Estimation of the average of arbitrary unknown phase delays with Heisenberg-scaling precision

Danilo Triggiani[a] and Vincenzo Tamma[a,b]

[a]School of Mathematics and Physics, University of Portsmouth, Portsmouth PO1 3QL, UK
[b]Institute of Cosmology and Gravitation, University of Portsmouth, Portsmouth PO1 3FX, UK

ABSTRACT

We show an estimation scheme which reaches the Heisenberg-scaling sensitivity in the estimation of the average of the optical phases along the two arms of a Mach-Zehnder interferometer, by using a single squeezed vacuum state and homodyne detection at a single output port. We show that, in order to achieve this quantum advantage, it is required only a classical prior knowledge about the two phases, namely obtainable with a classical estimation strategy with shot-noise limited precision.

Keywords: Quantum Metrology, Heisenberg limit

1. INTRODUCTION

In the early works by Caves[1] and Bondurant and Shapiro,[2] it has been shown that it is possible to surpass the classical limits in the estimation of physical properties, such as optical phases, refraction indices, temperatures and lengths, employing squeezed states of light.[3–6] In particular, the uncertainty $\delta\varphi$ in the estimation of a relative phase φ can reach a faster scaling in the mean number of photons N than the one achievable classically,[2,7] namely

$$\delta\varphi \propto \frac{1}{N}, \tag{1}$$

with an improvement of $N^{-1/2}$ in the scaling. The scaling of the uncertainty in Eq. (1) is often referred to as the Heisenberg scaling,[7] because of heuristic arguments which relate it with the Heisenberg uncertainty principle.[8] Moreover, it has been recently shown that this scaling is the ultimate limit in the precision for the estimation of a phase,[9] hence justifying the name of "*Heisenberg limit*" with which it is typically referred. Since these initial works, much effort has been put by the scientific community to conceive quantum estimation schemes able to overcome the classical scaling and achieve the more efficient Heisenberg limit, giving birth to a new field of research called *quantum metrology*.[7,10–18] Within the framework of single parameter estimation, remarkable results have been recently found, such as schemes that only require a classical knowledge on the unknown parameter to optimize the network and achieve the Heisenberg scaling,[19,20] or that do not require any adaptation of the network altogether.[21]

However, the estimation of a single phase is only the first step towards more complex and application-oriented estimation schemes. For example, in some cases we may be interested in the simultaneous estimation of multiple unknown parameters – e.g. a phase and the losses of a channel, or the components of the angular momentum.[14,22–24] In multi-parameter metrology, one of the main arguments becomes whether it is possible to gain any (quantum) advantage in the simultaneous estimation of such parameters, compared to a sequential strategy in which the parameters are measured one at the time. Toward this direction, distributed quantum metrology studies schemes in which we are interested in measuring a certain function or linear combination of multiple parameters, each one locally affecting a single node of a given optical network.[25–33] In particular, metrological protocols based on Gaussian states and on the squeezing of light have the advantage of employing

Further author information:
D.T.: E-mail: danilo.triggiani@port.ac.uk
V.T.: E-mail: vincenzo.tamma@port.ac.uk

