

Scale dependence of local f_{NL}

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Abstract

We consider possible scale-dependence of the non-linearity parameter f_{NL} in local and quasi-local models of non-Gaussian primordial density perturbations. In the simplest model where the primordial perturbations are a quadratic local function of a single Gaussian field then f_{NL} is scale-independent by construction. However scale-dependence can arise due to either a local function of more than one Gaussian field, or due to non-linear evolution of modes after horizon-exit during inflation. We show that the scale dependence of f_{NL} is typically first order in slow-roll. For some models this may be observable with experiments such as Planck provided that f_{NL} is close to the current observational bounds.

1 Introduction

Primordial density perturbations are traditionally described by a Gaussian distribution, characterised by an almost scale-invariant power spectrum. However the detailed information about the primordial density perturbations over a range of cosmological scales offers the opportunity to test in detail the nature of the primordial perturbations, both their scale-dependence and Gaussianity [1].

The local model for non-Gaussianity has proved a remarkably popular description of possible deviations from a purely Gaussian distribution of primordial perturbations. In the simplest case the primordial Newtonian potential includes a contribution from both the local value of a linear Gaussian field and a quadratic term proportional to the square of the local value of the Gaussian field:

$$\Phi(\vec{x}) = \varphi_G(\vec{x}) + f_{\text{NL}}^{\text{local}} (\varphi_G^2(\vec{x}) - \langle \varphi_G^2 \rangle), \quad (1)$$

where $f_{\text{NL}}^{\text{local}}$ is a dimensionless parameter characterising the deviations from Gaussianity [2]. $\langle \varphi_G^2 \rangle$ denotes the ensemble average, or equivalently the spatial average in a statistically homogeneous distribution. Note that following Komatsu and Spergel [2] we adopt a sign convention for the Newtonian potential Φ which is the opposite of that used, e.g., by Mukhanov et al [3].

In Fourier space we define the power spectrum and bispectrum as

$$\langle \Phi_{\vec{k}_1} \Phi_{\vec{k}_2} \rangle = (2\pi)^3 P_\Phi(k_1) \delta^3(\vec{k}_1 + \vec{k}_2), \quad (2)$$

$$\langle \Phi_{\vec{k}_1} \Phi_{\vec{k}_2} \Phi_{\vec{k}_3} \rangle = (2\pi)^3 B_\Phi(k_1, k_2, k_3) \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3), \quad (3)$$

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and the amplitude of the bispectrum relative to the power spectrum is conventionally given by the non-linearity parameter

$$f_{\text{NL}}(k_1, k_2, k_3) \equiv \frac{B_{\Phi}(k_1, k_2, k_3)}{2 [P_{\Phi}(k_1)P_{\Phi}(k_2) + P_{\Phi}(k_2)P_{\Phi}(k_3) + P_{\Phi}(k_3)P_{\Phi}(k_1)]}. \quad (4)$$

In the special case of the local model (1) we have $f_{\text{NL}} = f_{\text{NL}}^{\text{local}}$ and it is clear that f_{NL} is, by construction, a constant parameter independent of spatial position or scale.

This local model turns out to be a very good description of non-Gaussianity in some simple physical models for the origin of structure. In particular it can describe the primordial density perturbation on super-Hubble scales predicted, up to second-order, using the δN -formalism [4–6] if the local integrated expansion is a function of a Gaussian random field, $N(\sigma_g)$, at some initial time, $\sigma_g(\vec{x}) = \sigma(t_i, \vec{x})$. We then have

$$\Phi = \frac{3}{5}\zeta = \frac{3}{5} [N - \langle N \rangle]. \quad (5)$$

A good example is provided by the simplest curvaton scenario [7–9], where quantum fluctuations of a weakly-coupled field during inflation are well described by a Gaussian random field and the primordial density perturbation is determined by the density of the curvaton field when it decays, which is proportional to the square of the initial local field [10, 11].

In this paper we consider extensions of the simple local model (1). In particular we will characterise and quantify the scale-dependence of the parameter f_{NL} which arise in realistic inflationary models for the origin of structure. Scale-dependence arises due to two key features. Firstly we examine a multi-variate local model where the local expansion is a quadratic function of more than one Gaussian random field. In this case scale-dependent f_{NL} can arise if the Gaussian fields have differing scale-dependence, leading to a change with scale in the correlation of quadratic terms with the linear perturbation. Secondly, we show that scale-dependent f_{NL} can arise even when the expansion is a function of a single canonical scalar field. In this case, it is due to the development of intrinsic non-Gaussianity associated with the non-linear evolution of the initially Gaussian fluctuations after Hubble-exit. We will refer to this case as a quasi-local model.

In the case of the so-called equilateral f_{NL} , which arises for example from DBI inflation (see e.g. [12]), the scale dependence has already been quite well studied both in theoretical models [13–16] and forecasts have been made for future observational prospects [17, 18]. For a discussion of the different possible shape dependences of the bispectrum see [19]. However for local-type models there has been little previous consideration of a scale dependence. The scale dependence of local-type f_{NL} was first calculated in Byrnes et al [20] for a specific model of hybrid inflation with two-fields. It was found that although the scale dependence is slow-roll suppressed, it depends on a particular combination of slow-roll parameters which is not negligible and can easily be larger than the spectral index of the power spectrum. The scale dependence of f_{NL} was considered in the case of an ekpyrotic universe in [14]. In the case of an exact solution of two-field inflation, which can give rise to a large non-Gaussianity, it was shown that f_{NL} is scale independent [15]. The observational prospects for local-type models were considered in [18] which showed that the CMB data is sensitive to a scale dependence of f_{NL} . However they used a very simple Ansatz for the scale-dependent f_{NL} : here we calculate for the first time the full scale dependence of a scale-dependent quasi-local f_{NL} .

We note that higher-order contributions to the primordial perturbations are expected beyond quadratic order. These only affect the bispectrum at subleading order in the scalar field perturbations, although in some cases they might provide the dominant contribution to observables [21]. In the language of [22] they are loop corrections. The effective scale-dependence of f_{NL} due to higher-order terms is examined explicitly in [23] and was considered previously, for example, in [24]. Notice that the results depend at leading order on the infra-red cut-off, and that no infra-red complete theory is yet known [25]. It would be interesting to further develop this interesting issue in a case in which the infra-red effects are well-understood. Note that the loop term is not well described as having constant scale dependence, unlike the cases we consider in this paper. We also note that [23] only consider the scale dependence coming from the logarithmic term which depends on the cut-off, while the other terms which multiply this will also have a scale dependence in general. This scale dependence can be calculated using the methods presented in this paper.

This paper is organized as follow. In Section 2, we focus on multi-variate local models, which are models that contain contributions to the primordial curvature perturbation from more than one Gaussian field, in which the scale dependence of f_{NL} is due to the different scale dependences of the fields that drive local

expansion. As an example of this we study a mixed inflaton and curvaton scenario in sec. 2.1. In Section 3, we show that scale dependence of f_{NL} can arise also for single field models, due to the non-linear evolution of initially Gaussian fluctuations after Hubble exit. In Section 4 we extend the discussion to systems involving multiple fields, taking into account the non-linear evolution of fluctuations after horizon crossing in this context. We conclude in sec. 5.

2 Multi-variate local model

We will begin by showing how scale dependent f_{NL} can arise in an idealised model that can describe perturbations from multi-field inflation. A more systematic discussion on the scale dependence of this quantity, taking into account further second order-effects not considered here, will be developed in later sections.

A natural extension to the simplest local model given by Eq. (1) comes from considering a local function of more than one Gaussian field:

$$\Phi(\vec{x}) = \sum_I \varphi_G^I(\vec{x}) + \sum_{I,J} f_{IJ} (\varphi_G^I(\vec{x})\varphi_G^J(\vec{x}) - \langle \varphi_G^I \varphi_G^J \rangle) . \quad (6)$$

where $\varphi^I(\vec{x})$ describes n independent Gaussian random fields and $f_{IJ} = f_{JI}$ are $n(n+1)/2$ constant parameters.

In Fourier space we have

$$\langle \varphi_{\vec{k}}^I \varphi_{\vec{k}'}^J \rangle = (2\pi)^3 P_I(k) \delta^{IJ} \delta^3(\vec{k} + \vec{k}') \quad (7)$$

and all higher-order moments vanish for Gaussian fields. It is thus straightforward to construct the power spectrum

$$P_{\Phi}(k) = \sum_I P_{\varphi I}(k), \quad (8)$$

and the bispectrum

$$B_{\Phi}(k_1, k_2, k_3) = 2 \sum_{I,J} f_{IJ} [P_{\varphi I}(k_1)P_{\varphi J}(k_2) + P_{\varphi I}(k_2)P_{\varphi J}(k_3) + P_{\varphi I}(k_3)P_{\varphi J}(k_1)] . \quad (9)$$

Thus the non-linearity parameter defined in Eq. (4) is

$$f_{\text{NL}}(k_1, k_2, k_3) = \frac{\sum_{I,J} f_{IJ} [P_{\varphi I}(k_1)P_{\varphi J}(k_2) + P_{\varphi I}(k_2)P_{\varphi J}(k_3) + P_{\varphi I}(k_3)P_{\varphi J}(k_1)]}{[P_{\Phi}(k_1)P_{\Phi}(k_2) + P_{\Phi}(k_2)P_{\Phi}(k_3) + P_{\Phi}(k_3)P_{\Phi}(k_1)]} . \quad (10)$$

