Teleportation capability, distillability, and nonlocality on three-qubit states

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In this paper, we consider teleportation capability, distillability, and nonlocality on three-qubit states. In order to investigate some relations among them, we first find the explicit formulas of the quantities about the maximal teleportation fidelity on three-qubit states. We show that if any three-qubit state is useful for three-qubit teleportation then the three-qubit state is distillable into a Greenberger-Horne-Zeilinger state, and that if any three-qubit state violates a specific form of Mermin inequality then the three-qubit state is useful for three-qubit teleportation.

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I. INTRODUCTION

Teleportation capability, distillability, and nonlocality have been considered as significant features of quantum entanglement, and have been helpful to understand quantum entanglement. The three features have been known as a practical application of quantum entanglement, an important method to classify quantum entanglement with respect to the usefulness for quantum communication, and a physical property to explain the quantum correlation, respectively.

In the case of two-qubit states, it has been shown that there are two relations among the three features: If any two-qubit state is useful for teleportation then it is distillable into a pure entanglement, and if any two-qubit state violates the Bell inequality then it is useful for teleportation [1]. Then one could naturally ask what relations exist for multiqubit states.

In order to answer the question, a proper concept of teleportation capability over multiqubit states should be required, since distillability and nonlocality over multiqubit systems have already been presented, and their relations have been appropriately investigated [2–6]. In this paper, we present teleportation on three-qubit states, which could be generalized into the multiqubit case. We then define the meaningful quantities related to the teleportation capability on three-qubit states, and compare the quantities with distillability and nonlocality to look into the relations.

This paper is organized as follows. In Sec. II we properly define the quantities representing teleportation capability over three-qubit states, and explicitly compute the quantities. In Sec. III, we show that there are two relations among teleportation capability, distillability, and nonlocality, which are similar to the two-qubit case. Finally, in Sec. IV we summarize and discuss our results.

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II. TELEPORTATION CAPABILITY OVER THREE-QUBIT STATES

For teleportation over three-qubit states, we recall the Hillery-Bužek-Berthiaume [7] protocol, which is the splitting and reconstruction of quantum information over the Greenberger-Horne-Zeilinger (GHZ) state [8] by local quantum operations and classical communication (LOCC). The protocol can be modified into a teleportation protocol over a general three-qubit state in the compound system 123, as presented in [9]. The modified protocol is illustrated in Fig. 1 and is described as follows: Let i, j, and k be distinct in $\{1,2,3\}$. (i) Make a one-qubit orthogonal measurement on the system i. (ii) Prepare an arbitrary one-qubit state, and then make a two-qubit orthogonal measurement on the one qubit and the system j. (iii) On the system k, apply a proper unitary operation depending on the three-bit classical information of the two above measurement outcomes. We remark that the modified protocol is essentially equivalent to the original one with respect to the splitting and the reconstruction of quantum information.

As mentioned in [9], it is noted that any observable for a one-qubit measurement can be described as

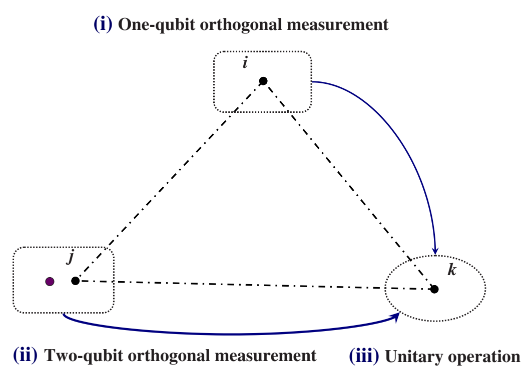

FIG. 1. (Color online) The modified teleportation protocol over a three-qubit state presented in [9]: The dotted boxes and ellipse represent performing the orthogonal measurements and applying the unitary operation, respectively. The arrows represent sending classical information corresponding to the measurement results.

