Extension and torsion of rubber-like hollow and solid circular cylinders for incompressible isotropic hyperelastic materials with limiting chain extensibility

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Abstract

In this paper we demonstrate the application of a newly proposed generalised neo-Hookean strain energy function within the family of limiting chain extensibility models to the problem of extension and torsion in incompressible isotropic rubber-like tubes and solid circular cylinders. We consider a general deformation involving extension and torsion in tubes, and subsequently specialise to the simple torsion of solid cylinders. Expressions for the twisting moment $\mathcal{M}$, axial load $\mathcal{N}$ and the inflation pressure P (for tubes) are derived and presented for all the considered deformations. Using the proposed model, solutions are obtained for the critical axial stretch λ_{cz} beyond which the specimens exhibit the reversal of the Poynting-type effect upon twisting for both stretched tubes and solid cylinders. Furthermore, it will be demonstrated that the response function of the model is compatible with Penn and Kearsley's *scaling law* in torsion and can be directly derived from the ensuing experimental data. While the problem of torsion in elastic tubes and cylinders has been well-studied, this work provides a contribution to nuanced aspects of this problem including the prediction of the critical axial stretch λ_{cz} at which Poynting-type effects reverse in stretched specimens and the demonstration of compatibility with the scaling law of Penn and Kearsley.

Keywords: Extension and torsion; Tubes and solid cylinders; Poynting-type effects; Constitutive modelling; Limiting chain extensibility; Scaling law.

1. Introduction

Our aim in this paper is to analyse the problem of extension and torsion in incompressible isotropic hyperelastic circular cylinders and tubes within the framework of nonlinear elasticity using a newly proposed generalised neo-Hookean strain energy function within the family of limiting chain extensibility models. The new model is (Anssari-Benam and Bucchi, 2021):

$$W = \mu N \left[\frac{1}{6N} (I_1 - 3) - \ln \left(\frac{I_1 - 3N}{3 - 3N} \right) \right], \qquad N > 1\,, \tag{1}$$

with the constraint:

$$I_1 < 3N \tag{2}$$

to ensure that the $\ln(\blacksquare)$ function is well defined. Here μ is a measure of shear modulus, N is the number of Kuhn segments and I_1 is the first principal invariant of the left Cauchy-Green deformation tensor **B**. Note that if the infinitesimal shear modulus is denoted by μ_0, on using $\mu_0 = 2\left(\frac{\partial W}{\partial I_1}\right)_{I_1=3}$ we have $\mu = \mu_0 \left(\frac{3-3N}{1-3N}\right)$. See also Puglisi and Saccomandi (2016) for the micromechanical interpretation of the parameters in generalised neo-Hookean strain energy functions based on the kinetic molecular theory. It can be shown that in the limit $N \to \infty$, the two-parameter model in equation (1) reduces to the classical one-parameter neo-Hookean model.

This model was originally devised by Anssari-Benam and Bucchi (2018) for application to the deformation of the elastin network in the aortic valve tissue, wherein a detailed account of the model derivation is presented. The application of this model was subsequently extended to in-plane homogeneous deformations of rubber-like materials (Anssari-Benam and Bucchi, 2021; Anssari-Benam et al., 2021a), and the problem of inflation of spherical and cylindrical rubber shells (Anssari-Benam et al., 2021b). It was shown that the model provides favourable fits to the experimental data and successfully captures the inflation instabilities. As demonstrated by Horgan (2021), this model is related to the celebrated Gent (1996) model and the first derivatives of W with respect to I_1 in these two models differ by just a constant. See, e.g., Horgan (2015) for a review of the Gent model and some of its applications. Compared with the Gent model, however, the model in equation (1) is a higher order constrained Padé approximation of the inverse Langevin function in terms of I_1. Employing the analysis provided by Horgan and Saccomandi (2002), the Gent model is a rational Padé approximant of order [0/1], while the model in equation (1) is of the higher [1/1] order (see Anssari-Benam (2021) for a detailed analysis). Note that, as a baseline, the neo-Hookean model corresponds to a [0/0] approximant. See the infographic in Table 1 for an illustration of this hierarchy of approximants for the respective response functions. Another well-known generalised neo-Hookean model in respect of the inverse Langevin function is the 8-chain model of Arruda and Boyce (1993); however, this model does not capture the singularity inherent in the inverse Langevin function (see, e.g., Anssari-Benam and Bucchi, 2018) and furthermore does not allow one to obtain analytic closed form solutions to boundary value problems (see, e.g., Horgan and Saccomandi, 2006). By contrast, the model in equation (1) does not suffer from these drawbacks and gives a closer approximation to the exact singularity point than the Gent model (see Anssari-Benam (2021) for further discussion). Indeed, the more accurate predictions of the model in equation (1) versus those of the Gent model have been demonstrated for pure homogeneous (Anssari-Benam and Bucchi, 2021) and inflation (Anssari-Benam et al., 2021b) deformations. Building on the foregoing results, we analyse here the problem of extension and torsion using the new model (1).

Table 1 – Hierarchy of the order of the rational Padé approximants of the response functions of the considered generalised neo-Hookean strain energy functions. Note that μ, C_1, N and μ_0 are model constants, with the latter being the infinitesimal shear modulus.