If we restrict our attention to equilateral triangles with $k_1 = k_2 = k_3 \equiv k$ we have

$$f_{\text{NL}}(k) = \sum_{I,J} w_I(k)w_J(k)f_{IJ}, \quad (11)$$

where the weight given to the non-linearity of each field depends on their contribution to the total power spectrum

$$w_I(k) = \frac{P_{\varphi I}(k)}{P_{\Phi}(k)} . \quad (12)$$

The scale dependence is then given by ⁵

$$n_{f_{\text{NL}}} \equiv \frac{d \ln |f_{\text{NL}}|}{d \ln k}, \quad (13)$$

$$= \frac{\sum_{I,J,K} (\tau_I + \tau_J - 2\tau_K) w_I(k)w_J(k)w_K(k) f_{IJ}}{\sum_{I,J} w_I(k)w_J(k) f_{IJ}}, \quad (14)$$

$$= \frac{\sum_{I,J} (\tau_I + \tau_J) w_I(k)w_J(k) f_{IJ}}{f_{\text{NL}}} - 2(n-1), \quad (15)$$

⁵Notice that we follow [18] but not most previous papers in defining $n_{f_{\text{NL}}} = 0$ (as opposed to -1) as corresponding to a scale independent f_{NL} . However we use the notation $n_{f_{\text{NL}}}$ rather than n_{NG} since this can be easily generalised to consider a scale dependence of the trispectrum.

where $\tau_I \equiv d \ln P_{\varphi_I} / d \ln k$ and the total spectral tilt is

$$n - 1 \equiv \frac{d \ln P_{\Phi}}{d \ln k} + 3 = \frac{d \ln P_{\zeta}}{d \ln k} + 3 = \sum_I w_I \tau_I + 3. \quad (16)$$

Although we have here calculated n_{fNL} in the case of an equilateral triangle, we discuss in sections 3.3 and B how to treat the more general case. Note that if all the fields share the same spectral index, $\tau_I = n - 4$ for all I , then $n_{\text{fNL}} = 0$. However we will see in Sec. 4 that f_{NL} could also acquire a scale dependence even if all of the τ_I are the same, when non-linear evolution of second order fluctuations spoils the Ansatz (6).

2.1 Mixed inflaton and curvaton scenario

As an example we consider an idealised model of an inflaton, ϕ , plus curvaton, σ , whose large-scale perturbations at some fixed initial time, after Hubble-exit during inflation, can be described by Gaussian random fields. The mixed inflaton-curvaton scenario has previously been studied by several authors [26–30]. Primordial density perturbations due to adiabatic fluctuations in the inflaton field during inflation, φ_{ϕ} , are necessarily very close to Gaussian, $f_{\phi\phi} = 0$, and independent of the curvaton perturbations, $f_{\phi\sigma} = 0$. However primordial density perturbations due to isocurvature curvaton perturbations may have significant non-Gaussianity [11], $f_{\sigma\sigma} \neq 0$. We thus have a bi-variate local model

$$\Phi(\vec{x}) = \varphi_{\phi}(\vec{x}) + \varphi_{\sigma}(\vec{x}) + f_{\sigma\sigma} (\varphi_{\sigma}^2(\vec{x}) - \langle \varphi_{\sigma}^2 \rangle). \quad (17)$$

The resulting non-linearity parameter for equilateral triangles (11) is

$$f_{\text{NL}}(k) = w_{\sigma}^2(k) f_{\sigma\sigma}, \quad (18)$$

where the scale-dependence arises solely due to the scale-dependence of the weighting function, that is given by $w_{\sigma}(k) = P_{\sigma}(k)/P_{\Phi}(k)$. We find (here $\tau_{\sigma} = d \ln P_{\sigma} / d \ln k$)

$$n_{\text{fNL}} = 2(\tau_{\sigma} + 3) - 2(n - 1). \quad (19)$$

Using the the results of Wands et al [31] for the scale dependence of the inflaton and curvaton perturbations, we obtain

$$n - 1 = -(6 - 4w_{\sigma})\epsilon + 2(1 - w_{\sigma})\eta_{\phi\phi} + 2w_{\sigma}\eta_{\sigma\sigma}, \quad (20)$$

$$n_{\text{fNL}} = 4(1 - w_{\sigma})(2\epsilon + \eta_{\sigma\sigma} - \eta_{\phi\phi}) \quad (21)$$

where, in the notation of [31], $\eta_{\sigma\sigma}$, $\eta_{\phi\phi}$ and ϵ are the usual slow-roll parameters and $w_{\sigma} = \cos^2 \Delta$ denotes the correlation between the curvaton perturbation, φ_{σ} , and the primordial curvature perturbation, Φ . Note that in the limit in which curvaton perturbations dominate over the inflaton in the primordial density perturbation ($w_{\sigma} = 1$) the scale-dependence vanishes.

It is interesting to estimate the size of n_{fNL} compared to the spectral index. Since both are a function of three slow-roll parameters as well as w_{σ} , it is not possible to make any definite relation in general. But in small field models of inflation one has $\epsilon \ll |\eta_{\phi\phi}|$ and it is reasonable to assume as well that $\eta_{\sigma\sigma} \ll |\eta_{\phi\phi}|$. Note that since the quadratic curvaton models requires $\eta_{\sigma\sigma} > 0$ we would otherwise need an unlikely cancellation between the two η parameters in order to have a red spectral index in agreement with observations. In the case that $\eta_{\phi\phi}$ provides the dominant contribution to both (20) and (21) we have independently of w_{σ}

$$n_{\text{fNL}} = -2(n - 1). \quad (22)$$

Observations [32] tell us that this implies $n_{\text{fNL}} \simeq 0.1$ which interestingly is roughly at the border of detectability with Planck, for a fiducial value $f_{\text{NL}} = 50$ [18]. It should be possible to observationally test the two conditions which are required for (22) to be valid, by considering additional observables. The tensor-to-scalar ratio is given by $r = 16\epsilon(1 - w_{\sigma})$ [31], and using the δN formalism it is possible to relate one term of the trispectrum with the bispectrum, as $\tau_{NL} = (6f_{\text{NL}}/5)^2/w_{\sigma}$ (using the notation and formulas of [33]). Hence there are in principle four separate observables which allow one to individually identify the values of the four model parameters in (20) and (21). Finally we note that g_{NL} , which parameterises a different term of the trispectrum, does not give an observationally competitive signature in this scenario [28].

3 Single field quasi-local models

We now move towards developing a systematic formalism to compute the scale-dependence of f_{NL} in inflationary models. We start by considering the simplest case where the primordial curvature perturbation is generated from fluctuations in one scalar field. We consider two examples. In the first one, we discuss a standard single slowly rolling inflaton field with canonical dynamics. While the non-Gaussianities generated in this model are unobservably small [34], this case serves as a useful example illustrating the physical origin for the scale dependence of f_{NL} . As a second example, we consider a curvaton scenario where the effect of inflaton perturbations can be neglected. This can generate observable non-Gaussianities. We show that while the simplest quadratic curvaton scenario predicts a scale-independent f_{NL} , models with interaction terms in the curvaton potential typically give rise to scale-dependence.

We assume the scalar field has the canonical Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) ,$$

and that it is light during the inflationary epoch: $V'' \ll H^2$. We assume the field obeys slow-roll dynamics during inflation but we do not require its energy density to dominate the universe. We set $m_{\text{Pl}} = (8\pi G)^{-1/2} = 1$ and introduce the slow-roll parameters

$$\epsilon = -\frac{\dot{H}}{H^2} , \quad \epsilon_\phi = \frac{1}{2} \left(\frac{V'}{3H^2} \right)^2 , \quad \eta_{\phi\phi} = \frac{V''}{3H^2} , \quad \xi^2 \equiv \frac{V'''V'}{9H^4} , \quad (23)$$

which are all assumed to be small during inflation. If the energy density of the scalar field ϕ dominates the universe during inflation, we can equate $\epsilon_\phi = \epsilon$, but in a more general set-up these two parameters can differ even at leading order in slow roll.

We will use the δN formalism to characterise the primordial curvature perturbation, given by (5), and analyze the scale dependence of f_{NL} . The curvature perturbation on superhorizon scales is given by the expansion

$$\zeta(\vec{x}) = N_\phi(t_i)\delta\phi(t_i, \vec{x}) + \frac{1}{2}N_{\phi\phi}(t_i) (\delta\phi(t_i, \vec{x})^2 - \langle\delta\phi(t_i, \vec{x})^2\rangle) + \dots , \quad (24)$$

where $N(t_i)$ denotes the number of e-foldings from an initial spatially flat hypersurface at t_i to a final uniform energy density hypersurface, which we assume is chosen such that the curvature perturbation has frozen to a constant value $\zeta = 0$. N_ϕ and $N_{\phi\phi}$ denote derivatives with respect to the initial value of the scalar field $\phi(t_i)$. For the actual computations, it is convenient to transform (24) into Fourier space writing

$$\zeta_{\vec{k}} = N_\phi(t_i)\delta\phi_{\vec{k}}(t_i) + \frac{1}{2}N_{\phi\phi}(t_i) (\delta\phi \star \delta\phi)_{\vec{k}}(t_i) + \dots , \quad (25)$$

where \star denotes convolution and subtraction of the zero mode $\zeta_{\vec{k}=0}$ from (25) is implicitly understood.