$$U^\dagger \sigma_3 U = U^\dagger |0\rangle\langle 0| U - U^\dagger |1\rangle\langle 1| U, \tag{1}$$

where $\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$ is one of Pauli matrices, and U is a 2×2 unitary matrix. Thus after the step (i) of the teleportation protocol over a given three-qubit state ρ_{123}, the resulting two-qubit state of the compound system jk becomes

$$\begin{aligned}\varrho^t_{jk} &\equiv \frac{\mathrm{tr}_i(U_i^\dagger|t\rangle\langle t|U_i \otimes I_{jk}\rho_{123}U_i^\dagger|t\rangle\langle t|U_i\otimes I_{jk})}{\langle t|U_i\rho_i U_i^\dagger|t\rangle}\\ &= \frac{\mathrm{tr}_i(|t\rangle\langle t|U_i \otimes I_{jk}\rho_{123}U_i^\dagger|t\rangle\langle t|\otimes I_{jk})}{\langle t|U_i\rho_i U_i^\dagger|t\rangle}\end{aligned} \tag{2}$$

with probability $\langle t|U_i\rho_i U_i^\dagger|t\rangle$ for each $t=0,1$, where U_i is a 2×2 unitary matrix of the system i, and $\rho_i = \mathrm{tr}_{jk}(\rho_{123})$.

We now review the properties of the teleportation fidelity [10], which represents the faithfulness of a teleportation over a two-qubit state, and the fully entangled fraction [1,11–13]. The teleportation fidelity is naturally defined as

$$F(\Lambda_\rho) = \int d\xi \langle \xi|\Lambda_\rho(|\xi\rangle\langle\xi|)|\xi\rangle, \tag{3}$$

where Λ_ρ is a given teleportation protocol over a two-qubit state ρ, and the integral is performed with respect to the uniform distribution $d\xi$ over all one-qubit pure states, and the fully entangled fraction of ρ is defined as

$$f(\rho) = \max \langle e|\rho|e\rangle, \tag{4}$$

where the maximum is over all maximally entangled states $|e\rangle$ of two qubits. It has been shown [12,13] that the maximal fidelity achievable from a given bipartite state ρ is

$$F(\Lambda_\rho) = \frac{2f(\rho)+1}{3}, \tag{5}$$

where Λ_ρ is the standard teleportation protocol over ρ to attain the maximal fidelity. We remark that $F(\Lambda_\rho) > 2/3$ [or $f(\rho) > 1/2$] if and only if ρ is said to be useful for teleportation, since it has been shown that the classical teleportation can have at most $F=2/3$ (or $f=1/2$) [1,10,14].

Let F_i be defined as the maximal teleportation fidelity on the resulting two-qubit state in the compound system jk after the measurement of the system i, and let f_i be the maximal average of the fully entangled fraction of the state in the compound system jk after the measurement of the system i, that is,

$$f_i = \max_{U_i}[\langle 0|U_i\rho_i U_i^\dagger|0\rangle f(\varrho^0_{jk}) + \langle 1|U_i\rho_i U_i^\dagger|1\rangle f(\varrho^1_{jk})], \tag{6}$$

where the maximum is over all 2×2 unitary matrices. Then, as in the two-qubit case, it can be obtained [9] that for $i \in \{1,2,3\}$

$$F_i = \frac{2f_i+1}{3}. \tag{7}$$

By the reason as in the two-qubit case, a given three-qubit state ρ_{123} can be said to be *useful for three-qubit teleportation* if and only if $F_i > 2/3$ (or $f_i > 1/2$) for every $i \in \{1,2,3\}$.

