Bipolar fuzzy integrals

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Abstract

In decision analysis, and especially in multiple-criteria decision analysis, several non-additive integrals have been introduced in the last 60 years. These include the Choquet integral, the Shilkret integral and the Sugeno integral, among others. Recently, a bipolar Choquet integral was proposed for cases in which the underlying scale is bipolar. In this paper we propose a bipolar Shilkret integral and a bipolar Sugeno integral. Moreover, we provide an axiomatic characterization of all three bipolar fuzzy integrals.

Keywords: Non-additive measures; Multiple criteria evaluation; Bi-capacity; Bipolar fuzzy integrals.

1 Introduction

In decision analysis, and especially in multiple-criteria decision analysis, several non-additive integrals have been introduced in the last 60 years [5, 7, 12]. These include the Choquet integral [4], the Shilkret integral [25] and the Sugeno integral [27], among others. More recently a bipolar Choquet integral has been proposed for cases in which the underlying scale is bipolar [10, 11, 16]. A further generalization is that of level-dependent integrals, which has led to definition of the level-dependent Choquet integral [15], the level-dependent Shilkret integral [3], the level-dependent Sugeno integral [20] and the bipolar level-dependent Choquet integral [15]. Very recently, on the basis of a minimal set of axioms, one concept of a universal integral giving a common framework to many of the above integrals has been proposed [18]. Here we provide a general framework for the case of bipolar fuzzy integrals, those integrals whose underlying scale is bipolar. For this purpose we propose a definition of bipolar Shilkret and bipolar Sugeno integrals. To provide a mathematical characterization of the three bipolar integrals mentioned, we give necessary and sufficient conditions for an aggregation function to be a bipolar Choquet integral, a bipolar Shilkret integral or a bipolar Sugeno integral. The bipolar fuzzy integrals admit a further generalization if the fuzzy measure (capacity) with respect to which the integrals are calculated can change from one level to another [15, 14]. For the sake of clarity, the characterization of bipolar Shilkret and Sugeno integrals with respect to a level-dependent capacity are addressed in a forthcoming paper, although such results have recently been published [14].

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The remainder of the paper is organized as follows. In Section 2 we present preliminaries and list some properties of an aggregation function useful in the characterization of the bipolar fuzzy integrals considered here. In Section 3 we review the definitions and characterizations of the classical Choquet integral, Shilkret integral and Sugeno integral and some of their symmetric extensions on a bipolar scale. Section 4 presents the main results. We propose bipolar versions of the Shilkret and Sugeno integrals and we characterize the bipolar Choquet, Shilkret and Sugeno integrals. All the proofs are presented in Section 5. Section 6 concludes.

2 Preliminaries

Consider a set of criteria $N = \{1, \dots, n\}$ and suppose that the range of evaluation for a given criterion is a real number interval $\mathcal{I}$. We denote $\alpha = \inf \mathcal{I}$ and $\beta = \sup \mathcal{I}$. An *alternative* can be identified with a score vector $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathcal{I}^n$, where x_i is the evaluation of such an alternative $\boldsymbol{x}$ with respect to the ith criterion. An alternative $\boldsymbol{x}$ dominates another $\boldsymbol{y}$ if for each criterion the evaluation of $\boldsymbol{x}$ is not less than the evaluation of $\boldsymbol{y}$; in other words, for all $i \in N$, $x_i \geq y_i$ and in this case we simply write $\boldsymbol{x} \geq \boldsymbol{y}$. The indicator function of any $A \subseteq N$ is the function that takes a value of 1 on A and 0 on $N \setminus A$ and can be identified using the vector $\mathbf{1}_A$ whose ith component is equal to 1 if $i \in A$ and 0 otherwise.

In general, an aggregation function is a function $G : \mathcal{I}^n \to \mathcal{I}$ such that

1. $G(\alpha, \dots, \alpha) = \alpha$ if $\alpha \in \mathcal{I}$ and $\lim_{x \to \alpha^+} G(x, \dots, x) = \alpha$ if $\alpha \notin \mathcal{I}$;
2. $G(\beta, \dots, \beta) = \beta$ if $\beta \in \mathcal{I}$ and $\lim_{x \to \beta^-} G(x, \dots, x) = \beta$ if $\beta \notin \mathcal{I}$; and
3. For all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$ such that $\boldsymbol{x} \geq \boldsymbol{y}$, $G(\boldsymbol{x}) \geq G(\boldsymbol{y})$.

In this paper we often denote the maximum and minimum of a set X by $\bigvee X$ and $\bigwedge X$, respectively. For any two alternatives $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$, the following definitions hold:

- $\boldsymbol{x} \wedge \boldsymbol{y}$ is the vector whose ith component is $(x \wedge y)_i = \bigwedge\{x_i, y_i\}$ for all $i = 1, \dots, n$ (in case $\boldsymbol{y} = (h, \dots, h)$ is a constant, then we can write $\boldsymbol{x} \wedge h$);
- $\boldsymbol{x} \vee \boldsymbol{y}$ is the vector whose ith component is $(x \vee y)_i = \bigvee\{x_i, y_i\}$ for all $i = 1, \dots, n$ (in case $\boldsymbol{y} = (h, \dots, h)$ is a constant, then we can write $\boldsymbol{x} \vee h$);
- $\boldsymbol{x}$ and $\boldsymbol{y}$ are comonotone (or comonotonic) if $(x_i - x_j)(y_i - y_j) \geq 0$ for all $i, j \in N$;
- $\boldsymbol{x}$ and $\boldsymbol{y}$ are bipolar comonotone if $(|x_i| - |x_j|)(|y_i| - |y_j|) \geq 0$ and $x_i y_i \geq 0$, for all $i, j \in N$.

The following properties of an aggregation function $G : \mathcal{I}^n \to \mathcal{I}$ are useful in the characterization of several integrals:

- Idempotency: for all $\mathbf{a} \in \mathcal{I}^n$ such that $\mathbf{a} = (a, \dots, a)$, $G(\mathbf{a}) = a$.
- Homogeneity: for all $\boldsymbol{x} \in \mathcal{I}^n$ and $c > 0$ such that $c \cdot \boldsymbol{x} \in \mathcal{I}^n$, $G(c \cdot \boldsymbol{x}) = c \cdot G(\boldsymbol{x})$.
- Stability w.r.t. the minimum: for all $\boldsymbol{x} \in \mathcal{I}^n$ and $\gamma \in \mathcal{I}$, $G(\boldsymbol{x} \wedge \gamma) = \bigwedge\{G(\boldsymbol{x}), \gamma\}$.
- Additivity: for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$ such that $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{I}^n$, $G(\boldsymbol{x} + \boldsymbol{y}) = G(\boldsymbol{x}) + G(\boldsymbol{y})$.
- Maxitivity: for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$ with $\alpha \geq 0$, $G(\boldsymbol{x} \vee \boldsymbol{y}) = \bigvee\{G(\boldsymbol{x}), G(\boldsymbol{y})\}$.
- Minitivity: for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$ with $\beta \leq 0$, $G(\boldsymbol{x} \wedge \boldsymbol{y}) = \bigwedge\{G(\boldsymbol{x}), G(\boldsymbol{y})\}$.

- Comonotonic additivity: for all comonotone $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$, $G(\boldsymbol{x} + \boldsymbol{y}) = G(\boldsymbol{x}) + G(\boldsymbol{y})$.
- Comonotonic maxitivity: for all comonotone $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$, $G(\boldsymbol{x} \vee \boldsymbol{y}) = \bigvee\{G(\boldsymbol{x}), G(\boldsymbol{y})\}$.
- Comonotonic minitivity: for all comonotone $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$, $G(\boldsymbol{x} \wedge \boldsymbol{y}) = \bigwedge\{G(\boldsymbol{x}), G(\boldsymbol{y})\}$.