The primordial power spectrum is given to leading order by

$$P_\zeta(k) = N_\phi^2(t_i)P_{\delta\phi}(t_i, k) . \quad (26)$$

The calculation of the bispectrum and the derivation of the general form for the f_{NL} parameter, defined in eq. (4), are straightforward. One obtains

$$\frac{6}{5}f_{\text{NL}}(k_1, k_2, k_3) = \frac{N_{\phi\phi}(t_i)}{N_\phi^2(t_i)} + \frac{(2\pi)^3 N_\phi^3(t_i) B_{\delta\phi}^c(k_1, k_2, k_3, t_i)}{P_\zeta(k_1)P_\zeta(k_2) + 2 \text{ perms}} . \quad (27)$$

The second term on the right is proportional to the connected part of the scalar field bispectrum $B_{\delta\phi}^c$, which is the three-point correlator of the scalar perturbations.

The initial time t_i in (24) or (25) can be chosen to be any time after the horizon exit of a given mode, $t_i > t_*(k)$ where $t_*(k)$ is determined by $k = a(t_*)H(t_*)$. The curvature perturbation is by construction independent of the choice of the initial time t_i [35], as discussed in more detail in Appendix A. However, statistical properties of the scalar field perturbations $\delta\phi_{\vec{k}}(t_i)$ do depend on the choice of t_i .

For a canonical scalar field, slowly rolling during inflation, the intrinsic non-Gaussianity is slow-roll suppressed at Hubble exit [34, 36]. In our quasi-local model we will thus set the scalar field bispectrum $B_{\delta\phi}^c(k_1, k_2, k_3, t_i) = 0$ when $t_i = t_*(k_i)$ for all k_i . However at later times the distribution in general develops non-Gaussian features because of the non-linearities of field equations, and it is this that leads to scale-dependence of f_{NL} .

3.1 Inflaton field

The case of single field slow-roll inflation is particularly instructive, since there exist general formulas connecting the derivatives of N with slow-roll parameters. Moreover, it is possible to appreciate the connection between the scale dependence of f_{NL} and evolution of the second-order curvature perturbations after horizon exit.

Before focusing on the properties of f_{NL} , we recall how in the familiar case of a single inflaton field the scale dependence of the power spectrum can be computed in the δN formalism in two different ways. Understanding this familiar example will be useful to tackle the more complicated case of the scale dependence of f_{NL} .

The solution of the first order equation of motion for the inflaton perturbations at a time $t > t_*(k)$ soon after horizon exit can be expressed as [37]

$$\delta\phi_{\vec{k}}(t) = \frac{iH(t)}{\sqrt{2k^3}} \left\{ 1 + \epsilon + \left[c + \ln \left(\frac{a(t)H(t)}{k} \right) \right] (3\epsilon - \eta_{\phi\phi}) \right\} a_{\vec{k}}, \quad (28)$$

where $\epsilon = \epsilon_\phi$ for the inflaton, $a_{\vec{k}}$ is a classical random variable satisfying $\langle a_{\vec{k}} a_{\vec{k}'}^\dagger \rangle = (2\pi)^3 \delta^3(\vec{k} + \vec{k}')$, and $c = 2 - \ln 2 - \gamma$, with γ being the Euler-Mascheroni constant. To leading order in slow roll, the time dependence of the slow-roll parameters defined in (23) can be neglected.

Using equations (25) and (28), one can work out the dimensionless power spectrum of ζ , defined by $\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle = (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2) 2\pi^2 \mathcal{P}_\zeta(k_1)/k_1^3$. To leading order in perturbations, and next-to-leading order in the slow-roll expansion, this gives

$$\mathcal{P}_\zeta(k) = N_\phi^2(t_i) \frac{H^2(t_i)}{4\pi^2} \left\{ 1 + 2\epsilon + 2 \left[c + \ln \left(\frac{a(t_i)H(t_i)}{k} \right) \right] (3\epsilon - \eta_{\phi\phi}) \right\}, \quad (29)$$

and the spectral index

$$n - 1 = \frac{d \ln \mathcal{P}_\zeta}{d \ln k}, \quad (30)$$

can be immediately computed by differentiating (29) with respect to $\ln k$. This yields,

$$n - 1 = 2\eta_{\phi\phi} - 6\epsilon. \quad (31)$$

In practice the same result for the spectral index is usually derived in a different way, that will turn out to be useful for the following discussion. In deriving (29) we have not specified the arbitrary initial time t_i in (25), except for the technical constraint that t_i needs to be soon after the horizon exit, $t_i \gtrsim t_*(k)$, for the solution (28) to be valid. A particularly simple choice of t_i is to set it equal to the horizon crossing for each mode separately $t_i(k) = t_*(k)$.

To obtain the spectral index we can use the identity [5]

$$H \left(\frac{\partial}{\partial \ln k} \right)_{t_i} = \left(\frac{\partial}{\partial t_i} \right)_{k/a} - \left(\frac{\partial}{\partial t_i} \right)_k \quad (32)$$

applied to eq. (30), choosing t_i at the epoch of Hubble crossing. The second term in the right hand side of (32) gives zero when applied to the power spectrum, due to the latter being independent of t_i at fixed k . Therefore, at leading order in the slow-roll expansion, we have

$$\begin{aligned} n - 1 &= 2 \frac{d \ln |N_\phi|}{H dt_i} + 2 \frac{d \ln H}{H dt_i} \\ &= 2n_\phi - 2\epsilon, \end{aligned} \quad (33)$$

after defining n_ϕ (and $n_{\phi\phi}$ that will be useful in the following) as

$$n_\phi = \frac{d \ln |N_\phi|}{H dt_i}, \quad (34)$$

$$n_{\phi\phi} = \frac{d \ln |N_{\phi\phi}|}{H dt_i}. \quad (35)$$

For the inflaton we have $N_\phi = -H/\dot{\phi}$ and hence

$$n_\phi = \eta - 2\epsilon. \quad (36)$$

Thus the expression (33) reduces to the usual formula (31) when focusing on single field slow-roll inflation.

Similarly, we can calculate f_{NL} in (27) and its scale-dependence using the fact that the curvature perturbation is independent of t_i , see Appendix A, provided that t_i is chosen after each of the scales k_j , $j = 1, 2, 3$ has crossed its Hubble scale. We concentrate here on equilateral triangles, $k_i = k$ postponing the detailed discussion of non-equilateral configurations until Appendix B (but see also Section 3.3).

If the field perturbations $\delta\phi_{\mathbf{k}}$ are Gaussian at horizon crossing, that is the bispectrum $B_{\delta\phi}^c(k, k, k, t_*)$ vanishes for any $k = a(t_*)H(t_*)$, then the second term in (27) vanishes at $t_i = t_*$. With this choice, all the scale dependence is encoded in the first term of (27), which acquires an implicit dependence on k through the function $t_i = t_*(k)$. Note that for the inflaton we have $d/Hdt_i = (\dot{\phi}/H)d/d\phi$ and hence

$$n_\phi = -\frac{N_{\phi\phi}}{N_\phi^2}. \quad (37)$$

Thus, using (36) we have

$$\frac{6}{5}f_{\text{NL}} = \frac{N_{\phi\phi}}{N_\phi^2} = 2\epsilon - \eta. \quad (38)$$

and the scale dependence is given by

$$n_{f_{\text{NL}}} = \frac{d \ln |N_{\phi\phi}/N_\phi^2|}{H dt_i} = n_{\phi\phi} - 2n_\phi. \quad (39)$$

To leading order in slow-roll, the result can be written explicitly as

$$n_{f_{\text{NL}}} = \frac{6\epsilon\eta - 8\epsilon^2 - \xi^2}{\eta - 2\epsilon}. \quad (40)$$

This shows that for single field inflation $n_{f_{\text{NL}}}$ is fully determined by the slow-roll parameters evaluated at horizon crossing and is first order in slow-roll. (Note that for $\eta - 2\epsilon = 0$ the logarithmic scale-dependence diverges simply because f_{NL} is zero.)

We can physically understand the result by analyzing the evolution of second order perturbations in this set-up, also using the results of Appendix A. For single field inflation, we can interpret the scale dependence of f_{NL} as being due to the evolution of second-order perturbations right after horizon exit. Indeed, by means of the δN formalism, splitting scalar perturbations into first and second order parts as $\delta\phi = \delta_1\phi + \frac{1}{2}\delta_2\phi$, we can expand the curvature perturbation up to second order in perturbations as

$$\zeta_{\vec{k}} = N_\phi \delta_1\phi_{\vec{k}} + \frac{1}{2} \frac{N_{\phi\phi}}{N_\phi^2} (N_\phi \delta_1\phi_{\vec{k}})^2 + \frac{1}{2} N_\phi \delta_2\phi_{\vec{k}} \quad (41)$$

The first two terms contain only the Gaussian first order perturbation $\delta_1\phi$. The last term, $\frac{1}{2}N_\phi \delta_2\phi$, is the second order contribution to the curvature perturbation that, evolving after the epoch of horizon crossing t_* , is responsible for the scale dependence of local f_{NL} in the single inflaton case. In Appendix A we determine the behavior of $\delta_2\phi$, expanding at leading order in slow roll and around t_* . Assuming that $\delta_2\phi$ vanishes at t_* , in accordance with $\delta\phi$ being Gaussian at this time, we find

$$\delta_2\phi(t) = \frac{H(t-t_*)}{\sqrt{2\epsilon}} (8\epsilon^2 - 6\epsilon\eta + \xi^2) (\delta_1\phi_*)^2 + \dots \quad (42)$$

We see that if $\delta_2\phi$ does not evolve at the time of horizon crossing, then n_{fNL} in eq. (40) vanishes as expected.