Model	**Response function $(2W_1)$**	**Order of the Padé approximant**	**Accuracy hierarchy**
Model in equation (1)	$\frac{1}{3}\mu\frac{I_1-9N}{I_1-3N}$	[1/1]	
Gent	$\mu_0\frac{J_m}{J_m-(I_1-3)}$	[0/1]	
Arruda-Boyce (n terms)	$C_1\left(1+\frac{I_1}{5N}+\frac{11I_1^2}{175N}+\cdots\right)$	[n-1/0]	
Neo-Hookean	μ_0	[0/0]	

Fundamental aspects of the problem of torsion in circular cylinders and tubes have been formulated in detail through the works of Rivlin (1948; 1949a; 1949b; 1956), and experimentally investigated by Rivlin and Saunders (1951) and Gent and Rivlin (1952). More recently Kanner and Horgan (2008) provided a comprehensive investigation of the effects of strain stiffening and limiting chain extensibility on the extension and torsion of solid circular cylinders. They found that in contrast to the classical neo-Hookean or Mooney-Rivlin models, limiting chain extensibility models predict a critical axial stretch λ_{cz} below which a stretched cylinder tends to elongate on twisting but above which the cylinder tends to shorten upon further twisting. A result of this type was experimentally observed by Gent and Hua (2004). Universal relationships that underpin the co-dependency of the twisting moment $\mathcal{M}$ and the axial force $\mathcal{N}$, independent of the functional form of the strain energy function, have also been studied and established through the works of, for example, Horgan and Saccomandi (1999), Wineman (2005) and Horgan and Smayda (2012).

There are, however, against this wealth of background literature, some aspects of the problem of torsion that remain less examined. First, a concerted analysis of the torsion in circular cylinders and tubes for strain-stiffening materials with limiting chain extensibility has not been presented in the literature. Second, the analysis of Kanner and Horgan (2008), i.e., the existence of a critical stretch in stretched solid circular cylinders under twisting, has not been extended to tubes. This analysis may be, *a fortiori*, of critical importance as the problem of extension and torsion in tubes is more intricate than in solid cylinders and may also have implications for biomechanical applications such as the stability analysis of arteries (see, e.g., Emuna and Durban, 2020). Here we extend the analysis of Kanner and Horgan (2008) to incompressible isotropic tubes and establish the critical value of the axial stretch λ_{cz} at which the elongation-to-shortening transition upon twisting occurs. Third, the work of Kanner and Horgan (2008) on limiting chain extensibility models was focused primarily on the Gent model. Here we extend their analysis to the new model in equation (1) and establish the critical conditions/values for this model. Fourth, as noted by Wineman and McKenna (1996), on use of the *scaling law* of Penn and Kearsley (1976), torsion experiments can be used to directly derive suitable strain energy functions W for application to the finite deformation of rubber-like materials. We will demonstrate that the functional form of the response function, i.e., $\partial W/\partial I_1$, of the model (1) may be recovered from Penn and Kearsley's results. To the best of our knowledge, the compatibility of other limiting chain extensibility models in the literature with Penn and Kearsley's analysis has not yet been demonstrated.

In proceeding towards the foregoing objectives of this paper, we start by presenting the preliminaries and basic definitions in §2. The torsion of a cylindrical tube will then be formulated for an incompressible isotropic strain energy function $W(I_1, I_2)$ under the general framework of combined simple extension and torsion and is subsequently specialised for subsets of simple torsion and to generalised neo-Hookean materials, respectively. A similar approach will be used in §3 for the torsion of solid circular cylinders where the expressions for the torsion of a solid circular cylinder are readily recovered from those of the general torsion of tubes, presented in §2. While much of the preceding material may be ascertained from the seminal work of Rivlin, it is summarised here for the convenience of the modern reader. The main new contributions of this paper are contained in §§4 to 6. In §4 we will derive the specialised relationships for the torsion of tubes using the model (1) and will employ Kanner and Horgan's approach to establish the value of λ_{cz}, demonstrating that the model predicts a reversal in the Poynting-type effect, i.e., elongation upon twisting for stretched tubes for axial stretch levels below λ_{cz} while the tendency changes to shortening above this level of stretch. In §5 we will present the results for the extension and torsion of solid circular cylinders using the same approach. In §6 we will briefly discuss and demonstrate the compatibility of the model with Penn and Kearsley's scaling law. Concluding remarks are provided in §7.

2. The problem of torsion in a circular cylindrical tube

Through the pioneering works of Rivlin (1949a, 1956) and Gent and Rivlin (1952) it is now well-known that in order to maintain a state of *simple torsion* in a circular cylindrical tube, in addition to the applied twisting couple (moment) $\mathcal{M}$, it is necessary to apply an axial force $\mathcal{N}$ to the plane ends of the tube as well as a homogeneous inflation pressure P to the tube's inner surface. In the case of a solid circular cylinder, this combination of loading reduces to a moment $\mathcal{M}$ and an axial force $\mathcal{N}$. If only the twisting moment $\mathcal{M}$ is applied, then the tube or the cylinder will experience torsion as well as change in length and diameter. In this spirit, a general experiment involving the torsion of tubes often consists of analysing and calculating $\mathcal{M}$, $\mathcal{N}$ and P, as is customary in investigating, for example, the mechanical behaviour of arteries (e.g., see Holzapfel et al., 2000). Accordingly, as our starting point we consider a deformation involving simple extension and torsion, with the pressure P only applied to maintain the deformation. While much of the material to be presented below for a general strain energy function may be ascertained from the seminal work of Rivlin, it is summarised here for the convenience of the modern reader.

