

Colourful components in k -caterpillars and planar graphs

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Abstract

A connected component of a vertex-coloured graph is said to be *colourful* if all its vertices have different colours. By extension, a graph is colourful if all its connected components are colourful. Given a vertex-coloured graph G and an integer p , the COLOURFUL COMPONENTS problem asks whether there exist at most p edges whose removal makes G colourful and the COLOURFUL PARTITION problem asks whether there exists a partition of G into at most p colourful components. In order to refine our understanding of the complexity of the problems on trees, we study both problems on k -caterpillars, which are trees with a central path P such that every vertex not in P is within distance k from a vertex in P . We prove that COLOURFUL COMPONENTS and COLOURFUL PARTITION are NP-complete on 4-caterpillars with maximum degree 3, 3-caterpillars with maximum degree 4 and 2-caterpillars with maximum degree 5. On the other hand, we show that the problems are linear-time solvable on 1-caterpillars. Hence, our results imply two complexity dichotomies on trees: COLOURFUL COMPONENTS and COLOURFUL PARTITION are linear-time solvable on trees with maximum degree d if $d \leq 2$ (that is, on paths), and NP-complete otherwise; COLOURFUL COMPONENTS and COLOURFUL PARTITION are linear-time solvable on k -caterpillars if $k \leq 1$, and NP-complete otherwise. We leave three open cases which, if solved, would provide a complexity dichotomy for both problems on k -caterpillars, for every non-negative integer k , with respect to the maximum degree. We also show that COLOURFUL COMPONENTS is NP-complete on 5-coloured planar graphs with maximum degree 4 and on 12-coloured planar graphs with maximum degree 3. Our results answer two open questions of Bulteau et al. mentioned in [30th Annual Symposium on Combinatorial Pattern Matching, 2019].

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1. Introduction

In this paper, all graphs are unweighted, undirected and simple (do not contain loops or multiple edges). A *vertex-coloured graph*, or simply a *coloured graph*, is a graph whose vertices are (not necessarily properly) coloured. A connected component of a coloured graph is a *colourful component* if all its vertices have different colours. A graph is said to be *colourful* if all its connected components are colourful.

We focus on two closely related decision problems where a coloured graph and an integer p are given as inputs: (i) the COLOURFUL COMPONENTS problem asks if there exist at most p edges whose removal makes the graph colourful; (ii) the COLOURFUL PARTITION problem is to decide if there exists a partition of the vertex set into at most p parts such that each part induces a colourful component in the graph.

The study of both COLOURFUL COMPONENTS and COLOURFUL PARTITION was initially motivated by problems raised in comparative genomics. One key problem is to partition a set of genes into orthologous genes, which are sets of genes in different species that have evolved through speciation events only, *i.e.* originated by vertical descent from a single gene in the last common ancestor. The problem has been modelled as a graph problem where orthologous genes translate into colourful components in the graph [1, 18]. Each vertex of the graph represents a gene, and the colour of a vertex illustrates the species of the corresponding gene. An edge between two vertices is present in the graph if the two corresponding genes are (sufficiently) similar. The quality of a partition of a set of genes into orthologous genes can be expressed in different ways. Minimising the number of similar genes in different subsets of the partition is a well studied variant [5, 6, 11, 15, 18], which corresponds to minimising the number of edges between the colourful components (as in the optimisation variant of COLOURFUL COMPONENTS). Alternatively, one can ask for a partition of minimum size, *i.e.* a partition containing the minimum number of orthologous genes, or equivalently the minimum number of colourful components [1, 6, 8] (as in the optimisation variant of COLOURFUL PARTITION). Another variant, however not studied in this paper, considers an objective function that maximises the number of edges in the transitive closure [1, 8, 15]. See Fig. 1 for an example of a graph partitioned into colourful components.

We give the formal definitions of the two problems considered in this paper.

COLOURFUL COMPONENTS

<i>Input:</i> A vertex-coloured graph G , a non-negative integer p .
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<i>Question:</i> Are there at most p edges of G whose removal makes G colourful?
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COLOURFUL PARTITION

Input: A vertex-coloured graph G , a positive integer p .

Question: Is there a partition of the vertex set of G with at most p parts such that each part induces a colourful component in G ?

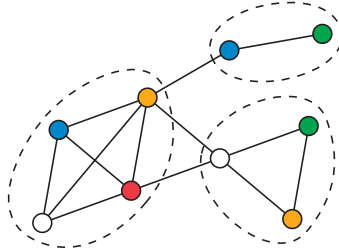


Figure 1: A vertex-coloured graph. The dashed parts represent a partition into colourful components. Removing the three edges between the dashed parts yields a colourful graph.

Observe that, on a tree, there is a solution to COLOURFUL COMPONENTS with p edges if and only if there is a solution to COLOURFUL PARTITION with $p+1$ parts. However, this is not the case on general graphs [6]. Both problems are known to be NP-complete on subdivided stars [8], trees of diameter at most 4 [5], and trees with maximum degree 6 [6]. Trees of diameter at most 4 are in fact a subclass of 2-caterpillars (defined later), so both problems are NP-complete on 2-caterpillars (when there is no restriction on the maximum degree).

It is interesting to notice the similarities between COLOURFUL COMPONENTS and the MULTICUT [4, 14] and MULTI-MULTIWAY CUT [2] problems. In the MULTICUT problem, a graph and a set of pairs of vertices are given and the goal is to minimise the number of edges to remove in order to disconnect each pair of vertices. In the MULTI-MULTIWAY CUT problem, a graph and sets of vertices are given and the goal is to minimise the number of edges to remove in order to disconnect all paths between vertices from the same vertex set. Thus, COLOURFUL COMPONENTS is a special case where such sets of vertices form a partition.

Another well-studied problem dealing with subgraphs of a vertex-coloured graph is the GRAPH MOTIF problem [3, 9, 10]. This problem takes a coloured graph and a multiset of colours M (the motif) as input, and the goal is to determine whether there exists a connected subgraph S such that the multiset of colours used by the vertices in S corresponds exactly to M . If M is a set (where each colour appears at most once), then M is said to be colourful.

Preliminaries. Given a graph G , we denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G . We assume that a coloured graph G is always associated with a colouring function c from $V(G)$ to a set of colours: for each vertex $u \in V(G)$, $c(u)$ is the colour of the vertex u . The *colour-multiplicity* of G

corresponds to the maximum number of occurrences of any colour in the graph. If G contains at most ℓ colours we say that G is an ℓ -coloured graph. Given a set of edges $S \subseteq E(G)$, we denote by $G - S$ the graph obtained from G by removing the edges in S ; if S contains a unique edge e , then we may simply write $G - e$. For a set $X \subseteq V(G)$ of vertices, $G[X]$ denotes the subgraph of G induced by X . The *(open) neighbourhood* of a vertex u , denoted by $N(u)$, is the set of vertices adjacent to u . The *degree* of a vertex u is the cardinality of its open neighbourhood. The *closed neighbourhood* of u is the set $N[u] = N(u) \cup \{u\}$. A *star* is a tree with at most one vertex with degree more than two. A *k -caterpillar* is tree containing a path P , called the *backbone*, such that every vertex that is not in P is within distance k from a vertex in P . Note that a 0-caterpillar is simply a path.

Overview of the results. Section 2 focuses on the complexity of COLOURFUL COMPONENTS and COLOURFUL PARTITION on trees, and in particular on k -caterpillars. In Subsection 2.1, we prove that both problems are NP-complete on 4-caterpillars with maximum degree 3, on 3-caterpillars with maximum degree 4, and on 2-caterpillars with maximum degree 5. This answers an open question of Bulteau et al., asked in [6], regarding the complexity of the problems on trees with maximum degree at most 5. Then, in Subsection 2.2, we give a linear-time algorithm for COLOURFUL COMPONENTS and COLOURFUL PARTITION on 1-caterpillars (without restriction on the maximum degree), even in the case where the backbone induces a cycle. This result improves the complexity of the previously known quadratic-time algorithm on paths [8] and applies to a wider class of graphs. We then extend our technique for 1-caterpillars to a larger class of graphs. Of particular interest are the following complexity dichotomies of the problems on trees which we derive from our results.

Corollary 1. COLOURFUL COMPONENTS and COLOURFUL PARTITION are linear-time solvable on paths, that is, on trees with maximum degree at most 2, and NP-complete on trees with maximum degree at least 3.

