Bipolar Semicopulas

Salvatore Greco[1,2]⋆, Radko Mesiar[3,4]⋆⋆, and Fabio Rindone[1]⋆⋆⋆

[1] Department of Economics and Business, University of Catania
Catania, Italy
[2] Portsmouth Business School, Operations & Systems Management University of Portsmouth
Portsmouth PO1 3DE, United Kingdom
[3] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering
Slovak University of Technology
Bratislava, Slovakia
[4] Institute for Research and Applications of Fuzzy Modeling
Division of University Ostrava
National Supercomputing Center IT4 Innovations
30. dubna 22, Ostrava 701 03
Czech Republic

Abstract. The concept of semicopula plays a fundamental role in the aggregation theory on interval $[0, 1]$. Semicopulas are applied, for example, in the definition of universal integrals. We present an extension of the notion of semicopula to the case of symmetric bipolar interval $[-1, 1]$. We call this extension bipolar semicopula. The last definition can be used to obtain a simplified definition of the bipolar universal integral. Moreover, bipolar semicopulas allow for an extension of theory of quasi-copulas to the interval $[-1, 1]$.

Keywords: Semicopula; bipolar semicopula; bipolar universal integrals.

1 Introduction

Integrals on unipolar scale $[0, 1]$ (often also called fuzzy integrals) aggregate the information hidden in measurable functions $f : X \to [0, 1]$ (fuzzy events) and $[0, 1]-$valued capacities (fuzzy measures) into a single value from $[0, 1]$ (fuzzy expectation). A crucial role in this integration is played by semicopulas $S : [0, 1]^2 \to [0, 1]$, corresponding to pseudo-multiplications characterizing universal integrals [18]. In the case of Choquet integral [2], $S = \Pi$ is the standard product. Similar is the case of the Shilkret integral [22]. On the other hand, Sugeno integral [23] is linked to the greatest semicopula $S = M\ (\min)$. Recall that semicopulas were introduced in [1], and further discussed and studied in several other papers, such as [5], and they play a fundamental role in the aggregation theory on the interval $[0, 1]$. As special subclasses of semicopulas we recall triangular norms (associative and symmetric semicopulas, see [16]) and copulas (supermodular semicopulas, see [19]).

⋆ E-mail: salgreco@unict.it
⋆⋆ E-mail: radko.mesiar@stuba.sk
⋆⋆⋆ Corresponding author. E-mail: frindone@unict.it

Several authors have observed that unipolar scales (such as $[0, 1]$) are often insufficient to describe how people process information in front of a choice and how they express their preferences. In expressing value judgments, people not only rank possible alternatives of choice, but also experiment an interior feeling of positiveness and/or negativeness regarding these alternatives, this is what is called *bipolarity*. These observations have driven several researchers towards bipolar representation of information and preferences. In decision theory bipolarity has firstly been modeled by Tversky and Kahneman [15, 24] (see also [14]). The reader is referred to [4] for a general treatment of bipolarity, and to [8] for discussion of bipolar and bivariate models in a multiple criteria framework. Bipolar integrals are those integrals useful to aggregate information based on bipolar scales (such as $[-1, 1]$). On this topic we refer the reader to literature [10, 12, 13].

Integration on bipolar scales requires another type of pseudo-multiplication. Considering measurable functions $f : X \to [-1, 1]$, and bi-capacities with values in $[-1, 1]$, one should consider an appropriate $[-1, 1]^2 \to [-1, 1]$ mapping, which will be called a bipolar semicopula. Note that for the Choquet integral with respect to bi-capacities [9, 11], the standard product $\Pi : [-1, 1]^2 \to [-1, 1]$ is considered. Simple bipolar semicopulas B_S, introduced for bipolar universal integrals in [12], are fully determined by standard semicopulas $S : [0, 1]^2 \to [0, 1]$, by means of $B_S(x, y) = (sign(xy))S(|x|, |y|)$. Observe that considering the greatest semicopula M, the related simple bipolar semicopula B_M is just the symmetric minimum introduced by Grabisch [6, 7] when generalizing the Sugeno integral to the bipolar scale $[-1, 1]$ (see also the recent paper [13] where the authors define the bipolar Sugeno integral, which is a bipolar universal integral with respect to the simple bipolar semicopula symmetric minimum).