2.1. Simple extension and torsion of tubes

For an incompressible, isotropic and hyperelastic tube with internal and external radii in the undeformed configuration denoted by R_i and R_o, respectively, we consider the following sequence of deformations as proposed by Rivlin (1949a): A simple axial extension with the extension ratio λ_z together with a simple torsion through an angle proportional to the distance of the plane of torsion from one end, where we denote the constant of proportionality by ϕ and call it the twisting angle measured per unit length of the deformed tube. See Figure 1 for the definition of dimensions, directions and the coordinates in the considered cylindrical tube.

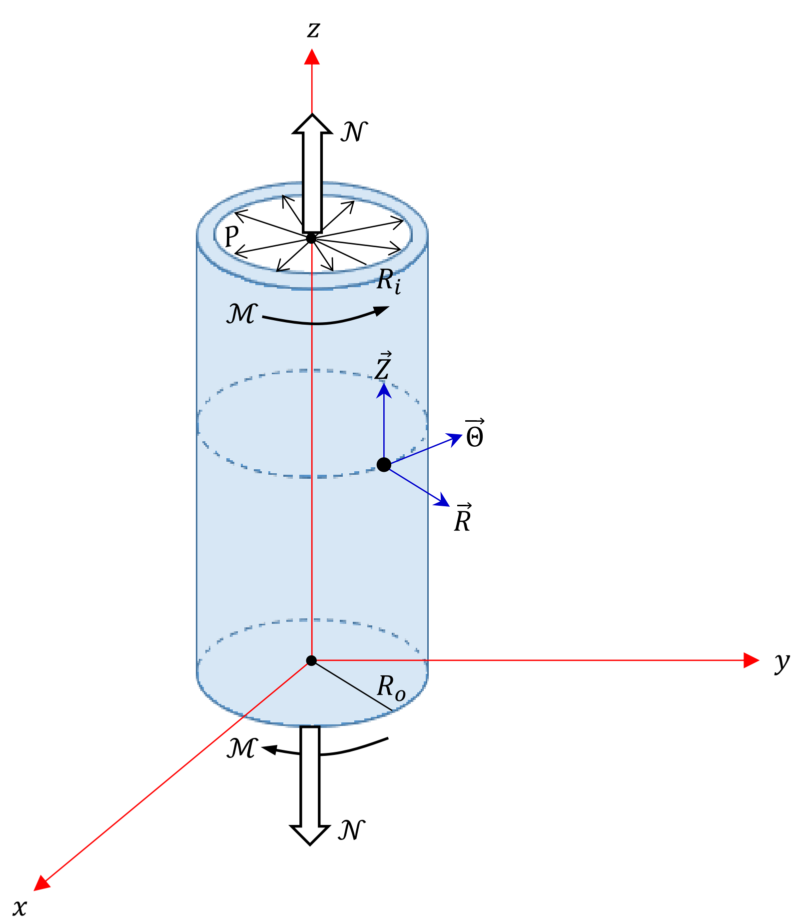

Figure 1 – Loads, dimensions and principal directions in the undeformed cylindrical tube: axial, circumferential and radial directions are denoted by Z, Θ and R, respectively; the resultant axial load, moment and internal pressure are denoted by $\mathcal{N}$, $\mathcal{M}$ and P. The internal and external undeformed radii are designated by R_i and R_o.

The final position of a point in the tube initially at (R, Θ, Z) will be located at (r, θ, z) under the foregoing sequence of deformations, rendering the description of the kinematics of this deformation as:

$$\begin{cases} r = \lambda_z^{-\frac{1}{2}} R \,, \\ \theta = \Theta + \phi \lambda_z Z \,, \\ z = \lambda_z Z \,. \end{cases} \tag{3}$$

Note that R is the radial distance of the point on the undeformed tube from the tube's central axis and the specified isochoric kinematics in equation (3) ensures incompressibility.

If an incompressible isotropic hyperelastic tube is subjected to an axial extension followed by a twist, it was shown by Rivlin (1949a) and Gent and Rivlin (1952) that the resultant moment, axial force and pressure at the inner surface are:

$$\mathcal{M} = 4\pi\phi \int_{R_i}^{R_o} R^3 (W_1 + \lambda_z^{-1} W_2)\, \mathrm{d}R \,,$$

$$\mathcal{N} = 4\pi(\lambda_z - \lambda_z^{-2}) \int_{R_i}^{R_o} R(W_1 + \lambda_z^{-1} W_2) \mathrm{d}R - 2\pi\phi^2 \int_{R_i}^{R_o} R^3 (W_1 + 2\lambda_z^{-1} W_2) \mathrm{d}R$$
$$+ 2\pi R_i^2 \phi^2 \int_{R_i}^{R_o} R W_1 \mathrm{d}R \,,$$

$$P = 2\phi^2 \lambda_z \int_{R_i}^{R_o} R W_1 \mathrm{d}R, \tag{4}$$

where the internal pressure P is required to maintain the deformation. The subscript notation on W in equation (4) denotes partial differentiation with respect to the appropriate invariant, and these derivatives are evaluated at the values of the invariants given by:

$$\begin{cases} I_1 = \lambda_z^2 + 2\lambda_z^{-1} + R^2\phi^2\lambda_z \, , \\ I_2 = \lambda_z^{-2} + 2\lambda_z + R^2\phi^2 \, . \end{cases} \tag{5}$$

The results for a generalised neo-Hookean material follow on formally setting $W_2 = 0$ in equation (4).

