POLYNOMIAL HAMILTONIAN SYSTEMS WITH MOVABLE ALGEBRAIC SINGULARITIES

By

THOMAS KECKER

Abstract. The singularity structure of solutions of a class of Hamiltonian systems of ordinary differential equations in two dependent variables is studied. It is shown that for any solution, all movable singularities obtained by analytic continuation along a rectifiable curve are at most algebraic branch points.

1 Introduction

Singularities of solutions of ordinary differential equations can be classed as being either fixed or movable. The set of fixed singularities consists of the points in the complex plane where the equation itself becomes singular in some sense. All other singularities of a solution are called movable, as their positions vary with the initial conditions of the equation.

In the 1900's, P. Painlevé [17] classified all rational second-order equations

$$y'' = R(z, y, y'), \tag{1.1}$$

for which all solutions are single-valued around their movable singularities, a property known as the **Painlevé property**. This classification, with some errors and gaps which have been filled by R. Fuchs, B. Gambier and others, led to the discovery of six non-linear equations now known as the six Painlevé equations. The result of the classification is that the solution of any equation of the form (1.1) that has the Painlevé property can be expressed by the solutions of classically known function (e.g., elliptic functions, the solutions of linear differential equations, or equations that are solvable by quadrature) and the solutions of the six non-linear Painlevé equations. Interestingly, to each Painlevé equation is associated an equivalent Hamiltonian system

$$\frac{dq}{dz} = \frac{\partial H_J}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial H_J}{\partial q}, \tag{1.2}$$

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol.
DOI

with Hamiltonians $H_J(z, p, q)$, $J = 1, \dots, 6$, polynomial in p and q. This was already noticed by J. Malmquist [10] and later extensively studied by K. Okamoto [12, 13, 14, 15].

A classification of systems of equations in two variables

$$y_1' = P(z, y_1, y_2), \quad y_2' = Q(z, y_1, y_2), \tag{1.3}$$

that possess the Painlevé property was given by R. Garnier [3] in the autonomous case, where $P = P(y_1, y_2)$ and $Q = Q(y_1, y_2)$ are homogeneous rational functions of y_1 and y_2. The solutions in this case are given by elliptic functions or a combination of rational and exponential functions. J. Goffar-Lombet [4] classified those systems (1.3) with the Painlevé property in which P and Q are certain polynomials of degree at most 3 in y_1 and y_2 with z-dependent analytic coefficients and showed that the solutions can be given in terms of classically known functions and Painlevé transcendents. T. Kimura and T. Matuda [8] extended this result to the case in which the degrees of P and Q are less than or equal to 5. They conjectured that any system (1.3) with the Painlevé property is equivalent to one of the systems (1.2).

Lifting the restriction of the Painlevé property, a number of authors have studied classes of second-order differential equations for which all movable singularities are at most algebraic branch points. In [19, 20], S. Shimomura considered the equations

$$y'' = \frac{2(2k+1)}{(2k-1)^2} y^{2k} + z, \quad k \in \mathbf{N},$$
$$y'' = \frac{k+1}{k^2} y^{2k+1} + zy + \alpha, \quad k \in \mathbf{N} \setminus \{2\},$$

of P_I-type and P_{II}-type, respectively, and showed that all movable singularities that can be reached by analytic continuation along a rectifiable curve are algebraic branch points. More generally, G. Filipuk and R. Halburd [2] studied equations of the form

$$y'' = P(z, y), \tag{1.4}$$

where P is a polynomial in y with analytic coefficients in z that satisfy one or two differential relations known as resonance conditions (these are equivalent to the existence of certain formal algebraic series solutions of (1.4)). Equation (1.4) can be viewed as a Hamiltonian system with Hamiltonian

$$H(z, y_1, y_2) = \frac{1}{2} y_2^2 - \hat{P}(z, y_1),$$

where $\hat{P}$ is a polynomial with $\hat{P}_y = P$ and we have the correspondence $y = y_1$, $y' = y_2$. In this article, we consider a much more general class of Hamiltonian systems in two variables. The way we prove that any movable singularity of a solution that can be reached by analytic continuation along a finite length curve is an algebraic branch point is based on a method used in certain proofs of the Painlevé property for the Painlevé equations. In particular, we mention the proofs in [7], [16], and [18]; see also [5].

2 Hamiltonian systems in two variables

Consider the Hamiltonian system given by

$$H(z, y_1, y_2) = \alpha_{M+1,0}(z)y_1^{M+1} + \alpha_{0,N+1}(z)y_2^{N+1} + \sum_{(i,j)\in I} \alpha_{ij}(z)y_1^i y_2^j, \tag{2.1}$$

where the set of indices I is defined by

$$I = \{(i, j) \in \mathbf{N}^2 : i(N + 1) + j(M + 1) < (N + 1)(M + 1)\} \tag{2.2}$$

and $\alpha_{ij}(z)$, $(i, j) \in I \cup \{(M + 1, 0), (0, N + 1)\}$, are analytic functions in some common domain $\Omega \subset \mathbf{C}$. The Hamiltonian equations are given by

$$\begin{aligned} y_1' &= (N + 1)\alpha_{0,N+1}(z)y_2^N + \sum_{(i,j)\in I} j\alpha_{ij}(z)y_1^i y_2^{j-1}, \\ y_2' &= -(M + 1)\alpha_{M+1,0}(z)y_1^M - \sum_{(i,j)\in I} i\alpha_{ij}(z)y_1^{i-1} y_2^j. \end{aligned} \tag{2.3}$$

The set I is chosen so that y_2^N and y_1^M turn out to be the dominant terms on the right hand sides of the system (2.3) in the vicinity of any singularity z_∞ for which $\alpha_{M+1,0}(z_\infty), \alpha_{0,N+1}(z_\infty) \neq 0$. We define the set

$$\Phi = \{z_0 \in \Omega : \alpha_{M+1,0}(z_0) = 0\} \cup \{z_0 \in \Omega : \alpha_{0,N+1}(z_0) = 0\}.$$

A singularity z_∞ of some solution $(y_1(z), y_2(z))$ of the system (2.3) is called **fixed** if $z_\infty \in \Phi$. A singularity $z_\infty \notin \Phi$ of a solution of (2.3) is called **movable**. Intuitively, the position of a movable singularity changes when the initial conditions of the system of differential equations are varied, whereas the fixed singularities are determined by the equation itself. A more general definition of fixed and movable singularities for first-order systems of differential equations can be found in [11]; see also [9] for the case of second-order differential equations.

To determine possible leading order behaviours of a solution (y_1, y_2) of (2.7) about a movable singularity z_∞, suppose that

$$y_1 \sim c_p(z - z_\infty)^p, \quad y_2 \sim c_q(z - z_\infty)^q.$$

Assuming that the leading order terms of the right hand side of (2.3) are y_2^N and y_1^M, respectively, we must have that

$$p - 1 = Nq \text{ and } q - 1 = Mp \text{ implies } p = -\frac{N+1}{MN-1} \text{ and } q = -\frac{M+1}{MN-1}.$$

A necessary condition for all solutions to have movable algebraic branch points is the existence of certain formal algebraic series solutions of (2.3):

$$y_1(z) = \sum_{k=-N-1}^{\infty} c_{1,k}(z-z_0)^{k/(MN-1)}, \quad y_2(z) = \sum_{k=-M-1}^{\infty} c_{2,k}(z-z_0)^{k/(MN-1)}, \tag{2.4}$$

about each point $z_0 \in \Omega \setminus \Phi$. The main result of this article is that the existence of a certain number of such formal series solutions is also a sufficient condition for every movable singularity of a solution of (2.3) to be of this form. This result should be compared to the fact that for an ODE, passing the Painlevé test is not equivalent to having the Painlevé property. The existence of the series solutions is equivalent to a number of differential relations between the coefficient functions $\alpha_{ij}(z)$, $(i, j) \in I$, of the Hamiltonian H, known as resonance conditions, which can be calculated algorithmically. They arise from the fact that an attempt to determine recursively the coefficients $c_{1,k}$, $k = -N-1, -N, \dots$ and $c_{2,k}$, $k = -M-1, -M, \dots$ by inserting the series (2.4) into the system (2.3) breaks down at certain stages known as resonances; and one is left with identities that need to be satisfied identically, leaving one coefficient at the resonance arbitrary.

