Transformation of Variable methods

in the numerical solution

of Fredholm integral equations

of the second kind

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Types of Fredholm integral equations (FIEs)

$$\begin{aligned} y(x) &= f(x) + \int_0^1 K(x, s, y(s))ds, \\ 0 &\leq x \leq 1, \end{aligned} \qquad (1)$$

$$\begin{aligned} y(x) &= f(x) + \int_{-1}^1 \frac{y(s)}{|x-s|^{\alpha}} ds, \\ 0 &< \alpha < 1, \quad -1 \leq x, s < 1. \end{aligned} \qquad (2)$$

Equation considered here: (1)

Idea of variable transformations in numerical integration

Consider

$$I = \int_0^1 f(x)dx,$$

and its approximation by the trapezoidal rule

$$I_N = h \sum_{j=0}^{N} w_j f(jh),$$

$$w_0 = w_N = 0.5, \quad w_j = 1, j = 1, 2, \ldots, N-1.$$

Euler - MacLaurin sum:

$$I - I_N = \sum_{k=1}^{\infty} B_k h^{2k} [F^{(2k)}(1) - F^{(2k)}(0)]$$

where

$$F(x) = \int f(x)dx$$

Idea of variable transformations in numerical integration ctd ...

By applying a change of variable of the form

$$x = \gamma(t), \gamma(0) = 0, \gamma(1) = 1,$$

the integral I is written as

$$I = \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

The function $\gamma(t)$ is increasing in $[0, 1]$ and such that a sufficient number of its derivatives vanishes at $t = 0$ and $t = 1$. We then apply the trapezoidal rule to the transformed integral giving

$$I_N = h \sum_{j=1}^{N-1} w_j f(\gamma(jh))\gamma'(jh)$$

The new rule has higher order due to the fact that a number of the derivatives appearing in the Euler - MacLaurin sum vanish.

Examples of transformations: Classes of transformations

Well known classes of variable transformations include:

- sigmoidal
- semi-sigmoidal
- monomial.

Elliott (1998a) gives the following definition for a real function γ to be a sigmoidal transformation.

Examples of transformations ctd: Sigmoidal-definition

Definition 1(Elliott (1998a, p. E.80).

(a) A real function γ is said to be a sigmoidal transformation if the following conditions are satisfied:

(i) $\gamma \in C^1[0,1] \cap C^\infty(0,1)$ with $\gamma(0) = 0$;

(ii) $\gamma(x) + \gamma(1-x) = 1, 0 \leq x \leq 1$;

(iii) γ is strictly increasing on $[0,1]$;

(iv) γ' is strictly increasing on $[0,1/2]$ with $\gamma'(0) = 0$.

Examples of transformations ctd: Sigmoidal-definition

(b) If in addition to (a) either

(i) $\gamma^{(j)}(x) = O(x^{r-j})$ near $x = 0$, for all $j \in N$ where $N_0 = \{0, 1, 2, \ldots\}$ and $r \geq 1$, then γ is said to be a sigmoidal transformation of order r; or

(ii) $\gamma^{(j)}(0) = 0$ for all $j \in N_0$, then γ is said to be a sigmoidal transformation of infinite order.

Because the graph of γ is like an elongated S, Elliott (1998a) has chosen to call such transformations *sigmoidal* transformations. They are also known as *periodizing* transformations (Laurie (1996).

Examples of transformations ctd ...: Semi-sigmoidal

Semi-sigmoidal transformations have been introduced by Johnston (2000), Johnston and Elliott (2000)

An r-order semi-sigmoidal transformation $\sigma_r(x)$ is defined in terms of an r-order sigmoidal transformation say $\gamma_r(x)$ as follows:

$$\sigma_r(x) = 2\gamma_r(\frac{x}{2}), 0 \leq x \leq 1. \qquad (3)$$

It has been proved by Johnston (2000), Theorem 1) that a semi-sigmoidal transformation in $[0, 1]$ is a sigmoidal in $[0, 2]$.

Examples of transformations ctd: Monomial

Johnston and Elliott (2002) introduced what they call *monomial* transformation given by

$$\gamma_r(x) = x^r, 0 \leq x \leq 1, \tag{4}$$

where the order r does not need to be integral. Although not a sigmoidal transformation, it has been obtained as the limit as $m \to \infty$ of the $(1/m)$th sigmoidal transformation $\gamma_{r,m}$, mapping $[0,1]$ onto itself and given by

$$\gamma_{r,m} = \frac{\gamma_r(x/m)}{\gamma_r(1/m)}, 0 \leq x \leq 1, m \in N. \tag{5}$$

Monomial transformations of the form (4) can be recovered from several other transformations which include these by Sato, Yoshioka and Tsukui (1988).