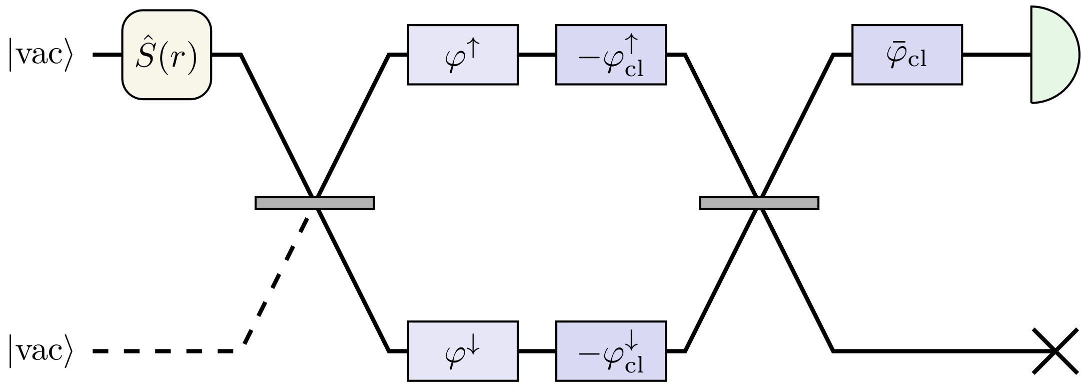

Figure 1. Diagram of the estimation scheme described in Section 2. A squeezed vacuum state is injected in the first input port of a Mach-Zehnder interferometer. This setup reaches the Heisenberg scaling in the estimation sensitivity of the average of two unknown phases $\varphi^{\uparrow,\downarrow}$. This enhanced level of precision can be achieved with only a preliminary classical knowledge $\varphi_{\uparrow/\downarrow,\mathrm{cl}}$ of the values of the unknown phases, and thus estimating $\bar{\varphi}_{\mathrm{cl}} \equiv \varphi^{\uparrow}_{\mathrm{cl}} + \varphi^{\downarrow}_{\mathrm{cl}}$, with shot-noise scaling precision with classical estimation strategies. Homodyne detection is performed at only the first output port, with the local oscillator phase tuned according to condition (8).

probes which are relatively easy to produce and more robust against decoherence compared to probes in entangled states.

In this work we present a scheme based on the use of a Mach-Zehnder interferometer which allows us to estimate the sum of two arbitrary unknown phases in the the two arms of the interferometer (see Figure 1), differently from most interferometric setup which are typically sensitive only to the relative phase between the two arms of the network.[34] We will show that, by employing a single squeezed vacuum state and a single homodyne detector, it is possible to reach Heisenberg-scaling sensitivity in the estimation of the sum of the phases, requiring only a prior information about the values of the two phases achievable with classical resources. The feasibility of this scheme, due to the use of squeezed states and homodyne detection, is further reinforced by the possibility to reach such a quantum-enhanced precision in the regime of large samples through maximum-likelihood estimation methods.[18,35,36] We refer to Ref.[32] for the demonstration of Heisenberg limited estimation of functions of an arbitrary number of parameters beyond the case addressed here of the sum of two phases.

2. SETUP

We here consider a 2-channel network $\hat{U}_{\boldsymbol{\varphi}}$ for the estimation of the average phase

$$\bar{\varphi} = \frac{\varphi^{\uparrow} + \varphi^{\downarrow}}{2} \tag{2}$$

of the values of the unknown phase delays $\boldsymbol{\varphi} = (\varphi^{\uparrow}, \varphi^{\downarrow})$ at the upper and lower arms of a Mach-Zehnder interferometer respectively, as depicted in Figure 1. Differently from typical interferometric applications of the Mach-Zehnder employing states with a fixed number of photons, in which only information on the relative phase $\varphi^{\uparrow} - \varphi^{\downarrow}$ between the two paths can be detected, we show that employing the squeezing of a single-mode squeezed-vacuum Gaussian state as metrological resource, and homodyne measurements at only a single output port, the unknown average value $\bar{\varphi}$ in Eq. (2) can be estimated with Heisenberg scaling precision. This requires only a classical prior knowledge $\varphi^{\uparrow,\downarrow}_{\mathrm{cl}}$ on the values of the two phase shifts $\varphi^{\uparrow,\downarrow}$, i.e. whose estimation error $\delta\varphi^{\uparrow,\downarrow} = \varphi^{\uparrow,\downarrow} - \varphi^{\uparrow,\downarrow}_{\mathrm{cl}}$ reaches the shot-noise scaling. In particular, our scheme employs a single squeezed-vacuum state with an average number $N = \sinh^2 r$ of photons injected in the first input port of the Mach-Zehnder interferometer, where r is the squeezing parameter of the probe, and homodyne detections at the first output channel allow us to to measure the quadrature field $\hat{x}_{\theta}$, where θ is the phase of the homodyne local oscillator.

The network is described by a unitary operator $\hat{U}_{\boldsymbol{\varphi}}$ which depends smoothly on the two phases $\boldsymbol{\varphi} = (\varphi^{\uparrow}, \varphi^{\downarrow})$, whose action on the M bosonic annihilation operators $\hat{a}_i$, for $i = 1, \ldots, M$, is given by the unitary matrix $U_{\boldsymbol{\varphi}}$

such that

$$\hat{U}^\dagger_{\boldsymbol{\varphi}} \hat{a}_i \hat{U}_{\boldsymbol{\varphi}} = \sum_{j=1}^{M} (U_{\boldsymbol{\varphi}})_{ij} \hat{a}_j. \tag{3}$$