In order to explicitly calculate the values of f_i, we remark that a three-qubit state ρ_{123} can be described as

$$\begin{aligned}&\frac{1}{8}I\otimes I\otimes I + \frac{1}{8}(\vec{s}_1\cdot\vec{\sigma}\otimes I\otimes I + I\otimes\vec{s}_2\cdot\vec{\sigma}\otimes I + I\otimes I\otimes\vec{s}_3\cdot\vec{\sigma})\\ &\quad + \frac{1}{8}\sum_{k,l=1}^{3}(b_1^{kl}I\otimes\sigma_k\otimes\sigma_l + b_2^{kl}\sigma_k\otimes I\otimes\sigma_l + b_3^{kl}\sigma_k\otimes\sigma_l\otimes I)\\ &\quad + \frac{1}{8}\sum_{j,k,l=1}^{3} t^{jkl}\sigma_j\otimes\sigma_k\otimes\sigma_l,\end{aligned} \tag{8}$$

where σ_i are Pauli matrices, $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$, $\vec{s}_i$ are real vectors in $\mathbb{R}^3$ satisfying $|\vec{s}_i| \leq 1$, and b_i^{kl} and t^{jkl} are real numbers.

For each $i=1,2,3$, let $\mathbf{b}_i$ be a 3×3 real matrix with (k,l) entry b_i^{kl}. Let $\mathbf{T}_1^j$, $\mathbf{T}_2^k$, and $\mathbf{T}_3^l$ be 3×3 real matrices with (k,l) entry t^{jkl}, (j,l) entry t^{jkl}, and (j,k) entry t^{jkl}, respectively. Then by way of the results in [1], it is obtained that for each $i=1,2,3$,

$$f_i = \frac{1}{4} + \frac{1}{8}\max\left[\left\|\mathbf{b}_i + \sum_{l=1}^{3}x_l\mathbf{T}_i^l\right\| + \left\|\mathbf{b}_i - \sum_{l=1}^{3}x_l\mathbf{T}_i^l\right\|\right], \tag{9}$$

where $\|\cdot\| = \mathrm{tr}|\cdot|$, and the maximum is taken over real numbers x_l satisfying $x_1^2+x_2^2+x_3^2=1$. By simple calculations of the Lagrange multiplier, we have the following formulas: For each $i=1,2,3$,

$$f_i = \frac{1}{4} + \frac{1}{8}\left[\left\|\mathbf{b}_i + \sum_{l=1}^{3}y_l\mathbf{T}_i^l\right\| + \left\|\mathbf{b}_i - \sum_{l=1}^{3}y_l\mathbf{T}_i^l\right\|\right], \tag{10}$$

where $y_l = \|\mathbf{T}_i^l\| / \sqrt{\Sigma_l \|\mathbf{T}_i^l\|^2}$.

For instance, we consider the values of f_i on the class of three-qubit states with four parameters presented by Dür *et al.* [15],

$$\begin{aligned}\rho_{\mathrm{GHZ}} &= \lambda_0^+|\Psi_0^+\rangle\langle\Psi_0^+| + \lambda_0^-|\Psi_0^-\rangle\langle\Psi_0^-|\\ &\quad + \sum_{j=1}^{3}\lambda_j(|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|),\end{aligned} \tag{11}$$

where $\lambda_0^+ + \lambda_0^- + 2\Sigma_j\lambda_j = 1$, and $|\Psi_j^\pm\rangle = (|j\rangle \pm |7-j\rangle)/\sqrt{2}$ are the GHZ-basis states. We note that any of three-qubit states can be transformed into a state ρ_{GHZ} in the class by LOCC (the so-called depolarizing process) [15,16].

Without loss of generality, we may assume that λ_0^+ is not less than λ_0^- and λ_j, since otherwise it can be adjusted by a local unitary operation. Then by Eq. (10), for four-parameter states ρ_{GHZ} with $\lambda_i + \lambda_j \leq 1/4$, we obtain

$$\begin{aligned}f_1 &= \lambda_0^+ + \lambda_3 = 1/2 + (\lambda_0^+ - \lambda_0^-)/2 - \lambda_1 - \lambda_2,\\ f_2 &= \lambda_0^+ + \lambda_2 = 1/2 + (\lambda_0^+ - \lambda_0^-)/2 - \lambda_1 - \lambda_3,\\ f_3 &= \lambda_0^+ + \lambda_1 = 1/2 + (\lambda_0^+ - \lambda_0^-)/2 - \lambda_2 - \lambda_3.\end{aligned} \tag{12}$$