The aim of this paper is to introduce and to study bipolar semicopulas, generalizing the standard product and the symmetric minimum acting on $[-1, 1]$. In the next section, after introduction of bipolar semicopulas, their representation by means of semicopulas is given, and some basic examples are added. Section 3 studies bipolar semicopulas with special properties. Several construction methods are discussed in Section 4. Some dualities in the class of bipolar semicopulas, including the invariants characterizations, are studied in Section 5. Finally, some concluding remarks are added.

2 Bipolar semicopula

Recall that a mapping $S : [0, 1]^2 \to [0, 1]$ is a semicopula [1, 5], whenever it is non-decreasing in both variables and 1 is the neutral element, i.e., $S(x, 1) = S(1, x) = x$ for all $x \in [0, 1]$. When considering the product $\Pi : [-1, 1]^2 \to [-1, 1]$, we see that 1 is its neutral element. More, it holds $\Pi(-1, x) = \Pi(x, -1) = -x$ for all $x \in [-1, 1]$.

Definition 1 *Let $A : [-1, 1]^2 \to [-1, 1]$ be a mapping such that $A(x, 1) = A(1, x) = x$ and $A(-1, x) = A(x, -1) = -x$ for all $x \in [-1, 1]$. Then 1 is called a bipolar neutral element for A.*

Intuitively, 1 is a bipolar neutral element for A if itself is a neutral element for A, while its “mirror-image”, -1, when composed with an element x, via A, yields $-x$, i.e. the mirror-image of x.

Definition 2 *A mapping* $B_S : [-1,1]^2 \to [-1,1]$ *is called a simple bipolar semicopula whenever it exists a semicopula* $S : [0,1]^2 \to [0,1]$*, such that for all* $(x,y) \in [-1,1]^2$*,* $B_S(x,y) = (sign(xy))S(|x|,|y|)$.

Observe that 1 is a bipolar neutral element for any simple bipolar semicopula B_S. Concerning the monotonicity required for semicopulas, observe that considering the product Π, or any simple bipolar semicopula B_S (note that $B_\Pi = \Pi$, abusing the notation Π both for the product on $[-1,1]$ and on $[0,1]$), these mappings are non-decreasing in both coordinates when fixing an element from the positive part of the scale, while they are non-increasing when fixing an element from the negative part of the scale $[-1,1]$.

Definition 3 *Let* $A : [-1,1]^2 \to [-1,1]$ *be a mapping such that the partial mappings* $A(x,\cdot)$ *and* $A(\cdot,y)$ *are non-decreasing for any* $x,y \in [0,1]$ *and they are non-increasing for any* $x,y \in [-1,0]$*. Then* A *will be called a bipolar increasing mapping.*

Now we are ready to introduce bipolar semicopulas.

Definition 4 *A mapping* $B : [-1,1]^2 \to [-1,1]$ *is called a bipolar semicopula whenever it is bipolar increasing and 1 is a bipolar neutral element of* B.

It is obvious that each simple bipolar semicopula B_S is a bipolar semicopula. Note also that 0 is a annihilator for any bipolar semicopula B, $B(x,0) = B(0,x) = 0$ for any $x \in [-1,1]$ (similarly 0 is a annihilator for any semicopula S).

The next representation result links bipolar semicopulas and semicopulas.

Theorem 1 *A mapping* $B : [-1,1]^2 \to [-1,1]$ *is a bipolar semicopula if and only if there is a quadruple* (S_1,S_2,S_3,S_4) *of semicopulas so that*

$$B(x,y) = \begin{cases} S_1(x,y) & \textit{if} \quad (x,y) \in [0,1]^2 \\ -S_2(-x,y) & \textit{if} \quad (x,y) \in [-1,0] \times [0,1] \\ S_3(-x,-y) & \textit{if} \quad (x,y) \in [-1,0]^2 \\ -S_4(x,-y) & \textit{if} \quad (x,y) \in [0,1] \times [-1,0]. \end{cases} \tag{1}$$

Proof. The necessity can be checked considering the properties of restrictions $B|_{[0,1]^2}$, $B|_{[-1,0]^2}$, $B|_{[-1,0]\times[0,1]}$ and $B|_{[0,1]\times[-1,0]}$. The sufficiency is a matter of verifying the bipolar monotonicity of B and the bipolar neutrality of 1. These properties can be checked by cases. For example

$$B(-1,x) = \begin{cases} S_3(1,-x) & \text{if } x \in [-1,0] \\ -S_2(1,x) & \text{if } x \in [0,1] \end{cases} = -x \tag{2}$$

for each $x \in [-1,1]$.

