

Scale-dependent non-Gaussianity probes inflationary physics

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ABSTRACT: We calculate the scale dependence of the bispectrum and trispectrum in (quasi) local models of non-Gaussian primordial density perturbations, and characterize this scale dependence in terms of new observable parameters. They can help to discriminate between models of inflation, since they are sensitive to properties of the inflationary physics that are not probed by the standard observables. We find consistency relations between these parameters in certain classes of models. We apply our results to a scenario of modulated reheating, showing that the scale dependence of non-Gaussianity can be significant. We also discuss the scale dependence of the bispectrum and trispectrum, in cases where one varies the shape as well as the overall scale of the figure under consideration. We conclude providing a formulation of the curvature perturbation in real space, which generalises the standard local form by dropping the assumption that f_{NL} and g_{NL} are constants.

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1. Introduction

Inflation is the simplest framework which explains the origin of the observed power spectrum of temperature fluctuations in the cosmic microwave background [1]. It is now widely accepted that non-Gaussianity is a powerful probe to discriminate between the many currently viable inflationary models [2, 3, 4, 5, 6, 7, 8]. It is usually parameterized in terms of a single constant parameter, f_{NL} , corresponding to the amplitude of the bispectrum normalized to the square of the power spectrum of primordial curvature fluctuations [9, 10]. More recently it has become common to further characterize local non-Gaussianity including the two non-linearity parameters associated to the trispectrum, called g_{NL} and τ_{NL} , again treating them as constants [11]. However, it has been recently pointed out, both from theoretical [12, 13, 14, 15] and observational viewpoints [16], that f_{NL} is not necessarily constant. We show the same holds true for g_{NL} and τ_{NL} . As happens with the power spectrum and the spectral index, they are characterized by a scale dependence, that which denote respectively with $n_{f_{\text{NL}}}$, $n_{g_{\text{NL}}}$ and $n_{\tau_{\text{NL}}}$. For example, if f_{NL} is large and positive on large scale structure scales [17, 18, 19, 20], but has a smaller value on the largest CMB scales then this would require that f_{NL} is scale dependent. Any

scale dependence of the non-linearity parameters provides a new and potentially powerful observational probe of inflationary physics.

In this work, we discuss a new approach to study the scale dependence of the non-linearity parameters, based on the δN -formalism [21, 22, 23]. This allows us to obtain an expression for the curvature perturbation that generalizes the local Ansatz introduced in [9, 10], and that contains the aforementioned scale-dependent parameters. For the single field case, the curvature perturbation can be schematically written as

$$\zeta_{\mathbf{k}} = \zeta_{\mathbf{k}}^{\text{G}} + \frac{3}{5} f_{\text{NL}}^{\text{p}} (1 + n_{f_{\text{NL}}} \ln k) (\zeta^{\text{G}} \star \zeta^{\text{G}})_{\mathbf{k}} + \frac{9}{25} g_{\text{NL}}^{\text{p}} (1 + n_{g_{\text{NL}}} \ln k) (\zeta^{\text{G}} \star \zeta^{\text{G}} \star \zeta^{\text{G}})_{\mathbf{k}} ,$$

where $\zeta_{\mathbf{k}}^{\text{G}}$ is a Gaussian variable, and f_{NL}^{p} and g_{NL}^{p} are constants. Our approach allows us to directly calculate $n_{f_{\text{NL}}}$, $n_{g_{\text{NL}}}$ and $n_{\tau_{\text{NL}}}$ in models with an arbitrary inflationary potential and an arbitrary number of fields, assuming slow-roll inflation. We also assume the field perturbations are Gaussian at Hubble exit. Our results depend only on the slow-roll parameters evaluated at Hubble exit, and on the derivatives of N , the e-folding number. In particular, we find that $n_{f_{\text{NL}}}$ and $n_{g_{\text{NL}}}$ are sensitive to third and fourth derivatives of the potential along the directions in field space that are responsible for generating non-Gaussianities. These do not in general coincide with the adiabatic direction (during inflation) and such features cannot therefore be probed by only studying the spectral index and its running [24]. In the case that a single field generates the curvature perturbation there is a consistency relation between $n_{f_{\text{NL}}}$ and $n_{\tau_{\text{NL}}}$ which is the derivative of the consistency relation between f_{NL} and τ_{NL} . We explicitly show how this consistency relation is violated in multiple field models.

In the framework of slow-roll inflation, there are various ways to generate large non-Gaussianity, in models in which more than one field play a role during the inflationary process. This is the case of multiple field inflation [25, 26, 27, 28, 29, 30, 31, 32, 33, 34], in which two or more fields contribute to the curvature perturbations. But there are also approaches in which, although more than one field is light during inflation, only one of them contributes significantly to the curvature perturbations (the most studied examples are the curvaton [35] and modulated reheating [36, 37] scenarios). In this work, we apply our general findings to this last class of models. We consider set-ups in which an isocurvature field remains subdominant during inflation (as required in order to have an observable level of non-Gaussianity [38]), but represents the main source of curvature fluctuations after inflation ends. In this case, neither the spectral index of the power spectrum of curvature perturbations, nor its running are sensitive to the third and fourth derivative of the potential of the subdominant field. Hence the scale dependence of non-Gaussian parameters provide a unique opportunity to probe self interactions in these scenarios. As an example, in the modulated reheating scenario, it is possible for any of the non-Gaussian parameters to be large. We show that if the modulation field has self interactions, for example a quartic potential, then all of f_{NL} , $n_{f_{\text{NL}}}$, g_{NL} and $n_{g_{\text{NL}}}$ can be large and provide novel information about the mechanism which generates curvature perturbations. We will also consider mixed scenarios in which the inflaton perturbations are not neglected [39, 40, 41, 42]. We note that the scale dependence of equilateral type non-Gaussianity is also of theoretical and observational interest [43, 44, 45, 46].

We have previously shown [14] that provided one scales all three sides of the bispectrum at the same rate then $n_{f_{\text{NL}}}$ is a constant (and hence it is simplest to focus on an equilateral configuration). We show a similar result for the trispectrum parameters. Since it may be of interest to consider more general variations in which one changes the shape of the figure under consideration, we also consider this case. We find the combination of shape and scale dependence which maximizes $n_{f_{\text{NL}}}$ and show that it is never significantly larger than the standard result, in which one keeps the shape

fixed. However we single out interesting limits in which there is no scale dependence, corresponding to squeezed figures.

While in most of the paper we work in momentum space, in the last part we also discuss how to describe our results in coordinate space. We provide an expression for the curvature perturbations in real space, that generalizes the simplest local Ansatz [10], and that exhibits directly in coordinate space the effect of scale dependence of non-Gaussian parameters.

The plan of our paper is as follows: In Sec. 2 we extend and simplify the results from our previous paper to give general results for the non-linearity parameters, including those which measure the trispectrum. In Sec. 3 we reduce the results to general single field models and derive a consistency relation. In Sec. 4 we consider simple one or two-field models in which one field, e.g. the curvaton, generates non-Gaussianity but we do not exclude the Gaussian perturbations from the inflaton. Many popular models in the literature fall into this class and the reader may choose to skip straight to this section where the results and notation are significantly simpler. As an explicit example we study modulated reheating. In Sec. 5 we consider in detail the scale dependence of the bispectrum and the trispectrum, and how this can be affected by the shape of the triangle or quadrilateral. In Sec. 6 we consider how to generalize the coordinate space expression of the curvature perturbation to include scale dependence. Finally we conclude in Sec. 7.

2. General results

In this section, we discuss a new approach to analyze the scale dependence of quasi-local non-Gaussianity, by means of a suitable implementation of the δN formalism. Using the δN formalism [21, 22, 23], the curvature perturbation for a system of n scalar fields ϕ_a is given by the expression

$$\zeta(t_f, \mathbf{x}) = \sum_a N_a(t_f, t_i) \delta\phi^a(t_i, \mathbf{x}) + \frac{1}{2} \sum_{ab} N_{ab}(t_f, t_i) \delta\phi^a(t_i, \mathbf{x}) \delta\phi^b(t_i, \mathbf{x}) + \dots, \quad (2.1)$$

where t_f labels a uniform energy density hypersurface and t_i denotes a spatially flat hypersurface. The result is valid on super-horizon scales where spatial gradients can be neglected. In this work, we do not consider secondary effects on curvature perturbations arising from late-time physics (see [47] for a review), nor the effects of possible isocurvature modes during the late universe. The quantities N_a and N_{ab} denote derivatives of the number of e-foldings along the scalar fields. We choose t_i as a time soon after the horizon crossing of all the modes of interest. Written in momentum space, Eq. (2.1) reads

$$\zeta_{\mathbf{k}}(t_f) = \sum_a N_a(t_f, t_i) \delta\phi_{\mathbf{k}}^a(t_i) + \frac{1}{2} \sum_{ab} N_{ab}(t_f, t_i) (\delta\phi^a(t_i) \star \delta\phi^b(t_i))_{\mathbf{k}} + \dots. \quad (2.2)$$

Here $k < a(t_i)H(t_i)$, since we focus on super-horizon scales, and \star denotes a convolution:

$$(\delta\phi^a(t_i) \star \delta\phi^b(t_i))_{\mathbf{k}} \equiv \int \frac{d\mathbf{q}}{(2\pi)^3} \delta\phi_{\mathbf{q}}^a(t_i) \delta\phi_{\mathbf{k}-\mathbf{q}}^b(t_i). \quad (2.3)$$

To analyze the statistical properties of the curvature perturbation it is useful to express the results in terms of scalar perturbations evaluated at horizon crossing $\delta\phi_{\mathbf{k}}^a(t_k)$. Assuming the fields ϕ_a obey slow roll dynamics during inflation and have canonical kinetic terms, $\delta\phi_{\mathbf{k}}^a(t_k)$ are Gaussian up to slow roll corrections [48, 49]. These corrections are irrelevant in cases where the non-Gaussianities are large, $|f_{\text{NL}}| \gg 1$ or $|g_{\text{NL}}| \gg 1$. Therefore, in our analysis we take the fields $\delta\phi_{\mathbf{k}}^a(t_k)$ to be Gaussian at horizon crossing, $k = a(t_k)H(t_k)$. In [14] the result (2.2) was expressed in terms of $\delta\phi_{\mathbf{k}}^a(t_k)$ by

setting $t_i \rightarrow t_k(k)$. This makes the coefficients $N_{ab\dots}$ implicitly dependent on k . In this work, we follow a different approach, choosing a fixed t_i for all observable k modes, and explicitly solving for the perturbations at t_i as a function of $\delta\phi_{\mathbf{k}}^a(t_k)$. Besides being more transparent, this method allows us to easily write down explicit results for the scale-dependence of non-linearity parameters. The two approaches are compared in detail in Appendix B.

We first note that, assuming slow roll, the evolution of super-horizon scale fluctuations $\delta\phi^a(t, \mathbf{x})$ from some initial spatially flat hypersurface at $t_0 < t_i$ to the spatially flat hypersurface at t_i can be expressed in terms of the Taylor expansion

$$\delta\phi^a(t_i, \mathbf{x}) = \sum_b \frac{\partial\phi^a(t_i)}{\partial\phi^b(t_0)} \delta\phi^b(t_0, \mathbf{x}) + \frac{1}{2} \sum_{bc} \frac{\partial^2\phi^a(t_i)}{\partial\phi^b(t_0)\partial\phi^c(t_0)} \delta\phi^b(t_0, \mathbf{x})\delta\phi^c(t_0, \mathbf{x}) + \dots \quad (2.4)$$

Here we have also assumed the fields have canonical kinetic terms, i.e. the metric in field space is flat. The result (2.4) follows directly from the application of the δN formalism where any super-horizon region, labeled by \mathbf{x} , evolves as a separate FRW universe with its own initial conditions. Since we assume that slow roll conditions are satisfied, the initial conditions are set by the field values $\{\phi^a(t_0)\}$ alone, i.e. any dependence on the field time derivatives can be neglected. Therefore,

$$\delta\phi^a(t_i, \mathbf{x}) = \phi^a(t_i)(\{\phi^b(t_0) + \delta\phi^b(t_0, \mathbf{x})\}) - \phi^a(t_i)(\{\phi^b(t_0)\}) \quad (2.5)$$

where $\phi^a(t_i)(\dots)$ denote FRW solutions with the initial conditions set at t_0 . Eq. (2.4) is obtained by expanding this with respect to $\{\delta\phi^b(t_0, \mathbf{x})\}$ while keeping fixed the number of e-foldings between t_0 and t_i . This corresponds to choosing t_0 and t_i as spatially flat hyper-surfaces, since it amounts to comparing different realizations of FRW universes that all undergo the same number of e-foldings between t_0 and t_i .