Indeed, instead of setting $t_i = t_*(k)$ in (27) we could have derived the result (40) by treating t_i as some arbitrary time soon after the horizon exit $t_i > t_*(k)$. Even though we have assumed the perturbations are Gaussian at horizon-crossing, equation (42) explicitly shows that non-linearities in the field evolution result in a non-vanishing value for $B_{\delta\phi}^c$ at later times, $t_i > t_*$. Since the second term in (27), proportional to $B_{\delta\phi}^c$, depends explicitly on the scale k , this makes f_{NL} scale-dependent. For single field inflation, this is also the only source of scale dependence since the first term in (27) is independent on time and, with the choice $t_i > t_*(k)$, does not depend on the scale. While this discussion provides a transparent physical explanation for the origin of the scale-dependence, in actual computations it is convenient to set $t_i = t_*(k)$. These two choices are perfectly equivalent, as discussed in Appendix B in more detail.

It is important to emphasize that in this subsection devoted to adiabatic perturbations of a slowly-rolling inflaton field we have neglected non-Gaussianity due to the non-linearity of the inflaton field at horizon-exit which is of the same order as that which arises due to non-linear evolution after horizon-exit. In any case f_{NL} in this model is suppressed by slow-roll parameters and thus we now turn our attention to models where non-adiabatic field perturbations can generate much larger non-Gaussianity after horizon-exit [30].

3.2 Curvaton scenario

In the curvaton scenario, primordial perturbations are generated by a curvaton field σ which is light and subdominant during inflation, and acquires a nearly scale-invariant spectrum of perturbations. After the end of inflation, the curvaton energy density dilutes slower than the dominant radiation component. Its contribution to the total energy density therefore increases and its perturbations start to affect the expansion history, which generates curvature perturbations. The curvature perturbations evolve in time, $\dot{\zeta} \neq 0$, until the curvaton eventually decays and the decay products thermalise with the rest of the universe. If the curvaton is still subdominant at the time of decay, the perturbations can be highly non-Gaussian. Here we concentrate on this limit and assume that the universe remains effectively radiation dominated until the curvaton decay, $a \propto t^{1/2}$. In addition, we assume the primordial perturbation is entirely generated by the curvaton fluctuations. In this limit the curvaton scenario effectively reduces to a single field model and it can be analysed using the formalism developed above.

We consider a generic curvaton potential containing a quadratic term plus self-interactions [29, 38–42],

$$V(\sigma) = \frac{1}{2}m^2\sigma^2 + \lambda\frac{\sigma^{n+4}}{M_{\text{P}}^n}, \quad (43)$$

where λ is some coupling and $n \geq 0$ is an even integer to keep the potential bounded from below. We assume the curvaton is long-lived enough so that the energy density stored in the interaction term eventually dilutes away and the curvaton ends up oscillating in an effectively quadratic potential before decaying. The curvaton energy density at this stage is approximatively given by

$$\rho_\sigma \simeq \frac{1}{2}m^2\bar{\sigma}^2(t, \sigma_*) \equiv \frac{1}{2}\frac{m^2\sigma_{\text{osc}}^2(\sigma_*)}{(mt)^{3/2}}, \quad (44)$$

where $\bar{\sigma}(t, \sigma_*)$ is the envelope of oscillations in the quadratic regime. It is proportional to $t^{-3/4}$ and to the function $\sigma_{\text{osc}}(\sigma_*)$ that depends on the initial value of the curvaton field σ_* . While σ_{osc} is a linear function of σ_* for a purely quadratic curvaton potential, it can acquire a highly non-linear form when self-interactions are switched on. As a representative example, at the end of this section we will discuss the curvaton model with a quartic self-interaction term in the potential and the corresponding form of the function $\sigma_{\text{osc}}(\sigma_*)$.

We assume the curvaton decays instantaneously at $H = \Gamma$, where Γ is the effective curvaton decay rate, and denote its contribution to the total energy density at this time by

$$r = \frac{3\rho_\sigma}{4\rho_\gamma}\Big|_{\text{dec}} \simeq \frac{\sqrt{2}}{4}\sigma_{\text{osc}}^2\left(\frac{m}{\Gamma}\right)^{1/2} \ll 1. \quad (45)$$

Setting $t_i = t_*(k)$ in (25) and working to leading order in r , one finds that the derivatives of N in (25) are given by [6]

$$N_\sigma(t_*(k)) = \frac{2r}{3}\frac{\sigma'_{\text{osc}}}{\sigma_{\text{osc}}}, \quad N_{\sigma\sigma}(t_*(k)) = \frac{2r}{3}\left(\frac{\sigma''_{\text{osc}}}{\sigma_{\text{osc}}} + \left(\frac{\sigma'_{\text{osc}}}{\sigma_{\text{osc}}}\right)^2\right), \quad (46)$$

where the primes denote derivatives with respect to σ_* . Using (46), the non-linearity parameter f_{NL} (27) becomes

$$f_{\text{NL}} = \frac{5}{6} \frac{N_{\sigma\sigma}}{N_\sigma^2} = \frac{5}{4r} \left(1 + \frac{\sigma_{\text{osc}} \sigma''_{\text{osc}}}{(\sigma'_{\text{osc}})^2} \right). \quad (47)$$

As before, we have assumed the curvaton perturbations are Gaussian at horizon crossing.

The scale dependence of f_{NL} can be computed using equation (39) with ϕ replaced by σ . To find the quantities n_σ and $n_{\sigma\sigma}$, defined by (34) and (35), it is convenient to first express the above results in terms of a reference curvaton value $\sigma_1(\sigma_*, t_*(k))$ measured at some time t_1 during inflation but after the horizon exit of all relevant modes, $t_1 > t_*(k)$. Since $\sigma_{\text{osc}}(\sigma_1)$, we can use the chain rule to write

$$\frac{d\sigma_{\text{osc}}(\sigma_1)}{dt_*} = \frac{\partial\sigma_{\text{osc}}}{\partial\sigma_1} \frac{d\sigma_1}{dt_*} = \frac{\partial\sigma_{\text{osc}}}{\partial\sigma_1} \left(\dot{\sigma}_* \left(\frac{\partial\sigma_1}{\partial\sigma_*} \right)_{t_*} + \left(\frac{\partial\sigma_1}{\partial t_*} \right)_{\sigma_*} \right). \quad (48)$$

Moreover, as t_1 is a time event during inflation, the curvaton evolution until this time is governed by the slow-roll dynamics

$$\int_{\sigma_*}^{\sigma_1} \frac{d\sigma}{V'(\sigma)} = - \int_{t_*}^{t_1} \frac{dt}{3H(t)}, \quad (49)$$

from which we get

$$\left(\frac{\partial\sigma_1}{\partial\sigma_*} \right)_{t_*} = \frac{V'(\sigma_1)}{V'(\sigma_*)}, \quad \left(\frac{\partial\sigma_1}{\partial t_*} \right)_{\sigma_*} = \frac{V'(\sigma_1)}{3H(t_*)}. \quad (50)$$

Substituting these into (48) and using $3H\dot{\sigma}_* = -V'(\sigma_*)$, we arrive at the result

$$\frac{d\sigma_{\text{osc}}}{d\log k} = H(t_*)^{-1} \frac{d\sigma_{\text{osc}}}{dt_*} = 0, \quad (51)$$

to leading order precision in slow-roll. This derivation can be immediately generalized for an arbitrary function of the form $f(\sigma_1)$, and we arrive at the useful generic result $df(\sigma_1)/d\log k = 0$.

Applying these results to (46), we can easily compute the parameters n_σ and $n_{\sigma\sigma}$. According to the discussion above, σ_{osc} and r do not depend on the wavenumber k . Similarly, using the chain rule, we can write $\sigma'_{\text{osc}} = (\partial\sigma_1/\partial\sigma_*)(\partial\sigma_{\text{osc}}/\partial\sigma_1)$ where the second term depends only on σ_1 and is constant under differentiation with respect to k . Therefore, we have

$$n_\sigma = \frac{d \ln \sigma'_{\text{osc}}}{d \log k} = H(t_*)^{-1} \frac{d}{dt_*} \ln \left(\frac{\partial\sigma_1}{\partial\sigma_*} \right) = -H(t_*)^{-1} \frac{d \ln V'(\sigma_*)}{dt_*} = \eta_{\sigma\sigma}, \quad (52)$$

where we have used (50) and the slow roll equation $3H\dot{\sigma} = -V'$. Similarly, we find

$$n_{\sigma\sigma} = 2\eta_{\sigma\sigma} + \frac{V'''(\sigma_*)}{3H_*^2} \left(\frac{\sigma_{\text{osc}} \sigma'_{\text{osc}}}{(\sigma'_{\text{osc}})^2 + \sigma_{\text{osc}} \sigma''_{\text{osc}}} \right). \quad (53)$$

Substituting these into (39), we find the scale dependence of f_{NL} is given by

$$n_{f_{\text{NL}}} = n_{\sigma\sigma} - 2n_\sigma = \frac{V'''(\sigma_*)}{3H_*^2} \left(\frac{\sigma_{\text{osc}} \sigma'_{\text{osc}}}{(\sigma'_{\text{osc}})^2 + \sigma_{\text{osc}} \sigma''_{\text{osc}}} \right). \quad (54)$$

It can be immediately seen that $n_{f_{\text{NL}}} = 0$ for a purely quadratic curvaton model. However, in the presence of interactions, $V''' \neq 0$, the non-linearity parameter becomes scale-dependent. Equation (54) gives a rough estimate $|n_{f_{\text{NL}}}| \sim |\eta_{\sigma\sigma}|$ but the precise prediction depends on the curvaton potential and the details of the dynamics encoded in the derivatives σ'_{osc} and σ''_{osc} .

