 - \mathcal{H}' = \frac{a^2}{2}g_0. \quad (67)$$

We then can apply Eqs. (58) and (59) where we take  $f = f_0(\eta) + \delta f(\eta, \mathbf{x})$  and  $g = g_0(\eta) + \delta g(\eta, \mathbf{x})$  and the local expansion rate is given, to first-order, by Eq. (64). Neglecting all spatial gradients, we can then write the first-order equations in terms of the lapse function  $A$ , its derivative, the curvature perturbation  $\psi$  and its first and second derivatives,

$$-3\mathcal{H}(\psi' + \mathcal{H}A) = \frac{a^2}{2}\delta f, \quad (68)$$

$$\psi'' - \mathcal{H}\psi' + \mathcal{H}A' + 2(\mathcal{H}' - \mathcal{H}^2)A = \frac{a^2}{2}\delta g. \quad (69)$$

Note that these equations are independent of two of the scalar metric perturbations,  $B$  and  $E$  in Eq. (12), which determine the anisotropic shear (65), which vanishes in this long-wavelength limit.

For adiabatic perturbations on large scales different patches of the inhomogeneous universe follow the same trajectory in phase space, and the adiabatic perturbations correspond to a perturbation back or forward with respect to this background trajectory<sup>60</sup>. In this case the hypersurfaces of uniform- $\theta$  and uniform- $(d\theta/d\tau)$  coincide. To first-order this requires  $\delta g/g_0 = \delta f/f_0'$ .

More generally, we can write any perturbation  $\delta g$  as a sum of its adiabatic and non-adiabatic parts,

$$\delta g = \frac{g_0'}{f_0'}\delta f + \delta g_{\text{nad}}, \quad (70)$$

where  $\delta g_{\text{nad}}$  is automatically gauge-invariant. Indeed, if we identify  $f$  with an effective density and  $g - f$  with an effective pressure, then  $\delta g_{\text{nad}} = 8\pi G[\delta p^{\text{eff}} - (p_0^{\text{eff}}/\rho_0^{\text{eff}})\delta\rho^{\text{eff}}] = 8\pi G\delta p_{\text{nad}}^{\text{eff}}$ . If we assume  $f = f(\rho)$  in Eq. (58) and impose energy conservation, so that  $d\rho/d\tau + \theta(\rho + p) = 0$  along comoving worldlines, then we would require from Eq. (59) that  $g = (df/d\rho)(\rho + p)$  and then  $\delta g_{\text{nad}} = (df/d\rho)\delta p_{\text{nad}}$ .

Using Eqs. (70), (66) and (67), we have from Eqs. (68) and (69) that

$$\begin{aligned} \psi'' + \frac{3\mathcal{H}\mathcal{H}' - \mathcal{H}'' - \mathcal{H}^3}{\mathcal{H}' - \mathcal{H}^2}\psi' \\ + \frac{\mathcal{H}\mathcal{H}' - \mathcal{H}^3}{\mathcal{H}' - \mathcal{H}^2}A' + \frac{2\mathcal{H}'^2 - \mathcal{H}\mathcal{H}''}{\mathcal{H}' - \mathcal{H}^2}A = \frac{a^2}{2}\delta g_{\text{nad}}. \end{aligned} \quad (71)$$

Equation (71) includes the two gauge-dependent metric perturbations  $\psi$  and  $A$ . If we work in the longitudinal

gauge then we have  $\Psi = \psi = A$  in the absence of any effective anisotropic pressure<sup>29</sup>. (More generally one can use the gauge freedom to work in a pseudo-longitudinal gauge<sup>57</sup> which is constructed such that  $\psi = A$ .) We then have

$$\begin{aligned} \Psi'' + \frac{4\mathcal{H}\mathcal{H}' - \mathcal{H}'' - 2\mathcal{H}^3}{\mathcal{H}' - \mathcal{H}^2}\Psi' \\ + \frac{2\mathcal{H}'^2 - \mathcal{H}\mathcal{H}''}{\mathcal{H}' - \mathcal{H}^2}\Psi = \frac{a^2}{2}\delta g_{\text{nad}}. \end{aligned} \quad (72)$$

For adiabatic perturbations the right-hand-side vanishes and we have a homogeneous second-order evolution equation for  $\Psi$ .