□

Representation Theorem 1 gives a simple characterization of simple bipolar semicopulas. Identifying B and the corresponding quadruple (S_1, S_2, S_3, S_4) we see that $B_S = (S, S, S, S)$, i.e., simple bipolar semicopulas correspond to constant quadruples of simple semicopulas. It is not difficult to check that the extremal bipolar semicopulas are related to extremal semicopulas M (the greatest semicopula given by $M(x, y) = \min(x, y)$) and Z (the smallest semicopula, called also the drastic product, and given by $Z(x, y) = \min(x, y)$ if $1 \in \{x, y\}$ and $Z(x, y) = 0$ else).

Proposition 1 *The class $\mathcal{B}$ of bipolar semicopulas is a complete bounded lattice (with point-wise supremum and infimum), with top element $B^* = (M, Z, M, Z)$ and bottom element $B_* = (Z, M, Z, M)$.*

Proof. The result follows from the fact that the class S of all semicopulas is a complete bounded lattice with top element M and bottom element Z.
Moreover for $B^{(i)} = (S_1^{(i)}, S_2^{(i)}, S_3^{(i)}, S_4^{(i)})$, $i \in I$,

$$\bigvee_{i\in I} B^{(i)} = (\bigvee_{i\in I} S_1^{(i)}, \bigwedge_{i\in I} S_2^{(i)}, \bigvee_{i\in I} S_3^{(i)}, \bigwedge_{i\in I} S_4^{(i)})$$

and

$$\bigwedge_{i\in I} B^{(i)} = (\bigwedge_{i\in I} S_1^{(i)}, \bigvee_{i\in I} S_2^{(i)}, \bigwedge_{i\in I} S_3^{(i)}, \bigvee_{i\in I} S_4^{(i)}).$$

□

As an example of a bipolar semicopula, consider $B = (M, \Pi, W, Z)$, where $W : [0, 1]^2 \to [0, 1]$ is the Lukasiewicz semicopula, given by $W(x, y) = \max(0, x+y-1)$. Then

$$B(x, y) = \begin{cases} \min(x, y) & \text{if} \quad (x, y) \in [0, 1]^2 \\ xy & \text{if} \quad (x, y) \in [-1, 0] \times [0, 1] \text{ or } \{x, y\} \cap \{-1, 1\} \neq \emptyset \\ -x - y - 1 & \text{if} \quad (x, y) \in [-1, 0]^2 \text{ and } x + y \leq -1 \\ 0 & \text{else.} \end{cases}$$

The visualization of this bipolar semicopula in the square of vertexes $(-1, 1)$, $(1, 1)$, $(1, -1)$ and $(-1, -1)$ can be seen in Figure 1.

3 Bipolar semicopulas with special properties

Recall that an element $x \in [0, 1]$ is an idempotent element of a semicopula $S \in \mathcal{S}$ whenever $S(x, x) = x$. Then for the corresponding simple bipolar semicopula $B_S \in \mathcal{B}$ it holds $B_S(x, x) = B_S(-x, -x) = x$ and $B_S(-x, x) = B_S(x, -x) = -x$.

Definition 5 *Let $B \in \mathcal{B}$ be a bipolar semicopula. An element $x \in [0, 1]$ is called a bipolar idempotent element of B whenever it satisfies $B_S(x, x) = B_S(-x, -x) = x$ and $B_S(-x, x) = B_S(x, -x) = -x$.*