With reference to Figure 1, we can write the matrices representing the action of a balanced beam-splitter and the unknown phase-shifts at each arm of the Mach-Zehnder as

$$U_{\mathrm{BS}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad U_{\mathrm{PS}}(\varphi^\uparrow, \varphi^\downarrow) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi^\uparrow} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi^\downarrow} \end{pmatrix}, \tag{4}$$

respectively. We then add to the upper (lower) inner arm of the Mach-Zehnder interferometer an additional phase-shift with value $-\varphi^\uparrow_{\mathrm{cl}}$ $(-\varphi^\downarrow_{\mathrm{cl}})$ corresponding, apart from the sign, to the classical prior estimate of the unknown phase $\varphi^\uparrow$ $(\varphi^\downarrow)$. Finally, we add the phase delay $\bar{\varphi}_{\mathrm{cl}} \equiv (\varphi^\uparrow_{\mathrm{cl}} + \varphi^\downarrow_{\mathrm{cl}})/2$ at the upper output port of the Mach-Zehnder interferometer, where $\bar{\varphi}_{\mathrm{cl}}$ is thus the classical estimate of the unknown average $\bar{\varphi}$ to be estimated at the Heisenberg scaling precision. Hence, the overall network $\hat{U}_{\boldsymbol{\varphi}}$ depicted in Figure 1 is represented by the matrix

$$\begin{aligned} U_{\boldsymbol{\varphi}} &= U_{\mathrm{PS}} \left(\frac{\varphi^\uparrow_{\mathrm{cl}} + \varphi^\downarrow_{\mathrm{cl}}}{2}, 0 \right) U^\dagger_{\mathrm{BS}} U_{\mathrm{PS}}(-\varphi^\uparrow_{\mathrm{cl}}, -\varphi^\downarrow_{\mathrm{cl}}) U_{\mathrm{PS}}(\varphi^\uparrow, \varphi^\downarrow) U_{\mathrm{BS}} \\ &= \exp{(\mathrm{i}\bar{\varphi})} \begin{pmatrix} \cos(\frac{\delta\varphi^\uparrow - \delta\varphi^\downarrow}{2}) & \mathrm{i}\sin(\frac{\delta\varphi^\uparrow - \delta\varphi^\downarrow}{2}) \\ \mathrm{i}\sin(\frac{\delta\varphi^\uparrow - \delta\varphi^\downarrow}{2}) & \cos(\frac{\delta\varphi^\uparrow - \delta\varphi^\downarrow}{2}) \end{pmatrix}, \end{aligned} \tag{5}$$

where $\delta\varphi^{\uparrow,\downarrow} = \varphi^{\uparrow,\downarrow} - \varphi^{\uparrow,\downarrow}_{\mathrm{cl}}$ is the error in the prior classical estimation of the unknown parameter $\varphi^{\uparrow,\downarrow}$, which is thus of order $1/\sqrt{N}$. Since the probe is injected in the first input port of the Mach-Zehnder interferometer, and homodyne detections are performed only at the first output port, we easily see from Eq. (3) that the only relevant transition amplitude is the matrix element

$$(U_{\boldsymbol{\varphi}})_{11} = \exp{(\mathrm{i}\bar{\varphi})} \cos(\frac{\delta\varphi^\uparrow - \delta\varphi^\downarrow}{2}) \tag{6}$$

We will now show that the Heisenberg scaling can be achieved in the estimation of the average $\bar{\varphi} = (\varphi^\uparrow + \varphi^\downarrow)/2$ of the two unknown phases. In fact, we can easily expand the transition probability $|(U_{\boldsymbol{\varphi}})_{11}|^2 = \cos^2(\frac{1}{2}(\delta\varphi^\uparrow - \delta\varphi^\downarrow))$ starting from the expression in Eq. (6) in powers of N, obtaining

$$\begin{aligned} |(U_{\boldsymbol{\varphi}})_{11}|^2 &= 1 - \left(\frac{\delta\varphi^\uparrow - \delta\varphi^\downarrow}{2} \right)^2 + \mathcal{O}(\delta\varphi^4) \\ &= 1 - \frac{1}{N} \left(\frac{k^\uparrow - k^\downarrow}{2} \right)^2 + \mathcal{O}(N^{-2}) \\ &\sim 1 - \frac{\ell}{N}, \end{aligned} \tag{7}$$