2.2. Simple torsion

To maintain a state of simple torsion in an incompressible isotropic hyperelastic tube, where no additional extension and change of tube diameter are desired, the moment $\mathcal{M}$, which may be obtained from equation $(4)_1$ by setting $\lambda_z = 1$, is:

$$\mathcal{M} = 4\pi\phi \int_{R_i}^{R_o} R^3(W_1 + W_2)\mathrm{d}R \, , \tag{6}$$

as well as the axial load $\mathcal{N}$, obtained from equation $(4)_2$ as:

$$\mathcal{N} = -2\pi\phi^2 \int_{R_i}^{R_o} R^3(W_1 + 2W_2)\,\mathrm{d}R + 2\pi\phi^2 R_i^2 \int_{R_i}^{R_o} RW_1\,\mathrm{d}R \, . \tag{7}$$

However, note that the inflation pressure P must also be applied in order to prevent the tube from decreasing in diameter and therefore to maintain the state of simple torsion. From equation $(4)_3$ one may obtain the expression for the required inflation pressure P as:

$$P = 2\phi^2 \int_{R_i}^{R_o} RW_1\mathrm{d}R \, . \tag{8}$$

In the foregoing, the derivatives of W are evaluated at:

$$I_1 = I_2 = 3 + R^2\phi^2 \, . \tag{9}$$

The expressions for P in equations (4) and (8) may be traced back first to the work of Rivlin (1956). Experimental results on vulcanized rubber tubes have verified the need for application of this inflation pressure if the state of simple torsion is to be maintained (see, e.g., Gent and Rivlin, 1952). The analogous results for a generalised neo-Hookean material may be obtained from the foregoing on formally setting $W_2 = 0$.

3. Extension and torsion of a solid circular cylinder

The relationships presented in §2 describe a general deformation involving extension and torsion in cylindrical tubes. From those equations, by formally setting $R_i = 0$ and discounting the inflation pressure P, the analogous relationships for torsion in an incompressible isotropic hyperplastic solid cylinder may be obtained.

3.1. Simple extension and torsion

On formally setting $R_i = 0$ in equation (4) we obtain:

$$\begin{cases} \mathcal{M} = 4\pi\phi \int_0^{R_o} R^3(W_1 + \lambda_z^{-1}W_2)\mathrm{d}R \,, \\ \mathcal{N} = 4\pi(\lambda_z - \lambda_z^{-2}) \int_0^{R_o} R(W_1 + \lambda_z^{-1}W_2)\mathrm{d}R - 2\pi\phi^2 \int_0^{R_o} R^3(W_1 + 2\lambda_z^{-1}W_2)\mathrm{d}R \,. \end{cases} \tag{10}$$

The above relationships have also been presented by Rivlin and Saunders (1951) and summarised in Kanner and Horgan (2008). The analogous results for a generalised neo-Hookean material may be obtained from the foregoing on formally setting $W_2 = 0$.

3.2. Simple torsion

A state of simple torsion in a solid circular cylinder may be maintained if the change of length in the cylinder is prevented. By setting $\lambda_z = 1$ in equation (10), we get:

$$\begin{cases} \mathcal{M} = 4\pi\phi \int_0^{R_o} R^3(W_1 + W_2)\mathrm{d}R \ , \\ \mathcal{N} = -2\pi\phi^2 \int_0^{R_o} R^3(W_1 + 2W_2)\mathrm{d}R \ . \end{cases} \tag{11}$$

These well-known relationships may be initially seen in Rivlin (1949b) and later alluded to in Rivlin (1956). The analogous results for a generalised neo-Hookean material may be obtained from the foregoing on formally setting $W_2 = 0$.

4. Specialisation of the problem of extension and torsion in tubes using the new model

In this section we present the expressions for the resultant couple, axial load and the inflation pressure specialised to the model (1). First we observe that the constraint (2) reads:

$$\lambda_z^2 + 2\lambda_z^{-1} + R^2\phi^2\lambda_z < 3N \ , \tag{12}$$

so that the axial stretch λ_z and twist ϕ are constrained in a coupled fashion. To ensure that the *local* constraint (12) holds for all R in $[R_i, R_o]$, we assume henceforth the stronger *global* constraint:

$$\lambda_z^2 + 2\lambda_z^{-1} + R_o^2\phi^2\lambda_z < \ 3N \ . \tag{13}$$

For a given extensibility parameter N and a given axial stretch λ_z, the inequality (13) constrains the total angle of twist $R_o\phi\lambda_z^{-\frac{1}{2}}$ that the tube can undergo. This, of course, is a reflection of the limiting chain extensibility inherent in the model (1). Two special cases for which the twist and axial stretch are uncoupled are worth noting:

(a) When $\phi = 0$, we have pure extension of the tube and (13) reduces to:

$$\lambda_z^2 + 2\lambda_z^{-1} < 3N \ , \tag{14}$$

which is a constraint on the total extension allowed in that deformation; and

(b) When $\lambda_z = 1$, we have pure torsion and in this case the total angle of twist is constrained by:

$$R_o\phi < \sqrt{3(N-1)}\,. \tag{15}$$

4.1. Simple extension and torsion

By noting that:

$$W_1 = \frac{\partial W}{\partial I_1} = \frac{1}{6}\mu\left(\frac{I_1 - 9N}{I_1 - 3N}\right), \tag{16}$$

where I_1 is given by equation $(5)_1$ and performing the integrals we find that the resultant moment $\mathcal{M}$, the axial load $\mathcal{N}$ and the inflation pressure P may be obtained from equation (4) as:

$$\mathcal{M} = \frac{\mu\pi}{6\phi^3\lambda_z^3}\left\{12N(\lambda_z^3 - 3N\lambda_z + 2)\ln\left[\frac{\lambda_z^3 + \lambda_z(R_o^2\phi^2\lambda_z - 3N) + 2}{\lambda_z^3 + \lambda_z(R_i^2\phi^2\lambda_z - 3N) + 2}\right]\right.$$

$$\left. + (R_o^2 - R_i^2)\phi^2\lambda_z^2[\lambda_z\phi^2(R_o^2 + R_i^2) - 12N]\right\},$$

$$\mathcal{N} = \frac{\mu\pi}{6}\left\{-\frac{6N}{\phi^2\lambda_z^3}[2(\lambda_z^3 - 1) + \lambda_z^3 - 3N\lambda_z + 2 + R_i^2\phi^2\lambda_z^2]\ln\left[\frac{\lambda_z^3 + \lambda_z(R_o^2\phi^2\lambda_z - 3N) + 2}{\lambda_z^3 + \lambda_z(R_i^2\phi^2\lambda_z - 3N) + 2}\right]\right.$$

$$\left. + (R_o^2 - R_i^2)\left[\frac{2(\lambda_z^3 - 1)}{\lambda_z^2} + \frac{6N}{\lambda_z} - \frac{1}{2}\phi^2(R_o^2 - R_i^2)\right]\right\},$$

$$P = \frac{1}{3}\mu\left\{-3N\ln\left[\frac{\lambda_z^3 + \lambda_z(R_o^2\phi^2\lambda_z - 3N) + 2}{\lambda_z^3 + \lambda_z(R_i^2\phi^2\lambda_z - 3N) + 2}\right] + \frac{1}{2}\phi^2\lambda_z(R_o^2 - R_i^2)\right\}. \tag{17}$$

Remark. The celebrated Poynting effect for *isotropic* materials describes the elongation of a solid cylinder on twisting, i.e., when it is subjected to simple torsion. Therefore, if one wishes to keep the axial length of the specimen constant, a compressive axial force (negative $\mathcal{N}$) in addition to the twisting moment $\mathcal{M}$ must be applied to the sample. However, Kanner and Horgan (2008) established an interesting prediction in the axial deformation of solid circular cylinders under simple extension and torsion when strain-stiffening energy functions with limiting chain extensibility are considered, e.g., the Gent model. They demonstrated that a threshold stretch exists below which the stretched cylinder tends to elongate but above which the cylinder will tend to shorten upon further twisting if no additional force is applied. Here we extend their approach to cylindrical tubes and establish the condition whereby this transitional Poynting-type effect is obtained.

As highlighted in Kanner and Horgan (2008), the sign of $\frac{\partial \mathcal{N}}{\partial \phi}$, i.e., whether $\mathcal{N}$ is a monotonically increasing or decreasing function of ϕ, determines if the additional axial force required upon further twisting should be positive or negative. For example, in the case of simple extension and torsion of a cylindrical tube composed of a neo-Hookean material we have:

$$\frac{\partial \mathcal{N}}{\partial \phi} = -\frac{\pi \mu_0 \phi}{2} \left(R_o^4 - R_i^4\right) , \qquad (18)$$

where $\mathcal{N}$ is obtained using equation $(4)_2$ by formally setting $W_2 = 0$. It is readily observed that $\frac{\partial \mathcal{N}}{\partial \phi} < 0$, and therefore the neo-Hookean model always predicts that the additional axial load should be compressive irrespective of the amount of axial λ_z stretch. For the model (1) it can be shown, on using equation $(17)_2$, that the corresponding condition holds if:

$$\lambda_z^2 + \lambda_z^{-1} < 3N \, . \qquad (19)$$

On solving the above cubic inequality, it is possible to establish values of λ_{cz} for each N as follows:

$$\lambda_{cz} = 2\sqrt{N} \cos\left[\frac{1}{3} \cos^{-1}\left(-\frac{1}{2} N^{-\frac{3}{2}}\right)\right] . \qquad (20)$$

The plot in Figure 2 shows this solution for λ_{cz} versus different values of N. For $\lambda_z < \lambda_{cz}$, the classical Poynting-type effect is predicted, i.e., the stretched tube will tend to elongate upon twisting. However, when $\lambda_z > \lambda_{cz}$, a reversal in the Poynting-type effect is predicted, so that the stretched tube will tend to shorten on twisting. We note that a similar qualitative trend between the limiting chain extensibility parameter J_m of the Gent model and the threshold axial stretch λ_{cz} has been demonstrated by Kanner and Horgan (2008); see figure 1 therein.