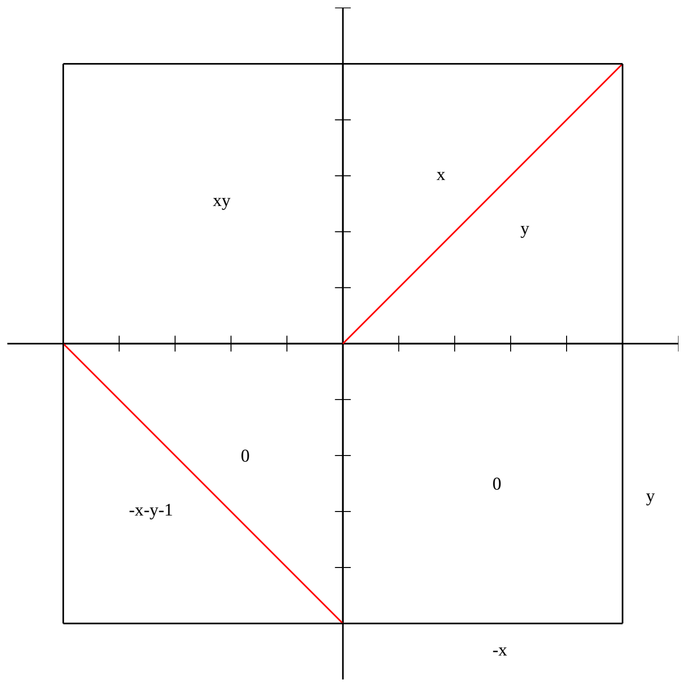

Fig. 1. An example of a bipolar semicopula

Proposition 2 *Let $B \in \mathcal{B}$ be a bipolar semicopula such that each $x \in [0,1]$ is its bipolar idempotent element. Then $B = B_M$ is the symmetric minimum introduced by Grabisch [6].*

Proof. Note that $x \in [0,1]$ is a bipolar idempotent element of $B = (S_1, S_2, S_3, S_4)$ if and only if x is a idempotent element of all four semicopulas S_1, S_2, S_3, and S_4. Now the result follows from the fact that M is the only idempotent semicopula, i.e., semicopula such that each $x \in [0,1]$ is its idempotent element.

□

Associativity of binary operations (binary functions) is a strong algebraic property, which, in the case of bipolar semicopulas characterizes a particular subclass of $\mathcal{B}$.

Theorem 2 *A bipolar semicopula $B \in \mathcal{B}$ is associative if and only if B is a simple bipolar semicopula, $B = B_S$, where $S \in \mathcal{S}$ is an associative semicopula.*

Proof. Let $B = (S_1, S_2, S_3, S_4) \in \mathcal{B}$ be associative. Then, necessarily, $S_1 = B|_{[0,1]^2}$ is associative. Moreover for any $y, z \in [0,1]$ it holds $B(-y, z) = B(B(-1, y), z) = B(-1, B(y, z)) = -B(y, z)$, i.e., $S_1 = S_2$. Similarly, for any $x, y \in [0,1]$ it holds $B(x, -y) = B(x, B(y, -1)) = B(B(x, y), -1) = B(x, y)$, i.e., $S_1 = S_4$. Finally, for any $x, y \in [-1, 0]$ the associativity of B implies that $S_3(-x, -y) = B(x, y) = B(B(-x, -1), y) = B(-x, B(-1, y)) = B(-x - y) = S_1(-x, -y)$, i.e., $S_1 = S_3$. Summarizing and ending $S_1 = S$, $B = B_S$ is a simple bipolar semicopula, linked to an associative semicopula S.

To see the sufficiency, recall that $B_S(x, y) = (sign(xy)) \cdot S(|x|, |y|)$ and thus $B_S(B_S(x, y), z) = (sign(xyz))S(S(|x|, |y|), |z|) = (sign(xyz))S(|x|, S(|y|, |z|)) = B_S(x, B_S(y, z))$, i.e., B_S is associative.

□

Typical examples of associative bipolar semicopulas are the product Π and the symmetric minimum B_M. Recall that a symmetric semicopula $S \in \mathcal{S}$, i.e., $S(x, y) = S(y, x)$ for all $x, y \in [0,1]$, which is also associative is called a triangular norm [21, 16].

Definition 6 *A symmetric associative bipolar semicopula $B \in \mathcal{B}$ is called a bipolar triangular norm.*

Due to Theorem 2 it is obvious that a bipolar semicopula $B \in \mathcal{B}$ is a bipolar triangular norm if and only if $B = B_T$, where $T : [0,1]^2 \to [0,1]$ is a triangular norm, i.e. if $B(x, y) = (sign(xy))T(|x|, |y|)$. Obviously, the product, Π, and the symmetric minimum, B_M, are bipolar triangular norms. The smallest semicopula Z is also a triangular norm and the corresponding bipolar triangular norm $B_Z : [-1,1]^2 \to [-1,1]$ is given by

$$B_Z(x, y) = \begin{cases} 0 & \text{if } (x, y) \in]-1, 1[^2, \\ xy & \text{else.} \end{cases} \tag{3}$$