where the terms $\mathcal{O}(\delta\varphi^4)$ and $\mathcal{O}(N^{-2})$ are of order N^{-2} or lower and can be for this reason neglected, we assumed without loss of generality that $\delta\varphi^{\uparrow,\downarrow} \sim k^{\uparrow,\downarrow}/\sqrt{N}$, with $k^{\uparrow,\downarrow}$ real constants independent of N, and introduced the term $\ell = (k^\uparrow - k^\downarrow)^2/4$. Thus, exploiting results demonstrated in Ref.,[32] the expansion of the probability transition shown in Equation (7), together with the minimum resolution requirement for the tuning of the local oscillator phase θ

$$\theta = \bar{\varphi} + \frac{\pi}{2} + \frac{k}{N}, \qquad k \neq 0 \tag{8}$$

with k an arbitrary real constant, guarantee Heisenberg scaling sensitivity in the estimation of the overall phase $\bar{\varphi}$ acquired by the probe after ν iterations of the measurement, with a precision[32]

$$\delta\left(\bar{\varphi}\right) = \frac{1}{\nu} \frac{1}{2\sqrt{2\varrho(k,\ell)}N}, \tag{9}$$

with

$$\varrho(k,\ell) = \left(\frac{8k}{1+16k^2+4\ell}\right)^2 \tag{10}$$

and ℓ and k introduced in Eqs. (7) and (8) respectively. The error in the estimation shown in Eq. (9) can be practically achieved in the regime of large samples (large ν) through maximum-likelihood estimation methods.[32]

3. CONCLUSIONS

We have presented a scheme for the estimation at the Heisenberg limit of the average of two unknown phases along the two paths of a Mach-Zehnder interferometer. Since it only employs a single-mode squeezed vacuum state and a homodyne detector at a single output port, this estimation scheme results feasible for experimental applications. No restriction is imposed on the values of the two phases, which can thus be left completely arbitrary. Remarkably, we have shown that, to carry out the estimation, it is only required a classical prior knowledge of the values of the two phases, obtainable with a standard coarse estimation with shot-noise scaling precision.

ACKNOWLEDGMENTS

This work was supported by the Office of Naval Research Global (N62909-18-1-2153).

REFERENCES

[1] Caves, C. M., "Quantum-mechanical noise in an interferometer," *Phys. Rev. D* **23**, 1693–1708 (1981).

[2] Bondurant, R. S. and Shapiro, J. H., "Squeezed states in phase-sensing interferometers," *Phys. Rev. D* **30**, 2548–2556 (1984).

[3] Schleich, W., [*Quantum Optics in Phase Space*], Wiley (2011).

[4] Weedbrook, C., Pirandola, S., García-Patrón, R., Cerf, N. J., Ralph, T. C., Shapiro, J. H., and Lloyd, S., "Gaussian quantum information," *Rev. Mod. Phys.* **84**, 621–669 (2012).

[5] Adesso, G., Ragy, S., and Lee, A. R., "Continuous variable quantum information: Gaussian states and beyond," *Open Systems & Information Dynamics* **21**(01n02), 1440001 (2014).

[6] Lvovsky, A. I., [*Squeezed Light*], ch. 5, 121–163, John Wiley and Sons, Ltd (2015).

[7] Giovannetti, V., Lloyd, S., and Maccone, L., "Quantum-enhanced measurements: Beating the standard quantum limit," *Science* **306**(5700), 1330–1336 (2004).

[8] Ou, Z. Y., "Fundamental quantum limit in precision phase measurement," *Phys. Rev. A* **55**, 2598–2609 (1997).

[9] Giovannetti, V., Lloyd, S., and Maccone, L., "Quantum measurement bounds beyond the uncertainty relations," *Phys. Rev. Lett.* **108**, 260405 (2012).

[10] Paris, M. G., "Quantum estimation for quantum technology," *International Journal of Quantum Information* **7**(supp01), 125–137 (2009).

[11] Giovannetti, V., Lloyd, S., and Maccone, L., "Advances in quantum metrology," *Nature Photonics* **5**(4), 222–229 (2011).

[12] Tóth, G. and Apellaniz, I., "Quantum metrology from a quantum information science perspective," **47**(42), 424006 (2014).

[13] Dowling, J. P. and Seshadreesan, K. P., "Quantum optical technologies for metrology, sensing, and imaging," *Journal of Lightwave Technology* **33**(12), 2359–2370 (2015).

[14] Szczykulska, M., Baumgratz, T., and Datta, A., "Multi-parameter quantum metrology," *Advances in Physics: X* **1**(4), 621–639 (2016).

[15] Schnabel, R., "Squeezed states of light and their applications in laser interferometers," *Physics Reports* **684**, 1–51 (2017). Squeezed states of light and their applications in laser interferometers.