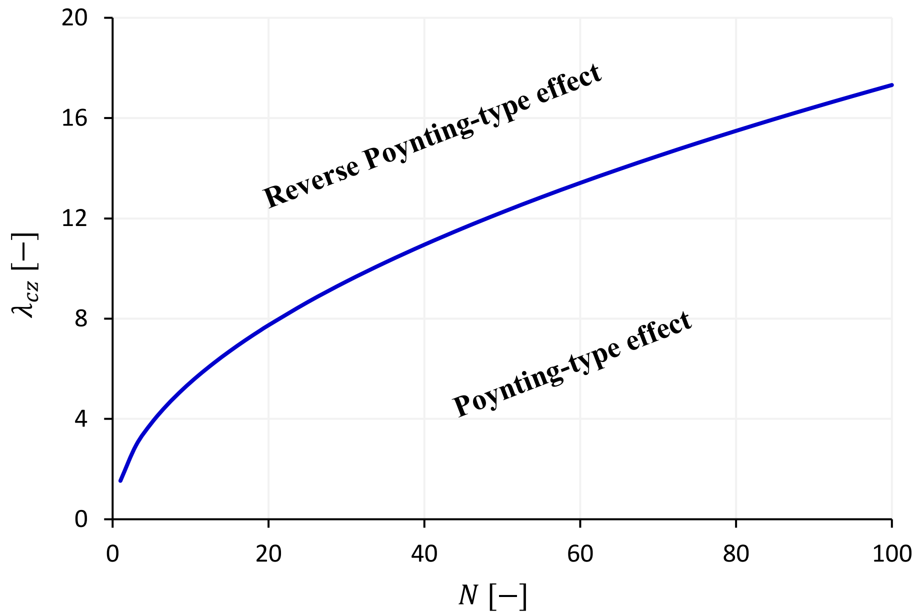

Figure 2 – The relationship between N and λ_{cz} as given by equation (20). For stretches λ below the curve at each N the classical Poynting-type effect is expected, while for λ above the curve the Poynting-type effect is predicted to be reversed.

4.2. Simple torsion

In the case of only a simple torsion in the tube, i.e., only twisting without extension and change of the tube diameter, we note that $I_1 = 3 + R^2\phi^2$ and on using equation (17) with $\lambda_z = 1$ we find that:

$$\mathcal{M} = \frac{\mu\pi}{6\phi^3}\left\{-36N(N-1)\ln\left(\frac{R_o^2\phi^2 - 3N + 3}{R_i^2\phi^2 - 3N + 3}\right) + \phi^2(R_o^2 - R_i^2)[\phi^2(R_o^2 + R_i^2) - 12N]\right\},$$

$$\mathcal{N} = \frac{\mu\pi}{6}\left\{6N\left(\frac{3(N-1)}{\phi^2} - R_i^2\right)\ln\left(\frac{R_o^2\phi^2 - 3N + 3}{R_i^2\phi^2 - 3N + 3}\right) + (R_o^2 - R_i^2)\left[6N - \frac{1}{2}\phi^2(R_o^2 - R_i^2)\right]\right\},$$

$$P = \frac{1}{3}\mu\left\{-3N\ln\left(\frac{R_o^2\phi^2 - 3N + 3}{R_i^2\phi^2 - 3N + 3}\right) + \frac{1}{2}\phi^2(R_o^2 - R_i^2)\right\}. \tag{21}$$

5. Extension and torsion in solid circular cylinders using the new model

Using the expressions for the resultant couple and the axial load for circular cylinders presented in §3, here we present the specialised relationships for the model (1). The constraints (13) and (15) due to limiting extensibility are still assumed to hold.

5.1. Simple extension and torsion

In the case of simple extension and torsion, noting from equation (5) that $I_1 = \lambda_z^2 + 2\lambda_z^{-1} + R^2\phi^2\lambda_z$, on using equation (10) we find:

$$\mathcal{M} = \frac{\mu\pi}{6\phi^3\lambda_z^3}\left\{12N\,(\lambda_z^3 - 3N\lambda_z + 2)\ln\left[\frac{\lambda_z^3 + \lambda_z(R_o^2\phi^2\lambda_z - 3N) + 2}{\lambda_z^3 - 3N\lambda_z + 2}\right]\right.$$
$$\left. + \lambda_z^2\phi^2 R_o^2(\lambda_z\phi^2 R_o^2 - 12N)\right\},$$

$$\mathcal{N} = \frac{\mu\pi}{6}\left\{-\frac{6N}{\phi^2\lambda_z^3}\,[2(\lambda_z^3 - 1) + \lambda_z^3 - 3N\lambda_z + 2]\,\ln\left[\frac{\lambda_z^3 + \lambda_z(R_o^2\phi^2\lambda_z - 3N) + 2}{\lambda_z^3 - 3N\lambda_z + 2}\right]\right.$$
$$\left. + R_o^2\left[\frac{2(\lambda_z^3 - 1)}{\lambda_z^2} - \frac{\phi^2 R_o^2}{2} + \frac{6N}{\lambda_z}\right]\right\}. \qquad (22)$$

Note that these expressions can also be obtained from the counterpart relationships for a tube in equation (17) by formally setting $R_i = 0$. As regards the Poynting-type effect, the condition in (19) also holds for the simple extension and torsion of solid circular cylinders, and the analysis presented in §4.1 is applicable in this case too – see equation (20) for the solution for λ_{cz} and the plot in Figure 2.