3 Fuzzy integrals

In this section we briefly review the three best-known fuzzy integrals, Choquet, Shilkret and Sugeno integrals, and some of their symmetric extensions. For each of them we shall discuss the restrictions to be imposed on the scale $\mathcal{I}$.

3.1 Choquet integral

Definition 1. *A capacity (or fuzzy measure) is a function $\mu : 2^N \to [0,1]$ satisfying the following properties:*

1. $\mu(\varnothing) = 0$, $\mu(N) = 1$; *and*
2. *For all* $A \subseteq B \subseteq N$, $\mu(A) \le \mu(B)$.

Definition 2. *The Choquet integral* [4] *of a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}^n \subseteq [0, +\infty[^n$ with respect to the capacity μ is given by*

$$Ch(\boldsymbol{x}, \mu) = \int_0^\infty \mu\left(\{i \in N : x_i \ge t\}\right) dt. \tag{1}$$

Schmeidler [24] extended the above definition to negative values and characterized the Choquet integral in terms of comonotonic additivity and idempotency.

Definition 3. [24] *The Choquet integral of a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}^n$ with respect to the capacity μ is given by*

$$Ch(\boldsymbol{x}, \mu) = \int_{-\infty}^0 \left(\mu\left(\{i \in N : x_i \ge t\}\right) - 1\right) dt + \int_0^\infty \mu\left(\{i \in N : x_i \ge t\}\right) dt. \tag{2}$$

Alternatively, (2) can be written as [15]

$$Ch(\boldsymbol{x}, \mu) = \int_{\min_i x_i}^{\max_i x_i} \mu\left(\{i \in N : x_i \ge t\}\right) dt + \min_i x_i. \tag{3}$$

Another formulation of (2) can be obtained by using summation:

$$Ch(\boldsymbol{x}, \mu) = \sum_{i=2}^{n} \left(x_{\sigma(i)} - x_{\sigma(i-1)}\right) \cdot \mu\left(\{j \in N : x_j \ge x_{\sigma(i)}\}\right) \ + \ x_{\sigma(1)}, \tag{4}$$

where $\sigma : N \to N$ is any permutation of indexes such that $x_{\sigma(1)} \le \ldots \le x_{\sigma(n)}$.

Theorem 1. [24] *An aggregation function $G : \mathcal{I}^n \to \mathcal{I}$ is idempotent and comonotone additive if and only if there exists a capacity μ such that, for all $\boldsymbol{x} \in \mathcal{I}^n$,*

$$G(\boldsymbol{x}) = Ch(\boldsymbol{x}, \mu).$$

The Šipoš integral [26] (or symmetric Choquet integral) of $\boldsymbol{x} \in \mathcal{I}^n$ with respect to the capacity μ is defined by

$$\check{\mathrm{C}}\mathrm{h}\,(\boldsymbol{x}, \mu) = Ch(\boldsymbol{x} \vee 0, \mu) - Ch(-(\boldsymbol{x} \wedge 0), \mu). \tag{5}$$

More generally, a functional $L : \mathcal{I}^n \to \mathcal{I}$ is a rank and sign-dependent functional [22] if there exist two fuzzy measures μ^+ and μ^- such that, for all $\boldsymbol{x} \in \mathcal{I}^n$,

$$L(\boldsymbol{x}) = Ch(\boldsymbol{x} \vee 0, \mu^+) - Ch(-(\boldsymbol{x} \wedge 0), \mu^-).$$

This function is used in cumulative prospect theory [28]. Clearly, when $\mu^+ = \mu^-$, the rank and sign-dependent functional L is exactly the symmetric Choquet integral. For further details on the rank and sign-dependent function and its use in cumulative prospect theory, readers are referred to the literature [28, 22]. Note that the Choquet integral was generalized and characterized by Benvenuti and coworkers [1, 2].

3.2 Shilkret integral

Definition 4. *The Shilkret integral* [25] *of a vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}^n \subseteq [0, +\infty[^n$ *with respect to the capacity* μ *is given by*

$$Sh(\boldsymbol{x}, \mu) = \bigvee_{i \in N} \{x_i \cdot \mu(\{j \in N : x_j \geq x_i\})\}. \tag{6}$$

A generalization of the Shilkret integral was introduced and characterized by Benvenuti and coworkers [1, 2]. From these studies we can obtain a characterization of the Shilkret integral in terms of idempotency, comonotonic maxitivity and homogeneity. For completeness, we report the proof of this characterization (Theorem 2) in Section 5.

Theorem 2. *Suppose that* $\alpha = \inf \mathcal{I} \geq 0$. *Then an aggregation function* $G : \mathcal{I}^n \to \mathcal{I}$ *is idempotent, comonotone maxitive and homogeneous if and only if there exists a capacity* μ *on* N *such that, for all* $\boldsymbol{x} \in \mathcal{I}^n$,

$$G(\boldsymbol{x}) = Sh(\boldsymbol{x}, \mu).$$

Although the Shilkret integral was formulated for non-negative functions [25], (6) also works for a generic $\boldsymbol{x} \in \mathcal{I}^n \subseteq \mathbb{R}^n$. However, in our opinion, if we allow for negative values too, the essence of the Shilkret integral is lost. We highlight this point with some examples. Suppose that an alternative is strongly negatively evaluated for each criterion except the last, where it has a low non-negative evaluation, such as $\boldsymbol{x} = (-100, -100, -100, 1)$. By applying (6), $Sh\,(\boldsymbol{x}, \mu) = \mu\,(\{4\})$ for every capacity μ. Thus, the negative evaluations and the weights that the capacity assigns to the relative criteria with respect to which of these negative evaluations are given have no influence on the evaluation of $\boldsymbol{x}$. In general, if we have simultaneously negative and positive evaluations on the various criteria for a given alternative $\boldsymbol{x}$, the negative ones have no influence and the Shilkret integral of $\boldsymbol{x}$ coincides with the Shilkret integral of $\boldsymbol{x} \vee 0$. In the case of $\boldsymbol{x} \in]-\infty, 0[^n$, this is straightforward, noting that $Sh\,(\boldsymbol{x}, \mu) = (\max_{i \in N} x_i) \cdot \mu\,(\{j \in N \mid x_j \geq \max_{i \in N} x_i\})$. Again, we note that for all capacities only the maximum evaluation of $\boldsymbol{x}$ matters. For vectors with a non-positive evaluation for each criterion, the logic of the Shilkret integral can be recovered if we substitute the maximum with the minimum and $\geq$ with $\leq$ in (6).

Definition 5. *The negative Shilkret integral of a vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}^n \subseteq]-\infty, 0]^n$ *with respect to the capacity* μ *is given by*

$$Sh^-(\boldsymbol{x}, \mu) = \bigwedge_{i \in N} \{x_i \cdot \mu(\{j \in N : x_j \leq x_i\})\} = -\bigvee_{i \in N} \{-x_i \cdot \mu(\{j \in N : -x_j \geq -x_i\})\} = -Sh(-\boldsymbol{x}, \mu). \tag{7}$$

Obviously, from Theorem 2 it follows that characterization of the negative Shilkret integral is in terms of idempotency, comonotonic minitivity and homogeneity.

