Generalized product

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Abstract. Aggregation functions on $[0,1]$ with annihilator 0 can be seen as a generalized product on $[0,1]$. We study the generalized product on the bipolar scale $[-1,1]$, stressing the axiomatic point of view. Based on newly introduced bipolar properties, such as the bipolar increasingness, bipolar unit element, bipolar idempotent element, several kinds of generalized bipolar product are introduced and studied. A special stress is put on bipolar semicopulas, bipolar quasi-copulas and bipolar copulas.

Keywords: Aggregation function; bipolar copula; bipolar scale; bipolar semicopula; symmetric minimum.

1 Introduction

Recall that an aggregation function $A : [0,1]^2 \to [0,1]$ is characterized by boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$, and by the increasingness of A, i.e., the sections $A(x,\cdot)$ and $A(\cdot,y)$ are increasing for each $x, y \in [0,1]$. For more details see [2, 5]. The product $\Pi : [0,1]^2 \to [0,1]$ has additionally 0 as its annihilator, and thus each aggregation function $A : [0,1]^2 \to [0,1]$ with annihilator 0 (i.e., $A(x,0) = A(0,y)$ for all $x, y \in [0,1]$) can be seen as a generalization of the product Π on the unipolar

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scale $[0,1]$. Observe that the class $\mathcal{P}$ of generalized products on the scale $[0,1]$ has the smallest aggregation function $A_* : [0,1]^2 \to [0,1]$ given by

$$A_*(x,y) = \begin{cases} 1 \text{ if } x = y = 1, \\ 0 \text{ else} \end{cases} \tag{1}$$

as its minimal element, and its maximal element $A^* : [0,1]^2 \to [0,1]$ is given by

$$A^*(x,y) = \begin{cases} 0 \text{ if } xy = 0, \\ 1 \text{ else.} \end{cases} \tag{2}$$

Moreover $\mathcal{P}$ is a complete lattice (with respect to pointwise suprema and infima), and it contains, among others, geometric mean G, harmonic mean H, minimum M, etc. The most distinguished subclasses of $\mathcal{P}$ are

– the class $\mathcal{S}$ of semicopulas, i.e., aggregation functions from $\mathcal{P}$ having $e = 1$ as neutral element, $S(x,1) = S(1,x) = x$ for all $x \in [0,1]$, see [1, 3];
– the class $\mathcal{T}$ of triangular norms, i.e., of associative and commutative semicopulas, [9, 13];
– the class $\mathcal{Q}$ of quasi-copulas, i.e. $1-$Lipschitz aggregation functions from $\mathcal{P}$ (observe that $\mathcal{Q} \subsetneq \mathcal{S}$), [12];
– the class $\mathcal{C}$ of copulas, i.e. supermodular functions from $\mathcal{P}$ (observe that $\mathcal{C} \subsetneq \mathcal{Q}$), [12].

Observe that the product Π belongs to any of mentioned classes, similarly as M. Among several applications of the generalized product functions, recall their role as conjunctions in fuzzy logic [8], or their role of multiplications in the area of general integrals [10, 14].

Integration on bipolar scale $[-1,1]$ requires a bipolar function $B : [-1,1]^2 \to [-1,1]$ related to the standard product $\Pi : [-1,1]^2 \to [-1,1]$ (we will use the same notation Π for the product independently of the actual scale). Up to standard product Π applied, e.g., in the case of Choquet integral on $[-1,1]$, or in the case of bipolar capacities based Choquet integral, Grabisch [4] has introduced a symmetric Sugeno integral on $[-1,1]$ based on the symmetric minimum $B_M : [-1,1]^2 \to [-1,1]$, $B_M(x,y) = sign(xy)\min(|x|,|y|)$. The aim of this paper is to generalize the bipolar product on $[-1,1]$ in a way similar to generalized product on $[0,1]$, and to study special classes of such generalizations.
The paper is organized as follows. In the next section, several properties of bipolar functions are proposed, and the generalized bipolar product is introduced. Section 3 is devoted to bipolar semicopulas and bipolar triangular norms, while in Section 4 we study bipolar quasi-copulas and bipolar copulas. Finally, some concluding remarks are added.