We can solve this equation by quadratures to find the general solution<sup>58,65</sup>

$$\Psi = D\frac{\mathcal{H}}{a^2} + C\left[-1 + \frac{\mathcal{H}}{a^2}\int a^2 d\eta\right], \quad (73)$$

where  $C$  and  $D$  are constants of integration. Although the differential equation (72) has a singular point when  $\mathcal{H}' - \mathcal{H}^2 = 0$ , the solution (73) is clearly regular through a bounce.

If we use equations (60) to identify an effective density and pressure on large scales, then one can show that our generalised perturbation equation (72) can be written in a ‘‘general relativistic’’ form

$$\begin{aligned} \Psi'' + 3(1 + c_s^{2\text{eff}})\mathcal{H}\Psi' \\ + [2\mathcal{H}' + (1 + 3c_s^{2\text{eff}})\mathcal{H}^2]\Psi = 4\pi G a^2 \delta p_{\text{nad}}^{\text{eff}}, \end{aligned} \quad (74)$$

where the effective adiabatic sound speed is

$$c_s^{2\text{eff}} = \frac{p_0^{\text{eff}'}}{\rho_0^{\text{eff}'}} = \frac{\mathcal{H}\mathcal{H}' + \mathcal{H}^3 - \mathcal{H}''}{3\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)}. \quad (75)$$

Our results are consistent with previous work<sup>60,66</sup> which pointed out that the curvature perturbation on uniform-density hypersurfaces<sup>60,67</sup>,

$$\zeta \equiv -\psi - \frac{\mathcal{H}}{\rho'}\delta\rho, \quad (76)$$

is conserved for adiabatic perturbations on large scales assuming only local conservation of energy (see also<sup>26,68,69</sup>). We can define a generalization

$$\zeta_f \equiv -\psi - \mathcal{H}\frac{\delta f}{f_0'} \quad (77)$$

which is the gauge-invariant definition of the curvature perturbation,  $-\psi$ , on uniform-expansion hypersurfaces, where  $\delta f = 0$ . Using Eqs. (66) and (68) we can write

$$\zeta_f = \frac{1}{\mathcal{H}' - \mathcal{H}^2} [\mathcal{H}\psi' + (\mathcal{H}^2 - \mathcal{H}')\psi + \mathcal{H}^2 A]. \quad (78)$$

In general relativity the uniform-density, uniform-expansion and comoving orthogonal hypersurfaces coincide in the long-wavelength limit and hence  $\zeta_f = \zeta$ . Using

Eqs. (68) and (69) for the evolution of perturbations on large scales we obtain

$$\zeta'_f = \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2} \frac{a^2}{2} \delta g_{\text{nad}}. \quad (79)$$

We see that  $\zeta_f$  is constant in the large-scale limit for modified gravitational field equations, even allowing for non-conservation of energy, if the perturbations are adiabatic, i.e.,  $\delta g_{\text{nad}} = 0$  in Eq. (70).

The growing mode solution (in an expanding universe) for the longitudinal gauge perturbation,  $\Psi_+ \propto C$  in Eq. (73), corresponds to  $\zeta_f = C$ , where  $C$  is a constant of integration. The decaying mode  $\Psi_- = D\mathcal{H}/a^2$ , which dominates during ekpyrotic collapse, does not contribute to the curvature perturbation  $\zeta_f$ . In a simple cosmological bounce model, assuming a specific *ansatz* for the background evolution, one can show<sup>57</sup> that the growing mode of the curvature perturbation after the bounce does not receive a contribution from the growing mode in the collapse phase.