Corollary 1. *Suppose that* $\beta = \sup \mathcal{I} \leq 0$. *Then an aggregation function* $G : \mathcal{I}^n \to \mathcal{I}$ *is idempotent, comonotone minitive and homogeneous if and only if there exists a capacity* μ *on* N *such that, for all* $\boldsymbol{x} \in \mathcal{I}^n$,

$$G(\boldsymbol{x}) = Sh^{-}(\boldsymbol{x}, \mu).$$

So far, we have a Shilkret integral for alternatives with all non-negative evaluations and one for alternatives with all non-positive evaluations. To obtain a suitable definition of the Shilkret integral for the mixed case, we propose two different approaches. In the first approach we define a *symmetric Shilkret integral* by applying the logic of Šipoš [26], that is, for all $\boldsymbol{x} \in \mathcal{I}$,

$$\check{\mathrm{S}}\mathrm{h}\,(\boldsymbol{x}, \mu) = Sh(\boldsymbol{x} \vee 0, \mu) + Sh^{-}(\boldsymbol{x} \wedge 0, \mu). \tag{8}$$

Note that (8) is called symmetric since $\check{\mathrm{S}}\mathrm{h}\,(\boldsymbol{x}, \mu) = -\ \check{\mathrm{S}}\mathrm{h}\,(-\boldsymbol{x}, \mu)$. A second, more general approach is to define a *bipolar Shilkret integral* (Section 4.3). This would be used directly for the bipolar scale, while it would coincide with the Shilkret integral and the negative Shilkret integral when restricted to $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively.

3.3 Sugeno integral

Definition 6. *A measure on* N *with scale* $\mathcal{I}$ *is any function* $\nu : 2^N \to \mathcal{I}$ *such that:*

1. $\nu(\varnothing) = \alpha = \inf \mathcal{I}$, $\nu(N) = \beta = \sup \mathcal{I}$; *and*
2. *For all* $A \subseteq B \subseteq N$, $\nu(A) \leq \nu(B)$.

Definition 7. *The Sugeno integral* [27] *of a vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}^n$ *with respect to the measure* ν *on* N *with scale* $\mathcal{I}$ *is given by*

$$Su(\boldsymbol{x}, \nu) = \bigvee_{i \in N} \bigwedge \{x_i, \nu(\{j \in N \mid x_j \geq x_i\})\}. \tag{9}$$

Alternatively, the Sugeno integral can be written as

$$Su(\boldsymbol{x}, \nu) = \bigvee_{A \subseteq N} \bigwedge \left\{\nu(A), \bigwedge_{i \in A} x_i\right\}. \tag{10}$$

The next theorem gives necessary and sufficient conditions for an aggregation function to be a Sugeno integral.

Theorem 3. [19] *An aggregation function* $G : \mathcal{I}^n \to \mathcal{I}$ *is idempotent, comonotone maxitive and stable with respect to the minimum if and only if there exists a measure* ν *on* N *with a scale* $\mathcal{I}$ *such that, for all* $\boldsymbol{x} \in \mathcal{I}^n$,

$$G(\boldsymbol{x}) = Su(\boldsymbol{x}, \nu).$$

Observe that the definition of the Sugeno integral only imposes that x_i and the $\nu(A)$ are measured on the same (possible only ordinal) scale $\mathcal{I}$. For a further generalization and characterization of the Sugeno integral, readers are referred to the literature [1, 2].

Consider the symmetric scale $[-1,1]$. The *symmetric maximum* of two elements $a, b \in [-1,1]$ [8, 9] is defined by the following binary operation:

$$a \circledvee b = \begin{cases} -(|a| \vee |b|) & \text{if } b \neq -a \text{ and either } |a| \vee |b| = -a \text{ or } = -b \\ 0 & \text{if } b = -a \\ |a| \vee |b| & \text{else.} \end{cases}$$

Alternatively, the symmetric maximum

$$a \circledvee b = \text{sign}(a+b)(|a| \vee |b|).$$

Suppose that $\mu : 2^N \to [0,1]$ is a capacity and $\boldsymbol{x} \in [-1,1]^n$ is a vector evaluated for each criterion on the symmetric scale $[-1,1]$. The *symmetric Sugeno integral* [8] of $\boldsymbol{x}$ is defined as

$$\check{\text{S}}\text{u}\,(\boldsymbol{x}, \mu) = (Su(\boldsymbol{x} \vee 0, \mu)) \circledvee (-Su((-\boldsymbol{x}) \vee 0, \mu)), \tag{11}$$

where, as in (8), symmetric means that $\check{\text{S}}\text{u}\,(\boldsymbol{x}, \mu) = -\check{\text{S}}\text{u}\,(-\boldsymbol{x}, \mu)$.
Clearly, if $x_i \geq 0$ for all $i \in N$, then $\check{\text{S}}\text{u}\,(\boldsymbol{x}, \mu) = Su(\boldsymbol{x}, \mu)$, while if $x_i \leq 0$ for all $i \in N$,

$$\check{\text{S}}\text{u}\,(\boldsymbol{x}, \mu) = \bigwedge_{i \in N} \bigvee \{x_i, -\nu(\{j \in N \mid x_j \leq x_i\})\}. \tag{12}$$

Equation (12) can be considered as a definition of a negative Sugeno integral for the case in which $\boldsymbol{x}$ is negatively evaluated for each criterion.
Pap and Mihailovic extended the notion of a symmetric Sugeno integral [23].

Definition 8. *A functional $L : [-1,1]^n \to [-1,1]$ is a fuzzy rank and sign-dependent functional if there exist two fuzzy measures μ^+ and μ^- such that, for all $\boldsymbol{x} \in [-1,1]^n$,*

$$L(\boldsymbol{x}) = (Su(\boldsymbol{x} \vee 0, \mu^+)) \circledvee (-Su((-\boldsymbol{x}) \vee 0, \mu^-)). \tag{13}$$

Clearly, when $\mu^+ = \mu^-$ the fuzzy rank and sign-dependent function L is exactly the symmetric Sugeno integral. For further details on the fuzzy rank and sign-dependent functional and on the symmetric Sugeno integral, readers are referred to the literature [8, 23].
In the next section we propose a more general approach, defining a *bipolar Sugeno integral*, which coincides with (7) and (12) when restricted to $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively.

4 Bipolar fuzzy integrals on the scale [-1,1]

This study focuses on bipolar fuzzy integrals, which are integrals that are useful when the scale underlying the alternatives is bipolar. For simplicity, in this section we adopt the bipolar scale $[-1,1]$ to present our results. However, without loss of generality, they can be extended to every other symmetric interval of $\mathbb{R}$, i.e. any of $[-\alpha, \alpha],]-\alpha, \alpha[,]-\infty, +\infty[$, where $\alpha \in \mathbb{R}^+$.

Consider the set $\mathcal{Q} = \{(A,B) \in 2^N \times 2^N : A \cap B = \varnothing\}$ of all disjoint pairs of subsets of N. With respect to the binary relation $(A,B) \precsim (C,D)$, iff $A \subseteq C$ and $B \supseteq D$, $\mathcal{Q}$ is a lattice, i.e. a partial ordered set in which any two elements have a unique supremum, $(A,B) \vee (C,D) = (A \cup C, B \cap D)$, and a unique infimum, $(A,B) \wedge (C,D) = (A \cap C, B \cup D)$. With a slight abuse of the notation we extend the relation of set inclusion to $\mathcal{Q}$ by defining $(A,B) \subseteq (C,D)$ if and only if $A \subseteq C$ and $B \subseteq D$ for all $(A,B), (C,D) \in \mathcal{Q}$. For all $(A,B) \in \mathcal{Q}$, the indicator function $1_{(A,B)} : N \to \{-1,0,1\}$ is the function that takes a value of 1 on A, -1 on B and 0 on $(A \cup B)^c$. Such a function can be identified using the vector $\mathbf{1}_{(A,B)}$ whose ith component is equal to 1 if $i \in A$, is equal to -1 if $i \in B$ and is equal to 0 otherwise.