The coefficients in Eq. (2.4) can easily be computed by solving the slow roll equations of motion, $3H\dot{\phi}_a = -V_a$. We find

$$\phi^a(t_i) = \phi^a(t_0) - \sqrt{2\epsilon_a} \ln(a_i/a_0) + \mathcal{O}\left(\epsilon^{3/2} \ln^2(a_i/a_0)\right) \quad (2.6)$$

where the slow roll parameters are evaluated at t_0 and defined as usual: $\epsilon_a = (V_a/(3H^2))^2/2$ and $\eta_{ab} = V_{ab}/(3H^2)$ (with $M_{\text{P}} \equiv 1$). In the following we neglect the slow-roll suppressed corrections $\mathcal{O}(\epsilon^{3/2} \ln^2(a_i/a_0))$, where $\mathcal{O}(\epsilon^{3/2})$ denotes terms involving powers of ϵ_a and η_{ab} up to $3/2$. The validity of this approximation is discussed in more detail below and also in Appendix A. Differentiating Eq. (2.6) once with respect to the initial field values, and keeping $\ln(a_i/a_0)$ fixed, we find

$$\frac{\partial\phi^a(t_i)}{\partial\phi^b(t_0)} = \delta_{ab} + \epsilon_{ab} \ln(a_i/a_0) \quad (2.7)$$

where we have defined

$$\epsilon_{ab} \equiv 2\sqrt{\epsilon_a\epsilon_b} - \eta_{ab} \quad (2.8)$$

The higher order derivatives in Eq. (2.6) can be computed in a similar way and the results are given in Appendix A.

By substituting Eq. (2.4) into the coordinate space expression for the curvature perturbation (2.1), taking the Fourier transform and thereafter setting $t_0 = t_k$, we arrive at the result

$$\zeta_{\mathbf{k}} = \sum_a \zeta_{\mathbf{k}}^{\text{G},a} + \sum_{ab} f_{ab}(k)(\zeta^{\text{G},a} \star \zeta^{\text{G},b})_{\mathbf{k}} + \sum_{abc} g_{abc}(k)(\zeta^{\text{G},a} \star \zeta^{\text{G},b} \star \zeta^{\text{G},c})_{\mathbf{k}} + \dots \quad (2.9)$$

Here $\zeta_{\mathbf{k}}^{G,a}$ are Gaussian fields defined as

$$\zeta_{\mathbf{k}}^{G,a}(t_i, t_k) = \sum_b N_a \delta\phi_{\mathbf{k}}^b(t_k) \left[\delta_{ab} + \epsilon_{ab} \ln \frac{a_i H_i}{k} \right] \theta(a_i H_i - k). \quad (2.10)$$

The Gaussianity of this quantity follows from our assumption of the perturbations $\delta\phi_{\mathbf{k}}^a(t_k)$ being Gaussian at the horizon crossing. For brevity, from now on we suppress the time arguments of the derivatives of N , denoting $N_{ab\dots} \equiv N_{ab\dots}(t_f, t_i)$. The theta function in Eq. (2.10) is included to constrain the convolutions to only include super-horizon scales, $k < a_i H_i$. The matrices $f_{ab}(k)$ and $g_{abc}(k)$ are given by

$$f_{ab}(k) = \frac{1}{2} \frac{N_{ab}}{N_a N_b} + \frac{1}{2} \sum_c \frac{N_c F_{cab}^{(2)}}{N_a N_b} \ln \frac{a_i H_i}{k}, \quad (2.11)$$

$$g_{abc}(k) = \frac{1}{6} \frac{N_{abc}}{N_a N_b N_c} + \frac{1}{6 N_a N_b N_c} \sum_d \left(3 N_{da} F_{dbc}^{(2)} + N_d F_{dabc}^{(3)} \right) \ln \frac{a_i H_i}{k}, \quad (2.12)$$

where $k < a_i H_i$ and $F_{ab_1\dots b_m}^{(m)}$ denotes the k -independent part of the m :th order coefficient in Eq. (2.4). They are proportional to combinations of slow roll parameters and their explicit expressions are given in Appendix A.

Our results are derived to first order in $\ln a_i H_i/k$. In Eqs. (2.11) and (2.12) the terms proportional to $\ln a_i H_i/k$ represent small corrections to the k -independent parts except in the cases where $f_{ab}(k_i)$ and $g_{abc}(k_i)$ are comparable to slow roll parameters or even smaller. Such components, however, do not generate observable non-Gaussianity and therefore do not play an important role in our discussion. For this reason, we can safely perform the expansion in $\ln a_i H_i/k$. Since the higher order terms arising in this expansion are further slow roll suppressed, and since we can choose t_i such that the logarithms never get larger than $\mathcal{O}(10)$ for the super-horizon modes in our observable universe, we can truncate the expansion at first order.

Instead of expanding in $\ln a_i H_i/k$, we can also choose one of our observable super-horizon modes as a pivot-scale, $k_p < a_i H_i$, and expand Eqs. (2.11) and (2.12) around this point. To first order in $\ln(k/k_p)$ the results are given by

$$f_{ab}(k) = f_{ab}(k_p) \left(1 + n_{f,ab} \ln \frac{k}{k_p} \right), \quad (2.13)$$

$$g_{abc}(k) = g_{abc}(k_p) \left(1 + n_{g,abc} \ln \frac{k}{k_p} \right), \quad (2.14)$$

where we have defined¹

$$n_{f,ab} \equiv \frac{d \ln |f_{ab}|}{d \ln k} = - \sum_c \frac{N_c F_{cab}^{(2)}}{N_{ab}}, \quad n_{g,abc} \equiv \frac{d \ln |g_{abc}|}{d \ln k} = - \sum_d \left(3 \frac{N_{da}}{N_{abc}} F_{dbc}^{(2)} + \frac{N_d}{N_{abc}} F_{dabc}^{(3)} \right). \quad (2.15)$$

Provided that $n_{f,ab}$ and $n_{g,abc}$ are not much larger than $\mathcal{O}(0.01)$, truncating the above series at first order leads to an error of a few per cents at most. Neglecting slow roll corrections, we can write $f_{ab}(k_p) = N_{ab}/(2N_a N_b)$ and $g_{abc}(k_p) = N_{abc}/(6N_a N_b N_c)$. This precision suffices when treating the k -independent terms in our expressions, since we are only interested in computing scale-dependencies

¹It is important to realise that $n_{f,ab}$ ($n_{g,abc}$) is only defined in the case where $f_{ab} \neq 0$ ($g_{abc} \neq 0$) in the limit $k \rightarrow a_i H_i$. If f_{ab} (g_{abc}) vanishes, it is convenient to define the derivative in Eq. (2.15) to be identically zero.

to leading order in slow roll. In what follows, we will therefore always write the constant terms to leading order precision in slow roll.

Finally, using Eqs. (2.13) and (2.14), we can express Eq. (2.9) as

$$\begin{aligned} \zeta_{\mathbf{k}} = & \sum_a \zeta_{\mathbf{k}}^{G,a} + \sum_{ab} f_{ab}(k_p) \left(1 + n_{f,ab} \ln \frac{k}{k_p} \right) (\zeta^{G,a} \star \zeta^{G,b})_{\mathbf{k}} \\ & + \sum_{abc} g_{abc}(k_p) \left(1 + n_{g,abc} \ln \frac{k}{k_p} \right) (\zeta^{G,a} \star \zeta^{G,b} \star \zeta^{G,c})_{\mathbf{k}} + \dots \end{aligned} \quad (2.16)$$

This result is the starting point for our analysis of the scale-dependence of non-linearity parameters. Explicit expressions for $n_{f,ab}$ and $n_{g,abc}$ are given in Appendix A and the scale-dependency arising from the fields $\zeta_{\mathbf{k}}^{G,a}$ can be computed using standard methods. Therefore, using Eq. (2.16) we can explicitly compute the scale-dependencies of f_{NL} , g_{NL} and τ_{NL} for any model with slow roll dynamics during inflation.

2.1 Two point function and power spectrum

Here we re-derive some well-known results for the scale dependence of the spectrum of curvature perturbations; they will be useful in what follows for analyzing the scale-dependence of bispectrum and trispectrum. The two point function of the scalar field perturbations $\delta\phi_{\mathbf{k}}^a(t_k)$ at horizon crossing is given by

$$\langle \delta\phi_{\mathbf{k}_1}^a(t_k) \delta\phi_{\mathbf{k}_2}^b(t_k) \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) \frac{2\pi^2}{k_1^3} \left(\frac{H(t_k(k_1))}{2\pi} \right)^2 [\delta_{ab} + 2c(1 - \delta_{ab})\epsilon_{ab}] , \quad (2.17)$$

where $c = 2 - \ln 2 - \gamma \simeq 0.73$ with γ being the Euler-Mascheroni constant. Both the diagonal $a = b$ and off-diagonal $a \neq b$ components are given to leading order in slow roll. Note that although the off-diagonal components are slow roll suppressed compared to the diagonal components, their scale dependence has no further suppression and therefore gives a contribution comparable to the scale-dependence of the diagonal components. Therefore, we need to retain the off-diagonal contributions in our analysis.

Using this together with Eqs. (2.10) and (2.16), we can express the power spectrum of ζ , defined by $\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle \equiv (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) P(k_1)$, in the form

$$P(k) = \frac{2\pi^2}{k^3} \mathcal{P}(k) = \frac{2\pi^2}{k^3} \sum_{ab} \mathcal{P}_{ab}(k) . \quad (2.18)$$

Here

$$\mathcal{P}_{ab}(k) \equiv \left(\frac{H(t_i)}{2\pi} \right)^2 N_a N_b \left[\delta_{ab} \left(1 - 2\epsilon_H \ln \frac{k}{k_p} \right) + 2\epsilon_{ab} \left(\tilde{c} - \ln \frac{k}{k_p} \right) \right] , \quad (2.19)$$

and we have defined $\epsilon_H = -\dot{H}/H^2$ and $\tilde{c} = c + \ln(a_i H_i / k_p)$. Subleading slow roll corrections are again neglected in the scale-independent terms.

Defining a quantity

$$n_{ab} - 1 \equiv \frac{d \ln \mathcal{P}_{ab}}{d \ln k} = -\delta_{ab}(2\epsilon_H + 2\epsilon_{ab}) - \frac{1}{\tilde{c}}(1 - \delta_{ab}) , \quad (2.20)$$

we can write the spectral index as

$$n_\zeta - 1 \equiv \frac{d \ln \mathcal{P}}{d \ln k} = \sum_{ab} \left(\frac{\mathcal{P}_{ab}}{\mathcal{P}} \right) n_{ab} - 1 = -2\epsilon_H - 2 \frac{\sum_{ab} \epsilon_{ab} N_a N_b}{\sum_c N_c^2} . \quad (2.21)$$

This agrees with the result given in [22].

2.2 Three point function and f_{NL}

We now proceed to apply our approach to derive the scale dependence of non-linearity parameters. Using previous definitions we can write f_{NL} in a general multiple field case as

$$\begin{aligned} f_{\text{NL}}(k_1, k_2, k_3) &\equiv \frac{5}{6} \frac{B(k_1, k_2, k_3)}{P(k_1)P(k_2) + 2 \text{ perms}} \\ &= \frac{5}{3} \frac{\sum_{abcd} (k_1 k_2)^{-3} \mathcal{P}_{ac}(k_1) \mathcal{P}_{bd}(k_2) f_{cd}(k_3) + 2 \text{ perms}}{(k_1 k_2)^{-3} \mathcal{P}(k_1) \mathcal{P}(k_2) + 2 \text{ perms}}, \end{aligned} \quad (2.22)$$

where the bispectrum is defined by $(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B(k_1, k_2, k_3) = \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle$.