Corollary 2. COLOURFUL COMPONENTS and COLOURFUL PARTITION are polynomial-time solvable on k -caterpillars if $k \leq 1$, and NP-complete otherwise.

In Section 3, we consider the complexity of COLOURFUL COMPONENTS on planar graphs with small maximum degree. It is known that the problem is NP-complete on 3-coloured graphs with maximum degree 6 [5], while COLOURFUL PARTITION is NP-complete on 3-coloured 2-connected (planar) graphs with maximum degree 3 [6]. However, the complexity of COLOURFUL COMPONENTS on ℓ -coloured graphs with maximum degree at most 5 was unknown and left as an open question in [6]. We answer that question by showing that COLOURFUL COMPONENTS is NP-complete on 5-coloured planar graphs with maximum degree 4 and on 12-coloured planar graphs with maximum degree 3.

2. Complexity on trees

This section is devoted to the study of the complexity of COLOURFUL COMPONENTS and COLOURFUL PARTITION on trees. We first derive NP-completeness results in [Subsection 2.1](#), and then present a linear-time algorithm on a superclass of 1-caterpillars in [Subsection 2.2](#). These results imply two complexity dichotomies for COLOURFUL COMPONENTS and COLOURFUL PARTITION on trees which we present in [Subsection 2.3](#): one dichotomy with respect to the maximum degree and the other with respect to the smallest integer k such that the input tree is a k -caterpillar, respectively.

2.1. NP-completeness results

In this subsection, we show that COLOURFUL COMPONENTS and COLOURFUL PARTITION are NP-complete on 4-caterpillars with maximum degree 3, 3-caterpillars with maximum degree 4, and 2-caterpillars with maximum degree 5. We propose a reduction from 3-SAT with at most three occurrence of each variable, denoted by 3,3-SAT, which is proved NP-complete in [\[16\]](#).

3,3-SAT

Input: A 3-CNF formula ϕ in which each variable occurs at most three times.

Question: Is there a satisfying assignment for ϕ ?

If a clause in a 3-CNF formula contains exactly one literal, we can set the value of the corresponding variable deterministically and either reduce the number of clauses or conclude that the formula is unsatisfiable. Hence, we assume that each clause of the input 3-CNF formula contains at least 2 literals. Furthermore we can suppose that each literal appears either one or two times; otherwise the formula can be simplified by setting the variable to the value that satisfies all clauses containing it.

Construction 1. Let ϕ be an instance of 3,3-SAT, that is, a set of m clauses C_1, C_2, \dots, C_m on n variables x_1, x_2, \dots, x_n , where each clause contains at most three literals and where each variable appears at most three times. We denote by m_3 the number of clauses containing three literals and by m_2 the number of clauses containing two literals. We construct a tree T in which each variable and each clause is represented by a gadget (a connected subtree of T).

For each variable x_i of ϕ , we construct a *variable gadget*: Firstly, create three vertices labelled r_{x_i} , x_i , \bar{x}_i , and connect x_i and \bar{x}_i to r_{x_i} . If a clause C_j contains the literal x_i , then create a vertex labelled $x_{i,j}$ and connect it to the vertex x_i . Similarly, if a clause C_j contains the literal \bar{x}_i , then create a vertex labelled $\bar{x}_{i,j}$ and connect it to the vertex \bar{x}_i . Since each literal appears at most two times, r_{x_i} is the root of a tree of depth 2 with maximum degree 3. Also, since each literal appears at least once, each leaf of the gadget corresponds to a vertex of the type $x_{i,j}$, that is, each leaf represents a literal in a clause (see [Fig. 2a](#)).

Then, for each clause C_j of ϕ , we construct a *clause gadget*. If C_j contains only two literals $\ell_{1,j}$ and $\ell_{2,j}$:

- *Clause gadget of type A:* Create two vertices z_j, z'_j , two vertices labelled $\ell_{1,j}, \ell_{2,j}$ representing the literals in C_j , and one extra vertex r_{C_j} . Then, connect z_j and z'_j to r_{C_j} , $\ell_{1,j}$ to z_j and $\ell_{2,j}$ to z'_j . The gadget is a tree of depth 2 with maximum degree 3 rooted in r_{C_j} (see Fig. 2b).

If C_j contains three literals ℓ_1, ℓ_2, ℓ_3 , then, depending on the desired properties of the final tree T , we propose two different kinds of clause gadgets:

- *Clause gadget of type B:* Create four vertices y_j, y'_j, z_j, z'_j , three vertices labelled $\ell_{1,j}, \ell_{2,j}, \ell_{3,j}$ representing the literals in C_j , and one extra vertex r_{C_j} . Then, add the edges $\{\ell_{1,j}, y_j\}, \{\ell_{2,j}, y'_j\}, \{\ell_{3,j}, z'_j\}, \{y_j, z_j\}, \{y'_j, z_j\}$, and the edges $\{z_j, r_{C_j}\}, \{z'_j, r_{C_j}\}$. The gadget is a tree of depth 3 with maximum degree 3 rooted in r_{C_j} (see Fig. 2c).
- *Clause gadget of type C:* Create three vertices z_j, z'_j, z''_j , three vertices labelled $\ell_{1,j}, \ell_{2,j}, \ell_{3,j}$ representing the literals in C_j , and one extra vertex r_{C_j} . Then, connect z_j, z'_j and z''_j to r_{C_j} , $\ell_{1,j}$ to z_j , $\ell_{2,j}$ to z'_j , and $\ell_{3,j}$ to z''_j . The gadget is a tree of depth 2 with maximum degree 4 rooted in r_{C_j} (see Fig. 2d).

We now explain which clause gadgets must be used, and how clause and variable gadgets must be connected together, so that the final tree T has the required properties. First, create a variable gadget for each variable. Also, for each clause containing two literals only, create a clause gadget of type A. Then, apply one of the following options:

- *To get T as a 4-caterpillar with maximum degree 3:* Create a clause gadget of type B for each clause containing three literals. Then, create a central path with $n + m_3 + m_2$ new vertices, and connect all r_{x_i} and r_{C_j} vertices to distinct vertices of the central path.
- *To get T as a 3-caterpillar with maximum degree 4:* Create a clause gadget of type B for each clause containing three literals. Then, connect all r_{x_i} and r_{C_j} vertices together to create a central path. Alternatively, create a clause gadget of type C for each clause containing three literals, create a central path with $n + m_3 + m_2$ new vertices and connect all r_{x_i} and r_{C_j} vertices to distinct vertices of the central path.
- *To get T as a 2-caterpillar with maximum degree 5:* Create a clause gadget of type C for each clause containing three literals. Then, connect all r_{x_i} and r_{C_j} vertices together to create a central path.

In all cases, the central path corresponds to the backbone of T . We set the root r of T such that it belongs to the backbone and has minimum degree, that is, two, three or four children if T has maximum degree 3, 4 or 5, respectively.

Lastly, we assign a colour to each vertex in T . For each variable gadget of a variable x_i , let $c(x_i) = c(\bar{x}_i)$ be a new colour. Also, for each vertex $\tilde{x}_{i,j} \in \{x_{i,j}, \bar{x}_{i,j}\}$, let $c(\tilde{x}_{i,j})$ be a new colour. Then, for each clause gadget of clause C_j , if the literal $\ell_{k,j} = x_i$ in C_j , then set $c(\ell_{k,j}) := c(x_{i,j})$, but if $\ell_{k,j} = \bar{x}_i$,