Mesiar et al. showed that the symmetric maximum $\circledvee : [-1,1] \times [-1,1] \to [-1,1]$ coincides with two recent symmetric extensions of the Choquet integral, the *balancing Choquet integral* and the *fusion Choquet integral*, when they are computed with respect to the strongest capacity (i.e. the capacity $\nu : 2^N \to [0,1]$ that takes a value of 0 on the empty set and 1 elsewhere) [21]. However, the symmetric maximum of a set X cannot be defined, since $\circledvee$ is non-associative. For example, suppose that $X = \{3, -3, 2\}$; then $(3 \circledvee -3) \circledvee 2 = 2$ or $3 \circledvee (-3 \circledvee 2) = 0$, depending on the order. Several possible extensions of the symmetric maximum for dimension $n, n > 2$ have been proposed [9, 13] and discussed [21]. One of these extensions is based on the splitting rule applied to the maximum and to the minimum as described in the following.

Given $X = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}$, the *bipolar maximum* of X, denoted by $\bigvee^b X$, is defined as

$$\bigvee^b X = \bigvee_i^{m}{}^b x_i = \left(\bigvee_i^m x_i\right) \circledvee \left(\bigwedge_i^m x_i\right). \tag{14}$$

The following definitions are closely related to the above discussion.

Definition 9. *Given $X = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}$, the positive bipolar maximum of X, denoted by $\bigvee^{b^+} X$, is the element with the greatest absolute value, with the convention that, in the case of two different opposite elements with this property, we choose the non-negative one.*

Definition 10. *Given $X = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}$, the negative bipolar maximum of X, denoted by $\bigvee^{b^-} X$, is the element with the greatest absolute value, with the convention that, in the case of two different opposite elements with this property, we choose the non-positive one.*

Following these definitions, if $X = \{9, -9, 7, -3\}$, then $\bigvee^b X = 0$, $\bigvee^{b^+} X = 9$ and $\bigvee^{b^-} X = -9$. Clearly the three operators just defined are linked by means of the relation $\bigvee^b X = \bigvee^b \left\{\bigvee^{b^+} X, \bigvee^{b^-} X\right\}$.

Given the vectors $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k \in [-1.1]^n$ with $K = \{1, \ldots, k\}$, $\bigvee^b_{j \in K} \boldsymbol{x}_j$ is the vector whose ith component is $\bigvee^b \{x_i^1, \ldots, x_i^k\}$ for all $i = 1, \ldots, n$. The following properties of an aggregation function $G : [-1,1]^n \to [-1,1]$ are useful for the characterization of several bipolar integrals.

- Bipolar comonotonic additivity: for all bipolar comonotone $\boldsymbol{x}, \boldsymbol{y} \in [-1,1]^n$,

$$G(\boldsymbol{x} + \boldsymbol{y}) = G(\boldsymbol{x}) + G(\boldsymbol{y}).$$

- Bipolar stability of the sign: for all $r, s \in]0,1]$ and for all $(A,B) \in \mathcal{Q}$,

$$G(r\mathbf{1}_{(A,B)})G(s\mathbf{1}_{(A,B)}) > 0 \qquad \text{or} \qquad G(r\mathbf{1}_{(A,B)}) = G(s\mathbf{1}_{(A,B)}) = 0,$$

  in other words, $G(r\mathbf{1}_{(A,B)})$ and $G(s\mathbf{1}_{(A,B)})$ have the same sign.

- Bipolar stability with respect to the minimum: for all $r, s \in]0,1]$ such that $r > s$ and for all $(A,B) \in \mathcal{Q}$,

$$|G(r\mathbf{1}_{(A,B)})| \geq |G(s\mathbf{1}_{(A,B)})|, \quad \text{and}$$
$$\text{if} \quad |G(r\mathbf{1}_{(A,B)})| > |G(s\mathbf{1}_{(A,B)})|, \quad \text{then} \quad |G(s\mathbf{1}_{(A,B)})| = s.$$

4.1 A specific property: bipolar comonotone maxitivity

Suppose we have k different levels, $l_1, \ldots, l_k \in \mathbb{R}$ with $0 < l_1 < l_2 < \ldots < l_k \leq 1$, and a sequence $\{(A_i, B_i)\}_{i=1,\ldots,k}$ such that $(A_i, B_i) \in \mathcal{Q}$ for all $i = 1, \ldots, k$ and $(A_{i+1}, B_{i+1}) \subseteq (A_i, B_i)$ for all $i = 1, \ldots, k-1$. The vectors $l_i \cdot \mathbf{1}_{(A_i,B_i)}$, $i = 1, \ldots, k$, are bipolar comonotonic. By ordering them with respect to the level l_i, then in the vector $l_i \cdot \mathbf{1}_{(A_i,B_i)}$, for each component the elements under level l_i are the opposite of that under level $-l_i$, as in the following four vectors, for example:

$$\begin{aligned} \boldsymbol{x} &= (7, -7,\ 0,\ \ 0) \\ \boldsymbol{y} &= (5, -5,\ 5,\ \ 0) \\ \mathbf{w} &= (3, -3,\ 3, -3) \\ \boldsymbol{z} &= (2, -2,\ 2, -2). \end{aligned}$$

An aggregation function G is said to be bipolar comonotone maxitive if it is maxitive on such a type of bipolar comonotonic *bi-constant*, that is, if for fixed $K = \{1, \ldots, k\}$ it holds that

$$G\left(\bigvee_{i\in K}^{b} l_i \cdot \mathbf{1}_{(A_i,B_i)}\right) = \bigvee_{i\in K}^{b} G\left(l_i \cdot \mathbf{1}_{(A_i,B_i)}\right). \tag{15}$$

G is said to be right bipolar comonotone maxitive if

$$G\left(\bigvee_{i\in K}^{b^+} l_i \cdot \mathbf{1}_{(A_i,B_i)}\right) = \bigvee_{i\in K}^{b^+} G\left(l_i \cdot \mathbf{1}_{(A_i,B_i)}\right). \tag{16}$$

G is said to be left bipolar comonotone maxitive if

$$G\left(\bigvee_{i\in K}^{b^-} l_i \cdot \mathbf{1}_{(A_i,B_i)}\right) = \bigvee_{i\in K}^{b^-} G\left(l_i \cdot \mathbf{1}_{(A_i,B_i)}\right). \tag{17}$$

Clearly, bipolar comonotonicity means that in (15)-(17),

$$\bigvee_{i\in K}^{b} l_i \cdot \mathbf{1}_{(A_i,B_i)} = \bigvee_{i\in K}^{b^+} l_i \cdot \mathbf{1}_{(A_i,B_i)} = \bigvee_{i\in K}^{b^-} l_i \cdot \mathbf{1}_{(A_i,B_i)}.$$

4.2 Bipolar Choquet integral

Definition 11. *A function $\mu_b : \mathcal{Q} \to [-1, 1]$ is a bi-capacity [10, 11, 16] on N if*