2 Generalized bipolar product

Considering the function $F : [-1,1]^2 \to [-1,1]$, several algebraic and analytic properties can be considered in their standard form, such as the commutativity, associativity, annihilator 0, continuity, Lipschitzianity, supermodularity, etc. Note that the bipolar product $\Pi : [-1,1]^2 \to [-1,1]$ as well as the symmetric minimum $B_M : [-1,1]^2 \to [-1,1]$ satisfy all of them. However, there are some properties reflecting the bipolarity of the scale $[-1,1]$.

Recall that a mapping $S : [0,1]^2 \to [0,1]$ is a semicopula [1, 3], whenever it is non-decreasing in both variables and 1 is the neutral element, i.e., $S(x,1) = S(1,x) = x$ for all $x \in [0,1]$. When considering the product $\Pi : [-1,1]^2 \to [-1,1]$, we see that 1 is its neutral element. More, it holds $\Pi(-1,x) = \Pi(x,-1) = -x$ for all $x \in [-1,1]$.

Definition 1 *Let $F : [-1,1]^2 \to [-1,1]$ be a mapping such that $F(x,1) = F(1,x) = x$ and $F(-1,x) = F(x,-1) = -x$ for all $x \in [-1,1]$. Then 1 is called a bipolar neutral element for F.*

Simple bipolar semicopulas B_S, introduced for bipolar universal integrals in [6], are fully determined by standard semicopulas $S : [0,1]^2 \to [0,1]$, by means of $B_S(x,y) = (sign(xy))S(|x|,|y|)$. Observe that 1 is a bipolar neutral element for any simple bipolar semicopula B_S. Concerning the monotonicity required for semicopulas, observe that considering the product Π, or any simple bipolar semicopula B_S (note that $B_\Pi = \Pi$, abusing the notation Π both for the product on $[-1,1]$ and on $[0,1]$), these mappings are non-decreasing in both coordinates when fixing an element from the positive part of the scale, while they are non-increasing when fixing an element from the negative part of the scale $[-1,1]$.

Definition 2 *Let $F : [-1,1]^2 \to [-1,1]$ be a mapping such that the partial mappings $F(x,\cdot)$ and $F(\cdot,y)$ are non-decreasing for any $x,y \in [0,1]$ and they are non-increasing for any $x,y \in [-1,0]$. Then F will be called a bipolar increasing mapping.*

Similarly, inspired by the symmetric minimum B_M, we introduce the notion of a bipolar idem potent element.

Definition 3 *Let $F : [-1,1]^2 \to [-1,1]$ be given. An element $x \in [0,1]$ is called a bipolar idempotent element of F whenever it satisfies $F(x,x) = F(-x,-x) = x$ and $F(-x,x) = F(x,-x) = -x$.*

Recall that the class $\mathcal{P}$ of generalized products on $[0,1]$ can be characterized as the class of all the increasing mappings $F : [0,1]^2 \to 0,1]$ such that $F|_{\{0,1\}^2} = \Pi|_{\{0,1\}^2}$. Inspired by this characterization, we introduce the class $\mathcal{BP}$ of all generalized bipolar products as follows.

Definition 4 *A function $B : [-1,1]^2 \to [-1,1]$ is a generalized bipolar product whenever it is bipolar increasing and $B|_{\{-1,0,1\}^2} = \Pi|_{\{-1,0,1\}^2}$.*

Theorem 1 *$B \in \mathcal{BP}$ if and only if there are $A_1, A_2, A_3, A_4 \in \mathcal{P}$ such that*

$$B(x,y) = \begin{cases} A_1(x,y) & \textit{if} \quad (x,y) \in [0,1]^2 \\ -A_2(-x,y) & \textit{if} \quad (x,y) \in [-1,0] \times [0,1] \\ A_3(-x,-y) & \textit{if} \quad (x,y) \in [-1,0]^2 \\ -A_4(x,-y) & \textit{if} \quad (x,y) \in [0,1] \times [-1,0]. \end{cases} \tag{3}$$

Due to Theorem 1, each $B \in \mathcal{BP}$ can be identified with a quadruple $(A_1, A_2, A_3, A_4) \in \mathcal{P}^4$.