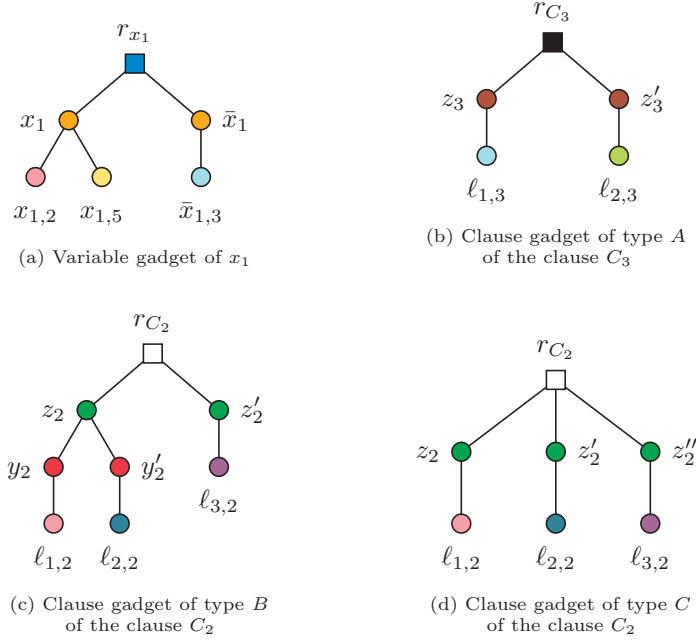


Figure 2: An example of variable and clause gadgets used in [Construction 1](#) given a 3,3-SAT instance ϕ . The variable x_1 appears twice as a positive literal (in C_2 and C_5) and once as a negative literal (in C_3) in ϕ . The clause C_2 (of size 3) in ϕ , which can be represented with a clause gadget of type B or C , contains x_1 as a positive literal (pink vertex \bullet corresponding to vertex $x_{1,2}$). The clause C_3 (of size 2) in ϕ , which must be represented with a clause gadget of type A , contains x_1 as a negative literal (light blue vertex \bullet corresponding to vertex $\bar{x}_{1,3}$). Squared vertices have a colour that appears only once in the obtained tree T .

then set $c(\ell_{k,j}) := c(\bar{x}_{i,j})$. If a clause gadget is of type B , then let $c(y_j) = c(y'_j)$ and $c(z_j) = c(z'_j)$ be two new colours. If a clause gadget is of type C , then let $c(z_j) = c(z'_j) = c(z''_j)$ be a new colour. If a clause gadget is of type A , then let $c(z_j) = c(z'_j)$ be a new colour. Finally, each remaining uncoloured vertex (root vertex of a variable or clause gadget, or vertex of the central path) is assigned a new (unused) colour. If T does not contain clause gadgets of type C , then its colour-multiplicity is 2, otherwise it is 3. An example of variable and clause gadgets with coloured vertices is given in [Fig. 2](#).

Note that [Construction 1](#) can be done in polynomial time.

Theorem 1. COLOURFUL COMPONENTS and COLOURFUL PARTITION are NP-complete on coloured :

- 2-caterpillars with maximum degree 5 and colour-multiplicity 3,
- 3-caterpillars with maximum degree 4 and colour-multiplicity 2, and
- 4-caterpillars with maximum degree 3 and colour-multiplicity 2.

Proof. Obviously, the problems are in NP. Since the problems are equivalent on trees, we prove the statement for COLOURFUL COMPONENTS. Let ϕ be an instance of 3,3-SAT with m clauses and n variables. We denote by m_3 the number of clauses containing 3 literals and by m_2 the number of clauses containing 2 literals. We transform ϕ into a coloured tree T as described in [Construction 1](#). Note that T is either a 2-caterpillar with maximum degree 5, a 3-caterpillar with maximum degree 4 or a 4-caterpillar with maximum degree 3. The colour-multiplicity of T depends on the use of clause gadgets of type C , and therefore is 3 for a 2-caterpillar with maximum degree 5 but can only be 2 for a 3-caterpillar with maximum degree 4 or a 4-caterpillar with maximum degree 3. We claim that there exists a satisfying assignment for ϕ if and only if there is a set of exactly $n + 2m_3 + m_2$ edges in T whose removal makes T colourful.

Let β be a satisfying assignment for ϕ . We define the set of edges S as follows:

- For each variable x_i , the set S contains the edge $\{r_{x_i}, x_i\}$ if $x_i = True$ in β , or $\{r_{x_i}, \bar{x}_i\}$ if $x_i = False$ in β .
- For each clause C_j with three literals:
 - If the clause gadget is of type B , then the set S contains two edges: one from the path between y_j and y'_j , and one from the path between z_j and z'_j . These edges are chosen in such a way that, in $G - S$, the leaf $\ell_{k,j}$ which belongs to the same connected component as the vertex r_{C_j} corresponds to (one of) the literal(s) satisfying the clause C_j in β .
 - If the clause gadget is of type C , then the set S contains two edges incident to the vertex r_{C_j} such that z_j , z'_j , and z''_j are in different connected component in $G - S$. These edges are chosen so that, in $G - S$, the leaf $\ell_{k,j}$ which belongs to the same connected component as the vertex r_{C_j} corresponds to (one of) the literal(s) satisfying the clause C_j in β .
- For each clause C_j with two literals, which is represented by a clause gadget of type A , the set S contains either the edge $\{r_{C_j}, z_j\}$ or the edge $\{r_{C_j}, z'_j\}$. Again, this edge is chosen in order that, in $G - S$, the leaf $\ell_{k,j}$ which belongs to the same connected component as the vertex r_{C_j} corresponds to (one of) the literal(s) satisfying the clause C_j in β .

Clearly, the set S contains $n + 2m_3 + m_2$ edges. We denote by F the forest $T - S$, and by T' the connected component in F containing the root r of T . Obviously, two vertices of the same colour from a same variable gadget do not both belong to a same connected component of F , and the same holds for a clause gadget. Also, note that two vertices of different variable gadgets do not have the same colour, and similarly for vertices of different clause gadgets. Lastly, observe that two vertices of two different gadgets belong to the same connected component if and only if they are connected through the backbone, which is in T' . Thus, if

there exist two vertices of the same colour in a same connected component of F , one is from a variable gadget and the other one from a clause gadget; so both vertices belong to T' . Without loss of generality, consider $x_{i,j}$ from the variable gadget of x_i and $\ell_{k,j}$ from the clause gadget of C_j , such that $x_{i,j}, \ell_{k,j} \in T'$. To derive a contradiction, suppose that $c(x_{i,j}) = c(\ell_{k,j})$. Note that the literal represented by $\ell_{k,j}$ is $x_{i,j}$, otherwise the two vertices would not have the same colour. Since $\ell_{k,j}$ is in T' , it is connected to the vertex r_{C_j} of the clause gadget, hence $\ell_{k,j}$ satisfies the clause C_j . Therefore, the variable $x_i = True$ in β . By construction, this implies that the edge $\{r_{x_i}, x_i\}$ belongs to S , and that the subtree T_{x_i} , containing $x_{i,j}$, is not part of T' , which is a contradiction.

Let S be a solution to COLOURFUL COMPONENTS for T . Observe that S must contain at least one edge per variable gadget to put the vertices x_i and \bar{x}_i , which have the same colour, into different connected components. Besides, since each clause gadget of type B contains two pairs of vertices with the same colours (the pairs z_i, z'_i and y_i, y'_i), and each clause gadget of type C contains three vertices of the same colour (z_i, z'_i and z''_i), S must contain at least two edges per clause gadget of type B or type C . Lastly, as each clause gadget of type A contains two vertices with the same colour, S must contain at least one edge per clause gadget of type A . Putting everything together, we get that $|S| \geq n + 2m_3 + m_2$. Therefore, if we suppose that $|S| = n + 2m_3 + m_2$, then S contains exactly one edge per variable gadget, exactly two edges per clause gadget of type B or type C , and exactly one edge per clause gadget of type A . Note that S does not contain any edge from the backbone of T . Let T' be the connected component of $T - S$ containing the root r . Notice that, for each variable gadget, either x_i or \bar{x}_i belongs to T' , but not both. Also, for each clause gadget, exactly one leaf $\ell_{k,j}$ belongs to T' . We construct an assignment β of ϕ such that, for each variable gadget, if $\{r_{x_i}, x_i\} \in S$, then we set $x_i := True$ in β , and if $\{r_{x_i}, \bar{x}_i\} \in S$, then we set $x_i := False$ in β . Suppose for a contradiction that there is a clause C_j which is not satisfied in ϕ with regard to β . Consider the leaf $\ell_{k,j} \in T'$ from the gadget clause of C_j , and assume without loss of generality that $\ell_{k,j} = x_i$. If C_j is not satisfied, then the variable $x_i := False$ in β . This means that S contains the edge $\{r_{x_i}, \bar{x}_i\}$, but not the edge $\{r_{x_i}, x_i\}$, and thus $x_{i,j} \in T'$. However, since $c(x_{i,j}) = c(\ell_{k,j})$, then S is not a solution for T , a contradiction. \square

Bulteau et al. asked in [6] what is the complexity of COLOURFUL COMPONENTS and COLOURFUL PARTITION on trees of maximum degree d for $3 \leq d \leq 5$. Theorem 1 answers the question completely, since both problem are NP-complete on 4-caterpillars with maximum degree 3.