- $\mu_b(\emptyset, \emptyset) = 0$, $\mu_b(N, \emptyset) = 1$ *and* $\mu_b(\emptyset, N) = -1$*;*
- $\mu_b(A, B) \leq \mu_b(C, D)\ \forall\ (A, B), (C, D) \in \mathcal{Q}$ *such that* $(A, B) \precsim (C, D)$*.*

Definition 12. *The bipolar Choquet integral of $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1, 1]^n$ with respect to the bi-capacity μ_b is given by* [10, 11, 16, 15]

$$Ch_b(\boldsymbol{x}, \mu_b) = \int_0^{\infty} \mu_b(\{i \in N : x_i > t\}, \{i \in N : x_i < -t\}) dt. \tag{18}$$

The bipolar Choquet integral of $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1, 1]^n$ with respect to the bi-capacity μ_b can be rewritten as

$$Ch_b(\boldsymbol{x}, \mu_b) = \sum_{i=1}^{n} \left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right) \mu_b(\{j \in N : x_j \geq |x_{\sigma(i)}|\}, \{j \in N : x_j \leq -|x_{\sigma(i)}|\}), \tag{19}$$

where $\sigma : N \to N$ is any index permutation such that $0 = |x_{\sigma(0)}| \leq |x_{\sigma(1)}| \leq \ldots \leq |x_{\sigma(n)}|$. Note that to ensure that the pair $(\{j \in N : x_j \geq |t|\}, \{j \in N : x_j \leq -|t|\})$ is an element of $\mathcal{Q}$ for all $t \in \mathbb{R}$, we adopt the convention (maintained throughout paper) that in the case of $t = 0$ the inequality $x_j \leq 0$ is understood as $x_j < 0$. Eq. (19) will be useful in proving some results, such as that in the next theorem.

Theorem 4. [16] *An aggregation function* $G : [-1,1]^n \to [-1,1]$ *is idempotent and bipolar comonotonic additive if and only if there exists a bi-capacity* μ_b *such that, for all* $\boldsymbol{x} \in [-1,1]^n$,

$$G(\boldsymbol{x}) = Ch_b(\boldsymbol{x}, \mu_b).$$

Remark 1. *Although the bipolar Choquet integral is trivially homogeneous, this condition does not appear in the theorem, since an aggregation function that is idempotent and bipolar comonotone additive is also homogeneous. Observe also that we could relax idempotency with the conditions* $G(\boldsymbol{1}_{(N,\varnothing)}) = 1$ *and* $G(\boldsymbol{1}_{(\varnothing,N)}) = -1$.

4.3 Bipolar Shilkret integral

Definition 13. *The bipolar Shilkret integral of* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1,1]^n$ *with respect to the bi-capacity* μ_b *is given by*

$$Sh_b(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b} \{|x_i| \cdot \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})\}. \tag{20}$$

Definition 14. *The right bipolar Shilkret integral of* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1,1]^n$ *with respect to the bi-capacity* μ_b *is given by*

$$Sh_b^+(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b^+} \{|x_i| \cdot \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})\}. \tag{21}$$

Definition 15. *The left bipolar Shilkret integral of* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1,1]^n$ *with respect to the bi-capacity* μ_b *is given by*

$$Sh_b^-(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b^-} \{|x_i| \cdot \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})\}. \tag{22}$$

Clearly the three definitions are linked via

$$Sh_b(\boldsymbol{x}, \mu_b) = \bigvee^{b} \{Sh_b^+(\boldsymbol{x}, \mu_b), Sh_b^-(\boldsymbol{x}, \mu_b)\}.$$

The condition $Sh_b(\boldsymbol{x}, \mu_b) = 0$ is equivalent to $Sh_b^+(\boldsymbol{x}, \mu_b) = -Sh_b^-(\boldsymbol{x}, \mu_b)$ and, in this case, the three integrals are all zero or they give three different results, one zero, one positive and one negative. We can think about them in terms of a neutral, an optimistic and a pessimistic aggregate evaluation of $\boldsymbol{x}$. The condition $Sh_b(\boldsymbol{x}, \mu_b) \neq 0$ implies that $Sh_b^+(\boldsymbol{x}, \mu_b) = Sh_b^-(\boldsymbol{x}, \mu_b) = Sh_b(\boldsymbol{x}, \mu_b)$.
The following theorems characterize the bipolar Shilkret integral.

Theorem 5. *An aggregation function* $G : [-1,1]^n \to [-1,1]$ *is idempotent, bipolar comonotone maxitive and homogeneous if and only if there exists a bi-capacity* μ_b *on* N *such that, for all* $\boldsymbol{x} \in [-1,1]^n$,

$$G(\boldsymbol{x}) = Sh_b(\boldsymbol{x}, \mu_b).$$

Remark 2. *Note that Theorem 5 implies, as a corollary, Theorem 2, since bipolar comonotone maxitivity restricted on* $\mathbb{R}^+$ *implies comonotone maxitivity.*

Theorem 6. *An aggregation function* $G : [-1,1]^n \to [-1,1]$ *is idempotent, positive bipolar comonotone maxitive and homogeneous if and only if there exists a bi-capacity* μ_b *on* N *such that, for all* $\boldsymbol{x} \in [-1,1]^n$,

$$G(\boldsymbol{x}) = Sh_b^+(\boldsymbol{x}, \mu_b).$$

Theorem 7. *An aggregation function $G:[-1,1]^n \to [-1,1]$ is idempotent, negative bipolar comonotone maxitive and homogeneous if and only if there exists a bi-capacity μ_b on N such that, for all $\boldsymbol{x} \in [-1,1]^n$,*

$$G(\boldsymbol{x}) = Sh_b^-(\boldsymbol{x}, \mu_b).$$

Remark 3. *Idempotency could be relaxed with the conditions $G(\boldsymbol{1}_{(N,\varnothing)}) = 1$ and $G(\boldsymbol{1}_{(\varnothing,N)}) = -1$; in fact, from these and from homogeneity, idempotency can be deduced.*

4.4 Bipolar Sugeno integral

Definition 16. *The bipolar Sugeno integral of a vector $\boldsymbol{x} = (x_1,\dots,x_n) \in [-1,1]^n$ with respect to the bi-capacity μ_b on N is given by*

$$\begin{aligned} Su_b(\boldsymbol{x},\mu_b) \;=\; & \bigvee_{i\in N}^{b} \Big\{ \text{sign}\left(\mu_b\left(\{j \in N : x_j \geq |x_i|\}, \{j\in N : x_j \leq -|x_i|\}\right)\right) \\ & \cdot \bigwedge \left\{ |\mu_b(\{j \in N : x_j \geq |x_i|\}, \{j\in N : x_j \leq -|x_i|\})|, |x_i| \right\} \Big\}. \end{aligned} \tag{23}$$

Definition 17. *The right bipolar Sugeno integral of a vector $\boldsymbol{x} = (x_1,\dots,x_n) \in [-1,1]^n$ with respect to the bi-capacity μ_b on N is given by*

$$\begin{aligned} Su_b^+(\boldsymbol{x},\mu_b) \;=\; & \bigvee_{i\in N}^{b^+} \Big\{ \text{sign}\left(\mu_b\left(\{j \in N : x_j \geq |x_i|\}, \{j\in N : x_j \leq -|x_i|\}\right)\right) \\ & \cdot \bigwedge \left\{ |\mu_b(\{j \in N : x_j \geq |x_i|\}, \{j\in N : x_j \leq -|x_i|\})|, |x_i| \right\} \Big\}. \end{aligned} \tag{24}$$