Theorem 1. *Suppose that at each point $z_0 \in \Omega \setminus \Phi$, the Hamiltonian system* (2.3) *admits formal series solutions of the form* (2.4) *for every pair of values $(c_{1,-N-1}, c_{2,-M-1})$ satisfying*

$$\begin{aligned} c_{1,-N-1}^{MN-1} &= -\left(\alpha_{0,N+1}(z_0)\alpha_{M+1,0}(z_0)^N (MN-1)^{N+1}\right)^{-1}, \\ c_{2,-M-1} &= (MN-1)\alpha_{M+1,0}(z_0)c_{1,-N-1}^M. \end{aligned}$$

Let $\gamma \subset \Omega$ be a finite length curve with endpoint $z_\infty \in \Omega \setminus \Phi$ such that a solution (y_1, y_2) can be continued analytically along γ up to, but not including, z_∞. Then the solution is represented by the series (2.4) *at $z_0 = z_\infty$:*

$$\begin{aligned} y_1(z) &= \sum_{k=-(N+1)/d}^{\infty} C_{1,k}(z-z_\infty)^{kd/MN-1}, \\ y_2(z) &= \sum_{k=-(M+1)/d}^{\infty} C_{2,k}(z-z_\infty)^{kd/MN-1}, \end{aligned} \tag{2.5}$$

where $d = \gcd\{M+1, N+1, MN-1\}$, convergent in some punctured, branched neighbourhood of z_∞.

Remark 1. As mentioned above, the existence of the formal series solutions (2.4) is an assumption on the form of the equations, each formal series being equivalent to a differential relation between the coefficients $\alpha_{ij}(z)$. Whereas the existence of the formal series is clearly necessary for the solution to be represented by (2.5) near a movable singularity, Theorem1 states that every movable singularity is of this form. If $(N+1) \nmid (MN-1)$, i.e., $d = 1$, there is really only one leading order behaviour for the solution (2.5), as the choice of branch for $c_{1,-N-1}$ can be absorbed into the choice of branch for $(z-z_\infty)^{1/(MN-1)}$. In general, there are d possible leading order behaviours.

We assume in the following and for the rest of the article that $N \geq M$. In the neighbourhood of any movable singularity, one can set

$$\begin{aligned}
\tilde{y}_1(z) &= \left(\alpha_{M+1,0}(z)^N \alpha_{0,N+1}(z)\right)^{\frac{1}{MN-1}} \left(y_1(z) + \frac{\alpha_{M,0(z)}}{\alpha_{M+1,0}(z)}\right), \\
\tilde{y}_2(z) &= \left(\alpha_{M+1,0}(z)\alpha_{0,N+1}(z)^M\right)^{\frac{1}{MN-1}} \left(y_2(z) + \frac{\alpha_{0,N}(z)}{\alpha_{0,N+1}(z)}\right),
\end{aligned}$$

to ensure that the transformed Hamiltonian $\tilde{H}$ is of the same form as in (2.1) but with $\tilde{\alpha}_{M+1} \equiv 1 \equiv \tilde{\alpha}_{0,N+1}$ and $\tilde{\alpha}_{0N} \equiv 0$ (and also $\tilde{\alpha}_{M0} \equiv 0$ if $N = M$). In the following, we assume that the Hamiltonian is already given in this normalised form and readily omit the tildes again:

$$H(z, y_1, y_2) = y_1^{M+1} + y_2^{N+1} + \sum_{(i,j)\in I'} \alpha_{ij}(z) y_1^i y_2^j, \tag{2.6}$$

where $I' = I \setminus \{(0, N)\}$. The Hamiltonian equations are

$$\begin{aligned}
y_1' &= (N+1)y_2^N + \sum_{(i,j)\in I'} j\alpha_{ij}(z) y_1^i y_2^{j-1}, \\
y_2' &= -(M+1)y_1^M - \sum_{(i,j)\in I'} i\alpha_{ij}(z) y_1^{i-1} y_2^j.
\end{aligned} \tag{2.7}$$

For $N \geq M$, condition (2.2) implies that $j \leq N-1$ for all $(i, j) \in I'$.

3 Preliminary lemmas

We make repeated use of the following lemma of Painlevé; see, e.g., [6].

Lemma 1. *Let $F_k(z, y_1, \ldots, y_m)$, $k = 1, \ldots, m$, be analytic functions in a neighbourhood of $(z_\infty, \eta_1, \ldots, \eta_m) \in \mathbb{C}^{m+1}$. Let γ be a curve with end point z_∞, and suppose that $(y_1, \ldots, y_m)$ are analytic on $\gamma \setminus \{z_\infty\}$ and satisfy*

$$y_k' = F_k(z, y_1, \ldots, y_m), \quad k = 1, \ldots, m.$$

If there exists a sequence $\{z_n\}_{n\in\mathbf{N}} \subset \gamma$ *such that* $z_n \to z_\infty$ *and* $y_k(z_n) \to \eta_k \in \mathbf{C}$ *as* $n \to \infty$ *for all* $k = 1, \dots, n$*, then the solution can be analytically continued to include the point* z_∞*.*

Proof. Choose r such that all F_k, $k = 1, \dots, m$, are analytic in the set $D = \{ |z - z_\infty| \le r, |y_k - \eta_k| \le r, k = 1, \dots, m\}$. Let

$$M = \max\{ |F_k(z, y_1, \dots, y_m)| : (z, y_1, \dots, y_m) \in D, \quad k = 1, \dots, m\}.$$

From large enough n,

$$\{ |z - z_n| < r/2, |y_k - y_k(z_n)| < r/2, k = 1, \dots, m\} \subset D.$$

By Cauchy's local existence and uniqueness theorem, a solution around z_n is defined at least in the disc of radius $\rho = \frac{r}{2}\left(1 - e^{-1/(m+1)M}\right)$. For some n, we have $z_\infty \in B(z_n, \rho)$. □

The next lemma is needed to show that an auxiliary function W, which is constructed from the Hamiltonian H in Section 5, is bounded along γ. We show that W satisfies a first-order linear differential equation of the form (3.1) below. The lemma converts this into an integral representation for W.

Lemma 2. *Let* γ *be a finite length curve in the complex plane; and let* $P(z)$*,* $Q(z)$ *and* $R(z)$ *be bounded functions on* γ*. Then any solution of*

$$W' = PW + Q + R', \tag{3.1}$$

is also bounded on γ*.*

Proof. Choose $z_0 \in \gamma$. The solution can then be written as

$$W(z) = R(z) + I(z)\left(C + \int_{z_0}^{z} (Q(\zeta) + P(\zeta)R(\zeta))I(\zeta)^{-1} d\zeta\right),$$

where $C = W(z_0) - R(z_0)$ is an integration constant and I is the integrating factor $I(z) = \exp\left(\int_{z_0}^{z} P(\zeta) d\zeta\right)$. Since P, Q and R are bounded on γ and γ has finite length, $I(z)$ and $I(z)^{-1}$ are bounded, and hence $W(z)$ is also bounded on γ. □

4 Curve modifications

In this section, we show that the curve γ leading up to a singularity can be modified to a curve $\tilde{\gamma}$, still of finite length, that avoids the zeros of a solution (y_1, y_2) of (2.3). This is a technical fact needed to show that the auxiliary function W, to be constructed in Section 5, is bounded on γ. The proof runs along the lines of a lemma by S. Shimomura [18], in which he showed that for a solution $y(z)$ of a second-order ODE of the form $y'' = E(z,y)(y')^2 + F(z,y)y' + G(z,y)$, one can modify a curve γ ending in a singularity to a curve $\tilde{\gamma}$ such that on $\tilde{\gamma}$, y is bounded away from some fixed value c for which the equation is non-singular.

Consider a differential system of two equations in y_1 and y_2 of the form

$$y_1' = F_1(z, y_1, y_2), \quad y_2' = F_2(z, y_1, y_2), \tag{4.1}$$

where $F_1, F_2 \in \mathcal{O}_D[y_1, y_2]$ are polynomials in y_1, y_2 with coefficients analytic in some domain D, which we take to be the disc $D = \{z \in \mathbf{C} : |z-a| \le R_0\}$. Assume that F_1, F_2 are of the form

$$\begin{aligned} F_1(z, y_1, y_2) &= \alpha_{10N_1} y_2^{N_1} + \sum_{j=0}^{M_1} \sum_{k=0}^{N_1-1} \alpha_{1jk}(z) y_1^j y_k^k, \\ F_2(z, y_1, y_2) &= \alpha_{2M_20} y_1^{M_2} + \sum_{j=0}^{M_2-1} \sum_{k=0}^{N_2} \alpha_{2jk}(z) y_1^j y_k^k, \end{aligned} \tag{4.2}$$

where $N_1 \ge N_2$, $M_2 \ge M_1$ and $\alpha_{10N_1}, \alpha_{2M_20}$ are constants with $|\alpha_{10N_1}| \ge 1$, $|\alpha_{2M_20}| \ge 1$. Let $K > 1$ be a constant such that $|\alpha_{ijk}(z)| < K$ for all i, j, k and $z \in D$. Also, let $N_1 := N, M_2 := M$ and $C := 2^{N+1}(M+1)(N+1)K$.