2.2. A linear-time algorithm for 1-caterpillars, and more

In this subsection, we show that COLOURFUL COMPONENTS and COLOURFUL PARTITION can be solved in linear time on coloured 1-pseudocaterpillars, without restriction on the maximum degree.

Definition 1 (Pseudocaterpillar). A *cyclic 1-caterpillar* is a connected graph with a unique cycle B , called the backbone, such that for any $e \in E(B)$, the graph $G - e$ is a 1-caterpillar. A *1-pseudocaterpillar* is either a 1-caterpillar or a cyclic 1-caterpillar. For the sake of simplicity, we refer to 1-pseudocaterpillars as *pseudocaterpillars*.

If a coloured pseudocaterpillar is not colourful, then it contains either one or at least two colours that appear more than once. We deal with these two cases independently in the following lemmas.

Lemma 1. COLOURFUL COMPONENTS *and* COLOURFUL PARTITION *can be solved in linear time on coloured pseudocaterpillars where at most one colour appears at least twice.*

Proof. Let $G = (V, E)$ be a coloured pseudocaterpillar such that there is at most one colour appearing at least twice in G . If each colour appears at most once, then G is colourful. So let γ be a colour that appears $n_\gamma \geq 2$ times in G .

Notice that if the backbone is a path, then an optimal solution to COLOURFUL PARTITION with n_γ parts can be found in linear time. Clearly, the $n_\gamma - 1$ edges with endpoints in different parts form an optimal solution to COLOURFUL COMPONENTS and can also be obtained in linear time.

Suppose that the backbone induces a cycle and let b_γ be the number of vertices of colour γ on the backbone. It is easy to see that if $b_\gamma \leq 1$, the removal of $n_\gamma - 1$ edges is necessary and sufficient. Assume now that $b_\gamma \geq 2$. Since at least two vertices of colour γ belong to the backbone (which is a cycle), any solution to COLOURFUL COMPONENTS must contain at least 2 edges from the backbone. Remove any edge e from the backbone and let G' be the resulting graph. Since the backbone of G' is a path, G' is a tree, and thus any optimal solution S' to COLOURFUL COMPONENTS on G' contains exactly $n_\gamma - 1$ edges. Of course, $S' \cup \{e\}$ is a solution for G that can be obtained in linear time and contains at most n_γ edges. We claim that such a solution is optimal. Let S be a solution to COLOURFUL COMPONENTS for G . To obtain a contradiction, suppose that $|S| < n_\gamma$. Let $e \in S$ be an edge on the backbone (there exist at least two such edges in S), and let G' be the graph obtained by removing e from G . Of course, G' is connected and has a path as backbone. Since S is a solution to COLOURFUL COMPONENTS for G , then $S' = S \setminus \{e\}$ is a solution for G' of size $|S'| = |S| - 1 < n_\gamma - 1$, which is impossible since G' is connected and contains n_γ vertices of colour γ .

Clearly, the partition of the vertices defined by each connected component is an optimal solution to COLOURFUL PARTITION since each component contains exactly one vertex of colour γ . \square

We now deal with the case where the input pseudocaterpillar contains at least two colours that appear more than once. We consider the vertices of the backbone as internal vertices of stars whose leaves are the adjacent vertices with degree 1. When the backbone is a path, we assume it is of minimum length. This in particular implies that, if the input pseudocaterpillar contains at least 3

vertices, then there is a bijection between vertices of the backbone and vertices with degree at least 2, and conversely a bijection between leaves of the stars and vertices with degree 1.

Definition 2 ($C[u]$). Given a coloured pseudocaterpillar G with backbone B and a vertex $u \in V(B)$, we denote by $C[u]$ the multiset of colours used by all the vertices of the star having u as internal vertex. Note that using a breadth-first search, the value of $C[u]$ can be computed in linear time for all $u \in V(B)$.

Remark 1. Consider a coloured pseudocaterpillar G with backbone B and a vertex $u \in V(B)$. If $C[u]$ contains a colour γ more than once, then at least one vertex with colour γ in the corresponding star is a leaf which must belong to a different colourful component than u . Hence, G can be preprocessed in such a way that, for each such leaf, we add its adjacent edge to a set S_p . This procedure is repeated until there is no such leaf in $G - S_p$. At the end of the preprocessing, each star of $G - S_p$ is a *colourful star*, that is, only contains vertices with different colours. See an example of such a preprocessing in Fig. 3.

A *coloured pseudocaterpillar with colourful stars* is a coloured pseudocaterpillar whose all stars are colourful.

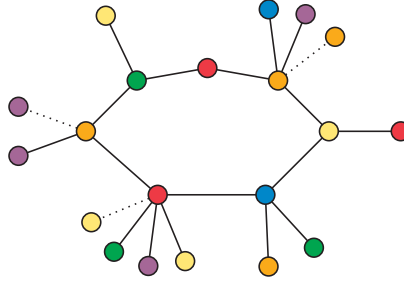


Figure 3: A 6-coloured pseudocaterpillar. Dotted edges belong to all solutions to COLOURFUL COMPONENTS (up to isomorphism) and are removed from the graph in the preprocessing.

Definition 3 (Bad path). A path P in a coloured graph G between two distinct vertices u and v of colour γ is called a γ -bad path. A bad path is a γ -bad path for some colour γ .

Note that a graph is colourful if and only if it does not contain a bad path.

We say that a solution to COLOURFUL COMPONENTS is *optimal* if it is a solution of minimum size.

Lemma 2. Let G be a coloured pseudocaterpillar with colourful stars such that at least two colours appear at least twice in G . Then there exists an optimal solution S to COLOURFUL COMPONENTS for G such that $S \subseteq E(B)$, where B is the backbone of G .

Proof. Note that, since every star of G is colourful and at least two colours appear at least twice in G , the backbone B must contain at least 2 vertices. For simplicity, let x_0, \dots, x_t be the vertices in $V(B)$ ordered such that $\{x_i, x_{i+1}\} \in E(B)$ for all $i \in \{0, \dots, t-1\}$, and $\{x_t, x_0\} \in E(B)$ when B is a cycle.

It is easily observed that if the backbone B is a path, then there exists an optimal solution S to COLOURFUL COMPONENTS for G such that $S \subseteq E(B)$. For instance, one can construct such an optimal solution S as follows: let $S = \emptyset$ and $Col = \emptyset$; traverse B starting at x_0 ; each time a vertex x_i is visited (x_0 being the first visited vertex), if $C[x_i] \cap Col \neq \emptyset$, then add the edge $\{x_{i-1}, x_i\}$ to S and set $Col := C[x_i]$; otherwise, set $Col := Col \cup C[x_i]$. In particular, all along the algorithm, Col contains the colours used by the vertices of the stars whose internal vertices are in the same connected component as the newly visited vertex x_i in $G - S$. Clearly, once all the vertices of B have been visited, the set S is an optimal solution for G .

So we can focus on the case where B is a cycle. Let S be an optimal solution to COLOURFUL COMPONENTS for G maximising $|S \cap E(B)|$. We assume that $S \not\subseteq E(B)$, otherwise the statement is true. We consider two cases.

Suppose first that there exists an edge $f \in S \cap E(B)$. Then the backbone B' of $G' = G - \{f\}$ is a path, and thus there exists an optimal solution S' for G' such that $S' \subseteq E(B')$. Obviously, $S' \cup \{f\}$ is a solution to COLOURFUL COMPONENTS for G . We claim that $S' \cup \{f\}$ is an optimal solution for G . If not, then $|S| \leq |S'|$. This implies that $S \setminus \{f\}$ is a solution for G' of size $|S| - 1 < |S'|$, contradicting the optimality of S' . So $S' \cup \{f\}$ is an optimal solution for G . However, as $S' \cup \{f\} \subseteq E(B)$, this contradicts the choice of S .