Definition 5 *Let $A \in \mathcal{P}$. then $B_A = (A, A, A, A) \in \mathcal{BP}$, given by $B_A(x, y) = sign(xy)A(|x|, |y|)$, is called a simple generalized bipolar product (simple GBP, in short).*

Evidently, B_M is a simple GBP related to M, while $B_\Pi = \Pi$. Observe that

$$B_{A_*}(x, y) = \begin{cases} \Pi(x, y) & \text{if} \quad (x, y) \in \{-1, 1\} \\ 0 & \text{else,} \end{cases}$$

and

$$B_{A^*}(x, y) = sign(xy).$$

However B_{A_*} and B_{A^*} are not extremal elements of $\mathcal{BP}$. The class $\mathcal{BP}$ is a complete lattice (considering pointwise sup and inf) with top element $B^* = (A^*, A_*, A^*, A_*)$ and bottom element $B_* = (A_*, A^*, A_*, A^*)$, given by

$$B^*(x, y) = \begin{cases} 1 & \text{if} \quad xy > 0 \\ -1 & \text{if} \quad xy = -1 \\ 0 & \text{else,} \end{cases} \tag{4}$$

and

$$B_*(x, y) = \begin{cases} -1 & \text{if} \quad xy < 0 \\ 1 & \text{if} \quad xy = 1 \\ 0 & \text{else.} \end{cases} \tag{5}$$

3 Bipolar semicopulas and bipolar t-norms

Based on the idea of a bipolar neutral element $e = 1$, we introduce now the bipolar semicopulas, compare also [7].

Definition 6 *A mapping $B : [-1, 1]^2 \to [-1, 1]$ is called a bipolar semicopula whenever it is bipolar increasing and 1 is a bipolar neutral element of B.*

Based on Theorem 1 we have the next result

Corollary 1 *A mapping $B : [-1, 1]^2 \to [-1, 1]$ is a bipolar semicopula if and only if there is a quadruple (S_1, S_2, S_3, S_4) of semicopulas so that*

$$B(x, y) = \begin{cases} S_1(x, y) & \textit{if} \quad (x, y) \in [0, 1]^2 \\ -S_2(-x, y) & \textit{if} \quad (x, y) \in [-1, 0] \times [0, 1] \\ S_3(-x, -y) & \textit{if} \quad (x, y) \in [-1, 0]^2 \\ -S_4(x, -y) & \textit{if} \quad (x, y) \in [0, 1] \times [-1, 0]. \end{cases} \tag{6}$$

It is not difficult to check that the extremal bipolar semicopulas are related to extremal semicopulas M (the greatest semicopula given by $M(x, y) = \min(x, y)$) and Z (the smallest semicopula, called also the drastic product, and given by $Z(x, y) = \min(x, y)$ if $1 \in \{x, y\}$ and $Z(x, y) = 0$ else).

We have also the next results.

Proposition 1 *Let $B \in \mathcal{B}$ be a bipolar semicopula such that each $x \in [0, 1]$ is its bipolar idempotent element. Then $B = B_M$ is the symmetric minimum introduced by Grabisch [4].*

Associativity of binary operations (binary functions) is a strong algebraic property, which, in the case of bipolar semicopulas characterizes a particular subclass of $\mathcal{B}$.

Theorem 2 *A bipolar semicopula $B \in \mathcal{B}$ is associative if and only if B is a simple bipolar semicopula, $B = B_S$, where $S \in \mathcal{S}$ is an associative semicopula.*

Typical examples of associative bipolar semicopulas are the product Π and the symmetric minimum B_M. Recall that a symmetric semicopula $S \in \mathcal{S}$, i.e., $S(x, y) = S(y, x)$ for all $x, y \in [0, 1]$, which is also associative is called a triangular norm [13, 9].