Lemma 3. *Let* $0 < \Delta < 1$ *and* $\theta := \min\{\Delta/C, R_0\}$. *Let* (y_1, y_2) *be a solution of* (4.1) *analytic at a point c for which* $|c-a| < R_0/2$. *Suppose that* $|y_1(c)| < \theta/8$ *and* $|y_2(c)| > C$. *Then* $(y_1(z), y_2(z))$ *is analytic on the disc* $|z-c| < \theta/|y_2(c)|$ *and satisfies* $|y_1(z)| \ge \theta/8$ *and* $|y_2(z)| \ge 1$ *on the circle* $|z-c| = \theta/2|y_2(c)|$.

Proof. Let $\rho = y_2(c)^N$, $\zeta = \rho(z-c)$, and define $\eta_i(\zeta) := y_i(z)$, $i = 1, 2$. We have $\dot{\eta}_i(\zeta) = \rho^{-1} y_i'(z)$ and

$$\eta_i(\zeta) = \eta_i(0) + \int_0^\zeta \dot{\eta}_i(\tilde{\zeta}) d\tilde{\zeta},$$

where $\eta_i(0) = y_i(c)$ and $\dot{} = d/d\zeta$. Define $M_i(r) = \max_{|\zeta| \le r} |\eta_i(\zeta)|$, $i = 1, 2$, and let $r_0 = \sup\{r : M_1(r) < \Delta, M_2(r) < 2|\rho|^{1/N}\}$. Clearly, $r_0 > 0$. Since $|z-a| \le |z-c| + |c-a| < R_0/|\rho| + R_0/2 \le R_0$ for $|\zeta| < \min\{r_0, R_0\}$, we have

$$\begin{aligned} |\eta_i(\zeta)| &\le |y_i(c)| + |\rho|^{-1} |\zeta| \sum_{j=0}^{M_i} \sum_{k=0}^{N_i} K \Delta^j 2^k |\rho|^{k/N} \\ &\le |y_i(c)| + |\zeta| 2^N K (N+1)(M+1). \end{aligned} \tag{4.3}$$

Now suppose that $r_0 < \theta$. Then, for $|\zeta| < r_0 < R_0$,

$$|\eta_1(\zeta)| < \theta(1/8 + 2^N(N+1)(M+1)K) < \Delta,$$
$$|\eta_2(\zeta)| < |y_2(c)| + \theta 2^N(M+1)(N+1)K < 2|y_2(c)|,$$

in contradiction to the definition of r_0. Therefore, $r_0 \geq \theta$, which shows that (4.3), $i = 1, 2$, is valid for $|\zeta| < \theta$, and therefore that η_1 and η_2 are analytic for $|\zeta| < \theta$.

We now obtain estimates for η_1 and η_2 in the opposite direction on the circle $|\zeta| = \theta/2$:

$$\begin{aligned}
|\eta_1(\zeta)| &\geq \left| \int_0^\zeta \rho^{-1}\alpha_{10N}\eta_2(\tilde{\zeta})^N \mathrm{d}\tilde{\zeta} \right| - \left| \int_0^\zeta \rho^{-1} \sum_{i=0}^{M_1} \sum_{j=0}^{N-1} \alpha_{1ij}(z)\eta_1^i\eta_2^j \mathrm{d}\tilde{\zeta} \right| - |\eta_1(0)| \\
&\geq \left| \int_0^\zeta \left(1 + \frac{\eta_2(\tilde{\zeta}) - \eta_2(0)}{\eta_2(0)}\right)^N \mathrm{d}\tilde{\zeta} \right| - \frac{\theta}{2}|\rho|^{-\frac{1}{N}} 2^{N-1}(M+1)NK - \frac{\theta}{8} \\
&\geq \left| \int_0^\zeta \left(1 + \sum_{n=1}^N \binom{N}{n} \left(\frac{\eta_2(\tilde{\zeta}) - \eta_2(0)}{\eta_2(0)}\right)^n\right) \mathrm{d}\tilde{\zeta} \right| - \frac{\theta}{4} \\
&\geq \frac{\theta}{2} - \frac{\theta}{2}\sum_{n=1}^N \binom{N}{n}\left(\frac{\Delta}{C}\right)^n - \frac{\theta}{4} \geq \frac{\theta}{8}, \\
|\eta_2(\zeta)| &\geq |y_2(c)| - \theta 2^N(M+1)(N+1)K \geq 1.
\end{aligned}$$

□

Remark 2. In Lemma 3, the roles of y_1 and y_2 can be interchanged if, in every expression, one simultaneously replaces $M \leftrightarrow N$.

Using Lemma 3 and Remark 2, we can now show that a curve ending in a movable singularity of a solution (y_1, y_2) of the system (4.1) can be modified by arcs of circles in such a way that both y_1 and y_2 are bounded away from 0 on the modified curve. The argument is very similar to that in [18].

Lemma 4 (First curve modification)**.** *Suppose* (y_1, y_2) *is a solution of* (4.1), *analytic on a finite length curve* $\gamma \subset D$ *up to, but not including, its endpoint* $z_\infty \in D$. *Then we can deform* γ, *if necessary, in the region where* (y_1, y_2) *is analytic, to a curve* $\tilde{\gamma}$, *still of finite length, such that* y_1 *and* y_2 *are bounded away from* 0 *on* $\tilde{\gamma}$ *in a neighbourhood of* z_∞.

Proof. Let γ be parametrised by arclength, so that $\gamma(0) = z_0$, $\gamma(l) = z_\infty$ where l is the length of γ. Define

$$S_i := \{s : 0 < s < l \text{ and } |y_i(\gamma(s))| \leq \theta/8\}, \quad i = 1, 2.$$

Assume that $\liminf_{s\to l^-} \min\{|y_1|, |y_2|\} = 0$; otherwise, there is nothing to show. Then $S_1 \cup S_2$ contains values arbitrarily close to l. There now exists some number $0 < s_0 < l$ with the following two properties:

(i) $S_1 \cap S_2 \cap [s_0, l) = \varnothing$,

(ii) $|y_{3-i}(\gamma(s))| > C$ whenever $s \in S_i$, $s > s_0$.

Were (ii) not the case, we could find a sequence $z_i = \gamma(s_i)$, $s_i \to l$, such that $(y_1(z_i), y_2(z_i))$ is bounded; and hence, by Lemma 1, the solution could be analytically continued to z_∞, in contradiction to assumption. Let $S = (S_1 \cup S_2) \cap [s_0, l)$, and let $s_1 = \inf\{s \in S : s > s_0\}$. Suppose that $s_1 \in S_i$, and let $r_1 = \theta/2|y_{3-i}(\gamma(s_1))|$. Lemma 3 now shows that y_1 and y_2 are analytic for $|z - \gamma(s_1)| < 2r_1$ and that $|y_i(z)| \geq \theta/8$ and $|y_{3-i}(z)| \geq 1$ on the circle $C_1 = \{z : |z-\gamma(s_1)| = r_1\}$. We now recursively define a sequence of points s_n and circles C_n with radii r_n as follows. Let $s_{n+1} = \inf\{s \in S : s > s_n + r_n\}$. If $s_{n+1} \in S_i$ ($i = 1$ or 2), let $r_{n+1} = \theta/2|y_{3-i}(\gamma(s_n))|$.

By Lemma 3, for every circle C_n, $n = 1, 2, \ldots$, we have $|y_1(z)|, |y_2(z)| \geq \theta/8$ for all $z \in C_n$. Also, $\sum_{n=1}^{\infty} r_n \leq \sum_{n=1}^{\infty} |s_{n+1} - s_n| \leq l$, which implies $r_n \to 0$ as $n \to \infty$. The centers s_n of the circles accumulate at z_∞. Indeed, were this not the case, we would have $s_n \to s_\infty$ for some $s_\infty < l$; but then

$$\lim_{n\to\infty} \max\{|y_1(\gamma(s_n))|, |y_2(\gamma(s_n))|\} \geq \lim_{n\to\infty} \frac{\theta}{2r_n} = \infty,$$

in contradiction to the fact that $(y_1(z), y_2(z))$ is analytic on $\gamma \setminus \{z_\infty\}$. We now define $\tilde{\gamma}$ as follows. Suppose, for convenience, that γ has no self-intersections (otherwise, we could shorten γ by omitting pieces between self-intersections). Let γ_{ext} be an infinite non-intersecting extension of γ such that $\gamma_{\text{ext}}(s) \to \infty$ for $s \to \pm\infty$ that divides the complex plane into parts $\mathbf{C}_+$ and $\mathbf{C}_-$ such that $\mathbf{C}_+$, γ_{ext} and $\mathbf{C}_-$ are pairwise disjoint and $\mathbf{C}_+ \cup \gamma_{\text{ext}} \cup \mathbf{C}_- = \mathbf{C}$. Now let $D = \gamma \cup \bigcup_{n=1}^{\infty} D_n$, where $D_n = \{z : |z - \gamma(s_n)| \leq r_n\}$, and define $\tilde{\gamma} = \partial D \cap (\mathbf{C}_+ \cup \gamma_{\text{ext}})$. Then (y_1, y_2) is analytic on $\tilde{\gamma}$, and $|y_1(z)|, |y_2(z)| \geq \theta/8$ for all $z \in \tilde{\gamma}$. Furthermore, $\tilde{\gamma}$ has length less than $(1 + 2\pi)l$. □