5.2. Simple torsion

In simple torsion, where we have $I_1 = 3 + R^2\phi^2$, we find that:

$$\mathcal{M} = \frac{\mu\pi}{6}\left[-\frac{36N(N-1)}{\phi^3}\ln\left(\frac{R_o^2\phi^2 - 3N + 3}{3-3N}\right) + R_o^2\left(\phi R_o^2 - \frac{12N}{\phi}\right)\right],$$

$$\mathcal{N} = \frac{\mu\pi}{6}\left[\frac{18N(N-1)}{\phi^2}\ln\left(\frac{R_o^2\phi^2 - 3N + 3}{3-3N}\right) - R_o^2\left(\frac{1}{2}\phi^2 R_o^2 - 6N\right)\right]. \qquad (23)$$

The expressions in equation (23) may be derived on formally setting $R_i = 0$ in equation (21), or alternatively from equation (17) on setting $\lambda_z = 1$. These results are the analogues of equations (4.19) and (4.20) in Kanner and Horgan (2008) obtained there for the Gent model.

6. Determination of the proposed strain energy function by torsion experiments in solid cylinders using the *scaling law*

As was shown in §3.2, equation (11), the twisting moment $\mathcal{M}$ and axial force $\mathcal{N}$ in the simple torsion of a solid circular cylinder are determined by the integrals of linear combinations of W_1 and W_2. Penn and Kearsley (1976) used an interesting approach to reformulate these expressions and derived what they referred to as *scaling laws* for the torsion of incompressible rubber cylinders. Their reformulation essentially expresses *reduced* forms of $\mathcal{M}$ and $\mathcal{N}$ in terms of W_1 and W_2, which only depends on the radius R_o and the twist ϕ. Therefore, by experimentally measuring $\mathcal{M}$, $\mathcal{N}$, R_o and ϕ, W_1 and W_2 may be determined. For the convenience of the reader, we briefly summarise their approach in the following.

Since in simple torsion $I_1 = I_2 = 3 + R^2\phi^2$, on using the change of variable $\zeta = R^2\phi^2$, both R and ϕ can be eliminated from the integrals of the expressions $\mathcal{M}$ and $\mathcal{N}$ in equation (11) as follows:

$$\begin{cases} \mathcal{M} = \dfrac{2\pi}{\phi^3}\displaystyle\int_0^{\chi} \zeta(W_1 + W_2)\mathrm{d}\zeta\,, \\ \mathcal{N} = -\dfrac{\pi}{\phi^2}\displaystyle\int_0^{\chi} \zeta(W_1 + 2W_2)\mathrm{d}\zeta\,, \end{cases} \qquad (24)$$

where $\chi = R_0^2\phi^2$. It may be observed that the results of the integrals in equation (24) will be only a function of χ, which we denote by $\mathscr{f}(\chi)$ and $\mathscr{g}(\chi)$, respectively. Using the relationships in equation (11), and the $\mathscr{f}(\chi)$ and $\mathscr{g}(\chi)$ functions in equation (24), we arrive at:

$$\begin{cases} \dfrac{\mathcal{M}}{R_0^4\phi} = \mathscr{f}(\chi)\,, \\ \\ \dfrac{\mathcal{N}}{R_0^4\phi^2} = \mathscr{g}(\chi)\,, \end{cases} \tag{25}$$

with the expression of $\mathcal{M}$ and $\mathcal{N}$ on the left-hand side of the above equation referred to as the *reduced* twisting moment and axial force, respectively. Given that $\mathscr{f}(\chi)$ and $\mathscr{g}(\chi)$ directly involve the integration of W_1 and W_2, experimentally measuring $\mathcal{M}$, $\mathcal{N}$, R_o and ϕ paves the way for the determination of W_1 and W_2. In the case of a generalised neo-Hookean material, $W_2 = 0$, and thus the response function W_1 may be established by experimentally measuring either $\mathcal{N}$ or $\mathcal{M}$ (along with R_o and ϕ) in simple torsion tests.

To this end, Penn and Kearsley (1976) performed a set of experiments on natural gum rubber samples and report the *reduced* values of $\mathcal{M}$ and $\mathcal{N}$ versus $R_0^2\phi^2$. We collated the reported experimental $\mathcal{N}/R_0^4\phi^2$ vs $R_0^2\phi^2$ data, to which we empirically fitted the following function:

$$\mathscr{g}(\chi) = a[\chi - \ln(\chi^b) + c]\,, \tag{26}$$

where *a*, *b*, *c* are dimensionless constants. The data versus the fitting result is shown in Figure 3. The favourability of the function in capturing the data is evident from the plot. Note that, from equations $(24)_2$ and $(25)_2$, $\mathscr{g}(\chi)$ is indeed the integral of W_1; therefore:

$$\frac{\partial \mathscr{g}(\chi)}{\partial \chi} = a\frac{\chi - b}{\chi} \equiv W_1\,. \tag{27}$$