III. RELATIONS WITH DISTILLABILITY AND NONLOCALITY OF THREE-QUBIT STATES

We now take the distillability over three-qubit states into account. Note that if a three-qubit state ρ_{123} has $\rho_{123}^{T_i} < 0$ for all $j=1,2,3$, where T_j represents the partial transposition for the system j, then one can distill a GHZ state from many copies of ρ_{123} by LOCC [15]. (We call such a state *GHZ distillable*.) Thus it can be obtained that a given three-qubit state ρ_{123} is GHZ distillable if $N_j(\rho_{123})>0$ for all $j=1,2,3$, where

$$N_j(\rho_{123}) = (\|\rho_{123}^{T_j}\| - 1)/2, \tag{13}$$

which is called the negativity, a bipartite entanglement measure [17,18]. We recall that for any two-qubit state ρ, its fully entangled fraction and its negativity satisfy the inequality $f(\rho)\leqslant 1/2+N(\rho)$, where N is the negativity [17]. Then since N is an entanglement monotone, it follows from the definition of f_i in Eq. (6) that for any three-qubit state ρ_{123}

$$\begin{aligned} f_i &= \max_{U_i} \sum_{t=0}^{1} \langle t|U_i\rho_i U_i^\dagger|t\rangle f(\varrho_{jk}^t) \\ &\leqslant \max_{U_i} \sum_{t=0}^{1} \langle t|U_i\rho_i U_i^\dagger|t\rangle [1/2 + N(\varrho_{jk}^t)] \\ &\leqslant 1/2 + N_j(\rho_{123}), \quad 1/2 + N_k(\rho_{123}), \end{aligned} \tag{14}$$

where i, j, and k are distinct in {1,2,3}. Therefore by the inequalities in Eq. (14), we obtain the following theorem.

Theorem 1. If a three-qubit state ρ_{123} is useful for three-qubit teleportation then it is GHZ distillable.

We remark that the converse of Theorem 1 is not true in general. For example, we consider ρ_{GHZ} with $\lambda_0^+=0.4$, $\lambda_0^-=0$, and $\lambda_1=\lambda_2=\lambda_3=0.1$. Then since

$$N_j(\rho_{\mathrm{GHZ}}) = \max\{0, (\lambda_0^+ - \lambda_0^-)/2 - \lambda_{4-j}\}, \tag{15}$$

we get $N_1(\rho_{\mathrm{GHZ}})=N_2(\rho_{\mathrm{GHZ}})=N_3(\rho_{\mathrm{GHZ}})=0.1>0$, that is, it is GHZ distillable. However, since $f_1=f_2=f_3=0.5$, it is not useful for three-qubit teleportation.

For the nonlocality over three-qubit states, we consider the Mermin inequality [2] on three-qubit states. Let $\mathcal{B}_M$ be the Mermin operator associated with the Mermin inequality as the following:

$$\begin{aligned} \mathcal{B}_M &= \vec{a}_1 \cdot \vec{\sigma} \otimes \vec{a}_2 \cdot \vec{\sigma} \otimes \vec{a}_3 \cdot \vec{\sigma} - \vec{a}_1 \cdot \vec{\sigma} \otimes \vec{b}_2 \cdot \vec{\sigma} \otimes \vec{b}_3 \cdot \vec{\sigma} \\ &\quad - \vec{b}_1 \cdot \vec{\sigma} \otimes \vec{a}_2 \cdot \vec{\sigma} \otimes \vec{b}_3 \cdot \vec{\sigma} - \vec{b}_1 \cdot \vec{\sigma} \otimes \vec{b}_2 \cdot \vec{\sigma} \otimes \vec{a}_3 \cdot \vec{\sigma}, \end{aligned} \tag{16}$$