As a specific example, we estimate $n_{f_{\text{NL}}}$ for the curvaton potential $V = 1/2 m^2 \sigma^2 + \lambda \sigma^4$. This case can be treated analytically using the results of [42]. In the limit where the interaction term dominates initially, $\lambda \sigma_*^2 \gtrsim m^2$, the function $\sigma_{\text{osc}}(\sigma_*)$ in (44) can be estimated by equation (4.9) of [42],

$$\sigma_{\text{osc}}(\sigma_*) \simeq \sigma_* \frac{1.3 e^{-0.80\sqrt{s}}}{|\Gamma(0.75 + i 0.51\sqrt{s})|} \equiv \sigma_* \xi(s), \quad (55)$$

where $s = \lambda\sigma_*^2/m^2$. As discussed in [42], this agrees well with numerical results for $s \gtrsim 1$ but becomes inaccurate when approaching the quadratic limit $s = 0$. The qualitative behaviour however remains correct even in this limit and we can therefore use (55) for arbitrary s , provided that we understand the results in the limit $s \ll 1$ as order of magnitude estimates only.

Substituting (55) in (54), we find

$$n_{\text{fNL}} = \frac{8\lambda\sigma_*^2}{H_*^2} \left(\frac{\xi(s)^2 + 2s\xi(s)\xi'(s)}{10s\xi(s)\xi'(s) + 4s^2(\xi(s)\xi''(s) + \xi'(s)^2) + \xi(s)^2} \right) \equiv \frac{8\lambda\sigma_*^2}{H_*^2} f(s) = \frac{24s}{1 + 12s} \eta_{\sigma\sigma} f(s). \quad (56)$$

The function $f(s)$ can be evaluated using (55) and its behaviour is illustrated in Fig. 1. From (56) and Fig.

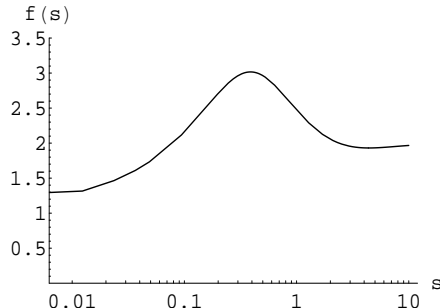


Figure 1: An estimate for the function $f(s)$ in equation (56) shown as a log-linear plot. For $s \lesssim 1$ only a qualitative agreement (error $\sim 20\%$) with the exact behaviour is expected.

1 we see that n_{fNL} vanishes in the quadratic limit $s = 0$ as expected. If the interaction is initially comparable or dominates over the quadratic term in the potential, $s \gtrsim 1$, we find $n_{\text{fNL}} \sim \eta_{\sigma\sigma}$ up to a factor of few.

Note that for the potential $V = 1/2 m^2 \sigma^2 + \lambda \sigma^4$ we have $\eta_{\sigma\sigma} > 0$ and since $n - 1 = -2\epsilon + 2\eta_{\sigma\sigma}$, it is assumed in [42] that ϵ gives the dominant contribution to the spectral index. In this case it is clear from (56) that n_{fNL} will be subdominant compared to the spectral index and hence unobservably small. However in the more general case (54) it is possible to have $\eta_{\sigma\sigma} < 0$ at the time of horizon exit but with a minimum, e.g. if the curvaton has an axionic type of potential [43]. In this case it might be possible to have a large n_{fNL} . Another case which could lead to a large n_{fNL} in the curvaton scenario is the non-perturbative curvaton decay [44, 45] but an investigation of these points is beyond the scope of this work.

3.3 Scale-dependence for generic single-field case

It is straightforward to generalize the result derived for n_{fNL} to any quasi-local model where the curvature perturbation is generated from a single field. Indeed, since equation (27) for f_{NL} is valid for all such models, we can simply set $t_i = t_*$ in it and read off the result from equations (34), (35) and (39) where ϕ is now understood as the generic scalar degree of freedom responsible for generating the primordial curvature perturbation. We show in Appendix B that the result $n_{\text{fNL}} = n_{\phi\phi} - 2n_\phi$ actually holds not only for the equilateral case but for arbitrary triangles that change their area while preserving their shape.

In Appendix B we also derive a result that allows to study variations that change the shape of the triangles. Considering a triangle with sides k_1, k_2, k_3 not too much different from each other and from a generic reference scale k_p , we can expand f_{NL} around k_p as

$$f_{\text{NL}}(k_1, k_2, k_3) = f_{\text{NL}}^{\text{pivot}} \frac{\left(\frac{k_1}{k_p}\right)^{n-4-n_\phi-2\epsilon_\phi+\eta_{\phi\phi}} \left(\frac{k_2}{k_p}\right)^{n-4-n_\phi-2\epsilon_\phi+\eta_{\phi\phi}} \left(\frac{k_3}{k_p}\right)^{n_\phi\phi+4\epsilon_\phi-2\eta_{\phi\phi}} + 2 \text{ perms}}{\left(\frac{k_1}{k_p}\right)^{n-4} \left(\frac{k_2}{k_p}\right)^{n-4} + 2 \text{ perms}}. \quad (57)$$

Here $f_{\text{NL}}^{\text{pivot}}$ corresponds to f_{NL} evaluated on an equilateral triangle with sides k_p and the result is given to leading order precision in $\mathcal{O}(n-1, \epsilon_\phi, \eta_{\phi\phi}, n_\phi, n_{\phi\phi}) \ln(k_i/k_p)$. Note that if ϕ is an isocurvature field during inflation, like in the curvaton scenario (or more generally in models characterized by a large f_{NL} [30]), then $\epsilon_\phi \sim 0$ in (57).

The result (57) allows one to generally study the leading order contributions to the scale dependence of local f_{NL} : $n_{f_{\text{NL}}}$ obtained from it reduces to equation (39) for arbitrary triangles that change area preserving their shape, while variations that allow the shape to change in general lead to a different scale dependence. For example, in the squeezed limit $k_3 \ll k_1 \simeq k_2 \equiv k$ we find the result

$$f_{\text{NL}}(k, k, k_3) \simeq f_{\text{NL}}^{\text{pivot}} \left(\frac{k_3}{k_p} \right)^{-n_\phi - 2\epsilon_\phi + \eta_{\phi\phi}} \left(\frac{k}{k_p} \right)^{n_{\phi\phi} - n_\phi + 2\epsilon_\phi - \eta_{\phi\phi}}. \quad (58)$$

If we take a derivative of f_{NL} with respect to the scale k , while keeping the ratio k_3/k fixed, we again have $n_{f_{\text{NL}}} = n_{\phi\phi} - 2n_\phi$. But it is clear from (58) that if we would instead take the derivative w.r.t. k_3 while keeping k fixed or vice versa we would get a different result, $-n_\phi - 2\epsilon_\phi + \eta_{\phi\phi}$ and $n_{\phi\phi} - n_\phi + 2\epsilon_\phi - \eta_{\phi\phi}$ respectively.

We note that (57) has a more complicated form than the simple Ansatz for a scale dependent f_{NL} used by Sefusatti et al [18] (Eq. II.6 of their paper). The two forms agree in the case of an equilateral triangle but in general it is not possible to factorise out the local scale dependence which they denote $F(k_1, k_2, k_3)$. It would be interesting to see how the forecasted constraints on $n_{f_{\text{NL}}}$ vary if one uses the full form for f_{NL} given in (57).

4 Multi field quasi-local models

A set-up in which more than one scalar field determines the properties of the curvature perturbation can be treated similarly to single field, but in this case there are generically two sources which contribute to a scale dependence of the quasi-local f_{NL} . The first, discussed in Section 2, is associated with the fact that the various Gaussian fields constituting the system can have different scale dependence, leading to a change with scale in the correlation of quadratic terms with the linear perturbations. The second, discussed in Section 3, is due to the non-linear evolution of second order perturbations after horizon exit.

The Lagrangian governing the system reads

$$\mathcal{L} = \frac{1}{2} \delta_{IJ} \partial_\mu \phi^I \partial^\mu \phi^J - V(\phi^K). \quad (59)$$

Notice that for simplicity we assume a flat metric in field space. In this situation, the expansion for the curvature perturbation, valid at super-horizon scales, reads in Fourier space

$$\zeta_{\vec{k}}(t_f) = N_I \delta \phi_{\vec{k}}^I(t_i) + \frac{1}{2} N_{IJ} (\delta \phi^I \star \delta \phi^J)_{\vec{k}}(t_i) \quad (60)$$

where we have truncated the expansion at second order. The quantities N_I, N_{IJ} indicate derivatives of the number of e-foldings $N = \int_{t_i}^{t_f} H dt$ with respect to the scalar fields. In the framework of multi-field inflation, they explicitly depend on t_f , while the dependence on t_i is again included in a dependence on ϕ^L , writing $N_I = N_I(t_f, \phi^L(t_i))$. Again, $\zeta(t_f, \vec{x})$, as defined in (60), does not depend on t_i .