We now specialise the results obtained so far in this section to the Hamiltonian system (2.7) of the form (4.1) with $N_1 = N$, $M_2 = M$. Lemma 4 does not quite suffice to show that the auxiliary function W defined in Section 5, which is rational in y_1 and y_2, is bounded. We need to show that certain terms of the form y_2^k/y_1^l are bounded. To do so, we apply a second curve modification, where we can now make use of the fact that y_1 and y_2 are bounded away from 0 on γ. We rewrite the system of equations (2.7) in the variables $u_1 = y_1 \cdot y_2^{-(N+1)/(M+1)}$ and $u_2 = y_2$ for some branch of $y_2^{1/(M+1)}$.

The system of equations in the variables u_1, u_2 becomes

$$
\begin{aligned}
u_1' &= (N+1)u_2^{N-\frac{N+1}{M+1}}\left(1+u_1^{M+1}\right) + \sum_{(i,j)\in I'}\left(j+i\frac{N+1}{M+1}\right)\alpha_{ij}u_1^i u_2^{(i-1)\frac{N+1}{M+1}+j-1} \\
u_2' &= -(M+1)u_1^M u_2^{M\frac{N+1}{M+1}} - \sum_{(i,j)\in I'} i\alpha_{ij}u_1^{i-1}u_2^{(i-1)\frac{N+1}{M+1}+j}.
\end{aligned}
\tag{4.4}
$$

Let $K > 1$ be a constant such that $|i\alpha_{ij}(z)| < K$ and $\left|\left(j+i\frac{N+1}{M+1}\right)\alpha_{ij}(z)\right| < K$ for all $(i,j) \in \tilde{I} = I' \cup \{(M+1,0),(0,N+1)\}$, $z \in D$. As before, let $C = 2^{N+1}K(M+1)(N+1)$. Suppose $(u_1(z), u_2(z))$ is a solution of (4.4), corresponding to a solution $(y_1(z), y_2(z))$ of (2.7) on a curve γ which, by Lemma 4, we assume to be such that y_1 and $y_2 = u_2$ are bounded away from 0 on γ. The following Lemma is somewhat similar to Lemma 3; its proof, however, requires some modifications.

Lemma 5. *Let* $0 < \Delta < 2^{-N-2}(N+1)^{-1} < 1$ *and* $\theta := \min\{\Delta/C, R_0\}$. *Let* (u_1, u_2) *be a solution of* (4.4)*, analytic at c, where* $|c-a| \leq R_0/2$*, and suppose that* $|u_1(c)| < \theta/8$ *and* $|u_2(c)| > (4C)^{M+1}$. *Then* $(u_1(z), u_2(z))$ *is analytic in the disc* $|z-c| < \theta/|u_2(c)|$. *Moreover,* $|u_1(z)| \geq \theta/8$ *and* $|u_2(z)| \geq 1$ *on the circle* $|z-c| = \theta/2|u_2(c)|$.

Proof. Let $\rho = u_2(c)^L$, where $L = N-(N+1)/(M+1) \leq N-1$. For $i = 1, 2$, let $\eta_i(\zeta) := u_i(z)$, where $\zeta = \rho(z-c)$, and define $M_i(r) = \max_{|\zeta|\leq r}|\eta_i(\zeta)|$, $m_i(r) = \min_{|\zeta|\leq r}|\eta_i(\zeta)|$. Let

$$
r_0 = \sup\left\{r : M_1(r) < \Delta, M_2(r) < 2|\rho|^{1/L}, m_2(r) > \frac{1}{2}|\rho|^{1/L}\right\},
\tag{4.5}
$$

which is positive, since $|\eta_1(0)| < \Delta$ and $|\eta_2(0)| = |\rho|^{1/L}$. Then

$$
\eta_i(\zeta) = \eta_i(0) + \int_0^\zeta \dot{\eta}_i(\zeta)d\zeta,
$$

where $\eta_i(0) = u_i(c)$ and $\dot{\eta}_i(\zeta) = \rho^{-1}u_i'(z)$. Since

$$
|z-a| \leq |z-c| + |c-a| < \frac{R_0}{|\rho|} + \frac{R_0}{2} < R_0,
$$

for $|\zeta| < \min\{r_0, R_0\}$, we have

$$
\begin{aligned}
|\eta_1(\zeta)| &\leq |u_1(c)| + |\rho|^{-1}|\zeta| \sum_{(i,j)\in\tilde{I}\setminus\{(0,0)\}} K\Delta^i 2^{|(i-1)\frac{N+1}{M+1}+j-1|}|\rho|^{((i-1)\frac{N+1}{M+1}+j-1)/L} \\
&\leq |u_1(c)| + |\zeta| 2^N K(M+1)(N+1),
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
|\eta_2(\zeta)| &\le |u_2(c)| + |\rho|^{-1}|\zeta| \sum_{(i,j)\in\tilde{I},\ i\neq 0} K\Delta^{i-1}2^{|(i-1)\frac{N+1}{M+1}+j|}|\rho|^{((i-1)\frac{N+1}{M+1}+j)/L} \\
&\le |u_2(c)|\left(1+|\zeta|2^N K(M+1)(N+1)\right), \qquad (4.7)\\
|\eta_2(\zeta)| &\ge |u_2(c)|\left(1-|\zeta|2^N K(M+1)(N+1)\right).
\end{aligned}
$$

We have used condition (2.2), which implies $(i-1)(N+1)/(M+1)+j-1 \le L$ for $(i,j)\in\tilde{I}\setminus\{(0,0)\}$, and therefore $|(i-1)(N+1)/(M+1)+j-1| \le N$. Now, if $r_0 < \theta$, it would follow that

$$
\begin{aligned}
|\eta_1(\zeta)| &\le \theta(1/8 + 2^N K(M+1)(N+1)) < \Delta,\\
|\eta_2(\zeta)| &\le |u_2(c)|\left(1+\theta 2^N K(M+1)(N+1)\right) < 2|\rho|^{1/L},\\
|\eta_2(\zeta)| &\ge |u_2(c)|\left(1-\theta 2^N K(M+1)(N+1)\right) > \frac{1}{2}|\rho|^{1/L},
\end{aligned}
$$

in contradiction to the definition (4.5) of r_0. Therefore, $r_0 \ge \theta$, which implies that the estimates (4.6), (4.7) are valid for $|\zeta| < \theta$ and that u_1, u_2 are analytic for $|\zeta| < \theta$. On the circle $|\zeta| = \theta/2$, we now have

$$
\begin{aligned}
|\eta_1(\zeta)| \ge{}& (N+1)\left|\int_0^\zeta \rho^{-1}\eta_2(\tilde\zeta)^L d\tilde\zeta\right| - \left|\int_0^\zeta \rho^{-1}(N+1)\eta_1^{M+1}\eta_2^{N-\frac{N+1}{M+1}} d\tilde\zeta\right|\\
&-\left|\int_0^\zeta \rho^{-1}\sum_{(i,j)\in I'}\left(j+i\frac{N+1}{M+1}\right)\alpha_{ij}\eta_1^i\eta_2^{(i-1)\frac{N+1}{M+1}+j-1}d\tilde\zeta\right| - |\eta_1(0)|\\
\ge{}& (N+1)\left|\int_0^\zeta\left(1+\frac{\eta_2(\tilde\zeta)-\eta_2(0)}{\eta_2(0)}\right)^L d\tilde\zeta\right| - \frac{\theta}{2}(N+1)\Delta^{M+1}2^L\\
&-\frac{\theta}{2}|\rho|^{-\frac{1}{L(M+1)}}2^N K(M+1)(N+1) - \frac{\theta}{8}\\
\ge{}& \left|\int_0^\zeta d\tilde\zeta\right| - \left|\int_0^\zeta\left(\left(1+\frac{\eta_2(\tilde\zeta)-\eta_2(0)}{\eta_2(0)}\right)^L - 1\right)d\tilde\zeta\right| - \frac{\theta}{4}\\
\ge{}& \frac{\theta}{4} - \frac{\theta}{2}\sum_{n=1}^N\binom{N}{n}\left(\frac{\Delta}{4C}\right)^n \ge \frac{\theta}{8},\\
|\eta_2(\zeta)| \ge{}& \frac{1}{2}|\rho|^{1/L} > 1.
\end{aligned}
$$