where $\vec{a}_j$ and $\vec{b}_j$ are unit vectors in $\mathbb{R}^3$. Then for a given three-qubit state ρ, the Mermin inequality is

$$\mathrm{tr}(\rho\mathcal{B}_M) \leqslant 2. \tag{17}$$

We take $\vec{a}_j=(0,-1,0)$ and $\vec{b}_j=(-1,0,0)$ for all $j=1,2,3$. Then after local phase redefinition [4], the Mermin operator $\mathcal{B}_M$ in Eq. (16) can be written as

$$\mathcal{B}_{M_0} = 4(|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|). \tag{18}$$

Note that any three-qubit state ρ_{123} can be transformed into a four-parameter state ρ_{GHZ} in Eq. (11) by the depolarizing process, and that $\lambda_0^\pm=\langle\Psi_0^\pm|\rho_{\mathrm{GHZ}}|\Psi_0^\pm\rangle=\langle\Psi_0^\pm|\rho_{123}|\Psi_0^\pm\rangle$ and $2\lambda_j=\langle\Psi_j^+|\rho_{\mathrm{GHZ}}|\Psi_j^+\rangle+\langle\Psi_j^-|\rho_{\mathrm{GHZ}}|\Psi_j^-\rangle=\langle\Psi_j^+|\rho_{123}|\Psi_j^+\rangle$ $+\langle\Psi_j^-|\rho_{123}|\Psi_j^-\rangle$. Thus for the Mermin operator $\mathcal{B}_{M_0}$ in Eq. (18), we obtain the following equalities:

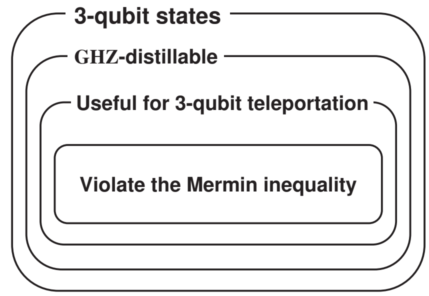

FIG. 2. The relations among the teleportation capability, distillability, and nonlocality for three-qubit states: The Mermin inequality we consider is the inequality with respect to the quantity (20).

$$\begin{aligned} \frac{1}{4}\mathrm{tr}(\rho_{123}\mathcal{B}_{M_0}) &= \langle\Psi_0^+|\rho_{123}|\Psi_0^+\rangle - \langle\Psi_0^-|\rho_{123}|\Psi_0^-\rangle \\ &= \langle\Psi_0^+|\rho_{\mathrm{GHZ}}|\Psi_0^+\rangle - \langle\Psi_0^-|\rho_{\mathrm{GHZ}}|\Psi_0^-\rangle = \lambda_0^+ - \lambda_0^-. \end{aligned} \tag{19}$$

We now assume that a given state ρ_{123} violates the Mermin inequality with the Mermin operator in Eq. (18). Then $\lambda_0^+ - \lambda_0^- > 1/2$, and hence $f_i(\rho_{\mathrm{GHZ}})=\lambda_0^+ + \lambda_{4-i} > 1/2$ for each $i=1,2,3$.

Since, by the definition of f_i in Eqs. (6) and (10), it can be easily shown that f_i is invariant under local unitary operations and f_i is convex, it can also be shown that f_i does not increase after applying operators in the depolarizing process [16], that is, $f_i(\rho_{\mathrm{GHZ}})\leqslant f_i(\rho_{123})$ for each $i=1,2,3$, and hence the given state ρ_{123} is useful for three-qubit teleportation.

Hence if we take $\vec{a}=\vec{a}_1=\vec{a}_2=\vec{a}_3$ and $\vec{b}=\vec{b}_1=\vec{b}_2=\vec{b}_3$ then we can readily show that if the quantity

$$\max_{\vec{a},\vec{b}} \mathrm{tr}(\rho_{123}\mathcal{B}_M) \tag{20}$$

is greater than 2 then ρ_{123} is useful for three-qubit teleportation. Therefore we have the following relation between nonlocality and teleportation on three-qubit states.