In the equilateral case, $k_i = k$ for $i = 1, 2$ and 3 , this simplifies to

$$f_{\text{NL}}(k) = \frac{5}{3} \frac{\sum_{abcd} \mathcal{P}_{ac}(k) \mathcal{P}_{bd}(k) f_{cd}(k)}{\mathcal{P}(k)^2}, \quad (2.23)$$

and using Eqs. (2.13) and (2.19), we find the scale dependence of $f_{\text{NL}}(k)$ is given by

$$n_{f_{\text{NL}}} \equiv \frac{d \ln |f_{\text{NL}}(k)|}{d \ln k} = \frac{1}{f_{\text{NL}}(k_p)} \sum_{ab} f_{\text{NL}}^{ab} (2n_{\text{multi},a} + n_{f,ab}). \quad (2.24)$$

Here we have defined [23]

$$f_{\text{NL}}^{ab} \equiv \frac{5}{6} \frac{N_a N_b N_{ab}}{(\sum_c N_c^2)^2}, \quad f_{\text{NL}}(k_p) = \sum_{ab} f_{\text{NL}}^{ab}, \quad (2.25)$$

and

$$n_{\text{multi},a} \equiv n_{aa} - n_\zeta - 2 \sum_c (1 - \delta_{ac}) \epsilon_{ac} \frac{N_c}{N_a} \quad (2.26)$$

$$= 2 \sum_{cd} \epsilon_{cd} \left(\frac{N_c N_d}{\sum_b N_b^2} - \delta_{ad} \frac{N_c}{N_a} \right). \quad (2.27)$$

All the quantities in Eq. (2.24) depend on combinations of slow roll parameters and on the constant coefficients N_a, N_{ab} in the δN expansion (recall that the explicit expression for $n_{f,ab}$ is given by Eq. (A.7)). For a given model these can all be regarded as known quantities and the scale-dependence of f_{NL} can therefore be directly read off from the above result without doing any further computations.

In Eq. (2.24) we can clearly identify two sources of scale dependence. The contribution proportional to $n_{f,ab}$ follows from the non-linear evolution of perturbations outside the horizon [14]. The part proportional to $n_{\text{multi},a}$ is associated with the scale dependence of factors of the form $\mathcal{P}_{ac}/\mathcal{P}$ in equation (2.23). It is present only in the multi-field case (indeed for a single field model this factor is equal to unity) and arises due to the presence of multiple unrelated Gaussian fields $\zeta^{\text{G},a}$ in the expansion of ζ in (2.16). This generates deviations from the local form and makes f_{NL} scale-dependent even if the perturbations would evolve linearly outside the horizon. Indeed, by setting $n_{f,ab} = 0$ we recover the results of a multi-local case analyzed separately in [14].

As shown in [14], the scale dependence of f_{NL} is given by the same result (2.24) for the class of variations where the sides are scaled by the same constant factor, $\mathbf{k}_i \rightarrow \alpha \mathbf{k}_i$. For such shape-preserving variations where only the overall scale of the triangle is varied, the result does not depend on the triangle shape. This holds at leading order in slow roll. Generic variations changing both the scale and the shape of the triangle are considered in Sec. 5.

2.3 Four point function, g_{NL} and τ_{NL}

The connected part of the four point correlator of ζ can be written in the form

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \zeta_{\mathbf{k}_4} \rangle = (2\pi)^3 \delta\left(\sum_{i=1}^4 \mathbf{k}_i\right) \left[\tau_{\text{NL}}(k_1, k_2, k_3, k_4, k_{13}) \left(P(k_1)P(k_2)P(|\mathbf{k}_1 + \mathbf{k}_3|) + 11 \text{ perm} \right) \right. \\ \left. + \frac{54}{25} g_{\text{NL}}(k_1, k_2, k_3, k_4) \left(P(k_1)P(k_2)P(k_3) + 3 \text{ perm} \right) \right], \quad (2.28)$$

where we have defined $k_{ij} \equiv |\mathbf{k}_i + \mathbf{k}_j|$. The functions τ_{NL} and g_{NL} are given by

$$\tau_{\text{NL}}(k_1, k_2, k_3, k_4, k_{13}) = 4 \frac{(k_1 k_2 k_{13})^{-3} \sum_{abcdef} \mathcal{P}_{ac}(k_1) \mathcal{P}_{be}(k_2) \mathcal{P}_{df}(k_{13}) f_{cd}(k_3) f_{ef}(k_4) + 11 \text{ perm}}{(k_1 k_2 k_{13})^{-3} \mathcal{P}(k_1) \mathcal{P}(k_2) \mathcal{P}(k_{13}) + 11 \text{ perm}}, \quad (2.29)$$

$$g_{\text{NL}}(k_1, k_2, k_3, k_4) = \frac{25 (k_1 k_2 k_3)^{-3} \sum_{abcdef} \mathcal{P}_{ad}(k_1) \mathcal{P}_{be}(k_2) \mathcal{P}_{cf}(k_3) g_{def}(k_4) + 3 \text{ perms}}{9 (k_1 k_2 k_3)^{-3} \mathcal{P}(k_1) \mathcal{P}(k_2) \mathcal{P}(k_3) + 3 \text{ perm}}. \quad (2.30)$$

In the case of a square, $k = k_i$ (notice that τ_{NL} , but not g_{NL} , is sensitive to the angles between the vectors and different equilateral figures in general yield different results), the above expressions reduce to

$$\tau_{\text{NL}}(k) = 4 \sum_{abcdef} \frac{\mathcal{P}_{ac}(k) \mathcal{P}_{be}(k) \mathcal{P}_{df}(\sqrt{2}k) f_{cd}(k) f_{ef}(k)}{\mathcal{P}(k)^3}, \quad (2.31)$$

$$g_{\text{NL}}(k) = \frac{25}{9} \sum_{abcdef} \frac{\mathcal{P}_{ad}(k) \mathcal{P}_{be}(k) \mathcal{P}_{cf}(k) g_{def}(k)}{\mathcal{P}(k)^3}. \quad (2.32)$$

The scale-dependence can be computed similarly to the analysis of the bispectrum above. Using Eqs. (2.13), (2.14) and (2.19), we find

$$n_{\tau_{\text{NL}}} \equiv \frac{d \ln |\tau_{\text{NL}}(k)|}{d \ln k} = \frac{1}{\tau_{\text{NL}}(k_{\text{p}})} \sum_{abcd} \tau_{\text{NL}}^{abcd} [(2n_{\text{multi},a} - (n_{\zeta} - 1) - 2\epsilon_H) \delta_{bc} - 2\epsilon_{bc} + 2n_{f,ab} \delta_{bc}], \quad (2.33)$$

$$n_{g_{\text{NL}}} \equiv \frac{d \ln |g_{\text{NL}}(k)|}{d \ln k} = \frac{1}{g_{\text{NL}}(k_{\text{p}})} \sum_{abc} g_{\text{NL}}^{abc} (3n_{\text{multi},a} + n_{g,abc}), \quad (2.34)$$

where $\epsilon_H = -\dot{H}/H^2$, and [28, 11]

$$\tau_{\text{NL}}^{abcd} = \frac{N_a N_{ab} N_{cd} N_d}{(\sum_e N_e^2)^3}, \quad \tau_{\text{NL}}(k_{\text{p}}) = \sum_{abcd} \tau_{\text{NL}}^{abcd} \delta_{bc}, \quad (2.35)$$

$$g_{\text{NL}}^{abc} = \frac{25}{54} \frac{N_a N_b N_c N_{abc}}{(\sum_d N_d^2)^3}, \quad g_{\text{NL}}(k_{\text{p}}) = \sum_{abc} g_{\text{NL}}^{abc}. \quad (2.36)$$

The scale-dependencies are fully determined by the constant coefficients N_a, N_{ab}, N_{abc} in the δN expression and by combinations of slow-roll parameters, which enter the results through Eqs. (A.7) and (A.8). Although the expressions appear lengthy in their general form, considerable simplifications typically occur when considering specific models. We will discuss examples in Sections 3 and 4.

Similarly to $n_{f_{\text{NL}}}$, we can again distinguish two physically different contributions in the expressions for $n_{\tau_{\text{NL}}}$ and $n_{g_{\text{NL}}}$. The parts proportional to $n_{f,ab}$ and $n_{g,abc}$ in Eqs. (2.33) and (2.34) respectively arise from the non-linear evolution outside the horizon. The other contributions describe deviations

from the local form due to the presence of multiple fields, similarly to what we discussed in the previous section.

The results (2.33) and (2.34) hold not only for the special case of a square, but for any variations where all the sides are scaled by the same constant factor, $\mathbf{k}_i \rightarrow \alpha \mathbf{k}_i$. These variations preserve the shape of the momentum space figure and change only its overall scale. We will prove this result in Sec. 5 where we also discuss generic variations that simultaneously change both the scale and the shape.

Having presenting our formalism and the general results, we will discuss in the next two sections applications to specific cases.

3. General single field case

We start by discussing models where the primordial curvature perturbation effectively arises from a single scalar field, which does not need to be the inflaton and we call σ . In this case, the functions $f_{\sigma\sigma}(k)$ and $g_{\sigma\sigma\sigma}(k)$ appearing in the expansion of ζ , Eq. (2.16), are up to numerical factors equal to $f_{\text{NL}}(k)$ and $g_{\text{NL}}(k)$, evaluated for the equilateral configurations. This can be seen directly from Eqs. (2.23) and (2.32). We can therefore rewrite Eq. (2.16) as

$$\zeta_{\mathbf{k}} = \zeta_{\mathbf{k}}^{\text{G}} + \frac{3}{5} f_{\text{NL}}(k) (\zeta^{\text{G}} \star \zeta^{\text{G}})_{\mathbf{k}} + \frac{9}{25} g_{\text{NL}}(k) (\zeta^{\text{G}} \star \zeta^{\text{G}} \star \zeta^{\text{G}})_{\mathbf{k}} + \dots \quad (3.1)$$

As we will discuss in Sec. 4, this result applies for example to the curvaton scenario and modulated reheating in the limit where the inflaton perturbations are negligible. We therefore call all the models where the curvature perturbation can be expressed in the form (3.1) as general single field models.

According to Eqs. (2.13) and (2.14), the non-linearity parameters $f_{\text{NL}}(k)$ and $g_{\text{NL}}(k)$ are now given by

$$f_{\text{NL}}(k) = \frac{5}{6} \frac{N''}{N'^2} \left(1 + n_{f_{\text{NL}}} \ln \frac{k}{k_{\text{p}}} \right), \quad (3.2)$$

$$g_{\text{NL}}(k) = \frac{25}{54} \frac{N'''}{N'^3} \left(1 + n_{g_{\text{NL}}} \ln \frac{k}{k_{\text{p}}} \right), \quad (3.3)$$

where the primes denote derivatives with respect to σ and $n_{f_{\text{NL}}} = n_{f, \sigma\sigma}$, $n_{g_{\text{NL}}} = n_{g, \sigma\sigma\sigma}$. Using the explicit expressions (A.7) and (A.8) in the Appendix A, we obtain

$$n_{f_{\text{NL}}} = \frac{N'}{N''} \left[\sqrt{2\epsilon_\sigma} (4\epsilon_\sigma - 3\eta_{\sigma\sigma}) + \frac{V'''}{3H^2} \right], \quad (3.4)$$

$$n_{g_{\text{NL}}} = 3 \frac{N''^2}{N''' N'} n_{f_{\text{NL}}} - \frac{N'}{N'''} \left[24\epsilon_\sigma^2 - 24\epsilon_\sigma \eta_{\sigma\sigma} + 3\eta_{\sigma\sigma}^2 + \frac{4\sqrt{2\epsilon_\sigma} V'''}{3H^2} - \frac{V''''}{3H^2} \right]. \quad (3.5)$$

The same results can of course be directly obtained from Eqs. (2.24) and (2.34). If σ is an isocurvature field during inflation, $\epsilon_\sigma = 0$ in the above expressions.

For the general single field case Eq. (2.31) further yields

$$\tau_{\text{NL}}(k) = \left(\frac{6f_{\text{NL}}(k)}{5} \right)^2, \quad (3.6)$$

up to scale-independent slow roll corrections. Therefore, the scale-dependencies of τ_{NL} and f_{NL} are related by

$$n_{\tau_{\text{NL}}} = 2n_{f_{\text{NL}}}. \quad (3.7)$$

This simple consistency relation is characteristic for general single field models. In multiple field models, the relation (3.6) is in general violated and consequently the result (3.7) is no longer true.