Suppose now that $S \cap E(B) = \emptyset$. Recall that at least two colours α and β appear at least twice in G , and thus there exists at least one α -bad path and at least one β -bad path in G . Since $S \cap E(B) = \emptyset$, no edge in S can belong to an α -bad path and a β -bad path. Thus, S contains two distinct edges $e = \{x_i, w\}$ and $f = \{x_j, y\}$ with $x_i, x_j \in V(B)$ and $w, y \notin V(B)$, and such that $c(w) = \alpha$ and $c(y) = \beta$ (note that we may have $x_i = x_j$). If $x_i = x_j$, then the set $S' = (S \setminus \{e, f\}) \cup \{e', f'\}$ where e' and f' are the two distinct edges of B incident to $x_i (= x_j)$ is also a solution for G . Besides, either $|S'| < |S|$, which contradicts the optimality of S , or $|S'| = |S|$ and $|S' \cap E(B)| > |S \cap E(B)|$, which contradicts the choice of S . So we conclude that $x_i \neq x_j$. Since S is an optimal solution, there exist exactly two distinct vertices $w', y' \in V(G) \setminus \{w, y\}$ in the same connected component as x_i and x_j in $G - S$ such that $c(w') = c(w) = \alpha$ and $c(y') = c(y) = \beta$. Denote by x_k the internal vertex of the star containing w' and by x_ℓ the internal vertex of the star containing y' . Note that we may have $x_k = x_\ell$; if $x_k \neq x_\ell$, then we may have $x_i = x_\ell$ or $x_j = x_k$. Let P_k be a path in B between x_i and x_j such that $x_k \in V(P_k)$, and $e' \in E(P_k)$ the edge incident to x_i . Similarly let P_ℓ be a path in B between x_i and x_j such that $x_\ell \in V(P_\ell)$, and $f' \in E(P_\ell)$ the edge incident to x_j . If $x_i = x_\ell$ or $x_j = x_k$, then the paths P_k and P_ℓ are not uniquely defined. In this case, we make sure to choose P_k and P_ℓ such that one is not a subgraph of the other (which is always possible since $x_i \neq x_j$ and B is a cycle). We claim that $S' = (S \setminus \{e, f\}) \cup \{e', f'\}$ is an optimal solution for G . First, observe that, for any colour $\gamma \notin \{\alpha, \beta\}$, there are no γ -bad paths in any connected component of $G - S'$. Indeed, if there were an α -bad path or a β -bad path P in $G - S'$, then, necessarily, w or y would be an endpoint of P . However, the fact that $e', f' \in S'$ guarantees that x_i and x_k are in two different connected components of $G - S'$, and the same holds for x_j and x_ℓ . In particular,

w is the only vertex of colour α in its connected component of $G - S'$ and, similarly, y is the only vertex of colour β in its connected component of $G - S'$. Hence, S' is a solution for G such that $|S'| \leq |S|$ and $|S' \cap E(B)| > |S \cap E(B)|$, a contradiction with the choice of S .

Thus, we conclude that $S \subseteq E(B)$. \square

In order to obtain a linear-time algorithm, we define a special type of bad paths, called *critical bad paths*.

Definition 4 (Critical bad path). Let G be a coloured pseudocaterpillar with backbone B and colourful stars. A γ -critical bad path P is a maximal path in B such that for all internal vertex w of P it holds $\gamma \notin C[w]$. A *critical bad path* is a γ -critical bad path for some colour γ . Note that two γ -critical bad paths do not have common edges.

Remark 2. Let G be a coloured pseudocaterpillar with colourful stars such that at least two colours appear twice. Let B be the backbone of G . Since G has colourful stars, each bad path, and therefore each critical bad path, contains at least one edge in B . It is easily observed that if a set of edges $S \subseteq B$ contains at least one edge from each critical bad path, then S contains at least one edge of each bad path, and thus S is a solution to COLOURFUL COMPONENTS. Moreover, since the existence of a bad path implies the existence of a critical bad path, if S is of minimum size, then according to Lemma 2 S is an optimal solution to COLOURFUL COMPONENTS.

We observe that, although the number of bad paths can be quadratic in the number of vertices, there is only a linear number of critical bad paths, and thus a linear number of edges from the backbone that have to be removed from the input graph to make it colourful. This observation together with Remark 2 allows us to develop a linear-time algorithm on coloured pseudocaterpillars.

The idea of the algorithm is to define a circular-arc graph H (an intersection graph of a collection of arcs on the circle) based on the critical bad paths of G . Hsu and Tsai showed that a minimum clique cover \mathcal{Q} of H , which is a partition of the vertex set into a minimum number of cliques, can be obtained in linear time [12]. We show that \mathcal{Q} can then be translated back into an optimal solution to COLOURFUL COMPONENTS and COLOURFUL PARTITION for G in linear time.

Lemma 3. *Let G be a coloured pseudocaterpillar with colourful stars, and A be the multiset of pairs returned by Algorithm 1. Then there is a bijection between the set of critical bad paths in G and the multiset A .*

Proof. Let B be the backbone of G . A critical-bad path P between two distinct vertices $x_j, x_i \in V(B)$ is detected in Algorithm 1 at Line 13, when the algorithm detects that there exist a colour $\gamma \in C[x_i]$ such that $L[\gamma] \notin \{NULL, x_i\}$. When x_i is considered in the algorithm, the pair $(L[\gamma], x_i)$ is added to A at Line 14, and since $L[\gamma] = x_j$, then $(x_j, x_i) \in A$. Note that P is a γ -critical bad path, since $x_j = L[\gamma]$ corresponds to the last visited vertex, before x_i , such that $\gamma \in C[x_j]$. Therefore, the arc with endpoints $(x_j, x_i) \in A$ corresponds to the critical bad path P between x_j and x_i in G .

Algorithm 1: From coloured pseudocaterpillar to ordered pairs.

Input: An ℓ -coloured pseudocaterpillar with colourful stars G and backbone B .

Output: A multiset of ordered pairs of vertices.

```

// Initialisation
1 let  $A$  be an empty multiset of ordered pairs of vertices;
2 let  $L$  be an array of length  $\ell$  initialised at  $NULL$ ;
3 let  $x_0, \dots, x_p \in V(B)$  s.t.  $\forall i \in \{1, \dots, p-1\}, \{x_i, x_{i+1}\} \in E(B)$ ;
4 let  $end := NULL$ ;
5 let  $proceed := True$ ;
6 let  $i := 0$ ;
// Main procedure
7 while  $proceed$  do //  $\mathcal{O}(n)$ 
8   if  $x_i = end$  then // no more critical bad path
9      $proceed := False$ ; //  $B$  is a cycle
10  foreach  $\gamma \in C[x_i]$  do
11    if  $L[\gamma] = NULL$  then // first time colour  $\gamma$  is seen
12       $end := x_i$ ;
13    else if  $L[\gamma] \neq x_i$  then // critical bad path detected
14      add  $(L[\gamma], x_i)$  to  $A$ ;
15       $L[\gamma] := x_i$ ;
16  if  $i = p$  and  $\{x_p, x_0\} \notin E(B)$  then
17     $proceed := False$ ; //  $B$  is a path
18     $i := (i + 1) \bmod (p + 1)$ ;
19 return  $A$ ;

```

An ordered pair (x_j, x_i) in A refers to two vertices x_i and x_j in $V(B)$. If such a pair exists, then there exists a colour γ such that $\gamma \in C[x_j] \cap C[x_i]$. Observe that when (x_j, x_i) is added to A , it holds $L[\gamma] = x_j$. This implies that the path P between x_j and x_i in B , with respect to the order in which the vertices in $V(B)$ are visited, does not contain an internal vertex x_k such that $\gamma \in C[x_k]$. Thus, P is a γ -critical bad path and it corresponds to the pair (x_j, x_i) in A . \square

Lemma 4. *Algorithm 1 runs in linear time.*

Proof. Let G be a coloured pseudocaterpillar with colourful stars and backbone B , and A the multiset of ordered pairs obtained by Algorithm 1 with input G .

In Algorithm 1, when a colour is seen for the first time at Line 11, the internal vertex u of the star is stored in the variable end . If the backbone induces a cycle, then the second time that the vertex end is considered in the main loop the algorithm sets the variable $proceed$ to $False$ at Line 9. If the backbone is a path, then algorithm considers each vertex exactly once and sets the variable $proceed$ to $False$ at Line 17. Thus, Algorithm 1 runs in linear time. \square

Before proving [Theorem 2](#), we first need to define the notions of *clique cover* and *circular-arc graph*.