In order to proceed, it is convenient to know some properties of the solutions for the first order scalar perturbations soon after horizon exit, that reads

$$\delta_{\vec{k}} \phi^I(t_i) = \frac{iH(t_i)}{\sqrt{2k^3}} \left\{ (1 - \epsilon) \delta^{IJ} + \left[c + \ln \left(\frac{a(t_i)H(t_i)}{k} \right) \right] \epsilon^{IJ} \right\} a_{\vec{k}}^J \quad (61)$$

where $c = 2 - \ln 2 - \gamma$ with γ the Euler-Mascheroni constant, while ϵ^{IJ} is a combination of slow-roll parameters [37]

$$\epsilon_{IJ} = \epsilon \delta_{IJ} + \frac{V_I V_J}{9H^4} - \frac{V_{IJ}}{3H^2}, \quad (62)$$

which is small at the epoch of horizon crossing. Finally, $a_{\vec{k}}^J$ is a classical, time independent random variable satisfying the orthogonality property $\langle a_{\vec{k}}^I a_{\vec{l}}^J \rangle = (2\pi)^3 \delta^{IJ} \delta^3(\vec{k} + \vec{l})$.

By proceeding as we did in Section 3, we can calculate the spectral index. By applying identity (32) to the power spectrum, one finds

$$n - 1 = -2\epsilon + \frac{2 \sum_I n_I N_I N_I}{\sum_J N_J N_J} \quad (63)$$

where we introduce the notation (as a natural extension of (34) and (35))

$$n_I = \frac{d \ln N_I}{H dt_i}, \quad n_{IJ} = \frac{d \ln N_{IJ}}{H dt_i}, \quad (64)$$

and all quantities are evaluated at time $t_i = t_*$ corresponding to horizon exit. We checked that this expression coincides with the one of [5], in the case in which the derivatives of N are explicitly expressed in terms of slow-roll parameters [46].

We can use the above equations and definitions to find an expression for the bispectrum associated with the curvature perturbations,

$$\begin{aligned} B_\zeta &= (2\pi)^3 N_I N_J N_L B_{\delta\phi}^c(\delta\phi_{k_1}^I, \delta\phi_{k_2}^J, \delta\phi_{k_3}^L) \\ &+ \frac{N_I N_J N_{LM} (\delta^{IL} + q^{IJ}) (\delta^{JM} + q^{JM})}{N_R N_S N_T N_Z (\delta^{TZ} + q^{TZ}) (\delta^{RS} + q^{RS})} (P_\zeta(k_1) P_\zeta(k_2) + \text{perms}), \end{aligned} \quad (65)$$

where the scalar perturbations are evaluated at a common time t_i , and $B_{\delta\phi}^c$ is proportional to the connected part of the three point function for the scalar perturbations. The parameter f_{NL} is obtainable from the definition (4), or equivalently

$$\frac{6}{5} f_{\text{NL}} = \frac{B_\zeta}{[P_\zeta(k_1) P_\zeta(k_2) + \text{perms}]}. \quad (66)$$

The tensor q^{IJ} is easily obtained from formula (61). For our purposes, it is enough to say that it can be expressed as

$$q^{IJ} = q_1^{IJ} + q_2^{IJ} \ln \left(\frac{a(t_i) H(t_i)}{k} \right), \quad (67)$$

where q_1 and q_2 are linear functions of slow-roll parameters, with scale-independent coefficients. At leading order in slow-roll, this quantity depends on k only by means of the logarithm, $\log k/(aH)$.

The quasi-local form of f_{NL} given in eq. (66) has various sources of scale dependence. In order to investigate them, we proceed as in the single-field case. We consider scale variations that preserve the triangle shape: here we focus on equilateral triangles, while the case of triangles with different shape is discussed in Appendix B. In order to calculate $n_{f_{\text{NL}}}$, we apply formula (32) to equation (66). Because f_{NL} is independent of t_i , the second term of eq (32) does not give contributions. The first term acts only on the first term of (66), since when applied on the second term it gives only contributions subdominant in slow-roll that can be neglected, analogously to what happens in single field case. Moreover we can neglect its action on q^{IJ} , since this provides contributions that are higher order in slow-roll parameters.

Then, to conclude, the expression for $n_{f_{\text{NL}}}$ in this context is

$$n_{f_{\text{NL}}} = \frac{d}{H dt_i} \ln \left[\left(\frac{N^I N^J N_{IJ}}{(N^L N_L)^2} \right) (t_i) \right] \quad (68)$$

which is a natural generalization of what happens in the single field case. We make a comparison with the explicit multi-variate local model discussed in Section 2, by expanding equation (68) by means of the quantities n_I and n_{IJ} defined in eq. (64). One finds

$$\begin{aligned} n_{f_{\text{NL}}} &= \frac{\sum_{I,J} (n_I + n_J + n_{IJ}) N_I N_J N_{IJ}}{\sum_{L,M} N_L N_M N_{LM}} - \frac{4 \sum_I n_I N_I N_I}{\sum_J N_J N_J}, \\ &= \frac{\sum_{I,J} (n_I + n_J + n_{IJ}) N_I N_J N_{IJ}}{\sum_{L,M} N_L N_M N_{LM}} - 2(n_\zeta - 1 + 2\epsilon). \end{aligned} \quad (69)$$

Eq. (69) can be seen as the analogue of equation (15), this time calculated by using the δN formalism. If not all of the n_I are the same, there is a source of scale dependence for f_{NL} , mainly due to the fact that

first order, Gaussian perturbations have different scale dependence (see Section 2). But, even if all the fields have the same n_I , implying $n_I = \frac{1}{2}(n_\zeta - 1 + 2\epsilon)$, n_{fNL} still will not generically vanish due to the evolution of second-order perturbations after horizon exit (see Section 3).

4.1 Multi curvaton scenario

In this section we consider the possibility that two curvaton fields contributed to the primordial curvature perturbation. Following [47] we assume that the curvatons are non-interacting and we neglect the inflaton field fluctuations (for a related scenario see [48]). In the limit that the non-Gaussianity is large, which is the observationally interesting case, we can write the primordial curvature perturbation in the multivariate local form

$$\zeta = r_a \zeta_a + r_b \zeta_b = r_a \left(\zeta_{G a} + \frac{3}{4} \zeta_{G a}^2 \right) + r_b \left(\zeta_{G b} + \frac{3}{4} \zeta_{G b}^2 \right), \quad (70)$$

where ζ_a is the curvature perturbation of the curvaton field a and $\zeta_{G a}$ is the Gaussian part of its perturbation. We denote the initial, horizon crossing curvaton field values a_* and b_* . Then the Gaussian part of the of the fluctuation due to the curvaton field a is given by

$$\zeta_{G a} = \frac{2}{3} \frac{\delta a_*}{a_*} \quad (71)$$

and it is useful to further define

$$\beta = \frac{a_*}{b_*}, \quad r = \frac{r_a}{r_b}. \quad (72)$$

Then

$$P_\zeta = r_a^2 P_{\zeta_a} + r_b^2 P_{\zeta_b} = (r_a^2 + r_b^2 \beta^2) P_{\zeta_a}, \quad (73)$$

$$f_{\text{NL}} = \frac{5}{4} \frac{r_a^3 + \beta^4 r_b^3}{(r_a^2 + \beta^2 r_b^2)^2}. \quad (74)$$

The full result for f_{NL} is given in [47], but the simplified expression above is a good approximation for $|f_{\text{NL}}| \gg 1$.

Since this form of the curvature perturbation satisfies Ansatz (6) we could have analysed this model using the results of section 2, but we use this model to provide an example of using the formalism of the previous section. To calculate the scale dependence we can use (69), and from (70) and (71) we can read off the derivatives of N as

$$N_a = \frac{2}{3} \frac{r_a}{a_*}, \quad N_{aa} = 2 \frac{r_a}{a_*^2}, \quad (75)$$

and similarly for the derivatives with respect to the b field. Thus, we can proceed as in Section (3.2) to find $n_a = \eta_{aa}$, $n_{aa} = 2\eta_{aa}$ and similarly for n_b . Putting all of this together we find

$$n - 1 = -2\epsilon + \frac{2r^2}{r^2 + \beta^2} \eta_{aa} + \frac{2\beta^2}{r^2 + \beta^2} \eta_{bb}, \quad (76)$$

$$n_{\text{fNL}} = 4(\eta_{aa} - \eta_{bb}) \left[\frac{r^3}{r^3 + \beta^4} \frac{\beta^2}{r^2 + \beta^2} - \frac{\beta^4}{r^3 + \beta^4} \frac{r^2}{r^2 + \beta^2} \right]. \quad (77)$$

Notice that in the limit $\beta^2 \rightarrow 0$ or $\beta^2 \rightarrow \infty$ only one of the fields contributes to ζ , since $P_{\zeta_b} = \beta^2 P_{\zeta_a}$. In that case we can see from (77) that $n_{\text{fNL}} \rightarrow 0$ in agreement with the expectation from section 3.2, because we are considering the non-interacting case. Also in agreement with Sec. 2 we see that $n_{\text{fNL}} = 0$ if $\eta_{aa} = \eta_{bb}$. Since both of the terms of $n - 1$ which depend on the η parameters must give a positive contribution to the spectral index and we observe a red spectral index the contribution from these terms must be small barring unlikely cancellations between these two terms and the -2ϵ . In addition each of the four fractions in (77) is less than unity, so we conclude that (77) is likely to be suppressed compared to the spectral index.

5 Conclusion

We have studied generalisations of the local model of non-Gaussianity, thereby seeing in which cases the standard assumption that it can be described by a single, scale-independent parameter is valid. We have shown that this is only strictly valid in specific models, where the primordial curvature perturbation is generated by a single scalar field which acts as an isocurvature (“test”) field during inflation and has a quadratic potential. An example of this is the simplest curvaton scenario. However as soon as the scalar field has interactions these generate non-Gaussianity of the field perturbations after Hubble exit, and these give rise to a scale dependent non-Gaussianity which we call quasi-local, meaning that in the limit that the scale dependence of f_{NL} goes to zero one recovers the local model.