Theorem 2. If a three-qubit state ρ_{123} violates the Mermin inequality with respect to Eq. (20), then $f_i>1/2$ for all $i=1,2,3$, and hence it is useful for three-qubit teleportation.

By Theorem 1 and Theorem 2, we have two relations among the teleportation capability, distillability, and nonlocality for three-qubit states as in two-qubit states. The relations are seen in Fig. 2.

We remark that if we consider the Mermin inequality with respect to the quantity

$$\max_{\vec{a}_j,\vec{b}_k} \mathrm{tr}(\rho_{123}\mathcal{B}_M), \tag{21}$$

where $\mathcal{B}_M$ is the Mermin operator in Eq. (16), then Theorem 2 does not hold in general. For instance, $|0\rangle(|00\rangle+|11\rangle)/\sqrt{2}$ violates the Mermin inequality with respect to Eq. (21), but it is clear that the state is not useful for three-qubit teleportation although $f_1=1$.

However, by direct calculations and the fact [2] that the value of Eq. (21) for the GHZ state is 4, for any four-parameter state ρ_{GHZ} in Eq. (11), we can explicitly find the maximum value in Eq. (21),

$$\max_{\vec{a}_j,\vec{b}_k} \mathrm{tr}(\rho_{\mathrm{GHZ}}\mathcal{B}_M) = 4(\lambda_0^+ - \lambda_0^-). \tag{22}$$

Therefore it can be obtained that if ρ_{GHZ} violates any form of the Mermin inequality, then it is useful for three-qubit teleportation.

IV. CONCLUSIONS

In conclusion, we have considered two relations among the maximal teleportation fidelity, the distillability of the GHZ state, and violation of the Mermin inequality. In order to investigate the relations, we have first presented teleportation capability over three-qubit states, and have found the explicit formula of the maximal teleportation fidelity as a quantity representing the teleportation capability. Then we have shown that if any three-qubit state is useful for three-qubit teleportation then the three-qubit state is GHZ distillable, and that if any three-qubit state violates a specific Mermin inequality then the three-qubit state is useful for three-qubit teleportation.

It is known that even though for $n \geqslant 4$ there exist n-qubit bound entangled states which violates the Mermin inequality [3], there exists at least one splitting of the n qubits into two groups such that pure-state entanglement can be distilled [4–6]. Therefore if we would consider quantum communications between two or three groups then our results could be generalized into multiqubit cases.

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[16] Let ρ be an arbitrary three-qubit state, and for unitary operators U, V, W in $U(2)$, let $\mathcal{S}(U,V,W)$ be a superoperator defined as

$$\mathcal{S}(U,V,W)(\rho) = \frac{1}{2}U \otimes V \otimes W\rho U^\dagger \otimes V^\dagger \otimes W^\dagger + \frac{1}{2}\rho, \tag{23}$$

that is, $U\otimes V\otimes W$ is performed to ρ with probability $1/2$, while no operation is performed with probability $1/2$. Then we first apply $\mathcal{S}(\sigma_x,\sigma_x,\sigma_x)$ to ρ, and then apply $\mathcal{S}(\sigma_z,\sigma_z,I)$, $\mathcal{S}(\sigma_z,I,\sigma_z)$. Then it can be easily obtained that the resulting state is diagonal in the GHZ basis. We now apply $\mathcal{S}(U_{3\pi/2},U_{\pi/2},I)$ and $\mathcal{S}(U_{3\pi/2},I,U_{\pi/2})$, subsequently. Here U_θ maps $|j\rangle$ to $e^{\iota(1-j)\theta}|j\rangle$ for $j=0,1$, when $\iota=\sqrt{-1}$. Then we can obtain the three-qubit state with four parameters.
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