Definition 5 (Clique cover). A *clique cover* of a graph H is a partition \mathcal{Q} of $V(H)$ such that for each $Q_i \in \mathcal{Q}$, Q_i is a clique. A *minimum clique cover* is a clique cover containing a minimum number of cliques.

Definition 6 (Circular-arc graph). A *circular-arc graph* H is the intersection graph of a set of arcs on the circle, that is, each vertex of H can be represented as an arc on the circle and there is an edge between two vertices if and only if the corresponding two arcs intersect.

Finally, we prove the main result of this section.

Theorem 2. COLOURFUL COMPONENTS and COLOURFUL PARTITION can be solved in linear time on coloured pseudocaterpillars.

Proof. We note that checking whether a given graph is colourful can be done in linear time, for instance with a breadth-first search.

Let G be a coloured pseudocaterpillar with backbone B . First, we prove that a solution to COLOURFUL COMPONENTS for G can be found in linear time. We apply the preprocessing to G , as defined in [Remark 1](#), and denote by S_p the set of edges that have been removed. Hence, $G - S_p$ contains colourful stars. If $G - S_p$ is colourful, then S_p is an optimal solution to COLOURFUL COMPONENTS. Otherwise, denote by G' the connected component of $G - S_p$ containing the backbone B . If G' contains exactly one colour that appears more than once, then according to [Lemma 1](#) COLOURFUL COMPONENTS and COLOURFUL PARTITION can be solved in linear time. Therefore, we assume that G' contains at least two colours that appear at least twice. Let A be the multiset of ordered pairs obtained by [Algorithm 1](#) with input G' . According to [Lemma 4](#), A can be obtained in linear time.

Following [Lemma 3](#), each ordered pair (x, y) in A corresponds to a critical bad path P from x to y in G . Using the algorithm of Hsu and Tsai [[12](#)], we obtain a minimum clique cover \mathcal{Q} of the circular-arc graph H represented by A in linear time. As mentioned in [[12](#)], there are two types of cliques in a circular-arc graph. The first type of cliques contains three arcs which do not contain a common point of the circle. The second type of cliques contain a common point of the circle and are called *linear cliques*. It is proved that in any circular-arc graph, there exists a minimum clique cover made of linear cliques only, unless the graph is a complete graph and the unique maximal clique is not linear. Hence, either all cliques in \mathcal{Q} are linear or the unique clique in \mathcal{Q} is not linear. In the following, we first deal with the case where \mathcal{Q} contains linear cliques and then consider the case where the unique clique in \mathcal{Q} is not linear.

Suppose that all cliques in \mathcal{Q} are linear. Let S' be an empty set of edges. Choose a clique $Q_i \in \mathcal{Q}$. From our construction of A , and thus of H , each vertex $z \in Q_i$ corresponds to a critical bad path P_z in G . Let $D_i := \bigcap_{z \in Q_i} P_z$, and notice that, since Q_i is linear, $|D_i \cap B| > 0$. Then, choose an edge $e \in D_i \cap B$, and add e to S' . We claim that, once each clique in \mathcal{Q} has been processed, that is,

when $|S'| = |\mathcal{Q}|$, the set S' is an optimal solution to COLOURFUL COMPONENTS on G . Notice that S' can be computed in linear time. As stated before, each critical bad path in G' maps to an ordered pair in A , which corresponds to a vertex of H . Hence, a linear clique Q_i of H corresponds to a set of critical bad paths sharing a common subpath D_i . The set S' contains an edge in $D_i \cap B$ for each $Q_i \in \mathcal{Q}$, and hence there is no critical bad path in $G' - S'$. Since $S' \subseteq B$ and each critical bad path has an edge in S' , then by Remark 2 S' is a solution to COLOURFUL COMPONENTS on G' . Moreover, since \mathcal{Q} is minimum, S' is an optimal solution for G' . Thus, the set $S := S_p \cup S'$ is an optimal solution to COLOURFUL COMPONENTS on G .

Now, suppose that the unique clique in \mathcal{Q} is not linear, which implies that H is a complete graph. As mentioned before, there exist at least three arcs that do not have a common point on the circle. Let z denote one of these arcs such that it does not strictly contain any other arc. Clearly, since H is a complete graph, z overlaps every other arc. Also, since z does not strictly contain any other arc, if one extremity of z does not overlap another arc z' , then its other extremity must overlap z' . Let P_z be the colour-minimal bad path, with endpoints u and v , corresponding to z . Denote by u' and v' the neighbours of u and v in P_z , respectively. Then, the set $S' := \{\{u, u'\}, \{v, v'\}\}$ is a solution to COLOURFUL COMPONENTS on G' . Since the intersection of all the critical bad paths in G' is empty, any solution to COLOURFUL COMPONENTS on G' contains at least 2 edges. Hence, S' is an optimal solution and $S := S_p \cup S'$ is an optimal solution to COLOURFUL COMPONENTS on G .

Finally, we can detect each connected component of $G - S$ in linear time (for instance, with a breadth-first search). Thus, we can construct the partition π of V such that each part corresponds to a connected component of $G - S$ in linear time. Obviously, π is a solution to COLOURFUL PARTITION on G . Since S is optimal, the partition π is optimal. \square

Note that, for any two critical bad paths P and P' , if $E(P') \subset E(P)$, then P does not have to be represented. While this observation can help decreasing the number of vertices in the circular-arc graph, it does not affect the overall worst-case linear-time complexity of the algorithm.

Figure 4 gives an example of a pseudocaterpillar G and its representation as circular-arc graph H where a minimum clique cover of H represents an optimal solution to COLOURFUL COMPONENTS on G .

Generalisation of the result.

Now, we slightly generalise our results on pseudocaterpillars to what we call *necklace graphs with colourful beads*. We say that a graph is a *necklace graph* if it can be obtained by adding edges between connected components, called *beads*, such that the newly added edges induce a path or a cycle, called the *backbone*, and exactly one vertex of each bead is contained in the backbone. Note that necklace graphs are, by definition, connected. We say that a graph is a necklace graph with colourful beads if every bead is colourful. The general idea is to transform the input necklace graph with colourful beads into a pseudocaterpillar

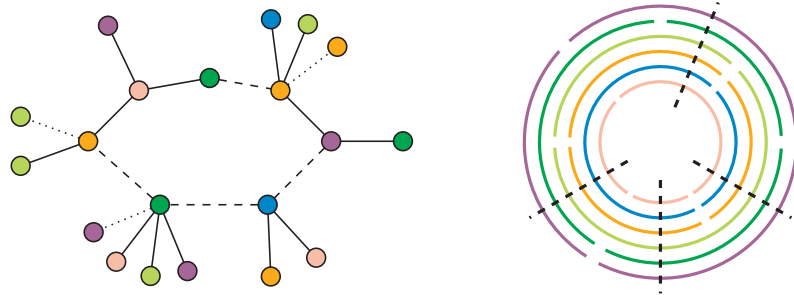


Figure 4: On the left, a pseudocaterpillar G : dotted edges are removed in the preprocessing; dashed edges are obtained from Algorithm 1; dotted and dashed edges form an optimal solution to COLOURFUL COMPONENTS. On the right, an arc representation of the circular-arc graph constructed from the critical bad paths in G (after preprocessing): dashed segments represent a minimum clique cover and correspond to the dashed edges in G .

with colourful stars, where beads will be represented as leaves, and then use our earlier results on pseudocaterpillars from to obtain a linear time algorithm for COLOURFUL COMPONENTS and COLOURFUL PARTITION.

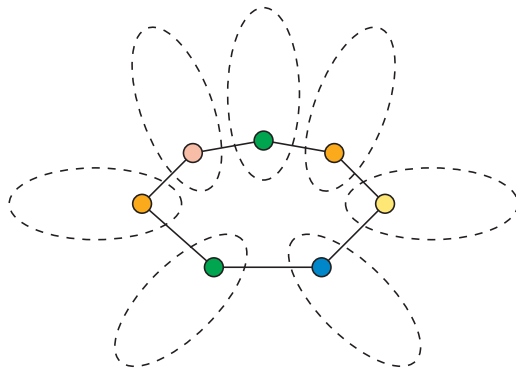


Figure 5: A schematic representation of a necklace graph with colourful beads. The backbone is represented with plain edges and the beads with dashed ellipses.