□

Lemma 6 (Second curve modification)**.** *Let (y_1, y_2) be a solution of the system (2.7), analytic on the finite length curve γ ending at a movable singularity z_∞, such that $1/y_1$ and $1/y_2$ are bounded on γ. Then γ can be deformed to $\tilde\gamma$ in the region where y_1, y_2 are analytic in such a way that y_2^k/y_1^l remains bounded on $\tilde\gamma$ for all $k, l \ge 0$ for which $l(N+1) - k(M+1) \ge 0$.*

Proof. Define the set $S = \{s : 0 < s < l \text{ and } |u_1(\gamma(s))| \leq \theta/8\}$. There exists $0 < s_0 < l$ such that $|u_2(z)| > (4C)^{M+1}$ on $S \cap [s_0, l]$. Indeed, were this not the case, there would exist a sequence $\{z_n\}$ on γ such that $z_n \to z_\infty$ as $n \to \infty$, $u_1(z_n)$ is bounded, and $u_2(z_n)$ is bounded and bounded away from zero. Lemma 1 applied to the system (4.4) would then imply that u_1, u_2 are analytic at z_∞, in contradiction to the assumption. By the same method as used in the proof of Lemma 4 one can now deform the curve γ by arcs of circles such that u_1 and u_2 are bounded away from 0 on the modified curve $\tilde{\gamma}$, i.e., $u_1^{-(M+1)} = y_2^{N+1}/y_1^{M+1}$ and $u_2^{-1} = 1/y_2$ are bounded on $\tilde{\gamma}$. Writing

$$\frac{y_2^k}{y_1^l} = \left(\left(\frac{y_2^{N+1}}{y_1^{M+1}} \right)^l \cdot \frac{1}{y_2^{l(N+1)-k(M+1)}} \right)^{1/(M+1)},$$

we conclude that y_2^k/y_1^l is bounded on $\tilde{\gamma}$ if $l(N+1) - k(M+1) \geq 0$. □

5 An approximate first integral

In this section, we show the existence of a function W that remains bounded whenever a solution $(y_1(z), y_2(z))$ develops a movable singularity by analytic continuation along a finite length curve. Formally inserting the series expansions (2.4) for y_1 and y_2 into

$$H' = \frac{dH}{dz} = \frac{\partial H}{\partial z} = \sum_{(i,j)\in I'} \alpha'_{ij}(z) y_1(z)^i y_2(z)^j, \tag{5.1}$$

yields a formal series expansion for H' in $(z - z_0)^{1/(MN-1)}$. Heuristically, W is constructed from H by adding certain terms, rational in y_1 and y_2, which would cancel all terms of H' with negative powers of $(z - z_0)^{1/MN-1}$. Note, however, that terms of order $(z - z_0)^{-1}$ cannot be cancelled in this way, since these would correspond terms of H that are logarithmic in $z - z_0$ and cannot be obtained by rational expressions in y_1 and y_2.

We define

$$W(z, y_1, y_2) = y_1^{M+1} + y_2^{N+1} + \sum_{(i,j)\in I'} \alpha_{ij}(z) y_1^i y_2^j + \sum_{(k,l)\in J} \beta_{kl}(z) \frac{y_2^k}{y_1^l}, \tag{5.2}$$

where the $\beta_{kl}(z)$ are certain analytic functions to be determined in terms of the $\alpha_{ij}(z)$ and their derivatives and the index set J is given by

$$J = \{(k, l) \in \mathbf{N}^2 : 1 \leq k \leq N + 1, 1 - MN < k(M + 1) - l(N + 1) < M + N + 2\}.$$

It can easily be seen by setting $k = j + 1$ and $l = M - i$ that the pairs of indices in the set J are in one-to-one correspondence with the elements of the set $I \setminus \{(0,0)\}$. Thus, for each unbounded term $\alpha'_{ij}(z)y_1^i y_2^j$ in (5.1), there exists a function β_{kl} to compensate for it. However, it turns out that not all the functions β_{kl} can be used.

The other essential ingredient is the existence of the formal series solutions (2.4), which ensure that the terms of order $(z - z_0)^{-1}$ vanish identically. We now show formally that W is bounded.

Lemma 7. *The coefficients $\beta_{kl}(z)$, $(k,l) \in J$, in* (5.2) *can be chosen so that the function W is bounded on the curve $\tilde{\gamma}$.*

Proof. Taking the total z-derivative of (5.2) yields

$$
\begin{aligned}
W' &= \sum_{(i,j)\in I'} \alpha'_{ij} y_1^i y_2^j + \sum_{(k,l)\in J} \left(\beta'_{kl} \frac{y_2^k}{y_1^l} + k\beta_{kl} \frac{y_2^{k-1} y_2'}{y_1^l} - l\beta_{kl} \frac{y_2^k y_1'}{y_1^{l+1}} \right) \\
&= \sum_{(i,j)\in I'} \alpha'_{ij} y_1^i y_2^j - \sum_{(i,j)\in I'} \sum_{(k,l)\in J} (ik + jl)\alpha_{ij}\beta_{kl} y_1^{i-l-1} y_2^{k+j-1} \\
&\quad + \sum_{(k,l)\in J} \left(\beta'_{kl} \frac{y_2^k}{y_1^l} - k(M+1)\beta_{kl} y_1^{M-l} y_2^{k-1} - l(N+1)\beta_{kl} \frac{y_2^{N+k}}{y_1^{l+1}} \right) \\
&= \sum_{(i,j)\in I'} \alpha'_{ij} y_1^i y_2^j + \sum_{(k,l)\in J} (l(N+1) - k(M+1))\beta_{kl} y_1^{M-l} y_2^{k-1} \\
&\quad + \sum_{(k,l)\in J} \left(\beta'_{kl} \frac{y_2^k}{y_1^l} - l(N+1)\beta_{kl} \frac{y_2^{k-1}}{y_1^{l+1}} W \right) \\
&\quad + \sum_{(i,j)\in I'} \sum_{(k,l)\in J} (l(N - j + 1) - ik)\alpha_{ij}\beta_{kl} y_1^{i-l-1} y_2^{k+j-1} \\
&\quad + \sum_{(k,l)\in J} \sum_{(k',l')\in J} l(N+1)\beta_{kl}\beta_{k'l'} \frac{y_2^{k+k'-1}}{y_1^{l+l'+1}},
\end{aligned}
\tag{5.3}
$$

where we have used (5.2). All terms in (5.3) are now either of the form $y_1^{i_0} y_2^{j_0}$ with $(i_0, j_0) \in I$, or of the form $y_2^{j_0}/y_1^{i_0}$ with $i_0 \geq 1$ and $j_0(M+1) - i_0(N+1) < (M+1)(N+1)$. Note also that for the coefficients y_2^{k-1}/y_1^{l+1} of W, $(k,l) \in J$, we have $(l+1)(N+1) - (k-1)(M+1) \geq 0$; i.e., by Lemma 6, these are bounded on $\tilde{\gamma}$. Repeating the process of replacing powers y_2^{N+1} using (5.2), one can achieve in a finite number of steps that the terms of the form $y_2^{j_0}/y_1^{i_0}$ either have $j_0 \geq N+1$ with $i_0(N+1) - j_0(M+1) \geq 0$ and are therefore bounded by Lemma 6, or have $j_0 \leq N$ and $j_0(M+1) - i_0(N+1) \leq MN - 1$, equality holding if and only if $(i_0, j_0) = (1, N)$. We now manipulate the terms of the form $y_2^{j_0}/y_1^{i_0}$, $j_0 \leq N$ as follows:

$$\begin{aligned}
(M+1)(j_0+1)\frac{y_2^{j_0}}{y_1^{i_0}} &= -(j_0+1)\frac{y_2' y_2^{j_0}}{y_1^{M+i_0}} - \sum_{(i,j)\in I'} i(j_0+1)\alpha_{ij}\frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}} \\
&= -\left(\frac{y_2^{j_0+1}}{y_1^{M+i_0}}\right)' - (M+i_0)\frac{y_2^{j_0+1}y_1'}{y_1^{M+i_0+1}} - \sum_{(i,j)\in I'} i(j_0+1)\alpha_{ij}\frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}} \\
&= -\left(\frac{y_2^{j_0+1}}{y_1^{M+i_0}}\right)' - (N+1)(M+i_0)\frac{y_2^{N+j_0+1}}{y_1^{M+i_0+1}} \\
&\quad - \sum_{(i,j)\in I'} \left(i(j_0+1)+j(M+i_0)\right)\alpha_{ij}\frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}} \\
&= -\left(\frac{y_2^{j_0+1}}{y_1^{M+i_0}}\right)' - (N+1)(M+i_0)\frac{y_2^{j_0}}{y_1^{M+i_0+1}}W \\
&\quad + \sum_{(i,j)\in I'} \left((N+1)(M+i_0)-j(M+i_0)-i(j_0+1)\right)\alpha_{ij}\frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}} \\
&\quad + \sum_{(k,l)\in J} (N+1)(M+i_0)\beta_{kl}\frac{y_2^{k+j_0}}{y_1^{M+l+i_0+1}} + (N+1)(M+i_0)\frac{y_2^{j_0}}{y_1^{i_0}}.
\end{aligned} \tag{5.4}$$