4. Two field models of inflation

After considering single field models, in this section we discuss some scenarios in which more than one field can play an important role in the inflationary process. We focus on a class of models that contains the most important examples of inflationary set-ups characterized by large non-Gaussianity.

Many models of inflation that generate sizeable non-Gaussianity are characterized by the presence of a field σ , with significant non-Gaussian perturbations, that is isocurvature during inflation. The inflaton field ϕ also has its own perturbations, which for convenience can be considered as Gaussian. When the inflaton perturbations provide non-negligible contributions to the curvature fluctuation spectrum, the scenario is called a mixed scenario [39, 40, 41, 42]. In order to generate large non-Gaussianity by means of the field σ , it is required that $\dot{\sigma} \ll \dot{\phi}$, and hence $\epsilon_\sigma \ll \epsilon_\phi$ [38]. From this relation, it follows that the trajectory in field space while observable modes exit the horizon is nearly straight. Therefore it is a good approximation to treat the fields as uncorrelated [50]. We also make the common assumption that the potential is separable,

$$W(\sigma, \phi) = U(\phi) + V(\sigma) . \quad (4.1)$$

Hence, the only potentially non-negligible slow roll parameters in such a scenario are the following

$$\epsilon_H = \epsilon_\phi = -\frac{\dot{H}}{H^2} , \quad \eta_\phi = \frac{U''}{3H^2} , \quad \eta_\sigma = \frac{V''}{3H^2} , \quad \xi_\phi^2 = \frac{U'''U'}{9H^4} , \quad \xi_\sigma^2 = \frac{V'''U'}{9H^4} . \quad (4.2)$$

In this case, the curvature perturbation reads²

$$\zeta(\mathbf{k}) = \zeta_{\mathbf{k}}^{G,\phi} + \zeta_{\mathbf{k}}^{G,\sigma} + f_\sigma(k) (\zeta^{G,\sigma} \star \zeta^{G,\sigma})_{\mathbf{k}} + g_\sigma(k) (\zeta^{G,\sigma} \star \zeta^{G,\sigma} \star \zeta^{G,\sigma})_{\mathbf{k}} . \quad (4.3)$$

Although the assumed form of ζ is simplified, in practice the vast majority of models in the literature, characterized by large quasi-local non-Gaussianity, satisfy the above Ansatz to a good enough accuracy for observational purposes. For this reason we will limit our attention to models with curvature perturbation satisfying Eq. (4.3) in this section.

In the limit that f_σ and g_σ are independent of k , we recover the multivariate local model [14]. In the case that $\zeta^{G,\phi} = 0$ we have the general single field model we have analyzed in section 3, but here we assume this field was an isocurvature mode during horizon crossing. We will consider these two cases in more detail later in this section.

The power spectrum is given by

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\phi(k) + \mathcal{P}_\sigma(k) = \mathcal{P}_\phi(k)(1 - w_\sigma(k))^{-1} , \quad (4.4)$$

where we have introduced the ratio

$$w_\sigma(k) = \frac{\mathcal{P}_\sigma}{\mathcal{P}_\zeta} . \quad (4.5)$$

Note that neglecting all the slow-roll corrections, and hence also the scale dependence, $w_\sigma = N_\sigma^2 / (N_\phi^2 + N_\sigma^2)$. To lowest order in slow roll, the spectral index $n_\zeta - 1$ and tensor-to-scalar ratio r_T satisfy the

²We have used a simplified notation for this section compared to the rest of the paper. Since all cross terms such as $P_{\phi\sigma}$ are negligibly small in this scenario we use only a single index ϕ or σ where appropriate, e.g. for $\eta_\sigma \equiv \eta_{\sigma\sigma}$ and $g_\sigma \equiv g_{\sigma\sigma\sigma}$.

following relations [51]

$$\begin{aligned} n_\zeta - 1 &= (n_\sigma - 1)w_\sigma + (n_\phi - 1)(1 - w_\sigma) \\ &= -(6 - 4w_\sigma)\epsilon_{\text{H}} + 2(1 - w_\sigma)\eta_\phi + 2w_\sigma\eta_\sigma, \end{aligned} \quad (4.6)$$

$$\begin{aligned} r_T &\equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} = \frac{8}{N_\sigma^2 + N_\phi^2} \\ &= 8N_\phi^{-2}(1 - w_\sigma), \end{aligned} \quad (4.7)$$

where $\mathcal{P}_T = 8H_k^2/(4\pi^2)$ is the power spectrum of tensor perturbations and we have defined

$$n_\sigma - 1 = \frac{d \ln \mathcal{P}_\sigma}{d \ln k}, \quad n_\phi - 1 = \frac{d \ln \mathcal{P}_\phi}{d \ln k}, \quad n_\zeta - 1 = \frac{d \ln \mathcal{P}_\zeta}{d \ln k}. \quad (4.8)$$

The non-Gaussianity parameter f_{NL} in the equilateral limit, and the trispectrum non-linearity parameters in the case of a square configuration, are given by

$$f_{\text{NL}}(k) = \frac{5}{3}w_\sigma^2(k)f_\sigma(k), \quad (4.9)$$

$$\tau_{\text{NL}}(k) = 4w_\sigma(k)^2w_\sigma(\sqrt{2}k)f_\sigma^2(k), \quad (4.10)$$

$$g_{\text{NL}}(k) = \frac{25}{9}w_\sigma^3(k)g_\sigma(k). \quad (4.11)$$

Therefore their scale dependence is given by

$$n_{f_{\text{NL}}} \equiv \frac{d \ln |f_{\text{NL}}|}{d \ln k} = 2(n_\sigma - n_\zeta) + \frac{d \ln |f_\sigma|}{d \ln k} \quad (4.12)$$

$$= 4(1 - w_\sigma)(2\epsilon_{\text{H}} + \eta_\sigma - \eta_\phi) + \frac{N_\sigma}{N_{\sigma\sigma}} \left(\frac{V'''}{3H^2} - \sqrt{2\epsilon_{\text{H}}} \eta_\sigma \left(\frac{1}{\omega_\sigma} - 1 \right)^{1/2} \right), \quad (4.13)$$

$$n_{\tau_{\text{NL}}} = 3(n_\sigma - n_\zeta) + 2 \frac{d \ln |f_\sigma|}{d \ln k} \quad (4.14)$$

$$= 6(1 - w_\sigma)(2\epsilon_{\text{H}} + \eta_\sigma - \eta_\phi) + \frac{2N_\sigma}{N_{\sigma\sigma}} \left(\frac{V'''}{3H^2} - \sqrt{2\epsilon_{\text{H}}} \eta_\sigma \left(\frac{1}{\omega_\sigma} - 1 \right)^{1/2} \right), \quad (4.15)$$

$$n_{g_{\text{NL}}} = 3(n_\sigma - n_\zeta) + \frac{d \ln |g_\sigma|}{d \ln k} \quad (4.16)$$

$$\begin{aligned} &= 6(1 - w_\sigma)(2\epsilon_{\text{H}} + \eta_\sigma - \eta_\phi) + \frac{3N_{\sigma\sigma}}{N_{\sigma\sigma\sigma}} \frac{V'''}{3H^2} \\ &\quad + \frac{N_\sigma}{N_{\sigma\sigma\sigma}} \left(\frac{V''''}{3H^2} - 3\eta_\sigma^2 + \sqrt{2\epsilon_{\text{H}}} \frac{V'''}{3H^2} \left(\frac{1}{\omega_\sigma} - 1 \right)^{1/2} \right), \end{aligned} \quad (4.17)$$

where we have used the results derived in Sec. 2 and the fact that $N_{\phi\phi}$, $N_{\phi\sigma}$ and their derivatives are negligible in the class of models we are considering, see Eq. (4.3). The quantities on the right hand side of each equation should be evaluated at an initial time t_i shortly after the horizon crossing time of all the modes of observational interest. Observe that our results for $n_{f_{\text{NL}}}$ and $n_{g_{\text{NL}}}$ depend on the derivatives of the potential, in combinations that do not correspond to traditional slow-roll parameters. This turns out to be useful to probe these quantities, that cannot be tested by the power spectrum and its derivatives. We are going to discuss this in detail in what follows.

Observational constraints on the bispectrum are given in [1] while constraints on g_{NL} are given in [52, 53] (see also [54]) and for both g_{NL} and τ_{NL} in [55]. Forecasts for future constraints on all three

parameters are given in [56, 57] while forecast constraints on $n_{f_{\text{NL}}}$ are given in [16]. There are currently no forecasts for how well the scale dependence of the trispectrum parameters could be constrained or measured. Observational constraints on a model with the form (4.3), without considering the scale-dependence of f_{NL} or w_σ , are given in [58].

4.1 Limiting cases

After presenting the general formulae for the two-field case, we discuss important examples of general single field inflation, that arise as limiting cases of the previous discussion of two-field inflation.

4.1.1 Isocurvature single field

In the case that a single field σ , which is subdominant to the inflaton during inflation, generates the primordial curvature perturbation, one has $w_\sigma = 1$ which implies $n_\sigma = n$, $r_T \simeq 0$ and $N_\sigma \gg 1$.

In this scenario, it is useful to express the spectral index and its running up to second order, to understand which parameters are currently constrained by observations. From [50], we have

$$n_\zeta - 1 = -2\epsilon_{\text{H}} + 2\eta_\sigma + \left(-\frac{22}{3} + 8c\right)\epsilon_{\text{H}}^2 + \frac{2}{3}\eta_\sigma^2 + \left(\frac{8}{3} - 4c\right)\epsilon_{\text{H}}\eta_\phi + \left(\frac{2}{3} - 4c\right)\epsilon_{\text{H}}\eta_\sigma, \quad (4.18)$$

$$\alpha_\zeta = -8\epsilon_{\text{H}}^2 + 4\epsilon_{\text{H}}\eta_\phi + 4\epsilon_{\text{H}}\eta_\sigma, \quad (4.19)$$

where $c = 2 - \ln 2 - \gamma \simeq 0.73$. Notice that, in the previous formulae, the slow-roll parameters ξ_σ and ξ_ϕ do not appear in the running of the spectral index, because they are weighted by negligible quantities. This implies that third and higher derivatives of the potential do not enter in the previous quantities.

The non-Gaussianity observables (which follow as special cases of the formulae discussed in the first part of this section, when taking the limit $w_\sigma \rightarrow 1$) are

$$f_{\text{NL}} = \frac{5}{3}f_\sigma = \frac{5}{6}\frac{N_{\sigma\sigma}}{N_\sigma^2}, \quad g_{\text{NL}} = \frac{25}{9}g_\sigma = \frac{25}{54}\frac{N_{\sigma\sigma\sigma}}{N_\sigma^3}, \quad (4.20)$$

$$n_{f_{\text{NL}}} = \frac{n_{\tau_{\text{NL}}}}{2} \simeq \frac{N_\sigma}{N_{\sigma\sigma}}\frac{V''''}{3H^2} \simeq \frac{5}{6}\frac{\text{sgn}(N_\sigma)}{f_{\text{NL}}}\sqrt{\frac{r_{\text{T}}}{8}}\frac{V''''}{3H^2}, \quad (4.21)$$

$$n_{g_{\text{NL}}} \simeq 3\frac{N_{\sigma\sigma}}{N_{\sigma\sigma\sigma}}\frac{V''''}{3H^2} + \frac{N_\sigma}{N_{\sigma\sigma\sigma}}\left(\frac{V'''''}{3H^2} - 3\eta_\sigma^2\right) \quad (4.22)$$

$$\simeq \frac{5}{3}\frac{\text{sgn}(N_\sigma)}{g_{\text{NL}}}\frac{f_{\text{NL}}}{f_{\text{NL}}}\sqrt{\frac{r_{\text{T}}}{8}}\frac{V''''}{3H^2} + \frac{25}{54}\frac{1}{g_{\text{NL}}}\frac{r_{\text{T}}}{8}\frac{V'''''}{3H^2} \simeq 2\frac{f_{\text{NL}}^2}{g_{\text{NL}}}n_{f_{\text{NL}}} + \frac{25}{54}\frac{1}{g_{\text{NL}}}\frac{\mathcal{P}_\zeta^{-1}}{6\pi^2}V'''''. \quad (4.23)$$

Here f_{NL} and g_{NL} denote the non-linearity parameters evaluated for equilateral configurations at some pivot scale, $k = k_{\text{p}}$. As discussed in Sec. 2, k_{p} can be chosen as any of the super-horizon modes in our observable universe and the results are independent on this choice, up to subleading slow-roll corrections.