Examples of transformations ctd ... : Sigmoidal types and examples

The sigmoidal transformations $\gamma(x)$, considered in the literature can be written in the form

$$\gamma(x) = \frac{g(x)}{g(x) + g(1-x)}, \tag{6}$$

where g is a real-valued function defined on $[0, 1]$ satisfying certain conditions stated in a theorem by Elliott (1998a, Theorem 1.2, p. E82) Elliott in the same paper classifies the sigmoidal transformations in two categories, the "algebraic" ones, and the "integral" ones. A g function example of an algebraic sigmoidal transformation is:

$$g(x) = x^r, \quad r > 1, \tag{7}$$

It is such that $l := \gamma'(1/2) = r$.

Examples of transformations ctd ... : Integral type sigmoidal transformations

Sigmoidal transformations of "integral" type, use a function g in the form of an integral in terms of some functions h, w. In particular, they use

$$g(x) = \int_0^x h(s)ds, \tag{8}$$

or

$$g(x) = \int_0^x (x-s)w(s)ds. \tag{9}$$

Conditions that the functions h, w need to satisfy so that $\gamma(x)$ as given by (6) is a sigmoidal transformation are given in Elliott (1998a, Theorems 3.1, 3.6).

Examples of transformations ctd ... : Examples of h functions

Examples of h functions include:

$$h(x) = (x(1-x))^{r-1}, r > 1, \qquad (10)$$
$$(Korobov(1963))$$

$$h(x) = exp[-(1/x + 1/(1-x))], \qquad (11)$$
$$(Iri,\ Moriguti\ and\ Takasawa(1987))$$

$$h(x) = (\sin(\pi x))^{r-1}, N \ni r \geq 2, \qquad (12)$$
$$(Sidi(1993))$$

$$h(x) = \sum_{j=r}^{3(r-1)/2} c_j (B_{2j}(x) - B_{2j}(0)), \qquad (13)$$
$$(Laurie(1996))$$

where r is odd, $c_{3(r-1)/2} = 1$ and $B_{2j}(x)$ is a $2j$-degree Bernoulli polynomial. The coefficients c_j are obtained from the solution of a certain linear system of equations.

Examples of transformations ctd ... : An example of a w function

An example of a w function for use in (9) is

$$w(x) = \sin^{2r-1}(2\pi x), r \in N, \qquad (14)$$
$$(Elliott(1998a))$$

Variable transformation idea applied to FIEs.

Consider the FIE (1), i.e. consider

$$y(x) = f(x) + \int_0^1 K(x, s, y(s))ds, \quad 0 \le x \le 1, \tag{15}$$

Apply the transformations $x = \gamma(w)$ and $s = \gamma(t)$:

$$y(\gamma(w)) = f(\gamma(w)) + \int_0^1 K(\gamma(w), \gamma(t), y(\gamma(t)))\gamma'(t)dt. \tag{16}$$

Setting $Y(w) = y(\gamma(w))$ and $\tilde{f}(w) = f(\gamma(w))$, we obtain

$$Y(w) = \tilde{f}(w) + \int_0^1 K(\gamma(w), \gamma(w), Y(t))\gamma'(t)dt. \tag{17}$$

Examples of particular sigmoidal transformations

- Laurie's with $r = 3$ (cf. Elliott (1998a, p. E99).

$$\gamma_3(t) = 7t^3 - 21t^5 + 21t^6 - 6t^7. \tag{18}$$

- Sidi's with $m = r - 1 = 2$ $(r = 3)$,

$$\gamma_3(t) = t - \frac{1}{2\pi}\sin(2\pi t) \tag{19}$$

- Sidi's with $m = r - 1 = 3$ $(r = 4)$,

$$\gamma_4(t) = \frac{8 - 9\cos(\pi t) + \cos(3\pi t)}{16}. \tag{20}$$

Test examples

(a) An integral:

$$I_1 = \int_0^1 e^t dt$$

(b) A FIE (cf. Galperin, Kansa, Makroglou, Nelson (2000)

$$y(x) = x^3 + \frac{1}{3}(\cos(1) - 1) + \int_0^1 s^2 \sin(y(s)) ds,$$

where $y(x) = x^3$

Some numerical results

e_S: Error using trap. rule
with the integral in given (standard) form.
e_L: Error using Laurie's transf. with $r = 3$
and trap. rule for the transformed integral

N	e_S	e_L
20	$0.36D-3$	$0.13D-7$
40	$0.89D-4$	$0.21D-9$
80	$0.22D-4$	$0.33D-11$
160	$0.56D-5$	$0.50D-13$
320	$0.14D-5$	$0.67D-15$

Some numerical results ctd ...:
The FIE (b)

e_S: Max abs error using Sidi's transf. with with $r = 3$, combined with trap. rule.

e_L: Max abs error using Laurie's transf. with $r = 3$, combined with trap. rule

N	e_S	e_L
10	$0.72D-4$	$0.23D-5$
20	$0.45D-5$	$0.34D-7$
40	$0.28D-6$	$0.53D-9$

Further work: • Results with more variable transformations and test problems.

• Apply to FIEs with weakly singular kernels.

• Convergence analysis

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