Thus, unless $j_0(M+1) - i_0(N+1) = MN-1$, one can solve (5.4) for $y_2^{j_0}/y_1^{i_0}$:

$$\begin{aligned}
\frac{y_2^{j_0}}{y_1^{i_0}} &= \frac{1}{MN-1+i_0(N+1)-j_0(M+1)}\Bigg((N+1)(M+i_0)\frac{y_2^{j_0}}{y_1^{M+i_0+1}}W \\
&\quad + \sum_{(i,j)\in I'} \left(i(j_0+1)+j(M+i_0)-(N+1)(M+i_0)\right)\alpha_{ij}\frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}} \\
&\quad - \sum_{(k,l)\in J} (N+1)(M+i_0)\beta_{kl}\frac{y_2^{k+j_0}}{y_1^{M+l+i_0+1}} + \left(\frac{y_2^{j_0+1}}{y_1^{M+i_0}}\right)'\Bigg).
\end{aligned} \tag{5.5}$$

Again, by Lemma 6, the coefficient $y_2^{j_0}/y_1^{M+i_0+1}$ in (5.5) of W is bounded, since $(M+i_0+1)(N+1) - j_0(M+1) > 0$. Also, by Lemma 6, the term $y_2^{j_0+1}/y_1^{M+i_0}$ is bounded, since $(M+i_0)(N+1) - (j_0+1)(M+1) > 0$. Therefore, the term $\left(y_2^{j_0+1}/y_1^{M+i_0}\right)'$ is bounded when integrated over the finite length curve $\tilde{\gamma}$. For the terms of type $y_2^{k+j_0}/y_1^{M+l+i_0+1}$, $(k,l)\in J$, we find

$$(M+l+i_0+1)(N+1) - (k+j_0)(M+1) \geq 0,$$

which are therefore all bounded; and for the terms $y_2^{j+j_0}/y_1^{M-i+i_0+1}$, $(i,j)\in I'$,

$$(j+j_0)(M+1) - (M-i+i_0+1)(N+1) < j_0(M+1) - i_0(N+1).$$

We can thus replace $y_2^{j_0}/y_1^{i_0}$ with terms which are bounded or proportional to W with bounded factor and a sum of terms of the form $y_2^{j_1}/y_1^{i_1}$ with $j_1 = j + j_0$, $i_1 = M - i + i_0 + 1$ such that $j_1(M+1) - i_1(N+1)$ is strictly decreasing. Performing this process iteratively a finite number of times, we eventually end up only with terms $y_2^{j_n}/y_1^{i_n}$ for which $j_n(M+1) - i_n(N+1) \leq 0$. Lemma 6 shows that they are bounded on $\tilde{\gamma}$.

We thus arrive at a first-order differential equation for W of the form

$$W' = P(z, y_1^{-1}, y_2)W + \sum_{(i,j)\in I} \gamma_{ij}(z) y_1^i y_2^j + \gamma_{-1N}(z)\frac{y_2^N}{y_1} + Q(z, y_1^{-1}, y_2) + \frac{d}{dz}R(z, y_1^{-1}, y_2),$$

where P, Q and R are polynomial in their last two arguments; and, for each monomial y_2^k/y_1^l, we have $l(N+1) - k(M+1) \geq 0$, i.e., they are bounded on $\tilde{\gamma}$. We now show that by a suitable choice of the β_{kl} and the existence of the formal series solutions (2.4), all the coefficients γ_{ij}, $(i, j) \in I$, as well as γ_{-1N}, are identically 0.

We determine the functions $\beta_{kl} = \beta_{j+1,M-i}$ recursively, starting with the pairs $(i, j) \in I$ for which $i(N+1) + j(M+1)$ is maximal. From (5.3), we see that

$$\gamma_{ij}(z) = \alpha'_{ij}(z) + (MN - 1 - i(N+1) - j(M+1))\beta_{j+1,M-i}(z) + \cdots, \tag{5.6}$$

where $\cdots$ stands for expressions involving only terms $\beta_{k'l'} = \beta_{j'+1,M-i'}$ for which $i'(N+1) + j'(M+1) > i(N+1) + j(M+1)$. We can thus determine $\beta_{kl} = \beta_{j+1,M-i}$ for all pairs $(i, j) \in I$ for which $i(N+1) + j(M+1) > MN - 1$. However, when $i(N+1) + j(M+1) = MN - 1$, the coefficient of $\beta_{j+1,M-i}$ in (5.6) vanishes. We now make use of the existence of the formal series solutions (2.4) to show that also $\gamma_{ij} \equiv 0$ in this case.

Let $d = \gcd\{M+1, N+1\}$, $n = (N+1)/d$, and $m = (M+1)/d$. Consider the d terms $\gamma_{-1,N}(z)y_2^N/y_1$, $\gamma_{m-1,N-n}(z)y_1^{m-1}y_2^{N-n}, \ldots, \gamma_{M-m,n-1}(z)y_1^{M-m}y_2^{n-1}$. When one inserts the formal series solutions (2.4) into these expressions they have leading order $(z - z_0)^{-1}$. But, as explained in Remark 1, there are essentially d formal series solutions corresponding to the different choices of the leading coefficients $c_{1,-N-1}, c_{2,-M-1}$ such that $c_{1,-N-1}^{MN-1} = -1/(MN-1)^{N+1}$. Inserting any of the series into (5.2))shows that W has a Laurent series expansion in powers of $(z - z_0)^{1/(MN-1)}$. Therefore, the coefficient of $(z - z_0)^{-1}$ in W' must vanish; otherwise, W would have logarithmic terms in its expansion. The coefficients of

$(z-z_0)^{-1}$ in W', for the different choices of $(c_{1,-N-1}, c_{2,-M-1})$, are

$$\frac{-1}{MN-1}\left(\gamma_{-1,N}(z_0)+\omega_1\gamma_{m-1,N-n}(z_0)+\cdots+\omega_1^{d-1}\gamma_{M-m,n-1}(z_0)\right)=0,$$
$$\frac{-1}{MN-1}\left(\gamma_{-1,N}(z_0)+\omega_2\gamma_{m-1,N-n}(z_0)+\cdots+\omega_2^{d-1}\gamma_{M-m,n-1}(z_0)\right)=0,$$
$$\vdots$$
$$\frac{-1}{MN-1}\left(\gamma_{-1,N}(z_0)+\omega_d\gamma_{m-1,N-n}(z_0)+\cdots+\omega_d^{d-1}\gamma_{M-m,n-1}(z_0)\right)=0,$$

where ω_i, $i=1,\dots,d$, are the d distinct roots of $\omega^d=-1$. This system of d equations gives $\gamma_{-1,N}(z_0)=\gamma_{m-1,N-n}(z_0)=\cdots=\gamma_{M-m,n-1}(z_0)=0$. However, the formal series expansions exist for all $\hat{z}$ in a neighbourhood of z_0. Therefore we have shown that, in fact, $\gamma_{-1,N}=\gamma_{m-1,N-n}=\cdots=\gamma_{M-m,n-1}\equiv 0$. The functions $\beta_{j+1,M-i}$ with $i(N+1)+j(M+1)=MN-1$ can be chosen arbitrarily; we henceforth set them to 0. The remaining functions $\beta_{j+1,M-i}$ with $i(N+1)+j(M+1)<MN-1$ can now all be determined recursively, so that $\gamma_{ij}\equiv 0$ for all $(i,j)\in I\cup\{(-1,N)\}$. We have thus arrived at a first-order linear differential equation for W of the form

$$W'=P(z,y_1^{-1},y_2)W+Q(z,y_1^{-1},y_2)+R'(z,y_1^{-1},y_2), \tag{5.7}$$

where P, Q and R are bounded on $\tilde{\gamma}$ near a movable singularity z_0 of a solution $(y_1(z),y_2(z))$. Lemma 2 now shows that W is bounded on $\tilde{\gamma}$. □