We have also shown that even in a multi-variate local model where more than one Gaussian field contributes to the primordial curvature perturbation the effective f_{NL} has a scale dependence unless all fields have the same scale dependence. An example of this is a mixed inflaton and curvaton scenario or a multi-curvaton model, where the curvatons have quadratic potentials with different masses.

We have developed a formalism, based on the δN approach, which allows us to obtain compact expressions for $n_{f_{\text{NL}}} = d \ln |f_{\text{NL}}| / d \ln k$, in both the single and multi-field case. We have also discussed more generally the validity of defining a scale dependence of f_{NL} with respect to a single scale when the bispectrum is generally a function of three variables, except in the case of an equilateral triangle. We have shown that the scale dependence of f_{NL} is independent of the shape of the triangle it describes, provided that we consider scale variations which preserve its shape. It is then appropriate to perform the simpler calculation of f_{NL} for an equilateral triangle, and use this to calculate $n_{f_{\text{NL}}}$ by taking its logarithmic scale dependence.

We applied our formalism to discuss several specific models. Our results suggest that the scale dependence is typically first order in slow roll. The precise value however depends quite sensitively on details of the model which, makes it in principle possible to use $n_{f_{\text{NL}}}$ to discriminate observationally between different models.

We have shown that while the simplest realisation of the curvaton model has an exactly scale independent f_{NL} , almost any extension to a more realistic scenario does give rise to a scale dependence. In the case of an interacting curvaton or a multiple-curvaton scenario the scale dependence is likely to be suppressed compared to the spectral index. This is because $n_{f_{\text{NL}}}$ in these scenarios depends schematically on η_{curvaton} which is normally positive and, in light of the observational preference for a red spectrum, this is likely to be small. A detection of both f_{NL} and $n_{f_{\text{NL}}}$ would therefore be a signal of non-trivial dynamics in the curvaton scenario, or that the inflaton perturbations have also made a significant contribution to the observed power spectrum.

In the case of a mixed scenario, in which the primordial curvature perturbation has contributions from both the inflaton and curvaton perturbations it is quite natural for there to be a consistency relation between the scale dependence of the power spectrum and the bispectrum, $n_{f_{\text{NL}}} \simeq -2(n - 1)$, and this could be observable with Planck assuming a large enough fiducial value for f_{NL} , close to the current observational bounds. In this case the bispectrum would be larger on small scales and this could make a detection of f_{NL} using large scale structure data more likely. In any scenario where one includes the Gaussian perturbations of the inflaton field as well as the perturbations of a second field which generates non-Gaussianities one will have a scale dependent f_{NL} , unless both fields have exactly the same spectral index.

For the case where non-Gaussianity is generated during slow-roll hybrid inflation one generally has a non-negligible $n_{f_{\text{NL}}}$ [20]. In this case, since the magnitude of f_{NL} can only grow during inflation, larger scales, which exit the horizon earlier, will be more non-Gaussian and one necessarily has $n_{f_{\text{NL}}} < 0$.

It should be straightforward to extend the formalism we have presented here to the primordial trispectrum (the four-point function) [33,49] and we expect that the two non-linearity parameters which model it, τ_{NL} and g_{NL} , would also have a first order in slow roll scale dependence. We have chosen the notation $n_{f_{\text{NL}}}$ in a way that is extendible in an obvious manner to study this. For example in the case that a single field generates the primordial curvature perturbation one has a non-Gaussianity consistency relation, $\tau_{NL} = (6f_{\text{NL}}/5)^2$. Then assuming our formalism can be extended in the obvious way it follows that $n_{\tau_{NL}} = 2n_{f_{\text{NL}}}$.

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Appendices

A Explicit proof that f_{NL} is independent of the initial time

In this section we show explicitly that f_{NL} is independent of t_i . In the first part, we focus on single field slow-roll inflation, in which the unique scalar plays the role of inflaton field, and we study the equations of motion for the perturbations. In the second part, we prove the same results using general properties of our definition of the curvature perturbation.

In the single field case, ζ is conserved on super horizon scales so we do not need to consider a further dependence on t_f . Then the δN formalism provides the following expression

$$\zeta = \frac{1}{\sqrt{2\epsilon}}\delta\phi + \frac{2\epsilon - \eta}{4\epsilon}\delta^2\phi. \quad (78)$$

All quantities on the right hand side are evaluated at a given time t_i . It is useful to split the scalar field perturbation into first and second order perturbations as $\delta\phi = \delta_1\phi + \frac{1}{2}\delta_2\phi$. If we choose t_i to coincide with Hubble crossing, $t_i = t_*$, we have

$$\frac{6}{5}f_{\text{NL}} = \eta - 2\epsilon|_{t_*} + \frac{6}{5}f_{\text{NL}}^{(3)}. \quad (79)$$

We are following the notation of [46] and $f_{\text{NL}}^{(3)}$ has a complicated k dependence but it is zero if we take $\delta_2\phi_* = 0$ at horizon crossing, a condition which we will assume from now on. We now consider what happens if we choose to use a time $t_i > t_*$. We first note that

$$\frac{d(\eta - 2\epsilon)}{dN} = 6\epsilon\eta - 8\epsilon^2 - \xi^2, \quad (80)$$

where $\xi^2 \equiv \frac{V''''V'}{V^2}$ is a second order slow-roll parameter. We can get the scalar field time dependence from [50]. From (2.13) of that paper it is easy to see that

$$\delta_1\phi(t_i) = \delta_1\phi_* [1 + H(t_i - t_*)(2\epsilon - \eta)], \quad (81)$$

We assume that $H(t_i - t_*)$, corresponding to the number of e -foldings N between t_* and t_i , is much smaller than the inverse of the slow-roll parameters. Since we only need to consider about ten e -foldings for observational purposes, this is an adequate approximation.

The evolution of second order perturbations is more complicated, and we need to consider (2.15) of [50], at leading order in a slow-roll expansion. We find

$$\frac{1}{H}(\delta_2\phi)' + (\eta - 2\epsilon)\delta_2\phi = \frac{V}{V'}(8\epsilon^2 - 6\epsilon\eta + \xi^2)\delta_1\phi^2. \quad (82)$$

We note that the $8\epsilon^2$ term above is not present in (2.19) of [50], which is because they use a non-standard definition of ϵ . For our purposes it is important to include it. Since we only need to solve this equation at leading order in slow-roll, we can neglect the additional time evolution of the slow-roll parameters, as well as the evolution of $\delta_1\phi^2$. This makes the integration simple, and we find

$$\delta_2\phi(t_i) = \delta_2\phi_* + \frac{N}{\sqrt{2\epsilon}}(8\epsilon^2 - 6\epsilon\eta + \xi^2)\delta_1\phi^2, \quad (83)$$

For the reason explained above, it does not matter here whether the terms which multiply N are evaluated at t_i or t_* . Using (66), it is then clear that the time dependence of the slow-roll parameters in (79) as given in (80) is compensated by the time dependence of $\delta_2\phi$ as given by (83), rendering f_{NL} independent of t_i . The previous discussion can be extended to the case of multi-fields.

One can arrive to the same result in a different way, that is easily generalizable the case of multiple scalar fields. Recall that the curvature perturbation ζ is defined by means of the spatial part of the metric

$$g_{ij} = a^2(t) e^{2\zeta(t, \vec{x})} \gamma_{ij}(t, \vec{x}), \quad (84)$$

where $a(t)$ and $\gamma_{ij}(t, \vec{x})$ indicate background quantities. In [35], it has been rigorously proved that, at all orders in the expansion of the perturbations, the curvature perturbation at superhorizon scales can be expressed in terms of a variation in the number of e-foldings:

$$\zeta = \mathcal{N}(t_f, t_i, \vec{x}) - N(t_f, t_i) \equiv \delta N(t_f, t_i, \vec{x}) \quad (85)$$

with

$$N(t_f, t_i) = \int_{t_i}^{t_f} H dt, \quad \mathcal{N}(t_f, t_i, \vec{x}) = -\frac{1}{3} \int_{t_i}^{t_f} dt \left(\frac{\dot{\rho}}{\rho + p} \right) \Big|_{\vec{x}}. \quad (86)$$

In the previous equations, the energy momentum tensor for the system is assumed to correspond to an ideal fluid: this condition is satisfied for a multiple field set-up.

It is simple to see that the curvature perturbation ζ given in eq (85) is *independent* of t_i , as long as t_i labels a flat hypersurface. Indeed, one has

$$\frac{\partial}{\partial t_i} \zeta = -H(t_i) + \frac{1}{3} \left(\frac{\dot{\rho}}{\rho + p} \right) (t_i, \vec{x}). \quad (87)$$

On the other hand, as discussed in eqs (18) and following of [35], as long as one is focusing on flat hypersurfaces, the following energy conservation equation holds

$$\frac{\dot{a}}{a} = -\frac{\dot{\rho}}{3(\rho + p)} \quad (88)$$

Using the previous relation, (87) ensures that ζ is independent of t_i , at all orders in the expansion, as we wanted to prove. Because the curvature perturbation is independent of t_i , f_{NL} is also independent of this quantity.