In the previous formulae, we have presented several different ways of expressing $n_{f_{\text{NL}}}$ and $n_{g_{\text{NL}}}$ (in Eq. (4.23) we have dropped the negligible contribution $\eta_\sigma^2 r_{\text{T}} \lesssim 10^{-6}$). This is in order to make it easier to estimate their magnitude in different ways, depending on the available quantities. We also note that in some cases the previous formulae might include terms at different orders in slow roll, in which case one should neglect the subleading terms (since additional terms at the same order might also have been neglected). In general, they are suppressed by some combination of the tensor-to-scalar ratio, divided by non-linearity parameters. But their size could be significant, if σ has either a large cubic or quartic self interaction. As we mentioned earlier, the power spectrum does not contain information on

these parameters, even if the running of the spectral index can be measured. Hence $n_{f_{\text{NL}}}$ appears to be the best way of probing the cubic self interaction, while in principle $n_{g_{\text{NL}}}$ could probe the quartic derivative of the isocurvature field.

Although we have written $n_{f_{\text{NL}}} \sim 1/f_{\text{NL}}$, the prefactor to $1/f_{\text{NL}}$ will in general depend on some of the same model parameters as f_{NL} so one should not view the two parameters as being inversely proportional (an explicit example is given in [12]). In the case that f_{NL} follows an exact power law behavior, $f_{\text{NL}} \propto k^{n_{f_{\text{NL}}}}$ then f_{NL} and $n_{f_{\text{NL}}}$ are of course independent. However if $f_{\text{NL}} = A \ln(k) + B$ where A and B are constants, then $n_{f_{\text{NL}}} = A/(A \ln(k) + B) = A/f_{\text{NL}}$. In this case $n_{f_{\text{NL}}}$ and f_{NL} are not independent. Nonetheless one can easily check that the running of $n_{f_{\text{NL}}}$ satisfies $\alpha_{f_{\text{NL}}} = -n_{f_{\text{NL}}}^2$ so it is a good approximation to treat $n_{f_{\text{NL}}}$ as constant provided that $|n_{f_{\text{NL}}}| \ll 1$.

Consider, as a first example, the curvaton scenario [35] in the pure curvaton limit. In this case all of the non-Gaussianity parameters will have some scale dependence unless the curvaton has exactly a quadratic potential, in which case it can be treated as a free test field during inflation. This is manifest from eqs. (4.21) and (4.22)³. In [14] we computed $n_{f_{\text{NL}}}$ for curvaton models with a quartic self-interaction term, $V = m^2 \sigma^2/2 + \lambda \sigma^4$, finding a scale dependence proportional to η_σ , which tends to be too small to be of observable interest. The result might be different for other type of interactions and it would be interesting to compute $n_{f_{\text{NL}}}$ and $n_{g_{\text{NL}}}$ for generic interacting curvaton models. This, however, requires a numerical study and is beyond the scope of the current work [59]. Here we will instead consider the modulated reheating scenario as an example of isocurvature single field models. In this case there is little constraint on the form of the modulator potential and we can use results derived in the literature to compute the scale dependencies.

4.1.2 Modulated reheating

In this scenario, an isocurvature field σ during inflation modulates the decay rate of the inflaton field into radiation. Because the expansion rate of the universe changes after the decay, this process can convert the initial isocurvature perturbations of the modulator field into the primordial curvature perturbation [36, 37]. This is closely related to the model of modulated preheating [60] and modulated trapping [61] (see also [62]). This process leads to some level of non-Gaussianity, which depends on the efficiency of the transfer, on the functional form of the decay rate $\Gamma(\sigma)$ and on the potential of the modulator field $V(\sigma)$. The form of the inflaton potential during horizon crossing is unconstrained, assuming the inflaton perturbations can be neglected, but its shape around the minimum does influence reheating and we assume it has a quadratic potential while it is oscillating.

For simplicity we will consider the case that $\Gamma \ll H_e$, where H_e is the Hubble parameter measured at the end of inflation t_e . Hence we are assuming that the inflaton decays long after the end of inflation. In this case, the curvature perturbation in real space can be written as [37, 42]

$$\zeta(t_f, \mathbf{x}) \simeq -\frac{1}{6} \frac{\Gamma_{\sigma_i}}{\Gamma} \delta\sigma(t_i, \mathbf{x}) + \frac{1}{2} \left(-\frac{1}{6} \frac{\Gamma_{\sigma_i}}{\Gamma} \right)_{\sigma_i} \times \delta\sigma(t_i, \mathbf{x})^2 + \frac{1}{6} \left(-\frac{1}{6} \frac{\Gamma_{\sigma_i}}{\Gamma} \right)_{\sigma_i \sigma_i} \times \delta\sigma(t_i, \mathbf{x})^3 + \dots, \quad (4.24)$$

where t_i is a time soon after the horizon crossing of modes of interest. Using Eq. (4.20) we find the

³For a quadratic model $N_{\sigma\sigma\sigma} = 0$ (when working to first order in $r = \rho_\sigma/(3H^2)$, i.e. considering the curvaton as a test field) and hence $g_{\text{NL}} = 0$. As explained in Sec. (2), in this case we define $n_{g_{\text{NL}}} = 0$ instead of using the formally divergent result (4.22).

constant parts of f_{NL} and g_{NL} are given by

$$f_{\text{NL}} = 5 \left(1 - \frac{\Gamma \Gamma_{\sigma_i \sigma_i}}{\Gamma_{\sigma_i}^2} \right), \quad (4.25)$$

$$g_{\text{NL}} = \frac{50}{3} \left(2 - 3 \frac{\Gamma \Gamma_{\sigma_i \sigma_i}}{\Gamma_{\sigma_i}^2} + \frac{\Gamma^2 \Gamma_{\sigma_i \sigma_i \sigma_i}}{\Gamma_{\sigma_i}^3} \right). \quad (4.26)$$

The scale dependencies of f_{NL} and g_{NL} can be computed using Eqs. (4.21) and (4.22). From these equations it is obvious that a potentially large scale dependence, accompanied with large values for f_{NL} and g_{NL} , can arise only if the modulator field σ has large self interactions. For the rest of this section we will consider the case of a quartic potential

$$V(\sigma) = \frac{\lambda}{4!} \sigma^4, \quad \lambda > 0. \quad (4.27)$$

In keeping with the previous literature, we neglect the energy density of the σ field after the end of reheating, as studying this goes beyond the realms of this project.

Using Eqs. (4.21) and (4.22) we find

$$n_{f_{\text{NL}}} \simeq -\frac{5}{f_{\text{NL}}} \frac{\Gamma}{\Gamma_{\sigma_i}} \frac{\lambda \sigma_i}{3H_i^2} \sim 0.1 \frac{\sqrt{\lambda} \eta_\sigma}{f_{\text{NL}} \mathcal{P}_\zeta^{1/2}}, \quad (4.28)$$

$$n_{g_{\text{NL}}} \simeq 2 \frac{f_{\text{NL}}^2}{g_{\text{NL}}} n_{f_{\text{NL}}} + \frac{50}{3g_{\text{NL}}} \frac{\Gamma^2}{\Gamma_{\sigma_i}^2} \frac{\lambda}{3H_i^2} \sim 2 \frac{f_{\text{NL}}^2}{g_{\text{NL}}} n_{f_{\text{NL}}} + 4 \times 10^{-3} \frac{\lambda}{g_{\text{NL}} \mathcal{P}_\zeta}, \quad (4.29)$$

where $\eta_\sigma = \lambda \sigma_i^2 / (6H_i^2)$. In the expression for $n_{g_{\text{NL}}}$ we have neglected the contribution proportional to η_σ^2 in Eq. (4.22) which is negligible compared to $\lambda / (3H_i^2)$ because $\lambda \sigma_i^4 \ll H_i^2$ by construction.

The quantities $n_{f_{\text{NL}}}$ and $n_{g_{\text{NL}}}$ could be large, by making a suitable choice of the parameters. At first sight, it seems easy to obtain values for these parameters of order 10^{-1} , large enough to be detectable, and at the same time compatible with the assumptions that underlie our analysis of Section 2. This is correct, but we have to ensure that the parameters satisfy stringent constraints in order to obtain acceptable values for the tilt of the spectral index. Indeed, assuming inflation lasted considerably longer than 60 efoldings, a natural initial value for the field σ is [63]

$$\sigma_i \sim \left(\frac{3}{\pi^2} \right)^{\frac{1}{4}} \frac{H_i}{\lambda^{1/4}}, \quad (4.30)$$

(a different argument changes the power of λ from $1/4$ to $1/3$ and the numerical factors [64], but the difference is not very important here). Plugging the previous estimate in the expression for $\eta_\sigma = \lambda \sigma_i^2 / (6H_i^2)$, and requiring that this parameter is less than 10^{-2} , we find the following bound for the coupling λ :

$$\eta_\sigma \simeq \left(\frac{\lambda}{12\pi^2} \right)^{1/2} \lesssim 10^{-2} \Rightarrow \lambda \lesssim 10^{-2}. \quad (4.31)$$

The condition $\Gamma \ll H_e$ can place further bounds on η_σ since the modulator is assumed to remain nearly frozen until the inflaton decay. We will not further address this issue here.

Plugging the previous results in (4.28) and (4.29), and using $\mathcal{P}_\zeta = 2.5 \times 10^{-9}$ for the normalization of the power spectrum, we find

$$|n_{f_{\text{NL}}}| \sim \lambda^{3/4} \frac{600}{|f_{\text{NL}}|} \lesssim \frac{20}{|f_{\text{NL}}|}, \quad (4.32)$$

$$|n_{g_{\text{NL}}}| \sim \lambda \frac{2 \times 10^6}{|g_{\text{NL}}|} \lesssim \frac{2 \times 10^4}{|g_{\text{NL}}|}. \quad (4.33)$$

where the inequalities are saturated for $\lambda \sim 10^{-2}$. In the estimate for $n_{g_{\text{NL}}}$, we have neglected the first term in Eq. (4.29),

$$2 \frac{f_{\text{NL}}^2}{|g_{\text{NL}}|} |n_{f_{\text{NL}}}| \sim \lambda^{3/4} \frac{10^3 |f_{\text{NL}}|}{|g_{\text{NL}}|}, \quad (4.34)$$

which is subdominant compared to the second term if $\lambda^{1/4} \gtrsim 5 \times 10^{-4} |f_{\text{NL}}|$. For $|f_{\text{NL}}| \sim 100$, this corresponds to $\lambda \gtrsim 6 \times 10^{-6}$. In the opposite case, $\lambda^{1/4} \lesssim 5 \times 10^{-4} |f_{\text{NL}}|$, the estimate for $n_{g_{\text{NL}}}$ is given by Eq. (4.34) instead of Eq. (4.33).

We conclude that both $n_{f_{\text{NL}}}$ and $n_{g_{\text{NL}}}$ could acquire relatively large values, even if the values of f_{NL} and g_{NL} saturate their current observational bounds. It is however important to emphasize that $|n_{f_{\text{NL}}}|$ or $|n_{g_{\text{NL}}}| \gg 0.01$ is outside the regime of validity of our formulae, since the accuracy of the expansions performed in Sec. 2 starts to become inadequate.

4.2 Two-field local case

As a last example, we briefly discuss the so called two field local case, for which f_σ and g_σ are independent of k . This demonstrates an explicit violation of the relation (3.7). In this scenario, the formulae at the beginning of this section provide

$$\tau_{\text{NL}} = \left(\frac{6}{5} f_{\text{NL}} \right)^2 \frac{1}{w_\sigma}, \quad (4.35)$$

which shows that, in principle, the parameter w_σ is an observable. The scale dependences of the non-linearity parameters satisfy the following relation

$$n_{\tau_{\text{NL}}} = n_{g_{\text{NL}}} = \frac{3}{2} n_{f_{\text{NL}}}. \quad (4.36)$$

So, as previously stated, we have a different consistency relation between $n_{f_{\text{NL}}}$ and $n_{\tau_{\text{NL}}}$ in this case compared to the single field case, Eq. (3.7). Furthermore there is an additional consistency relation from $n_{g_{\text{NL}}}$. However two-field local models are likely to arise from a test field with a quadratic potential [14], in which case the amplitude of g_{NL} tends to be too small to be observable.