It is evident from equation (27) that W_1 is reminiscent of a [1/1] rational Padé approximation function in χ, as is the response function of our model in equation (1) – see Anssari-Benam (2021) for a detailed analysis. Indeed, by noting that $\chi = I_1 - 3$ and the procedure described in Anssari-Benam (2021) to introduce the number of links N for the limit of extensibility and imposing the condition that as $N \to \infty$ one must recover the shear modulus, the W_1 is readily equal to:

$$W_1 = \frac{1}{2n}\mu\frac{I_1 - 3nN}{I_1 - 3N}, \tag{28}$$

for integer values of $n = 1, 2 \ldots,$ which for $n = 3$ recovers the exact response function of the model, i.e., equation (16). This result further corroborates the suitability of the proposed strain energy function (1) for application to the torsion of rubber-like materials. To the best of our knowledge, such experimental validation has not been yet presented for other limiting chain extensibility models in the literature.

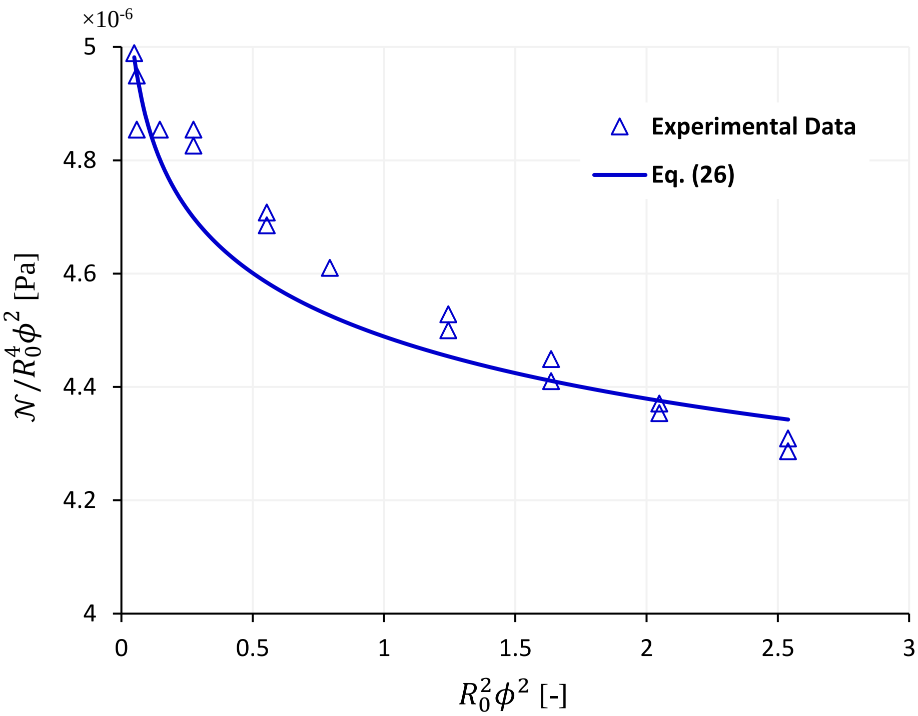

Figure 3 – The reduced axial load $\mathcal{N}$ versus $R_0^2\phi^2$ for a natural gum rubber sample. The markers represent the collated experimental datapoints, and the continuous line shows the model prediction using the empirical function in equation (26). The R^2 value for this fit is 0.98, with $a = 0.005$, $b = 974.87$ and $c = 35.83$. The experimental data is due to Penn and Kearsley (1976).

7. Concluding remarks

We have considered the application of a newly proposed model (1) within the family of the generalised neo-Hookean strain energy functions with limiting chain extensibility to the torsion of rubber-like tubes and cylinders. Expressions for the twisting moment $\mathcal{M}$, axial load $\mathcal{N}$ and the inflation pressure P were derived for the general combined deformations of a tube under extension and torsion and subsequently specialised to results for solid cylinders. Theoretical predictions of the existence of a critical axial stretch λ_{cz} beyond which the Poynting-type effect is reversed upon twisting were established and formulated for both stretched tubes and stretched solid cylinders under torsion. Furthermore, it was demonstrated that the response function of the proposed model is compatible with the scaling law of Penn and Kearsley (1976) and may be determined directly from their experimental data. Such verification has not been yet demonstrated for other limiting chain extensibility models in the literature.

While the problem of torsion in rubber-like materials has been extensively studied, some features analysed in this paper were previously less well-known. These include the critical axial stretch λ_{cz} for the reversal of the Poynting-type effect in tubes with limiting chain extensibility under torsion and demonstration of compatibility with the *scaling law* of Penn and Kearsley (1976). The application of our proposed model to the problem of torsion and its capability in capturing such nuance aspects of this inhomogeneous deformation was also presented herein for the first time.

With the foregoing preliminary results in analysing the problem of torsion using the limiting chain extensibility model (1) presented in this work, some problems of interest merit further investigation. It would, for example, be interesting to experimentally verify the critical axial stretch λ_{cz} in various rubber tube samples. Another interesting scope of investigation is the potential merit of the model for application to the biomechanics of arteries, and thus the generalisation of the model (1) to include anisotropy. The potential capability of the model in providing a more accurate characterisation of the biomechanical properties of arteries involving torsion tests merits a detailed exploration.

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