The information that ζ is independent of t_i , allows also to straightforwardly obtain the equations of motion for first and second order scalar perturbations $\delta_1\phi$ and $\delta_2\phi$. Expanding ζ at first order in perturbations, one has

$$\zeta(\vec{x}) = N_\phi(t_i) \delta_1\phi(t_i, \vec{x}) \quad (89)$$

The fact that the right hand side is independent of t_i provides the equation (we use the notation of Section 3)

$$(\delta_1\phi)' + n_\phi H \delta_1\phi = 0 \quad (90)$$

leading to eq. (81) for single field slow roll inflation. Analogously, expanding ζ up to second order and proceeding in the same way, one finds the following equation of motion for second order perturbations, valid at leading order in slow roll

$$(\delta_2\phi)' + n_\phi H \delta_2\phi = H N_\phi n_\phi (n_{\phi\phi} - 2n_\phi) (\delta_1\phi)^2, \quad (91)$$

that corresponds to equation (82). It should be possible to extend the present method to the multi-field case, even with non-canonical kinetic terms, possibly reproducing the results of [51].

B Non-equilateral triangles

In this appendix, we explain how to calculate the running of f_{NL} for generic triangle configurations. We directly discuss a multi-field system, the single field case is easily obtained as special limit of our computations. To perform the calculation, it is convenient to set the initial time t_i , in the expansion (60) for $\zeta_{\mathbf{k}}$,

equal to the horizon crossing time $t_i = t_*(k)$. With this choice and using Wick's theorem, the bispectrum can be written as

$$B_\zeta(k_1, k_2, k_3) = N_{(1)I}N_{(2)J}N_{(3)LM} \int \frac{d\mathbf{q}}{(2\pi)^3} \langle \delta\phi_{\mathbf{k}_1}^I(t_1)\delta\phi_{\mathbf{q}}^L(t_3) \rangle \langle \delta\phi_{\mathbf{k}_2}^J(t_2)\delta\phi_{\mathbf{k}_3-\mathbf{q}}^M(t_3) \rangle + 2 \text{ perms} , \quad (92)$$

since at horizon crossing we set the connected three and four point functions of $\delta\phi$ to zero, thereby assuming the perturbations are Gaussian at this time. In the previous formula, we introduced the notation $N_{(j)I} \equiv N_I(t_j, t_j)$ with t_j corresponding to the horizon crossing epoch for the mode k_j . Notice that the two point functions in (92) involve arguments evaluated at different times. In general, to first order in slow-roll, from (61) it follows that

$$\delta\phi_{\mathbf{k}}^L(t_a) = \left[\delta^L_M + \left(\ln \frac{a(t_a)H(t_a)}{a(t_b)H(t_b)} \right) \hat{\epsilon}^L_M \right] \delta\phi_{\mathbf{k}}^M(t_b) . \quad (93)$$

Here we have defined a quantity

$$\hat{\epsilon}_{LM} = \frac{V_L V_M}{9H^4} - \frac{V_{LM}}{3H^2} , \quad (94)$$

which differs by a factor $2\epsilon\delta_{LM}$ from ϵ_{LM} defined in (62) since we have explicitly included the time evolution of H in (93). Using the relation between time and scale at horizon crossing, $k = aH$, to leading order in slow-roll the two-point functions with unequal time arguments can be written as

$$\begin{aligned} \langle \delta\phi_{\mathbf{k}_1}^I(t_1)\delta\phi_{\mathbf{q}}^L(t_3) \rangle &= \left[\delta^L_M + \left(\ln \frac{k_3}{k_1} \right) \hat{\epsilon}^L_M \right] \langle \delta\phi_{\mathbf{k}_1}^I(t_1)\delta\phi_{\mathbf{q}}^M(t_1) \rangle \\ &= \left[\delta^L_M + \left(\ln \frac{k_3}{k_1} \right) \hat{\epsilon}^L_M \right] (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{q}) \frac{(\delta^{IM} + 2c\hat{\epsilon}^{IM})P_\zeta(k_1)}{N_{(1)R}N_{(1)S}(\delta^{RS} + 2c\hat{\epsilon}^{RS})} \\ &= (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{q}) \frac{\left[\delta^{IL} + \left(2c + \ln \frac{k_3}{k_1} \right) \hat{\epsilon}^{IL} \right] P_\zeta(k_1)}{N_{(1)R}N_{(1)S}(\delta^{RS} + 2c\hat{\epsilon}^{RS})} , \end{aligned} \quad (95)$$

and similarly for the other terms in (92). Recall that $c = 2 - \ln 2 - \gamma$, with γ the Euler-Mascheroni constant. The result is independent of the time at which the slow-roll matrix $\hat{\epsilon}^{IJ}$ is evaluated since we work to leading order in slow-roll. Using (95), the bispectrum (92) can be written as

$$\begin{aligned} B_\zeta(k_1, k_2, k_3) &= \left\{ \frac{N_{(1)I}N_{(2)J}N_{(3)LM} \left[\delta^{IL} + \left(2c + \ln \frac{k_3}{k_1} \right) \hat{\epsilon}^{IL} \right] \left[\delta^{JM} + \left(2c + \ln \frac{k_3}{k_2} \right) \hat{\epsilon}^{JM} \right]}{N_{(1)R}N_{(1)S}N_{(2)T}N_{(2)Z} (\delta^{RS} + 2c\hat{\epsilon}^{RS}) (\delta^{TZ} + 2c\hat{\epsilon}^{TZ})} \right\} \\ &\quad \times P_\zeta(k_1)P_\zeta(k_2) + 2 \text{ perms} , \end{aligned} \quad (96)$$

corresponding to equation (65) rewritten using the choice $t_i = t_*(k_i)$ in (60) for each of the three modes k_i . From (66) one finds the corresponding non-linearity parameter f_{NL} .

To evaluate the scale dependence of f_{NL} , we perform an expansion around a pivot scale k_p not too different from k_i 's. Let us start by considering scale variations that preserve the (arbitrary) shape of the triangles: we then write the wavenumbers k_i in terms of dimensionless parameters α_i and a common scale $k_i = k\alpha_i$. Here we treat the parameters α_i as constants, concentrating on variations of the scale k that preserve the shape of the triangle. Denoting the derivatives of N at the pivot scale by

$$n_I = \left. \frac{d \ln N_I}{H dt} \right|_{a=k_p/H} , \quad n_{IJ} = \left. \frac{d \ln N_{IJ}}{H dt} \right|_{a=k_p/H} , \quad (97)$$

and defining

$$\hat{n} = \frac{\sum_I n_I N_I N_I}{\sum_J n_J N_J N_J} , \quad (98)$$

we can expand f_{NL} around k_p as

$$f_{\text{NL}}(k) = f_{\text{NL}}(k_p) \left[1 + \left(\frac{\sum_{I,J} (n_I + n_J + n_{IJ}) N_I N_J N_{IJ}}{\sum_{L,M} N_L N_M N_{LM}} - 4\hat{n} \right) \ln \left(\frac{k}{k_p} \right) + \dots \right] . \quad (99)$$

Here the coefficient of the logarithm is given to leading order in slow-roll parameters and to the same precision the non-linearity parameter at the pivot scale reads $f_{\text{NL}}(k_p) = (5/6)N_I N_J N_{IJ} / (N_L N_L)^2$. From equation (99) we see that for a generic non-equilateral configuration with a fixed shape, the scale dependence of the non-linearity parameter is given by

$$n_{f_{\text{NL}}} = \frac{\sum_{I,J} (n_I + n_J + n_{IJ}) N_I N_J N_{IJ}}{\sum_{L,M} N_L N_M N_{LM}} - 4\hat{n}, \quad (100)$$

to leading order precision in slow-roll and in the expansion in $\ln(k/k_p)$. To this precision, the result coincides exactly with the one obtained for the equilateral case, equation (69), and is independent of the constants α_i parameterising the shape of the triangle.

However, if both the scale and the shape of the triangle vary simultaneously, the result will in general be different. In principle, it is straightforward to work out the scale dependence for an arbitrary variation, starting from the general equation (96). As an example we discuss the single field case. The bispectrum (96) depends on $\hat{\epsilon}_{IJ}$ which in the single field case has only one component $\hat{\epsilon}_{\phi\phi} = 2\epsilon_\phi - \eta_{\phi\phi}$. Equation (96) can be expanded around the pivot scale k_p as

$$B(k_1, k_2, k_3) = \frac{N_{\phi\phi}}{N_\phi^2} \left[1 + (n_{\phi\phi} + 2\hat{\epsilon}_{\phi\phi}) \ln\left(\frac{k_3}{k_p}\right) - (n_{\phi\phi} + \hat{\epsilon}_{\phi\phi}) \ln\left(\frac{k_1}{k_p}\right) - (n_{\phi\phi} + \hat{\epsilon}_{\phi\phi}) \ln\left(\frac{k_2}{k_p}\right) \right] P_\zeta(k_1) P_\zeta(k_2),$$

plus permutations. To the same order of accuracy as in (99), we can then expand f_{NL} around the pivot scale as

$$f_{\text{NL}}(k_1, k_2, k_3) = f_{\text{NL}}^{\text{pivot}} \frac{\left(\frac{k_1}{k_p}\right)^{n_\zeta - 4 - n_\phi - 2\epsilon_\phi + \eta_{\phi\phi}} \left(\frac{k_2}{k_p}\right)^{n_\zeta - 4 - n_\phi - 2\epsilon_\phi + \eta_{\phi\phi}} \left(\frac{k_3}{k_p}\right)^{n_{\phi\phi} + 4\epsilon_\phi - 2\eta_{\phi\phi}} + 2 \text{ perms}}{\left(\frac{k_1}{k_p}\right)^{n_\zeta - 4} \left(\frac{k_2}{k_p}\right)^{n_\zeta - 4} + 2 \text{ perms}}. \quad (101)$$

This expression allows to study the scale dependence of f_{NL} in full generality for arbitrary variations of the triangle.

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