As an explicit example we consider the mixed inflaton-curvaton scenario, assuming the curvaton field has a quadratic potential. We discussed this model previously at the level of the bispectrum in [14], and found that in a natural limit $n_{f_{\text{NL}}} = -2(n_\zeta - 1)$. It therefore follows that for this model $n_{\tau_{\text{NL}}}$ is even larger,

$$n_{\tau_{\text{NL}}} = -3(n_\zeta - 1). \quad (4.37)$$

In this model $g_{\text{NL}} \sim f_{\text{NL}}$ [65] which is too small to be of observational interest [57].

5. Shape dependence

In the previous sections, we concentrated our analysis on the scale dependence of equilateral figures (triangles and quadrilatera). Moreover, we only considered the possibility of varying simultaneously all of the sides of the figure by the same proportion. In this section, we study more general situations in which scale dependence can arise in parameters characterizing local non-Gaussianity. In particular, we consider the case in which the figure under consideration is not equilateral, and the case in which we vary the size of only one side, keeping the lengths of the other sides fixed.

We start by studying these issues for the parameter f_{NL} , generalizing the arguments developed in Sec. 2.2, and using the same quantities introduced there. Expanding Eq. (2.22), around a pivot scale k_p using Eq. (2.13), we obtain

$$f_{\text{NL}}(k_1, k_2, k_3) = \sum_{ab} f_{\text{NL}}^{ab} \left(1 + \frac{k_3^3 \left(n_{\text{multi},a} \ln \frac{k_1 k_2}{k_p^2} + n_{f,ab} \ln \frac{k_3}{k_p} \right) + 2 \text{ perms} }{k_1^3 + k_2^3 + k_3^3} \right). \quad (5.1)$$

As in the previous sections, the result is given to first order in $\ln(k_i/k_p)$ and both the scale-dependent and scale-independent parts are given to leading order in slow roll.

In this approximation, setting for simplicity $k_p = 1$, equation (5.1) can be re-expressed in a more elegant way as

$$f_{\text{NL}}(k_1, k_2, k_3) = \sum_{ab} f_{\text{NL}}^{ab} \frac{(k_1 k_2)^{n_{\text{multi},a}} k_3^{3+n_{f,ab}} + 2 \text{ perms}}{k_1^3 + k_2^3 + k_3^3}. \quad (5.2)$$

Indeed, since both $n_{\text{multi},a}$ and $n_{f,ab}$, for each a, b , are proportional to slow-roll parameters, an expansion of Eq. (5.2) at first order in slow-roll provides Eq. (5.1). For general single field models, it reduces to

$$f_{\text{NL}}(k_1, k_2, k_3) = f_{\text{NL}}^{\text{p}} \frac{k_1^{3+n_{f_{\text{NL}}}} + k_2^{3+n_{f_{\text{NL}}}} + k_3^{3+n_{f_{\text{NL}}}}}{k_1^3 + k_2^3 + k_3^3}, \quad (5.3)$$

where f_{NL}^{p} denotes $5f_{\sigma\sigma}/3$ evaluated at the pivot scale and $n_{f_{\text{NL}}} = n_{f,\sigma\sigma}$. These simple ways of expressing the parameter f_{NL} are particularly suitable to analyze how the triangle shape affects the scale dependence. The single field expression (5.3) is equivalent to the analogous result given in Section 3.3 of [14], as one can easily check using Appendix B. Eq. (5.3) however takes a much simpler form than the result in [14] as a consequence of cancellations that occur when explicitly writing out the results in terms of slow roll parameters.

We note that, although (5.3) is not of the form $f_{\text{NL}} \propto (k_1 k_2 k_3)^{n_{f_{\text{NL}}}/3}$ which [16] used in order to make observational forecasts for $n_{f_{\text{NL}}}$, the bispectrum is a sum of three simple, product separable terms

$$B_{\zeta}(k_1, k_2, k_3) \propto (k_1 k_2)^{n_{\zeta}-4} k_3^{n_{f_{\text{NL}}}} + 2 \text{ perms}, \quad (5.4)$$

and that it only depends on one new parameter $n_{f_{\text{NL}}}$. In the multiple field case the bispectrum will typically depend on more parameters than just $n_{f_{\text{NL}}}$, see (5.2). An exception is the two-field local model discussed in Sec. 4.2, in which case (we also use Eq. (4.12))

$$B_{\zeta}(k_1, k_2, k_3) \propto (k_1 k_2)^{n_{\zeta}+(n_{f_{\text{NL}}}/2)-4} + 2 \text{ perms}. \quad (5.5)$$

Notice that it therefore follows from (5.4) and (5.5) that models with the same f_{NL} and $n_{f_{\text{NL}}}$ can have different bispectral shapes which generalise in different ways the local shape. It is possible that observations may distinguish between these shapes and that we could therefore learn whether $n_{f_{\text{NL}}}$ arises due to single or multi-field effects (or a combination of the two)⁴.

In Sec. 2, we limited our considerations to the scale dependence of f_{NL} for equilateral triangles. On the other hand, by means of Eq. (5.2) one can observe that, considering a common rescaling for all the three vectors, say $\mathbf{k}_i \rightarrow \alpha \mathbf{k}_i$, our previous results remains valid *regardless* of the triangle shape. Namely,

$$\left. \frac{\partial \ln f_{\text{NL}}(\alpha k_1, \alpha k_2, \alpha k_3)}{\partial \ln \alpha} \right|_{\alpha=1} = \sum_{ab} f_{\text{NL}}^{ab} (2n_{\text{multi},a} + n_{f,ab}). \quad (5.6)$$

⁴CB thanks Sarah Shandera for pointing this out.

which is exactly our previous result.

While the scale dependence of f_{NL} , when simultaneously varying the triangle sides, does not depend on the triangle shape, there are other situations in which it does. We might indeed be interested on the scale dependence of f_{NL} , when varying the size of only one of the triangle sides, keeping the other two fixed (and the triangle closed). In this case, the result does depend on the triangle shape. We focus for simplicity on the single-field case, for which the analysis is particularly simple, we do not expect our results to change much when considering multiple fields. When varying $\mathbf{k}_1 \rightarrow \alpha \mathbf{k}_1$, equation (5.2) becomes

$$f_{\text{NL}}(\alpha k_1, k_2, k_3) = f_{\text{NL}}^{\text{P}} \frac{\alpha^{3+n_{f_{\text{NL}}}} k_1^{3+n_{f_{\text{NL}}}} + k_2^{3+n_{f_{\text{NL}}}} + k_3^{3+n_{f_{\text{NL}}}}}{\alpha^3 k_1^3 + k_2^3 + k_3^3} . \quad (5.7)$$

It is clear that the dependence on α , in this expression, goes to zero in the limit in which k_1 vanishes. This is because the coefficients of the terms depending on α , in equation (5.2), become very suppressed with respect to the remaining terms. This situation corresponds to a squeezed triangle: for this shape, we then learn that the value of f_{NL} does not change when varying the length of the triangles shortest side.

In order to determine the triangle shape that leads to maximal scale dependence, one is then lead to focus on the opposite limit. That is, on configurations for which k_1^3 is as large as possible, with respect to $k_2^3 + k_3^3$. In this case, indeed, the coefficients of the terms depending on α , in equation (5.2), become dominant with respect to the other terms.

This expectation is correct, as shown by the following more detailed analysis. Taking the logarithmic derivative of f_{NL} along α , we find, at leading order in slow-roll:

$$\left. \frac{\partial \ln f_{\text{NL}}(\alpha k_1, k_2, k_3)}{\partial \ln \alpha} \right|_{\alpha=1} = \frac{n_{f_{\text{NL}}}}{1+x^3+y^3} \left(1 - \frac{3x^3 \ln x + 3y^3 \ln y}{1+x^3+y^3} \right) , \quad (5.8)$$

where we have defined $x = k_2/k_1$, $y = k_3/k_1$. We have checked that the terms inside the parenthesis are not important for determining the location of the maxima of the previous expression. The maxima of Eq. (5.8) are therefore determined by the prefactor $(1+x^3+y^3)^{-1}$, which is maximized for triangles that minimize the combination x^3+y^3 . This corresponds, as anticipated from our previous expectation, to the shape for which k_1^3 is as large as possible, with respect to the combination $k_2^3 + k_3^3$. Calling θ the angle between k_1 and k_2 , we have $y^2 = (1-x)^2 + 2x(1-\cos\theta)$. So we can write

$$x^3 + y^3 = x^3 + \left[(1-x)^2 + 2x(1-\cos\theta) \right]^{\frac{3}{2}} . \quad (5.9)$$

It is easy to see that the previous expression is minimized for $\theta = 0$ and $x = 1/2$, that is for a folded triangle for which $k_2 = k_3 = k_1/2$. Plugging these values in (5.8), we find

$$\left. \frac{\partial \ln f_{\text{NL}}}{\partial \ln \alpha} \right|_{\alpha=1} \simeq 1.1 n_{f_{\text{NL}}} , \quad (5.10)$$

so we learn that, for the shape that maximizes the scale dependence, we gain around ten per cent with respect to the case in which we vary simultaneously all the sides of the triangle. Plots in Fig. 5 represent the logarithmic derivative of f_{NL} along α , and graphically show the results discussed so far. Notice that the shape which maximizes the scale dependence is indeed given by folded triangles.

A similar procedure, that generalizes what we have done in Sec. 2.3, can be applied to analyze g_{NL} and τ_{NL} . Expanding Eqs. (2.29) and (2.30) around a pivot scale k_{p} using Eqs. (2.13), (2.14) and

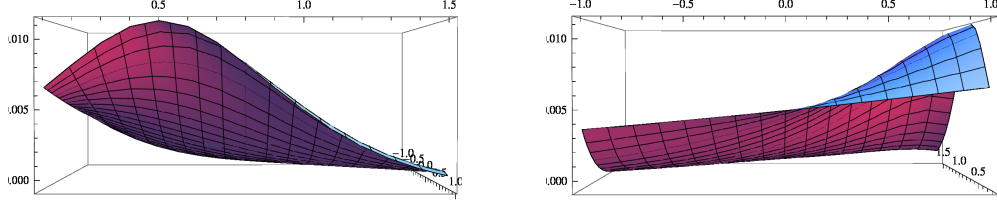


Figure 1: Behavior of the quantity $\partial \ln f_{\text{NL}} / \partial \ln \alpha$, as a simultaneous function of x (taken between 0 and 1.5) and of $\cos \theta$. The two plots represent the same figure from two different points of view, that emphasize respectively the dependence on x and on $\cos \theta$. We have chosen $n_{f_{\text{NL}}} = 0.01$.

(2.19), we obtain the results

$$\begin{aligned}
 g_{\text{NL}}(k_1, k_2, k_3, k_4) &= \sum_{abc} g_{\text{NL}}^{abc} \left(1 + \frac{k_4^3 \left(n_{\text{multi},a} \ln \frac{k_1 k_2 k_3}{k_{\text{p}}^3} + n_{g,abc} \ln \frac{k_4}{k_{\text{p}}} \right) + 4 \text{ perms}}{k_1^3 + k_2^3 + k_3^3 + k_4^3} \right), \quad (5.11) \\
 \tau_{\text{NL}}(k_1, k_2, k_3, k_4, k_{13}) &= \sum_{abcd} \tau_{\text{NL}}^{abcd} \left(\delta_{bc} + \left(\frac{\delta_{bc} \left(n_{\text{multi},a} \ln \frac{k_1 k_2}{k_{\text{p}}^2} + (n_{bb} - n_{\zeta}) \ln \frac{k_{13}}{k_{\text{p}}} \right)}{k_1^3 k_2^3 k_{13}^3} \right. \right. \\
 &\quad \left. \left. + \frac{\delta_{bc} \left(n_{f,ab} \ln \frac{k_3 k_4}{k_{\text{p}}^2} \right) - (1 - \delta_{bc}) (4\sqrt{\epsilon_b \epsilon_c} - 2\eta_{bc}) \ln \frac{k_{13}}{k_{\text{p}}}}{k_1^3 k_2^3 k_{13}^3} + 11 \text{ perms} \right) \right. \\
 &\quad \left. \times \left(\frac{1}{k_1^3 k_2^3 k_{13}^3} + 11 \text{ perms} \right)^{-1} \right). \quad (5.12)
 \end{aligned}$$

We can then proceed with arguments very similar to the ones developed for f_{NL} . Writing $\mathbf{k}_i \rightarrow \alpha \mathbf{k}_i$ in Eqs. (5.11) and (5.12), taking a logarithmic derivative with respect to α and finally setting $\alpha = 1$, we immediately recover the results (2.33) and (2.34), derived in Sec. 2 for $n_{\tau_{\text{NL}}}$ and $n_{g_{\text{NL}}}$ for equilateral configurations. This shows that for the class of shape preserving variations, $\mathbf{k}_i \rightarrow \alpha \mathbf{k}_i$, the results are independent of the figure shape.