6 A regular initial value problem

To show that a movable singularity is an algebraic branch point, we introduce coordinates u and v for which there exists a regular initial value problem. The coordinate u is defined by

$$y_1=u^{-(N+1)/d}, \tag{6.1}$$

where a choice of branch is made. We also define

$$w=y_2u^{(M+1)/d}. \tag{6.2}$$

From (5.2), one obtains the algebraic equation for w

$$\begin{aligned}0=w^{N+1}&+\sum_{(i,j)\in I'}\alpha_{ij}(z)u^{((M+1)(N+1)-i(N+1)-j(M+1))/d}w^j\\&+\sum_{(k,l)\in J}\beta_{kl}(z)u^{((M+1)(N+1)+l(N+1)-k(M+1))/d}w^k\\&+1-Wu^{(M+1)(N+1)/d},\end{aligned} \tag{6.3}$$

all the exponents of u being positive integers. Denote solutions of this equation for w by $w_1, \dots, w_{N+1}$. The w_i are analytic functions of u, z, and W in some neighbourhood of $u = 0$, $z = z_\infty$, and $W = W_0$ for all $W_0 \in \mathbf{C}$. We express the w_n as power series in u and W with coefficients analytic in z:

$$w_n = F_n(z, u, W) = \omega_n \sum_{j,k=0}^{\infty} a_{jkn}(z) u^j W^k,$$

where ω_n, $n = 1, \dots, N+1$, are the distinct roots of $\omega^{N+1} = -1$, $a_{00n} \equiv 1$, and the first monomial containing W is of the form $-u^{(M+1)(N+1)/d} W/(N+1)$. We write $\bar{F}_n(z, u) = \sum_{j=0}^{(M+1)(N+1)/d} a_{j0n}(z) u^j$ and define the functions v_n by

$$w_n = \omega_n \left(\bar{F}_n(z, u) - \frac{1}{N+1} u^{(M+1)(N+1)/d} v_n \right), \tag{6.4}$$

so that in the limit as $u \to 0$, v_n agrees to leading order with W. From the definition (6.2) of w, we see that the choice of branch for ω_n can partially be absorbed into the original choice of branch for u if $1 < d < M+1$, and be absorbed completely if $d = 1$; thus there are essentially only d inequivalent choices for (u, v_n). From (6.1) and (2.7), we obtain the following differential equation satisfied by u:

$$\begin{aligned} u' = -\frac{d}{N+1} u^{\frac{N+1}{d}+1} \Bigg[& (N+1)\omega_n^N \left(u^{-(M+1)/d} \bar{F}_n(z, u) - \frac{1}{N+1} u^{(M+1)N/d} v_n \right)^N \\ & + \sum_{(i,j)\in I'} j\alpha_{ij}(z) u^{-i(N+1)/d} \omega_n^{j-1} \left(u^{-(M+1)/d} \bar{F}_n(z, u) - \frac{1}{N+1} u^{(M+1)N/d} v_n \right)^{j-1} \Bigg]. \end{aligned} \tag{6.5}$$

Taking the reciprocal of (6.5), changing the role of the dependent and independent variables u and z, and extracting the highest power of u on the right hand side, we obtain, for $v_0 \in \mathbf{C}$, an initial value problem of the form

$$\frac{dz}{du} = u^{\frac{MN-1}{d}-1} A(u, z, v), \quad A(0, z_\infty, v_0) = \omega_n/d, \tag{6.6}$$

where $A(u, z, v)$ is analytic in (u, z, v) at $(0, z_\infty, v_0)$. We drop the index n from now on. Reinserting (6.4) into (6.3) yields an expression in terms of u and v for W of the form

$$W = v + G(z, u, v), \tag{6.7}$$

where G is a polynomial in v of degree $N+1$, analytic in z and u near $u = 0$, and satisfying $G(z, 0, v) = 0$. We differentiate (6.7) with respect to z to obtain

$$W' = v' + G_z + G_u u' + G_v v', \tag{6.8}$$

and compare this with equation (5.7), which can be written in the form

$$
\begin{aligned}
W' &= \tilde{P}(z,u,v)W + \tilde{Q}(z,u,v) + \frac{d}{dz}\tilde{R}(z,u,v) \\
&= \tilde{P}(v+G) + \tilde{Q} + \tilde{R}_z + \tilde{R}_u u' + \tilde{R}_v v',
\end{aligned} \tag{6.9}
$$

where $\tilde{P}$, $\tilde{Q}$, and $\tilde{R}$ are polynomial in u and v. Solving (6.8) and (6.9) for v' yields an equation of the form

$$
v' = B(z,u,v)u' + C(z,u,v), \tag{6.10}
$$

where B and C are analytic in their arguments. Multiplying (6.10) by (6.6), we obtain the following equation for v as function of u:

$$
\frac{dv}{du} = \frac{dv}{dz}\frac{dz}{du} = B(z,u,v) + u^{\frac{MN-1}{d}-1}A(z,u,v)C(z,u,v). \tag{6.11}
$$

Equations (6.6) and (6.11) together form a regular initial value problem for z and v as functions of u near $u=0$ with $z(0)=z_\infty$ and $v(0)=v_0$.

7 Proof of Theorem 1

We can now complete the proof of Theorem 1.

Proof. By Lemma 7, the auxiliary function W is bounded along γ or a modification of it, which we also call γ. Consider a sequence $\{z_n\} \subset \gamma$ such that $z_n \to z_\infty$ as $n \to \infty$. Suppose that the sequence $\{y_1(z_n)\}$ is bounded. Then the functional form of $W(z, y_1, y_2)$ implies that the sequence $\{y_2(z_n)\}$ is also bounded. However, Lemma 1 now implies that the solution (y_1, y_2) can be analytically continued to z_∞, in contradiction to the assumption in the theorem. Therefore, $\{y_1(z_n)\} \to \infty$, since otherwise it would have a bounded subsequence. In the coordinates u, v introduced in the previous section, we therefore have that $u(z_n) \to 0$ and $v(z_n)$ is bounded. Hence there exists some subsequence $\{z_{n_k}\}$ such that $v(z_{n_k}) \to v_0$ for some $v_0 \in \mathbf{C}$. Equations (6.6) and (6.11) now form a regular initial value problem for z and v as functions of u with initial values z_∞ and v_0 at $u=0$. Lemma 1 then shows that z and v are analytic at $u=0$. Since $A(0, z_\infty, v_0) \neq 0$ in (6.6), z has a convergent power series expansion of the form $z = z_\infty + \sum_{k=0}^{\infty} \xi_k u^{k+\frac{MN-1}{d}}$, valid in a neighbourhood of $u=0$. Taking the $(MN-1)/d$-th root, we obtain $(z-z_\infty)^{d/(MN-1)} = \sum_{k=1}^{\infty} \eta_k u^k$; and inverting the power series, we see that u has a convergent series expansion $u = \sum_{k=1}^{\infty} \zeta_k (z-z_\infty)^{kd/(MN-1)}$. The definition (6.1) of u yields a series expansion for y_1 $y_1(z) = \sum_{k=-(N+1)/d}^{\infty} C_{1,k}(z-z_\infty)^{kd/(MN-1)}$, which is convergent in a branched punctured neighbourhood of z_∞. Also, from the definition (6.2), we have $y_2(z) = \sum_{k=-(M+1)/d}^{\infty} C_{2,k}(z-z_\infty)^{kd/(MN-1)}$, since $w \neq 0$ at $z = z_\infty$. □

8 Lowest degree examples

If $M = 1$, the Hamiltonian (2.6) can essentially be reduced to the form

$$H(z, y_1, y_2) = \frac{1}{2}y_1^2 + P(z, y_2).$$

The Hamiltonian system thus corresponds to the second-order differential equation $y'' = P_y(z, y)$, which was treated in [2]. This case includes the Painlevé equations P_I (for $N = 2$) and P_{II} (for $N = 3$). For $N \geq 4$, the equation has genuinely branched solutions. Let us now consider Hamiltonian systems where both $M, N \geq 2$.