First, we prove a similar result as Lemma 1 in the context of necklace graphs.

Lemma 5. COLOURFUL COMPONENTS and COLOURFUL PARTITION can be solved in linear time on coloured necklace graphs with colourful beads where at most one colour appears at least twice.

Proof. Let G be a coloured necklace graph with colourful beads and backbone B . If no colour appears more than once, then G is colourful and it can be checked in linear time.

Suppose that there exists a unique colour γ that appears at least twice in G . Fix a vertex u of colour γ and let X be the set of vertices in the same bead as u . Let v be the vertex in $V(B) \cap X$ (note that $u = v$ may hold) and consider a

minimal solution S to COLOURFUL COMPONENTS on G . Observe that if $u = v$, that is, if u is on the backbone, then S does not contain any edge in $G[X]$ (or else S is not minimum). If the size of a minimum u, v -cut (a minimum set of edges intersecting every path between u and v in G) is at least 2, then it is always possible to find S such that S does not contain any edge from $G[X]$, for instance by removing the two edges in B incident to v instead of the edges in the u, v -cut. So in this case, we can assume that $u \in V(B)$. On the other hand, if the u, v -cut is of size exactly one, then it is equivalent as if v were the only neighbour of u (the edge $\{u, v\}$ being a minimum cut of size one). It is now easy to see the parallel with colourful pseudocaterpillars where exactly one colour appears at least twice, and thus we can use a simple reduction and obtain a solution to COLOURFUL COMPONENTS and COLOURFUL PARTITION for G in linear time using [Lemma 1](#). \square

Now, we can focus on the case where the input graph contains at least two colours appearing at least twice. The main idea consists in finding a reduction from COLOURFUL COMPONENTS on coloured necklace graphs with colourful beads to COLOURFUL COMPONENTS on coloured pseudocaterpillars with colourful stars.

The following lemma can be proved similarly as [Lemma 2](#), using the fact that all beads are colourful and generalising the notation $C[u]$, for some vertex u in the backbone, to be the set of colours of the vertices of the colourful bead containing u .

Lemma 6. *Let G be a coloured necklace graph with colourful beads such that at least two colours appear at least twice in G . Then there exists an optimal solution S to COLOURFUL COMPONENTS for G such that $S \subseteq B$, where B is the backbone of G .*

Theorem 3. COLOURFUL COMPONENTS and COLOURFUL PARTITION can be solved in linear time on coloured necklace graphs with colourful beads.

Proof. Let G be a coloured necklace graph with colourful beads with backbone B . We assume that G contains at least two colours that appear at least twice, otherwise we use [Lemma 5](#) to obtain a solution in linear time. We create a coloured graph G' with vertex set $V(G') := V(G)$ (the vertices keep their original colour) and backbone $B' := B$. Then, for each $u \in V(B')$, we connect u to all the vertices v such that in G the vertices u and v belong to the same bead. Since the beads are colourful, we obtain that G' is a coloured pseudocaterpillar with colourful stars. By [Lemma 2](#), there exists an optimal solution $S_{G'}$ to COLOURFUL COMPONENTS on G' such that $S_{G'} \subseteq B'$. It is clear that $S_{G'}$ is also a solution for G . Moreover, following [Lemma 6](#), we know that there exists an optimal solution S_G for G such that $S_G \subseteq B$. Suppose that $|S_G| < |S_{G'}|$ and observe that S_G is also a solution for G' . A contradiction with $S_{G'}$ being optimal.

Also, note that an optimal solution to COLOURFUL PARTITION can be obtained in linear time from $S_{G'}$ (since $S_{G'} \subseteq B'$). According to [Theorem 2](#), $S_{G'}$ can be obtained in linear time. \square

2.3. Complexity dichotomies

Our results from Subsections 2.1 and 2.2 imply two complexity dichotomies on trees: one with respect to the maximum degree and the other with respect to the smallest value k such that the input tree is a k -caterpillar (and thus not a $(k - 1)$ -caterpillar).

Corollary 1. COLOURFUL COMPONENTS and COLOURFUL PARTITION are linear-time solvable on paths, that is, on trees with maximum degree at most 2, and NP-complete on trees with maximum degree at least 3.

Proof. If the input graph is a path, then Theorem 2 implies that both COLOURFUL COMPONENTS and COLOURFUL PARTITION are linear-time solvable. On the other hand, if the input graph is a tree that can contain vertices with degree 3 or more, then by Theorem 1 both problems are NP-complete. \square

Corollary 2. COLOURFUL COMPONENTS and COLOURFUL PARTITION are polynomial-time solvable on k -caterpillars if $k \leq 1$, and NP-complete otherwise.

Proof. If $k \leq 1$, then Theorem 2 implies that both problems can be solved in linear-time. However, if $k \geq 2$, then Theorem 1 shows that the problem is NP-complete. \square

Note that these results naturally extend to the case when the input graph can be disconnected by considering each connected component separately. In particular, both problems are linear-time solvable on forest if every connected component is a 1-caterpillar (which includes the case when the component is a path) and NP-complete otherwise.

3. Colourful Components on planar graphs

In [5], the authors prove that COLOURFUL COMPONENTS is NP-complete even when restricted to 3-coloured graphs with maximum degree 6. Using a similar reduction from PLANAR 3-SAT, we show how the vertices of degree 6 can be replaced with gadgets only containing vertices of degree 4, or 3, if we relax the number of colours from 3 to 5, or to 12, respectively.

PLANAR 3-SAT

Input: A 3-CNF formula ϕ in which the bipartite graph of variables and clauses is planar.

Question: Is there a satisfying assignment for ϕ ?

Note that PLANAR 3-SAT is NP-complete [13] and it can be shown that it remains so even if each clause contains exactly 3 literals [17]. In the following, we consider the latter version.

Construction 2. Given an instance ϕ of PLANAR 3-SAT, that is a set of m clauses C_1, C_2, \dots, C_m on n variables, we construct the graph G such that:

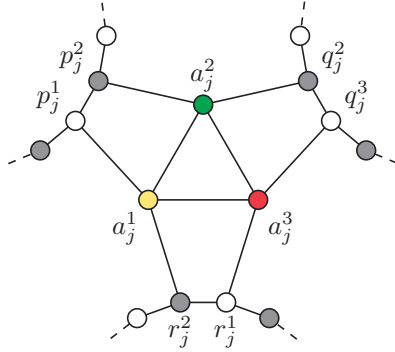
- For each variable x in ϕ , let m_x denote the number of clauses in which x appears. We construct a *variable cycle* of length $4m_x$ in G with vertices $V_x := \{x_j^1, x_j^2, x_j^3, x_j^4 \mid x \in C_j\}$ with an arbitrary fixed cyclic ordering of the clauses containing x . The vertices are coloured alternatively with two colours c_o and c_e such that $c(x_j^1) = c(x_j^3) = c_o$ and $c(x_j^2) = c(x_j^4) = c_e$, for all j such that $x \in C_j$. For simplicity, edges $\{x_j^1, x_j^2\}$ and $\{x_j^3, x_j^4\}$ are called *odd edges*, and the other edges in the variable cycle are called *even edges*.
- For each clause C_j containing three variables p, q and r , we construct a *clause gadget*. We propose two types of gadgets (see Fig. 6 for an example):
 - (i) The gadget \mathcal{A}_j^4 is made of a cycle of length 3, with vertices a_j^1, a_j^2 and a_j^3 such that each a_j^i is given colour i , different from c_o and c_e . We define how the vertices from V_p are connected to \mathcal{A}_j^4 . If the variable p appears as a positive literal in C_j , connect the vertices p_j^1 to a_j^1 and p_j^2 to a_j^2 . Otherwise, if p occurs as a negative literal, connect the vertices p_j^2 to a_j^1 and p_j^3 to a_j^2 . Do the same for the variables q and r by connecting the corresponding vertices in V_q to a_j^2 and a_j^3 , and the corresponding vertices in V_r to a_j^3 and a_j^1 . Notice that the vertices in \mathcal{A}_j^4 have degree 4.
 - (ii) The gadget \mathcal{A}_j^3 is made of a cycle of length 9 with vertices labelled a_j^1, \dots, a_j^9 and an additional vertex a_j^{10} connected to a_j^2, a_j^5 and a_j^8 . We set the colour i to each vertex a_j^i , different from c_o and c_e . We define how the vertices from V_p are connected to \mathcal{A}_j^3 . If the variable p appears as a positive literal in C_j , connect the vertices p_j^1 to a_j^1 and p_j^2 to a_j^3 . Otherwise, if p occurs as a negative literal, connect the vertices p_j^2 to a_j^1 and p_j^3 to a_j^3 . Do the same for the variables q and r by connecting the corresponding vertices in V_q to a_j^4 and a_j^6 , and the corresponding vertices in V_r to a_j^7 and a_j^9 . Notice that the vertices in \mathcal{A}_j^3 have degree 3.