In the single field case, if we vary only one of the sides, say the one labeled by k_1 , then the scale dependence vanishes when $k_1 \rightarrow 0$. We numerically analyzed for which shapes the scale dependence is maximal. For the case of g_{NL} , the analysis is a straightforward generalization of what we did for f_{NL} . The shape associated with maximal scale dependence corresponds to a folded polygon, in which three of the sides lie over the side whose length is varied. That is,

$$k_1 = 3k_2 = 3k_3 = 3k_4. \quad (5.13)$$

Again, for maximal scale dependence we gain order ten per cent with respect to the case in which we vary all the sides simultaneously.

We also performed a numerical analysis to study τ_{NL} , finding again maximal scale dependence for the folded shape of Eq. (5.13). For this parameter, we gain around 20 – 25 percent with respect to

the case in which we vary all the sides by the same amount (this result resonates with the consistency relation (3.7)).

6. Curvature perturbation in coordinate space

An additional important feature of our approach to the scale dependence of local non-Gaussianity, is that it allows one to express the results in coordinate space. In this section, we show how the scale dependent coefficients appearing in the momentum space expansion of curvature perturbation (2.16) manifest themselves in coordinate space. It is clear that scale dependence will cause deviations from the local form, for which $\zeta(\mathbf{x})$ can be expressed as a power series of a Gaussian variable $\zeta^{\text{G}}(\mathbf{x})$ with constant coefficients. Using the general single field case as an example, we work out the expression for $\zeta(\mathbf{x})$ in the scale-dependent case and quantify how it deviates from the local form.

In the general single field case, Eq. (2.9) can be written as

$$\begin{aligned} \zeta_{\mathbf{k}} = & \zeta_{\mathbf{k}}^{\text{G}} + \frac{3}{5}f_{\text{NL}}(k_{\text{p}}) \left(1 + n_{f_{\text{NL}}} \theta(k_i - k) \ln \frac{k}{k_i} \right) (\zeta^{\text{G}} \star \zeta^{\text{G}})_{\mathbf{k}} \\ & + \frac{9}{25}g_{\text{NL}}(k_{\text{p}}) \left(1 + n_{g_{\text{NL}}} \theta(k_i - k) \ln \frac{k}{k_i} \right) (\zeta^{\text{G}} \star \zeta^{\text{G}} \star \zeta^{\text{G}})_{\mathbf{k}} + \dots, \end{aligned} \quad (6.1)$$

which coincides with Eq. (3.1) up to slow roll corrections for constant terms. This form is useful for our analysis since the horizon scale $k_i = a_i H_i$ appears explicitly. We have inserted the theta functions $\theta(k_i - k)$ to explicitly indicate that the result holds only for super-horizon modes $k < k_i$. $k_i > k_p$ should correspond to a physically smaller scale than any of the modes of interest. Recall that similar theta functions are included in our definition of $\zeta_{\mathbf{k}}^{\text{G}}$, Eq. (2.10). Therefore $\zeta_{\mathbf{k}}^{\text{G}}$ can be viewed as a smoothed quantity; in Fourier space the window function is simply a top hat with the cutoff set at the horizon scale k_i .

Taking the inverse Fourier transform of (6.1) we find

$$\begin{aligned} \zeta(\mathbf{x}) = & \zeta^{\text{G}}(\mathbf{x}) + \frac{3}{5}f_{\text{NL}}(k_{\text{p}})\zeta^{\text{G}}(\mathbf{x})^2 + \frac{9}{25}g_{\text{NL}}(k_{\text{p}})\zeta^{\text{G}}(\mathbf{x})^3 \\ & + \frac{3}{5}f_{\text{NL}}(k_{\text{p}})n_{f_{\text{NL}}} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \theta(k_i - k) (\zeta^{\text{G}} \star \zeta^{\text{G}})_{\mathbf{k}} \ln \frac{k}{k_i} \\ & + \frac{9}{25}g_{\text{NL}}(k_{\text{p}})n_{g_{\text{NL}}} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \theta(k_i - k) (\zeta^{\text{G}} \star \zeta^{\text{G}} \star \zeta^{\text{G}})_{\mathbf{k}} \ln \frac{k}{k_i} + \dots. \end{aligned} \quad (6.2)$$

The two integrals describe deviations from the local form. They can be written more explicitly by performing the following manipulations

$$\begin{aligned} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \theta(k_i - k) (\zeta^{\text{G}} \star \zeta^{\text{G}})_{\mathbf{k}} \ln \frac{k}{k_i} &= \int d\mathbf{y} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \theta(k_i - k) \zeta^{\text{G}}(\mathbf{y})^2 \ln \frac{k}{k_i} \\ &= \int d\mathbf{y} \zeta^{\text{G}}(\mathbf{y})^2 \frac{1}{2\pi^2} \frac{\sin(k_i|\mathbf{x}-\mathbf{y}|) - \text{Si}(k_i|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|^3} \\ &\equiv \int d\mathbf{y} \zeta^{\text{G}}(\mathbf{y})^2 I(|\mathbf{x}-\mathbf{y}|), \end{aligned} \quad (6.3)$$

and similarly for the second integral. Using this we can rewrite Eq. (6.2) as

$$\begin{aligned} \zeta(\mathbf{x}) = & \zeta^{\text{G}}(\mathbf{x}) + \frac{3}{5}f_{\text{NL}}(k_{\text{p}}) \left(\zeta^{\text{G}}(\mathbf{x})^2 + n_{f_{\text{NL}}} \int d\mathbf{y} I(|\mathbf{x}-\mathbf{y}|) \zeta^{\text{G}}(\mathbf{y})^2 \right) \\ & + \frac{9}{25}g_{\text{NL}}(k_{\text{p}}) \left(\zeta^{\text{G}}(\mathbf{x})^3 + n_{g_{\text{NL}}} \int d\mathbf{y} I(|\mathbf{x}-\mathbf{y}|) \zeta^{\text{G}}(\mathbf{y})^3 \right) + \dots. \end{aligned} \quad (6.4)$$

This result clearly shows how the scale dependence of $f_{\sigma\sigma}$ and $g_{\sigma\sigma\sigma}$ in Eq. (2.16) renders $\zeta(\mathbf{x})$ a nonlocal function of $\zeta^G(\mathbf{x})$. Because of the integrals in Eq. (6.4), the curvature perturbation $\zeta(\mathbf{x})$ can not be expressed in terms of $\zeta^G(\mathbf{x})$ evaluated at the same point \mathbf{x} but one needs to know $\zeta^G(\mathbf{y})$ in the entire region where $I(|\mathbf{x} - \mathbf{y}|)$ is non-vanishing. The behavior of $I(|\mathbf{x} - \mathbf{y}|)$ is depicted in Fig. 6 which also displays the inverse Fourier transform,

$$W(|\mathbf{x} - \mathbf{y}|) = \frac{\sin(k_i |\mathbf{x} - \mathbf{y}|) - k_i |\mathbf{x} - \mathbf{y}| \cos(k_i |\mathbf{x} - \mathbf{y}|)}{2\pi^2 |\mathbf{x} - \mathbf{y}|^3}, \quad (6.5)$$

of the top hat window function $\theta(k_i - k)$ included in the definition of $\zeta_{\mathbf{k}}^G$, Eq. (2.10). Both $W(x)$

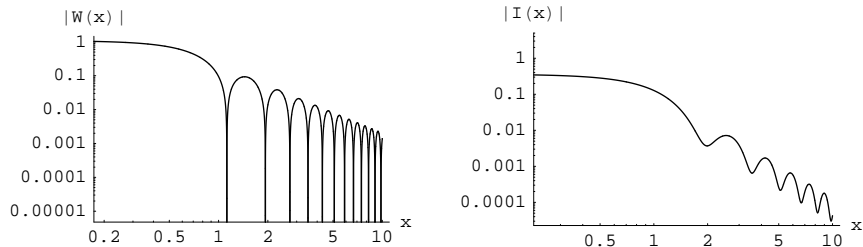


Figure 2: Absolute values of the functions W and I plotted on logarithmic scales for the choice $k_i = 4$ (in arbitrary units).

and $I(x)$ are approximatively constant at scales $x \lesssim (k_i)^{-1}$. They both fall off for $x \gtrsim (k_i)^{-1}$ but $I(x)$ remains negative definite unlike $W(x)$ which starts to oscillate rapidly. Keeping in mind that a smoothing over $W(x)$ is implicit in the definition of $\zeta^G(\mathbf{x})$, we therefore see that the convolutions of $\zeta^G(\mathbf{x})$ with $I(x)$ in Eq. (6.4) pick up a non-trivial contribution from the superhorizon modes $x \gtrsim (k_i)^{-1}$ where $W(x)$ effectively falls off faster than $I(x)$. This contribution makes Eq. (6.4) deviate from the local form.

The analysis can in principle be straightforwardly generalized to the multi-field case. The difference compared to the general single field case is the appearance of several unrelated Gaussian fields $\zeta_{\mathbf{k}}^{G,a}$ in Eq. (2.16). This in general makes it impossible to write $\zeta(\mathbf{x})$ as a series of a single Gaussian field even if the coefficients in Eq. (2.16) would be constants.

7. Conclusions

We have discussed a new approach, based on the δN -formalism, for studying the scale dependence of non-Gaussianity parameters. We have obtained explicit expressions for the scale dependence of the quantities f_{NL} , τ_{NL} and g_{NL} associated with the bispectrum and trispectrum of primordial curvature perturbations. Our results depend on the slow-roll parameters evaluated at horizon exit, and on the derivatives of the number of e-foldings and the inflationary potential. The parameters controlling the scale dependence of non-Gaussianity depend on properties of the the inflationary potential, namely its third and fourth derivatives, which in all observationally interesting cases cannot be probed by only studying the spectral index of the power spectrum and its running.

As a consequence, the scale dependence of non-Gaussianity provides additional powerful observables, able to offer novel information about the mechanism which generates the curvature perturbations. We demonstrated these features in the concrete example of modulated reheating. In models

with a quartic potential for the modulating field, we have shown that the associated non-linearity parameters, and their scale dependence, can be large enough to be observable.

While in most of the discussion we worked in momentum space, in the last part we also discussed how to describe our results in coordinate space. We provided an expression for curvature perturbations in coordinate space, that generalizes the frequently used local Ansatz, and that exhibits directly in real space the effects of scale dependence of non-Gaussian parameters.

Our results allow us to put onto a firm basis the phenomenological parameterizations of the scale dependence of non-Gaussian observables. In many models of observational interest, our formulae are relatively simple and depend on a single new parameter, the scale dependence of the non-linearity parameter. It would be interesting to use these results for analysing or simulating non-Gaussian data. At the same time, our investigation allows us to identify which properties inflationary models have to satisfy, in order to obtain large non-Gaussianity with sizeable scale dependence. It would be interesting to apply it to analyse further models, for example those in which multiple fields interact during inflation or where the non-Gaussianity is generated by an inhomogeneous end of inflation.

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Appendices

A. Explicit expressions for $n_{f,ab}$ and $n_{g,ab}$

For a system of slowly rolling scalar fields ϕ_a , the equations of motion are given by $3H\dot{\phi}_a = -V_a$ and $3H^2 = V$, to leading order in slow roll. Here we are interested in the evolution during a short time interval from t_0 to t_i . The slow roll equations can easily be solved for ϕ_a as

$$\phi^a(t_i) = \phi^a(t_0) - \sqrt{2\epsilon_a} \ln(a_i/a_0) + \mathcal{O}\left(\epsilon^{3/2} \ln^2(a_i/a_0)\right), \quad (\text{A.1})$$

where we have used the identity $Hdt = d \ln a$, which holds to leading order in slow roll. In the following we use the notation $\mathcal{O}(\epsilon^n)$ to denote the combinations of the slow roll parameters ϵ_a, η_{ab} of order ϵ^n .