Definition 18. *The left bipolar Sugeno integral of a vector $\boldsymbol{x} = (x_1,\dots,x_n) \in [-1,1]^n$ with respect to the bi-capacity μ_b on N is given by*

$$\begin{aligned} Su_b^-(\boldsymbol{x},\mu_b) \;=\; & \bigvee_{i\in N}^{b^-} \Big\{ \text{sign}\left(\mu_b\left(\{j \in N : x_j \geq |x_i|\}, \{j\in N : x_j \leq -|x_i|\}\right)\right) \\ & \cdot \bigwedge \left\{ |\mu_b(\{j \in N : x_j \geq |x_i|\}, \{j\in N : x_j \leq -|x_i|\})|, |x_i| \right\} \Big\}. \end{aligned} \tag{25}$$

Clearly the three definitions are linked via

$$Su_b(\boldsymbol{x},\mu_b) = \bigvee^b \left\{ Su_b^+(\boldsymbol{x},\mu_b), Su_b^-(\boldsymbol{x},\mu_b) \right\}.$$

The condition $Su_b(\boldsymbol{x},\mu_b) = 0$ is equivalent to $Su_b^+(\boldsymbol{x},\mu_b) = -Su_b^-(\boldsymbol{x},\mu_b)$ and, in this case, the three integrals are all zero or they give three different results, one zero (neutral), one positive (optimistic) and one negative (pessimistic). The condition $Su_b(\boldsymbol{x},\mu_b) \neq 0$ implies that $Su_b^+(\boldsymbol{x},\mu_b) = Su_b^-(\boldsymbol{x},\mu_b) = Su_b(\boldsymbol{x},\mu_b)$.
The following theorems characterize the bipolar Sugeno integral.

Theorem 8. *An aggregation function $G : [-1,1]^n \to [-1,1]$ is idempotent, bipolar comonotone maxitive, bipolar stable with respect to the sign and bipolar stable with respect to the minimum if and only if there exists a bi-capacity μ_b on N such that, for all $\boldsymbol{x} \in [-1,1]^n$,*

$$G(\boldsymbol{x}) = Su_b(\boldsymbol{x},\mu_b).$$

Theorem 9. *An aggregation function $G : [-1,1]^n \to [-1,1]$ is idempotent, positive bipolar comonotone maxitive, bipolar stable with respect to the sign and bipolar stable with respect to the minimum if and only if there exists a bi-capacity μ_b on N such that, for all $\boldsymbol{x} \in [-1,1]^n$,*

$$G(\boldsymbol{x}) = Su_b^+(\boldsymbol{x}, \mu_b).$$

Theorem 10. *An aggregation function $G : [-1,1]^n \to [-1,1]$ is idempotent, negative bipolar comonotone maxitive, bipolar stable with respect to the sign and bipolar stable with respect to the minimum if and only if there exists a bi-capacity μ_b on N such that, for all $\boldsymbol{x} \in [-1,1]^n$,*

$$G(\boldsymbol{x}) = Su_b^-(\boldsymbol{x}, \mu_b).$$

5 Theorem proofs

Proof of Theorem 2.
First we prove the necessary part. Suppose there exists a capacity μ on N such that, for all $\boldsymbol{x} \in \mathcal{I}^n$, $G(\boldsymbol{x}) = Sh(\boldsymbol{x}, \mu)$. In this case it is trivial to prove that the Shilkret integral is idempotent, comonotone maxitive and homogeneous by definition and we leave the proof to the reader. Now we prove the sufficient part of the theorem. We define

$$\mu(A) = G(\mathbf{1}_A), \quad \text{for all} \quad A \in 2^N. \tag{26}$$

Because G is an idempotent aggregation function, we obtain $\mu(\varnothing) = 0$, $\mu(N) = 1$ and $\mu(A) \le \mu(B)$ whenever $A \subseteq B$. Thus, μ is a capacity on N. Every $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathcal{I}^n$ can be written as

$$\boldsymbol{x} = \bigvee_{i \in N} x_{\sigma(i)} \cdot \mathbf{1}_{\{j \in N \mid x_j \ge x_{\sigma(i)}\}},$$

where $\sigma : N \to N$ is any index permutation such that $x_{\sigma(1)} \le \dots \le x_{\sigma(n)}$. Because vectors $x_{\sigma(i)} \cdot \mathbf{1}_{\{j \in N \mid x_j \ge x_{\sigma(i)}\}}$ are comonotonic, we obtain the hypothesis by applying comonotonic maxitivity, the homogeneity of G and the definition of μ according to (26):

$$G(\boldsymbol{x}) = G\left(\bigvee_{i \in N} x_{\sigma(i)} \cdot \mathbf{1}_{\{j \in N \mid x_j \ge x_{\sigma(i)}\}}\right) = \bigvee_{i \in N} G\left(x_{\sigma(i)} \cdot \mathbf{1}_{\{j \in N \mid x_j \ge x_{\sigma(i)}\}}\right)$$

$$= \bigvee_{i \in N} x_{\sigma(i)} \cdot G\left(\mathbf{1}_{\{j \in N \mid x_j \ge x_{\sigma(i)}\}}\right) = \bigvee_{i \in N} x_{\sigma(i)} \cdot \mu\left(\{j \in N \mid x_j \ge x_{\sigma(i)}\}\right) = Sh(\boldsymbol{x}, \mu).$$

□

Proof of Theorem 4.
First we prove the necessary part. Suppose that there exists a bi-capacity μ_b such that, for all $\boldsymbol{x} \in [-1,1]^n$, $G(\boldsymbol{x}) = Ch_b(\boldsymbol{x}, \mu_b)$. The idempotency of the bipolar Choquet integral follows by definition, because if $\lambda \ge 0$, then $Ch_b\left(\lambda \cdot \mathbf{1}_{(N,\varnothing)}, \mu_b\right) = \int_0^{\lambda} \mu_b(N, \varnothing)\, dt = \lambda$, while if $\lambda < 0$, then $Ch_b\left(\lambda \cdot \mathbf{1}_{(N,\varnothing)}, \mu_b\right) = \int_0^{-\lambda} \mu_b(\varnothing, N)\, dt = \lambda$. If $\boldsymbol{x}$ and $\boldsymbol{y} \in [-1,1]^n$ are bipolar comonotone, then there exists a permutation of indexes $\sigma : N \to N$ such that $0 = |x_{\sigma(0)}| \le |x_{\sigma(1)}| \le \dots \le |x_{\sigma(n)}|$ and $0 = |y_{\sigma(0)}| \le |y_{\sigma(1)}| \le \dots \le |y_{\sigma(n)}|$, and thus

$$Ch_b(\boldsymbol{x}, \mu_b) = \sum_{i=1}^{n} \left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right) \cdot \mu_b\left(\{j \in N : x_j \ge |x_{\sigma(i)}|\}, \{j \in N : x_j \le -|x_{\sigma(i)}|\}\right),$$

and

$$Ch_b(\boldsymbol{y},\mu_b) = \sum_{i=1}^{n} \left(|y_{\sigma(i)}| - |y_{\sigma(i-1)}|\right) \cdot \mu_b\left(\{j \in N : y_j \geq |y_{\sigma(i)}|\}, \{j \in N : y_j \leq -|y_{\sigma(i)}|\}\right).$$

Since $\boldsymbol{x}$ and $\boldsymbol{y}$ are absolutely comonotonic and cosigned, for every $i = 1,\ldots,n$,

$$\mu_b\left(\{j \in N : x_j \geq |x_{\sigma(i)}|\}, \{j \in N : x_j \leq -|x_{\sigma(i)}|\}\right) = \mu_b\left(\{j \in N : y_j \geq |y_{\sigma(i)}|\}, \{j \in N : y_j \leq -|y_{\sigma(i)}|\}\right). \quad (27)$$