[16] Braun, D., Adesso, G., Benatti, F., Floreanini, R., Marzolino, U., Mitchell, M. W., and Pirandola, S., "Quantum-enhanced measurements without entanglement," *Rev. Mod. Phys.* **90**, 035006 (2018).

[17] Pirandola, S., Bardhan, B. R., Gehring, T., Weedbrook, C., and Lloyd, S., "Advances in photonic quantum sensing," *Nature Photonics* **12**(12), 724–733 (2018).

[18] Polino, E., Valeri, M., Spagnolo, N., and Sciarrino, F., "Photonic quantum metrology," *AVS Quantum Science* **2**(2), 024703 (2020).

[19] Gramegna, G., Triggiani, D., Facchi, P., Narducci, F. A., and Tamma, V., "Heisenberg scaling precision in multi-mode distributed quantum metrology," *New Journal of Physics* **23**(5), 053002 (2021).

[20] Gramegna, G., Triggiani, D., Facchi, P., Narducci, F. A., and Tamma, V., "Typicality of heisenberg scaling precision in multimode quantum metrology," *Phys. Rev. Research* **3**, 013152 (2021).

[21] Triggiani, D., Facchi, P., and Tamma, V., "Non-adaptive heisenberg-limited metrology with multi-channel homodyne measurements," *The European Physical Journal Plus* **137**(1), 125 (2022).

[22] Baumgratz, T. and Datta, A., "Quantum enhanced estimation of a multidimensional field," *Phys. Rev. Lett.* **116**, 030801 (2016).

[23] Nichols, R., Liuzzo-Scorpo, P., Knott, P. A., and Adesso, G., "Multiparameter gaussian quantum metrology," *Phys. Rev. A* **98**, 012114 (2018).

[24] Demkowicz-Dobrzanski, R., Górecki, W., and Guţă, M., "Multi-parameter estimation beyond quantum fisher information," *Journal of Physics A: Mathematical and Theoretical* **53**(36), 363001 (2020).

[25] Proctor, T. J., Knott, P. A., and Dunningham, J. A., "Multiparameter estimation in networked quantum sensors," *Phys. Rev. Lett.* **120**, 080501 (2018).

[26] Zhuang, Q., Zhang, Z., and Shapiro, J. H., "Distributed quantum sensing using continuous-variable multipartite entanglement," *Phys. Rev. A* **97**, 032329 (2018).

[27] Qian, K., Eldredge, Z., Ge, W., Pagano, G., Monroe, C., Porto, J. V., and Gorshkov, A. V., "Heisenberg-scaling measurement protocol for analytic functions with quantum sensor networks," *Phys. Rev. A* **100**, 042304 (2019).

[28] Gatto, D., Facchi, P., Narducci, F. A., and Tamma, V., "Distributed quantum metrology with a single squeezed-vacuum source," *Phys. Rev. Research* **1**, 032024 (2019).

[29] Guo, X., Breum, C. R., Borregaard, J., Izumi, S., Larsen, M. V., Gehring, T., Christandl, M., Neergaard-Nielsen, J. S., and Andersen, U. L., "Distributed quantum sensing in a continuous-variable entangled network," *Nature Physics* **16**(3), 281–284 (2020).

[30] Oh, C., Lee, C., Lie, S. H., and Jeong, H., "Optimal distributed quantum sensing using gaussian states," *Phys. Rev. Research* **2**, 023030 (2020).

[31] Grace, M. R., Gagatsos, C. N., and Guha, S., "Entanglement-enhanced estimation of a parameter embedded in multiple phases," *Phys. Rev. Research* **3**, 033114 (2021).

[32] Triggiani, D., Facchi, P., and Tamma, V., "Heisenberg scaling precision in the estimation of functions of parameters in linear optical networks," *Phys. Rev. A* **104**, 062603 (2021).

[33] Gatto, D., Facchi, P., and Tamma, V., "Heisenberg-limited estimation robust to photon losses in a mach-zehnder network with squeezed light," *Phys. Rev. A* **105**, 012607 (2022).

[34] Pezzé, L. and Smerzi, A., "Phase sensitivity of a mach-zehnder interferometer," *Phys. Rev. A* **73**, 011801 (2006).

[35] Cramér, H., [*Mathematical Methods of Statistics (PMS-9)*], Princeton University Press (1946).

[36] Rohatgi, V. K. and Saleh, A. M. E., [*An introduction to probability and statistics*], John Wiley and Sons (2015).