Differentiating Eq. (A.1) with respect to $\phi^a(t_0)$ and keeping the number of e-foldings $\ln(a_i/a_0)$ fixed, we can compute the coefficients appearing in Eq. (2.4). We choose the initial time t_0 as the time t_k of horizon crossing of a mode k , defined by $a_k H_k = k$. Using $\ln(a_i/a_k) = \ln(a_i H_i/k)$, which is valid at leading order in slow roll, the three first coefficients in Eq. (2.4) can be written as

$$\frac{\partial \phi^a(t_i)}{\partial \phi^b(t_k)} = \delta_{ab} + \epsilon_{ab} \ln \frac{a_i H_i}{k} + \mathcal{O}\left(\left\{\epsilon^2, \frac{\epsilon^{1/2} V'''}{3H^2}\right\} \left(\ln \frac{a_i H_i}{k}\right)^2\right), \quad (\text{A.2})$$

$$\frac{\partial^2 \phi^a(t_i)}{\partial \phi^b(t_k) \partial \phi^c(t_k)} = F_{abc}^{(2)} \ln \frac{a_i H_i}{k} + \mathcal{O}\left(\left\{\epsilon^{5/2}, \frac{\epsilon V'''}{3H^2}, \frac{\epsilon^{1/2} V''''}{3H^2}\right\} \left(\ln \frac{a_i H_i}{k}\right)^2\right), \quad (\text{A.3})$$

$$\frac{\partial^3 \phi^a(t_i)}{\partial \phi^b(t_k) \partial \phi^c(t_k) \partial \phi^d(t_k)} = F_{abcd}^{(3)} \ln \frac{a_i H_i}{k} + \mathcal{O}\left(\left\{\epsilon^3, \frac{\epsilon^{3/2} V'''}{3H^2}, \frac{\epsilon V''''}{3H^2}, \frac{\epsilon^{1/2} V'''''}{3H^2}\right\} \left(\ln \frac{a_i H_i}{k}\right)^2\right), \quad (\text{A.4})$$

where the primes in $\mathcal{O}(V''')$ etc. denote derivatives with respect any of the fields ϕ_a , and

$$F_{abc}^{(2)} = \sqrt{2} \left(-4\sqrt{\epsilon_a \epsilon_b \epsilon_c} + \eta_{ab} \sqrt{\epsilon_c} + \eta_{bc} \sqrt{\epsilon_a} + \eta_{ca} \sqrt{\epsilon_b} - \frac{1}{\sqrt{2}} \frac{V_{abc}}{3H^2} \right), \quad (\text{A.5})$$

$$\begin{aligned} F_{abcd}^{(3)} &= -4(\eta_{ad} \sqrt{\epsilon_b \epsilon_c} + \eta_{bd} \sqrt{\epsilon_c \epsilon_a} + \eta_{cd} \sqrt{\epsilon_a \epsilon_b}) \\ &+ \sqrt{\epsilon_d} \left(24\sqrt{\epsilon_a \epsilon_b \epsilon_c} - 4\eta_{ab} \sqrt{\epsilon_c} - 4\eta_{bc} \sqrt{\epsilon_a} - 4\eta_{ca} \sqrt{\epsilon_b} + \sqrt{2} \frac{V_{abc}}{3H^2} \right) \\ &+ \sqrt{2\epsilon_c} \frac{V_{abd}}{3H^2} + \sqrt{2\epsilon_a} \frac{V_{bcd}}{3H^2} + \sqrt{2\epsilon_b} \frac{V_{cad}}{3H^2} + \eta_{ab} \eta_{cd} + \eta_{bc} \eta_{ad} + \eta_{ca} \eta_{bd} - \frac{V_{abcd}}{3H^2}. \end{aligned} \quad (\text{A.6})$$

Substituting these into Eq. (2.15),

$$n_{f,ab} = - \sum_c \frac{N_c F_{cab}^{(2)}}{N_{ab}}, \quad (\text{A.7})$$

$$n_{g,abc} = - \sum_d \left(3 \frac{N_{da}}{N_{abc}} F_{dbc}^{(2)} + \frac{N_d}{N_{abc}} F_{dabc}^{(3)} \right), \quad (\text{A.8})$$

we obtain fully explicit results for the parameters $n_{f,ab}$ and $n_{g,abc}$. The results are derived retaining only terms up to first order in $\ln(a_i H_i/k)$ in Eqs. (A.2) - (A.4). Higher order terms are suppressed by slow-roll parameters and their combinations with the derivatives of the potential. The former are small by construction and the latter also naturally remain small, provided that the flatness of the scalar field potential during inflation is not a result of extreme fine-tuning. Furthermore, since the logarithms never grow very large for the observable super-horizon modes, $\ln(a_i H_i/k) \lesssim \mathcal{O}(10)$, we conclude that the higher order contributions can indeed be neglected at first order.

B. On the different formulations of the δN approach

The δN expression for the super-horizon-scale curvature perturbation

$$\zeta_{\mathbf{k}}(t_f) = \sum_a N_a(t_f, t_i) \delta\phi_{\mathbf{k}}^a(t_i) + \frac{1}{2} \sum_{ab} N_{ab}(t_f, t_i) \int \frac{d\mathbf{q}}{(2\pi)^3} \delta\phi_{\mathbf{q}}^a(t_i) \delta\phi_{\mathbf{k}-\mathbf{q}}^b(t_i) + \dots, \quad (\text{B.1})$$

is by construction independent of the choice of the initial spatially flat hypersurface $t_i \geq t_k$. This property follows from the definition of the curvature perturbation [66] as demonstrated in [14]. A commonly used choice is to set t_i equal to the time t_k of horizon crossing of the mode k . The analysis in [14] was performed using this choice. In our current work, we have instead chosen t_i as a time soon after t_k following e.g. [22]. Here we explicitly compare the two choices. For simplicity, we consider only terms up to second order in Eq. (B.1). Generalization to higher orders is straightforward.

Using the chain rule it is easy to switch between the coefficients $N_{ab..}(t_f, t_i)$ and $N_{ab..}(t_f, t_k)$ in the two different formulations. For the first and second order terms shown in Eq. (B.1) we obtain

$$N_a(t_f, t_i) \equiv \frac{\partial N(t_f, t_i)}{\partial \phi^a(t_i)} = \sum_b \frac{\partial N(t_f, t_k)}{\partial \phi^b(t_k)} \frac{\partial \phi^b(t_k)}{\partial \phi^a(t_i)}, \quad (\text{B.2})$$

$$\begin{aligned} N_{ab}(t_f, t_i) &\equiv \frac{\partial^2 N(t_f, t_i)}{\partial \phi^a(t_i) \partial \phi^b(t_i)} = \sum_c \frac{\partial N(t_f, t_k)}{\partial \phi^c(t_k)} \frac{\partial^2 \phi^c(t_k)}{\partial \phi^a(t_i) \partial \phi^b(t_i)} \\ &+ \sum_{cd} \frac{\partial \phi^c(t_k)}{\partial \phi^a(t_i)} \frac{\partial \phi^d(t_k)}{\partial \phi^b(t_i)} \frac{\partial^2 N(t_f, t_k)}{\partial \phi^c(t_k) \partial \phi^d(t_k)}. \end{aligned} \quad (\text{B.3})$$

In the second equality of both equations we have replaced the time argument t_i in $N(t_f, t_i)$ by t_k making use of the fact that t_i and t_k both label spatially flat hypersurfaces. This implies that $N(t_i, t_k)$, the number of e-foldings from t_k to t_i , is a constant under differentiation with respect to the fields. Writing $N(t_f, t_i) = N(t_f, t_k) - N(t_i, t_k)$, we thus immediately see that $N(t_f, t_i)$ can be replaced by $N(t_f, t_k)$ in Eqs. (B.2) and (B.3).

Substituting Eqs. (B.2) and (B.3) into Eq. (B.1), we obtain

$$\begin{aligned} \zeta_{\mathbf{k}}(t_f) = & \sum_a N_a(t_f, t_k) \left(\frac{\partial \phi^a(t_k)}{\partial \phi^b(t_i)} \delta \phi_{\mathbf{k}}^b(t_i) + \frac{1}{2} \frac{\partial^2 \phi^a(t_k)}{\partial \phi^b(t_i) \partial \phi^c(t_i)} \int \frac{d\mathbf{q}}{(2\pi)^3} \delta \phi_{\mathbf{q}}^b(t_i) \delta \phi_{\mathbf{k}-\mathbf{q}}^c(t_i) \right) \\ & + \frac{1}{2} \sum_{ab} N_{ab}(t_f, t_k) \left(\frac{\partial \phi^a(t_k)}{\partial \phi^c(t_i)} \frac{\partial \phi^b(t_k)}{\partial \phi^d(t_i)} \int \frac{d\mathbf{q}}{(2\pi)^3} \delta \phi_{\mathbf{q}}^c(t_i) \delta \phi_{\mathbf{k}-\mathbf{q}}^d(t_i) \right) + \dots \end{aligned} \quad (\text{B.4})$$

On the other hand, according to Eq. (2.4) we have

$$\delta \phi_{\mathbf{k}}^a(t_k) = \frac{\partial \phi^a(t_k)}{\partial \phi^b(t_i)} \delta \phi_{\mathbf{k}}^b(t_i) + \frac{1}{2} \frac{\partial^2 \phi^a(t_k)}{\partial \phi^b(t_i) \partial \phi^c(t_i)} \int \frac{d\mathbf{q}}{(2\pi)^3} \delta \phi_{\mathbf{q}}^b(t_i) \delta \phi_{\mathbf{k}-\mathbf{q}}^c(t_i) + \dots \quad (\text{B.5})$$

In arriving at this result we have first taken the Fourier transform of Eq. (2.4) and only thereafter set one of the time arguments equal to t_k . Using Eq. (B.5) we can rewrite Eq. (B.4) as

$$\zeta_{\mathbf{k}}(t_f) = \sum_a N_a(t_f, t_k) \delta \phi_{\mathbf{k}}^a(t_k) + \frac{1}{2} \sum_{ab} N_{ab}(t_f, t_k) \int \frac{d\mathbf{q}}{(2\pi)^3} \delta \phi_{\mathbf{q}}^a(t_k) \delta \phi_{\mathbf{k}-\mathbf{q}}^b(t_k) + \dots \quad (\text{B.6})$$

This way of writing $\zeta_{\mathbf{k}}(t_f)$ is equivalent to Eq. (B.1) and the two expressions differ formally only by the choice of the initial time t_i , as expected. The relation between the coefficients in the two formulations is given by Eqs. (B.2) and (B.3), and the field perturbations are related by Eq. (B.5). These results explicitly show how to switch from one formulation to another.

In [14], the result for $n_{f_{\text{NL}}}$, measuring the scale dependence of f_{NL} , was expressed in terms of the parameters $n_I = d \ln N_I(t_f, t_k) / d \ln k$ and $n_{IJ} = d \ln N_{IJ}(t_f, t_k) / d \ln k$, see e.g. Eq. (69) in that paper. (Here we follow the notation of [14] and label the scalar field species ϕ_I by capital letters. This also serves to distinguish the parameters n_I and n_{IJ} from the quantities defined in the current work.) Using Eqs. (B.2) and (B.3) together with the results derived in Appendix A, we readily obtain explicit expressions for these parameters

$$n_I = \frac{d \ln N_I(t_f, t_k)}{d \ln k} = - \sum_J \frac{N_J}{N_I} \epsilon_{IJ} \ , \quad (\text{B.7})$$

$$n_{IJ} = \frac{d \ln N_{IJ}(t_f, t_k)}{d \ln k} = n_{f, IJ} - \sum_K \left(\frac{N_{IK} \epsilon_{KJ}}{N_{IJ}} + \frac{N_{JK} \epsilon_{KI}}{N_{IJ}} \right) \ . \quad (\text{B.8})$$

In the rightmost expressions we have suppressed the time arguments t_i for brevity, e.g. $N_I \equiv N_I(t_f, t_i)$. Using these results, it is straightforward to check that the general expression given for $n_{f_{\text{NL}}}$ in Eq. (69) of [14] agrees with our Eq. (2.24).

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