Definition 7 *A symmetric associative bipolar semicopula $B \in \mathcal{B}$ is called a bipolar triangular norm.*

Due to Theorem 2 it is obvious that a bipolar semicopula $B \in \mathcal{B}$ is a bipolar triangular norm if and only if $B = B_T$, where $T : [0, 1]^2 \to [0, 1]$ is a triangular norm, i.e. if $B(x, y) = (sign(xy))T(|x|, |y|)$. Obviously,the product, Π, and the symmetric minimum, B_M, are bipolar triangular norms. The smallest semicopula Z is also a triangular norm and the corresponding bipolar triangular norm $B_Z : [-1, 1]^2 \to [-1, 1]$ is given by

$$B_Z(x, y) = \begin{cases} 0 & \text{if } (x, y) \in]-1, 1[^2, \\ xy & \text{else.} \end{cases} \tag{7}$$

Observe that the genuine n-ary extension $B_Z : [-1, 1]^n \to [-1, 1]$, $n > 2$, is given by

$$B_Z(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } \#\{i \mid x_i \in]-1, 1[\} \geq 2, \\ \Pi_{i=1}^n x_i & \text{else.} \end{cases} \tag{8}$$

Also $W : [-1, 1]^2 \to [-1, 1]$ given by $W(x, y) = \max(0, x + y - 1)$ is a triangular norm, and consequently also $B_W : [-1, 1]^2 \to [-1, 1]$ given by $B_W(x, y) = (sign(xy)) \max(0, x+y-1)$ is a bipolar triangular norm. Moreover, its n-ary extension $B_W : [-1, 1]^n \to [-1, 1]$, $n > 2$, is given by

$$B_W(x_1, \ldots, x_n) = (sign(\Pi_{i=1}^n x_i)) \max(0, \sum x_i - n + 1).$$

Note that several construction methods for bipolar semicopulas were proposed in [7].

4 Bipolar quasi-copulas and copulas

In this section, we extend the notion of quasi-copulas and copulas acting on the unipolar scale $[0, 1]$ to the bipolar scale $[-1, 1]$.

Definition 8 *Let $B \in \mathcal{BP}$ be $1-$Lipschitz, i.e.,*

$$|B(x_1, x_2) - B(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|, \quad \textit{for all } x_1, x_2, y_1, y_2 \in [-1, 1].$$

Then B is called a bipolar quasi-copula.

Based on theorem 1 we have the following result.

Corollary 2 *$B \in \mathcal{GP}$ is a bipolar quasi-copula if and only if $B = (Q_1, Q_2, Q_3, Q_4) \in \mathcal{Q}^4$.*

Evidently, each bipolar quasi-copula is also a bipolar semicopula.

Definition 9 *Let $B \in \mathcal{GP}$ has a bipolar neutral element $e = 1$ and let B be supermodular, i.e.,*

$$B(x_1 \vee x_2, y_1 \vee y_2) + B(x_1 \wedge x_2, y_1 \wedge y_2) \geq B(x_1, y_1) + B(x_2, y_2),$$

for all $x_1, x_2, y_1, y_2 \in [-1, 1]$. Then B is called a bipolar copula.

Corollary 3 *$B \in \mathcal{GP}$ is a bipolar copula if and only if $B = (C_1, C_2, C_3, C_4) \in \mathcal{C}^4$.*

Observe that each bipolar copula B is also a bipolar quasi-copula, and that the class of all bipolar quasi-copulas $\mathcal{BQ}$ is a $\sup - (\inf -)$ closure of the class $\mathcal{BC}$ of all bipolar copulas. Π and B_M are typical example of simple bipolar copulas.
As an example of a bipolar copula B which is not simple, we consider the function $B : [-1, 1]^2 \to [-1, 1]$ given by

$$B(x, y) = xy + |xy|(1 - |x|)(1 - |y|).$$

Then $B = (C_1, C_2, C_1, C_2)$ where $C_1, C_2 \in \mathcal{C}$ are Farlie-Gumbel-Morgenstern copulas [12] given by

$$C_1(x, y) = xy + xy(1 - x)(1 - y)$$

and

$$C_2(x, y) = xy - xy(1 - x)(1 - y).$$

5 Concluding remarks

We have introduced and discussed bipolar generalizations of the product, including bipolar semicopulas, bipolar triangular norms, bipolar quasi-copulas and bipolar copulas. Observe that our approach to bipolar aggregation can be seen as a particular case of the multi-polar aggregation proposal as given in [11] for dimension n=2. We expect application of our results in multicriteria decision support when considering bipolar scales, especially when dealing with bipolar capacities based integrals. Observe that simple bipolar semicopulas were already applied when introducing universal integrals on the bipolar scale $[-1, 1]$, see [6].

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