8.1 Case $M = N = 2$. The Hamiltonian here is of the form

$$H(z, y_1, y_2) = \frac{1}{3}y_1^3 + \frac{1}{3}y_2^3 + \alpha(z)y_1y_2 + \beta(z)y_1 + \gamma(z)y_2,$$

where we have chosen a slightly different normalisation than that in (2.6). The resonance conditions in this case are $\alpha'' \equiv 0$, $\beta' \equiv 0$ and $\gamma' \equiv 0$. One is therefore essentially left with

$$H(z, y_1, y_2) = \frac{1}{3}y_1^3 + \frac{1}{3}y_2^3 + zy_1y_2 + \beta y_1 + \gamma y_2,$$

the corresponding system of differential equations being

$$\begin{aligned} y_1' &= y_2^2 + zy_1 + \gamma, \\ y_2' &= -y_1^2 - zy_2 - \beta. \end{aligned} \tag{8.1}$$

About any movable singularity z_∞, a solution is represented by

$$y_1(z) = \sum_{k=-1}^{\infty} C_{1,k}(z - z_\infty)^k, \quad y_2(z) = \sum_{k=-1}^{\infty} C_{2,k}(z - z_\infty)^k,$$

with $C_{1,-1}^3 = -1$ and $C_{2,-1} = C_{2,-1}^2$; i.e., there are three possible leading order behaviours about any movable singularity which, in this case, are simple poles. Theorem 1 in this case states that every local solution (y_1, y_2) extends to a meromorphic function in the whole complex plane, i.e., the system (8.1) has the Painlevé property. It is therefore of interest to know how its solutions can be expressed in terms of the six Painlevé transcendents. To answer this, we let $y = y_1$ and eliminate y_2 from (8.1), obtaining the scalar differential equation

$$\left(y'' + zy' - (1 - 2z^2)y - 2\gamma z\right)^2 = 4\left(y^2 + \beta\right)^2 \left(y' - zy - \gamma\right), \tag{8.2}$$

which is of second order and second degree in y. A Painlevé type classification for equations of second order and second degree has been done by C. Cosgrove and G. Scoufis in [1]. They found six inequivalent types of equations in the class $(y'')^2 = F(z, y, y')$, which they denoted by SD-I – SD-VI. All of these equations can be solved in terms of the Painlevé transcendents $P_I - P_{VI}$. In fact, equation (8.2) is of the modified form denoted by SD-IV'.A [1, (5.87)]), which is solved in terms of P_{IV}.

8.2 Case $M = 2, N = 3$. In this case, the normalised Hamiltonian (2.6) is

$$H = y_1^3 + y_2^4 + \alpha_{21} y_1^2 y_2 + \alpha_{12} y_1 y_2^2 + \alpha_{11} y_1 y_2 + \alpha_{20} y_1^2 + \alpha_{02} y_2^2 + \alpha_{10} y_1 + \alpha_{01} y_2.$$

The only resonance condition is

$$\left(3\alpha_{12} - \alpha_{21}^2\right)'' = 0;$$

and, if it is satisfied, the solutions near a movable singularity z_∞ are given by

$$y_1(z) = \sum_{k=-4}^{\infty} C_{1,k}(z - z_\infty)^{k/5}, \quad y_2(z) = \sum_{k=-3}^{\infty} C_{2,k}(z - z_\infty)^{k/5},$$

with $C_{1,-4}^5 = -5^{-4}$, $C_{2,-3} = 5C_{1,-4}^3$, where the choice for $C_{1,-4}$ can completely be absorbed into the choice of branch for $(z - z_\infty)^{1/5}$.

8.3 Case $M = N = 3$. The normalised Hamiltonian is given by

$$H = y_1^4 + y_2^4 + \alpha_{21} y_1^2 y_2 + \alpha_{12} y_1 y_2^2 + \alpha_{20} y_1^2 + \alpha_{11} y_1 y_2 + \alpha_{02} y_2^2 + \alpha_{10} y_1 + \alpha_{01} y_2.$$

In order for the solutions to have only movable algebraic singularities, the conditions

$$\left(2\alpha_{20} - \alpha_{12}^2\right)' = 0, \quad \alpha_{11}' = 0, \quad \left(2\alpha_{02} - \alpha_{21}^2\right)' = 0$$

need to be satisfied. The solutions are given by the series

$$y_1(z) = \sum_{k=-1}^{\infty} C_{1,k}(z - z_\infty)^{k/2}, \quad y_2(z) = \sum_{k=-1}^{\infty} C_{2,k}(z - z_\infty)^{k/2},$$

about any movable singularity z_∞, where $C_{1,-1}^8 = -1/16$, $C_{2,-1} = 2C_{1,-1}^3$; however, the choice for $C_{1,-1}$ can be only partially absorbed into the choice of branch for $(z - z_\infty)^1/2$, i.e., there are 4 possible leading order behaviours of the solution near any movable singularity.

9 Summary and outlook

For a class of Hamiltonian systems of ordinary differential equations, we have found that the only movable singularities obtained by analytic continuation along finite length curves are algebraic branch points; in particular, these singularities are isolated, and the solutions are locally finitely branched. The possibility of movable singularities obtained by analytic continuation along an infinite length curve is discussed by R. Smith in [21] for certain second-order differential equations. There, it is shown that a singularity of this type is non-isolated. More specifically, it is an accumulation point of algebraic singularities, and they cannot be ruled out at this stage for the systems presented here. It remains of interest to classify the structure of movable singularities for wider classes of differential equations.

Acknowledgements. The author expresses his sincere gratitude to Professor R. Halburd for his invaluable support and many interesting discussions.

REFERENCES

[1] C. M. Cosgrove and G. Scoufis, *Painlevé classification of a class of differential equations of the second order and second degree*, Stud. Appl. Math. **88** (1993), 25–87.

[2] G. Filipuk and R. G. Halburd, *Movable algebraic singularities of second-order ordinary differential equations*, J. Math. Phys. **50** (2009), 023509.

[3] R. Garnier, *Sur des systèmes différentiels du second ordre dont l'intégrale générale est uniforme*, Ann. Sci˙École Norm. Sup. (3) **77** (1960), 123–144.

[4] J. Goffar-Lombet, *Sur des systémes polynomiaux d'équations différentielles dont l'intégrale générale est à points critiques fixes*, Acad. Roy. Belg. Cl. Sci. Mém. Coll. in -8° (2) **41**, no. 3 (1974).

[5] V. I. Gromak, I. Laine, and S. Shimomura, *Painlevé Differential Equations in the Complex Plane*, De Gruyter, Berlin, 2002.

[6] E. Hille, *Ordinary Differential Equations in the Complex Domain*, Wiley-Interscience, New York, 1976.

[7] A. Hinkkanen and I. Laine, *Solutions of the first and second Painlevé equations are meromorphic*, J. Anal. Math. **79** (1999), 345–77.

[8] T. Kimura and T. Matuda, *On systems of differential equations of order two with fixed branch points*, Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), 445–449.

[9] T. Kimura, *Sur les points singuliers essentiels mobiles des équations différentielles du second ordre*, Comment. Math. Univ. St. Paul **5** (1956), 81–94.

[10] J. Malmquist, *Sur les équations différentielles du second ordre, dont l'intégrale générale a ses points critiques fixes*, Ark. för Mat., Astron. och Fys. **17** (1923), 1–89.

[11] Y. Murata, *On fixed and movable singularities of systems of rational differential equations of order n*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **35** (1988), 439–506.

[12] K. Okamoto, *Studies on the Painlevé equations. I. Sixth Painlevé equation* P_{VI}, Ann. Mat. Pura Appl. (4) **146** (1987), 337–381.

[13] K. Okamoto, *Studies on the Painlevé equations. II. Fifth Painlevé equation* P_{V}, Japan. J. Math. (N.S.) **13** (1987), 47–76.

[14] K. Okamoto, *Studies on the Painlevé equations. III. Second and fourth Painlevé equations, P_{II} and P_{IV}*, Math. Ann. **275** (1986), 221–255.

[15] K. Okamoto, *Studies on the Painlevé equations. IV. Third Painlevé equation P_{III}*, Funkcial. Ekvac. **30** (1987), 305–332.

[16] K. Okamoto and K. Takano, *The proof of the Painlevé property by Masuo Hukuhara*, Funkcial. Ekvac. **44** (2001), 201–217.

[17] P. Painlevé, *Mémoire sur les équations différentielles dont l'intégrale générale est uniforme*, Bull. Soc. Math. France **28** (1900), 201–261.

[18] S. Shimomura, *Proofs of the Painlevé property for all Painlevé equations*, Japan. J. Math. (N.S.) **29** (2003), 159–180.

[19] S. Shimomura, *A class of differential equations of PI-type with the quasi-Painlevé property*, Ann. Mat. Pura Appl. (4) **186** (2007), 267–280.

[20] S. Shimomura, *Nonlinear differential equations of second Painlevé type with the quasi-Painlevé property along a rectifiable curve*, Tohoku Math. J. (2) **60** (2008), 581–595.

[21] R. A. Smith, *On the singularities in the complex plane of solutions of $y'' + y'f(y) + g(y) = P(x)$*, Proc. London Math. Soc. (3) **3** (1953), 498–512.

Thomas Kecker
DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE LONDON
GOWER STREET
LONDON WC1E 6BT
UNITED KINGDOM
email: tkecker@math.ucl.ac.uk

(Received and in revised form)