## VI. CONCLUSIONS

It is an intriguing possibility that the large-scale structure of our Universe today could originate from vacuum fluctuations in a preceding collapse phase. Cosmological models including a bounce from collapse to expansion are certainly speculative as the end-point of gravitational collapse remains one of the outstanding challenges for quantum theories of gravity. We might hope that gravitational collapse should be non-singular, but there is no guarantee that our notions of a semi-classical spacetime will be preserved. Loop quantum cosmology offers one framework in which using the symmetries of FRW cosmology provides non-singular solutions for the homogeneous background<sup>70</sup>, but the dynamical evolution of an inhomogeneous universe is not known.

While the process of the bounce remains uncertain, I have argued that assuming our semi-classical framework still holds through the bounce, then the dynamics of a sudden bounce should affect field perturbations above some critical scale in a scale-independent way. And though the gravitational field equations controlling the evolution of metric perturbations through the bounce may be unknown, we can infer the general form of equations governing the behaviour of metric perturbations in a long-wavelength limit in which the inhomogeneous universe can be described locally in terms of homogeneous patches. This shows that the comoving curvature perturbation is constant on large scales for adiabatic perturbations. Non-adiabatic pressure perturbations can change the comoving curvature on large scales by changing the local equation of state<sup>60</sup> and this could imprint the scale-dependence of isocurvature field perturbations generated during collapse onto the comoving curvature in the hot big bang phase.

The decrease of the comoving Hubble length  $|H|^{-1}/a$  during accelerated collapse, like inflation in an expanding universe, leads to sub-Hubble vacuum fluctuations producing a spectrum on perturbations on super-Hubble scales. However there is no unique limit in which one can obtain a scale invariant power spectrum. I have highlighted three possibilities: (1) comoving curvature perturbations acquire a scale-invariant spectrum in a pressureless collapse, (2) isocurvature perturbations in axion fields in the pre big bang scenario, or (3) isocurvature perturbations in a two-field model of ekpyrotic collapse. In all cases we require a significant non-adiabatic pressure perturbation to lead to a change in the comoving curvature perturbations on super-Hubble scales either during the collapse phase or subsequently.

In (1) and (3) we require an instability of the background solution so that field perturbations can grow rapidly on super-Hubble scales to keep pace with the growing Hubble rate during collapse. However in (2) this may be avoided because the axion is non-minimally coupled to the dilaton and so the amplitude of its vacuum fluctuations is controlled by the Hubble rate in a conformally related axion frame<sup>28</sup>. The axion can acquire a scale-invariant spectrum if the axion frame is undergoing inflation, while in the Einstein frame the universe is collapsing.

The rapid growth of the field perturbations is likely to also lead to the growth of second- and higher-order perturbations which could lead to a non-Gaussian distribution of primordial density perturbations. This has recently been calculated in multi-field ekpyrotic scenarios<sup>71,72,73</sup>, case(3), and the current observational limits provide significant constraints on the allowed parameter values. Future observational limits should be able to effectively rule out such models as small non-Gaussianity seems to be incompatible with these fast-roll potentials.

In the pre big bang or ekpyrotic collapse the gravitational waves, like the comoving curvature perturbation, acquire a steep blue spectrum during collapse. Thus there are essentially no gravitational waves on large scales. A detection of gravitational waves on the Hubble scale at last scattering of the CMB would rule out these models. On the other hand during a pressureless collapse phase the gravitational waves acquire a scale-invariant power spectrum, like the comoving curvature perturbation, and unlike during slow-roll inflation, their relative amplitude is not suppressed by slow-roll parameters. The large relative amplitude of gravitational waves rules out this model unless the bounce phase strongly boosts the relative scalar perturbation, which may be possible in some cases<sup>38</sup>.

Ultimately we should be able to use observational evidence to confirm or rule out these very different models for the origin of large scale structure from before the Big Bang.

### Acknowledgements

I am grateful to numerous collaborators for many discussions and collaborations upon which much of the work presented here is based, including Laura Allen, Bruce

Bassett, Antonio Cardoso, Ed Copeland, Richard Easther, Kazuya Koyama, Andrew Liddle, Jim Lidsey, David Lyth, Karim Malik, Shuntaro Mizuno, Shinji Tsujikawa, Carlo Ungarelli and Filippo Vernizzi. DW is supported by STFC.

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