Moreover, again because $\boldsymbol{x}$ and $\boldsymbol{y}$ are absolutely comonotonic and cosigned, for every $i = 1,\ldots,n$, $|x_{\sigma(i)} + y_{\sigma(i)}| = |x_{\sigma(i)}| + |y_{\sigma(i)}|$, and consequently

$$0 = |x_{\sigma(0)} + y_{\sigma(0)}| \leq |x_{\sigma(1)} + y_{\sigma(i)}| \leq \ldots \leq |x_{\sigma(n)} + y_{\sigma(n)}| \quad \text{for every} \quad i = 1,\ldots,n. \quad (28)$$

By (27) and (28), we obtain $Ch_b(\boldsymbol{x},\mu_b) + Ch_b(\boldsymbol{y},\mu_b) = Ch_b(\boldsymbol{x}+\boldsymbol{y},\mu_b)$.
Now we prove the sufficient part of the theorem. We define

$$\mu_b(A,B) = G\left(\mathbf{1}_{(A,B)}\right), \quad \text{for all} \quad (A,B) \in \mathcal{Q}, \quad (29)$$

where μ_b represents a bi-capacity, because by the idempotency of G we obtain that $\mu_b(N,\varnothing) = G\left(\mathbf{1}_{(N,\varnothing)}\right) = 1$, $\mu_b(\varnothing,N) = G\left(\mathbf{1}_{(\varnothing,N)}\right) = -1$, and $\mu_b(\varnothing,\varnothing) = G\left(\mathbf{1}_{(\varnothing,\varnothing)}\right) = 0$. Moreover, if $(A,B) \precsim (A',B')$ for all $i \in N$, the ith component of the vector $\mathbf{1}_{(A,B)}$ is not greater than the ith component of the vector $\mathbf{1}_{(A',B')}$, and since G is an aggregation function (and thus monotone), then $\mu_b(A,B) \leq \mu_b(A',B')$. Observe that any vector $\boldsymbol{x} = (x_1,\ldots,x_n) \in [-1,1]^n$ can be rewritten as

$$\boldsymbol{x} = \sum_{i=1}^{n} \left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right) \cdot \mathbf{1}_{\left(\{j\in N: x_j \geq |x_{\sigma(i)}|\}, \{j \in N: x_j \leq -|x_{\sigma(i)}|\}\right)}, \quad (30)$$

where $\sigma : N \to N$ is any permutation of indexes such that $0 = |x_{\sigma(0)}| \leq |x_{\sigma(1)}| \leq \ldots \leq |x_{\sigma(n)}|$. Note that for all $(A,B),\ (A',B') \in \mathcal{Q}$ such that $(A,B) \subseteq (A',B')$ and for all $a,b \in [0,1]$, vectors $a \cdot \mathbf{1}_{(A,B)}$ and $b \cdot \mathbf{1}_{(A',B')}$ are bipolar comonotone. Consequently, (30) shows that any vector $\boldsymbol{x} \in [-1,1]^n$ can be decomposed as a sum of bipolar comonotonic vectors. Remembering that an aggregation function that is idempotent and bipolar comonotone additive is also homogeneous, to prove the hypothesis it is sufficient to apply bipolar comonotone additivity, the homogeneity of G, and the definition of bi-capacity μ_b according to (29):

$$G(\boldsymbol{x}) = G\left(\sum_{i=1}^{n} \left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right) \cdot \mathbf{1}_{\left(\{j\in N: x_j \geq |x_{\sigma(i)}|\}, \{j \in N: x_j \leq -|x_{\sigma(i)}|\}\right)}\right)$$

$$= \sum_{i=1}^{n} \left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right) \cdot G\left(\mathbf{1}_{\left(\{j\in N: x_j \geq |x_{\sigma(i)}|\}, \{j \in N: x_j \leq -|x_{\sigma(i)}|\}\right)}\right) = Ch_b(\boldsymbol{x},\mu_b).$$

□

Proof of Theorem 5.
First we prove the necessary part. Suppose there exists a bi-capacity μ_b such that, for all $\boldsymbol{x} \in [-1,1]^n$, $G(\boldsymbol{x}) = Sh_b(\boldsymbol{x},\mu_b)$. The bipolar Shilkret integral is, trivially, idempotent and homogeneous and we only need to demonstrate the bipolar comonotonic maxitivity. Consider a set of indexes $K = \{1,\ldots,k\}$ for k increasing levels $l_1,\ldots,l_k \in \mathbb{R}$ with $0 < l_1 < l_2 < \ldots < l_k \leq 1$ and a sequence $\{(A_i,B_i)\}_{i\in K}$ such that $(A_i,B_i) \in \mathcal{Q}$ and $(A_{i+1},B_{i+1}) \subseteq (A_i,B_i)$ for all $i \in K$. The jth component of the vector $\bigvee_{i\in K}^{b}\{l_i \cdot \mathbf{1}_{(A_i,B_i)}\}$ is equal to l_i if $j \in A_i \setminus A_{i+1}$, is equal to $-l_i$ if $j \in B_i \setminus B_{i+1}$, and is equal to zero if $j \in N \setminus (A_1 \cup B_1)$ for all $i \in K$ and taking $A_{k+1} = B_{k+1} = \varnothing$. Clearly, such a vector has a component greater than or equal to l_i for indexes in A_i and has a component less than or equal to $-l_i$ for indexes in B_i. Thus, by definition,

$$Sh_b\left(\bigvee_{i\in K}^{b}\{l_i \cdot \mathbf{1}_{(A_i,B_i)}\}, \mu_b\right) = \bigvee_{i\in K}^{b}\{l_i \cdot \mu_b\left((A_i,B_i)\right)\} = \bigvee_{i\in K}^{b}\{Sh_b\left(l_i \cdot \mathbf{1}_{(A_i,B_i)},\mu_b\right)\}. \quad (31)$$

Now we prove the sufficient part of the theorem. We define

$$\mu_b(A,B) = G\left(\mathbf{1}_{(A,B)}\right), \quad \text{for all} \quad (A,B) \in \mathcal{Q}, \tag{32}$$

where μ_b represents a bi-capacity (proof of Theorem 4). Note that each $\boldsymbol{x} \in [-1,1]^n$ can be rewritten as

$$\boldsymbol{x} = {\bigvee_{i\in N}}^b |x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\},\{j \mid x_j \leq -|x_i|\})} \tag{33}$$

and observe that vectors $|x_i| \cdot \mathbf{1}_{(\{j\in N \mid x_j \geq |x_i|\},\{j\in N \mid x_j \leq -|x_i|\})}$, $i = 1\ldots,n$, are bipolar comonotone. Consequently, for any $\boldsymbol{x} \in [-1,1]^n$, by bipolar comonotone maxitivity, homogeneity and the definition of bi-capacity μ_b according to (32), we obtain

$$G(\boldsymbol{x}) = G\left({\bigvee_{i\in N}}^b |x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\},\{j \mid x_j \leq -|x_i|\})}\right) = {\bigvee_{i\in N}}^b G\left(|x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\},\{j \mid x_j \leq -|x_i|\})}\right)$$

$$= {\bigvee_{i\in N}}^b |x_i| \cdot G\left(\mathbf{1}_{(\{j \mid x_j \geq |x_i|\},\{j \mid x_j \leq -|x_i|\})}\right) = {\bigvee_{i\in N}}^b |x_i| \cdot \mu_b\left(\{j \mid x_j \geq |x_i|\},\{j \mid x_j \leq -|x_i|\}\right) = Sh_b(\boldsymbol{x},\mu_b).$$

□

Proof of Theorems 6 and 7. The proofs are analogous to that of *Theorem* 5.