Since the bipartite graph of variables and clauses of ϕ is planar and each vertex can be replaced by a clause or vertex gadget, with a correct cyclic ordering of the clauses for each variable, the resulting graph G is planar.

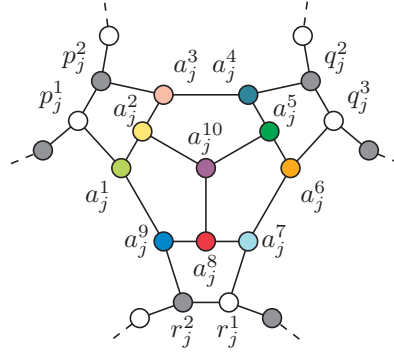
Note that [Construction 2](#) can be done in polynomial time.

Theorem 4. COLOURFUL COMPONENTS *is NP-complete on 5-coloured planar graphs with maximum degree 4 and on 12-coloured planar graphs with maximum degree 3.*

Proof. Let ϕ be an instance of PLANAR 3-SAT with m clauses and n variables and G be the graph obtained through [Construction 2](#), using either \mathcal{A}_j^4 gadgets or \mathcal{A}_j^3 gadgets. Notice that, depending on the type of gadget used, G is either a 5-coloured graph with maximum degree 4 or a 12-coloured graph with maximum degree 3. We claim that there exists a satisfying assignment β for ϕ if and only if there exists a solution S to COLOURFUL COMPONENTS such that $|S| = 10m$.



Gadget with vertices of degree 4.



Gadget with vertices of degree 3.

Figure 6: Two clause gadgets \mathcal{A}_j^4 (left) and \mathcal{A}_j^3 (right) of a clause $C_j := (p \vee \bar{q} \vee r)$. White vertices have colour c_o , grey vertices have colour c_e .

Let β be a satisfying assignment for ϕ . If $x = \text{False}$ in β , then remove the $2m_x$ odd edges in its variable cycle (of length $4m_x$), that is, edges of the type $\{x_j^1, x_j^2\}$ and $\{x_j^3, x_j^4\}$, for every clause C_j containing x . Similarly, if $x = \text{True}$ in β , then remove the $2m_x$ even edges in its variable cycle. Since every clause contains exactly 3 literals, $\sum_{1 \leq i \leq n} 4m_i/2 = 4 \times 3m/2 = 6m$ edges have been removed and G does not have any bad paths that contain only vertices from the variable cycles. Now, fix a clause C_j and denote its variables by p , q and r . Without loss of generality assume that p satisfies C_j . Remove the 4 edges between the clause gadget of C_j and the vertices in V_q and V_r . Without loss of generality, assume that C_j contains p as a positive literal. Hence the gadget is connected to the vertices p_j^1 and p_j^2 . Since p is set to true in β , the even edges of the variable cycle have been removed. Therefore, the vertices p_j^1, p_j^2 , together with the vertices in the clause gadget form a colourful component. The other case where C_j contains p as a negative literal is similar, and the vertices p_j^2, p_j^3 and those in the clause gadget form a colourful component. If the clause gadget is \mathcal{A}_j^4 , then the colourful component is of size 5. If the clause gadget is \mathcal{A}_j^3 , then the colourful component is of size 12. Since 4 edges are removed for each clause, a total of $4m$ edges between variable cycles and clause gadgets are removed. Let S be the set of removed edges, and notice that S is a solution to COLOURFUL COMPONENTS such that $|S| = 6m + 4m = 10m$.

Let S be a solution to COLOURFUL COMPONENTS for G such that $|S| = 10m$. Since the vertices of the variable cycles are coloured alternatively with c_o and c_e , for each variable cycle, S contains all its odd edges or all its even edges. Let $S' \subseteq S$ be a set of pairwise non-adjacent edges in S with maximum size whose edges have both endpoints in a variable cycle. In particular, notice that if a variable cycle has all its odd edges in S but not all its even edges, then all its odd edges belong to S' , and similarly if all its even edges belong to S ; if all edges of the variable cycle belong to S , that is, both odd and even edges, then

S' contains either all its odd edges or all its even edges. Hence, S' contains at least $\sum_{1 \leq i \leq n} 4m_i/2 = 6m$ edges. For a fixed clause C_j , we denote by H_j the connected component containing the clause gadget of C_j in $G - S'$. Note that no two clause gadgets belong to a same connected component of $G - S'$, and so for any two clauses C_k and C_ℓ we have $E(H_k) \cap E(H_\ell) = \emptyset$. Let $F_j = S \cap E(H_j)$. Observe that, independently on whether the odd or even edges of the variable cycles connected to the clause gadget of C_j belong to S' , if $|F_j| \leq 3$, then $H_j - F_j$ contains two vertices with the same colour c_o or c_e (from two variable cycles), and thus $H_j - F_j$ is not colourful; a contradiction with the fact that $G - S$ is colourful. Hence, $|F_j| \geq 4$. Since $|S| = 10m$ and $|S'| \geq 6m$, we get that $|S'| = 6m$ and that $|F_k| = 4$ for each clause C_k . Note that this also implies that, for each variable cycle, S' contains exactly either all of its odd edges or all of its even edges. Also, notice that the 4 edges in $S \cap F_k$ must be between the clause gadget and exactly two of the variable cycles attached to it; otherwise, at least two vertices with the same colour c_o or c_e (from two variable cycles) would belong to a same connected component of $G - S$. This implies that each clause gadget is adjacent to vertices of exactly one variable cycle in $G - S$. Let β be an assignment for ϕ described as follows. For each variable x , if S' contains the odd edges of the variable cycle of x , then set $x := False$ in β . Otherwise, if S' contains the even edges, then set $x := True$ in β . Recall that each clause gadget is adjacent to vertices of exactly one variable cycle in $G - S$. To prove that β is a satisfying assignment for ϕ , we claim that if vertices of the variable cycle of a variable x is connected to the clause gadget of a clause C_j in $G - S$, then C_j is satisfied by x . Suppose the contrary and assume, without loss of generality, that x appears as a positive literal in ϕ but $x = False$ in β . This means that the odd edge $\{x_j^1, x_j^2\}$ belongs to S' , and hence to S , and that the even edge $\{x_j^2, x_j^3\}$ belongs to $G - S$. Therefore, x_j^1 and x_j^3 belong to a same colourful component of $G - S$. However, $c(x_j^1) = c(x_j^3)$, a contradiction. We conclude that β is a satisfying assignment for ϕ . \square

4. Conclusion

We provide two complexity dichotomies on trees: one with respect to the maximum degree (see [Corollary 1](#)) and the other with respect to the smallest value k such that the input tree is a k -caterpillar (see [Corollary 2](#)). In order to complete the dichotomy on k -caterpillars with respect to the maximum degree, for every $k \in \mathbb{N}$, we ask the following question.

Question 1. What is the complexity of COLOURFUL COMPONENTS and COLOURFUL PARTITION on 3-caterpillars with maximum degree 3, 2-caterpillars with maximum degree 3 and 2-caterpillars with maximum degree 4?

We also prove that COLOURFUL COMPONENTS remains NP-complete on 5-coloured planar graphs with maximum degree 4 and on 12-coloured planar graphs with maximum degree 3. A natural question is to ask whether this result is tight with respect to the number of colours and the maximum degree.

Question 2. What is the complexity of COLOURFUL COMPONENTS on 4-coloured (planar) graphs with maximum degree 4 and on 11-coloured (planar) graphs with maximum degree 3?

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