□

Proof of Theorem 8. First we prove the necessary part. Suppose there exists a bi-capacity μ_b such that, for all $\boldsymbol{x} \in [-1,1]^n$, $G(\boldsymbol{x}) = Su_b(\boldsymbol{x},\mu_b)$. The Sugeno integral is idempotent by definition. Bipolar stability with respect to the sign and to the minimum is trivially verified once we consider that for all $r > 0$ and for all $(A,B) \in \mathcal{Q}$,

$$Su_b\left(r \cdot \mathbf{1}_{(A,B)},\mu_b\right) = sign\left(\mu_b(A,B)\right) \bigwedge \{r, |\mu_b(A,B)|\}.$$

Consider a set of indexes $K = \{1,\ldots,k\}$ for k increasing levels $l_1,\ldots,l_k \in \mathbb{R}$ with $0 < l_1 < l_2 < \ldots < l_k \leq 1$ and a sequence $\{(A_i,B_i)\}_{i\in K}$ such that $(A_i,B_i) \in \mathcal{Q}$ and $(A_{i+1},B_{i+1}) \subseteq (A_i,B_i)$ for all $i \in K$. Thus, by definition,

$$\begin{aligned} Su_b\left({\bigvee_{i\in K}}^b \{l_i \cdot \mathbf{1}_{(A_i,B_i)}\},\mu_b\right) &= {\bigvee_{i\in K}}^b \left\{sign\left[\mu_b\left((A_i,B_i)\right)\right] \bigwedge \{l_i, |\mu_b\left((A_i,B_i)\right)|\}\right\} \\ &= {\bigvee_{i\in K}}^b \{Su_b\left(l_i \cdot \mathbf{1}_{(A_i,B_i)},\mu_b\right)\}. \end{aligned} \tag{34}$$

Now we prove the sufficient part of the theorem. We define $\mu_b(A,B) = G\left(\mathbf{1}_{(A,B)}\right)$ for all $(A,B) \in \mathcal{Q}$, where μ_b represents a bi-capacity (proof of Theorem 4). Using the bipolar stability with respect to the minimum and the idempotency of G, we have that for all $r > 0$ and for all $(A,B) \in \mathcal{Q}$,

$$\left|G\left(r \cdot \mathbf{1}_{(A,B)}\right)\right| = \bigwedge\left\{r,\ \left|G\left(\mathbf{1}_{(A,B)}\right)\right|\right\}. \tag{35}$$

Equation (35) is obvious if $r = 0$ or $r = 1$. If $0 < r < 1$ and $\left|G\left(\mathbf{1}_{(A,B)}\right)\right| > \left|G\left(r \cdot \mathbf{1}_{(A,B)}\right)\right|$, then using the stability with respect to the minimum, $\left|G\left(r \cdot \mathbf{1}_{(A,B)}\right)\right| = r$ and (35) is again true. If $\left|G\left(\mathbf{1}_{(A,B)}\right)\right| = \left|G\left(r \cdot \mathbf{1}_{(A,B)}\right)\right|$, by the monotonicity and idempotency of G, $\left|G\left(r \cdot \mathbf{1}_{(A,B)}\right)\right| \leq \left|G\left(r \cdot \mathbf{1}_{(N,\emptyset)}\right)\right| = r$, which means that (35) is also true in this case. Finally, note that each $\boldsymbol{x} \in [-1,1]^n$ can be rewritten as

$$\boldsymbol{x} = {\bigvee_{i\in N}}^b |x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\},\{j \mid x_j \leq -|x_i|\})} \tag{36}$$

and observe that vectors $|x_i|\cdot\mathbf{1}_{(\{j\in N\ |\ x_j\geq|x_i|\},\{j\in N\ |\ x_j\leq-|x_i|\})}$, $i=1\ldots,n$, are bipolar comonotone. Consequently, for any $\boldsymbol{x}\in[-1,1]^n$, by the bipolar comonotone maxitivity we have

$$G(\boldsymbol{x})=G\left(\bigvee_{i\in N}^{b}|x_i|\cdot\mathbf{1}_{(\{j\ |\ x_j\geq|x_i|\},\{j\ |\ x_j\leq-|x_i|\})}\right)=\bigvee_{i\in N}^{b}G\left(|x_i|\cdot\mathbf{1}_{(\{j\ |\ x_j\geq|x_i|\},\{j\ |\ x_j\leq-|x_i|\})}\right)$$

(by bipolar stability with respect to the sign)

$$=\bigvee_{i\in N}^{b}\left\{\mathrm{sign}\left[G\left(\mathbf{1}_{(\{j\ |\ x_j\geq|x_i|\},\{j\ |\ x_j\leq-|x_i|\})}\right)\right]\left|G\left(|x_i|\cdot\mathbf{1}_{(\{j\ |\ x_j\geq|x_i|\},\{j\ |\ x_j\leq-|x_i|\})}\right)\right|\right\}$$

$$=\bigvee_{i\in N}^{b}\left\{\mathrm{sign}\left[\mu_b\left(\{j\ |\ x_j\geq|x_i|\},\{j\ |\ x_j\leq-|x_i|\}\right)\right]\left|G\left(|x_i|\cdot\mathbf{1}_{(\{j\ |\ x_j\geq|x_i|\},\{j\ |\ x_j\leq-|x_i|\})}\right)\right|\right\}$$

(by bipolar stability with respect to the minimum)

$$=\bigvee_{i\in N}^{b}\left\{\mathrm{sign}\left[\mu_b\left(\{j\ |\ x_j\geq|x_i|\},\{j\ |\ x_j\leq-|x_i|\}\right)\right]\bigwedge\left\{|x_i|,\ \left|G\left(\mathbf{1}_{(\{j\ |\ x_j\geq|x_i|\},\{j\ |\ x_j\leq-|x_i|\})}\right)\right|\right\}\right\}$$

$$=\bigvee_{i\in N}^{b}\left\{\mathrm{sign}\left[\mu_b\left(\{j\ |\ x_j\geq|x_i|\},\{j\ |\ x_j\leq-|x_i|\}\right)\right]\bigwedge\left\{|x_i|,\ |\mu_b\left(\{j\ |\ x_j\geq|x_i|\},\{j\ |\ x_j\leq-|x_i|\}\right)|\right\}\right\},$$

which is the bipolar Sugeno integral $Su_b(\boldsymbol{x},\mu_b)$.

□

Proof of Theorems 9 and 10. The proofs are analogous to that of Theorem 8.

□

6 Concluding remarks

In recent years there has been increasing interest in the development of new integrals useful in decision analysis or modeling of engineering problems. An interesting line of research is that of bipolar fuzzy integrals for cases in which the underlying scale is bipolar. An exhaustive survey of bipolarity and its possible applications has been published [6]. Here we axiomatically characterized the bipolar Choquet integral and defined and axiomatically characterized the bipolar Shilkret integral and the bipolar Sugeno integral. The results clarify and enrich the field of bipolar fuzzy integrals. A further direction of research is that of level-dependent bipolar fuzzy integrals, for which the fuzzy measure with respect to which the bipolar integrals are calculated can change from one level to another [15, 14]. It should be noted that a recent paper introduced the concept of a bipolar universal integral [17], which generalizes the Choquet, Shilkret and Sugeno bipolar integrals presented here.

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