Observe that due to the associativity of B_Z there is unique distinguished n-ary extension (often called "genuine"), $B_Z : [-1,1]^n \to [-1,1]$, $n > 2$ given by

$$B_Z(x_1,\ldots,x_n) = \begin{cases} 0 & \text{if } \#\{i \mid x_i \in]-1,1[\} \geq 2, \\ \Pi_{i=1}^n x_i & \text{else.} \end{cases} \tag{4}$$

Also W is a triangular norm, and thus also $B_W : [-1,1]^2 \to [-1,1]$ given by $B_W(x,y) = (sign(xy))\max(0, x+y-1)$ is a bipolar triangular norm. Moreover, its n-ary extension $B_W : [-1,1]^n \to [-1,1]$, $n > 2$, is given by

$$B_W(x_1,\ldots,x_n) = (sign(\Pi_{i=1}^n x_i))\max(0, \sum x_i - n + 1).$$

4 Construction methods for bipolar semicopulas

Some construction methods for bipolar semicopulas can be derived from construction methods for semicopulas. For example, similarly as in the case of semicopulas, for any $B \in \mathcal{B}$ also the mapping $\widetilde{B} : [-1,1]^2 \to [-1,1]$ given by $\widetilde{B}(x,y) = B(y,x)$ is a bipolar semicopula. We introduce some other construction methods. First of all, for any semicopula $S \in \mathcal{S}$ and any strictly increasing mapping $\varphi : [0,1] \to [0,1]$, $\varphi(0) = 0$ and $\varphi(1) = 1$, also the mapping $S_\varphi : [0,1]^2 \to [0,1]$ given by $S_\varphi(x,y) = \varphi^{(-1)}(S(\varphi(x),\varphi(y)))$ is a semicopula. Here $\varphi^{(-1)} : [0,1] \to [0,1]$ is a pseudo-inverse of φ [17] given by $\varphi^{(-1)}(t) = \sup\{x \in [0,1] \mid \varphi(x) < t\}$. Note that $\varphi^{(-1)}$ is continuous and $\varphi^{(-1)}(\varphi(x)) = x$ for all $x \in [0,1]$. For more details see [5].

Proposition 3 *Let $B \in \mathcal{B}$ be a bipolar semicopula and let $\eta : [-1,1] \to [-1,1]$ be an odd strictly increasing mapping such that $\eta(1) = 1$. Then the mapping $B_\eta : [-1,1]^2 \to [-1,1]$ given by*

$$B_\eta(x,y) = \eta^{(-1)}(B(\eta(x),\eta(y)))$$

is also a bipolar semicopula, where the pseudo-inverse $\eta^{(-1)} : [-1,1] \to [-1,1]$ is given by $\eta^{(-1)}(t) = \sup\{x \in [-1,1] \mid \eta(x) < t\}$.

Proof. Note that $\varphi = \eta|_{[0,1]}$ is strictly increasing, $\varphi(0) = 0$ and $\varphi(1) = 1$. Moreover for all $x \in [-1,0]$ it holds $-\eta(x) = \varphi(-x)$. Moreover, $\eta^{(-1)}|_{[0,1]} = \varphi^{(-1)}$ and $\eta^{(-1)}$ is a continuous odd mapping. Let $B = (S_1,S_2,S_3,S_4)$. It is not difficult to check that $B_\eta = ((S_1)_\varphi,(S_2)_\varphi,(S_3)_\varphi,(S_4)_\varphi)$, i.e., $B_\eta \in \mathcal{B}$. For example, if $(x,y) \in [-1,0]\times[0,1]$, then $B_\eta(x,y) = \eta^{(-1)}(B(\eta(x),\eta(y))) = \eta^{(-1)}(-S_2(-\eta(x),\eta(y))) = \eta^{(-1)}(-S_2(\varphi(-x),\varphi(y))) = -\varphi^{(-1)}(S_2(\varphi(-x),\varphi(y))) = -(S_2)_\varphi(-x,y)$. And so on for the other cases.

□

From proof of Proposition 3, we see that $B_\eta = ((S_1)_\varphi,(S_2)_\varphi,(S_3)_\varphi,(S_4)_\varphi)$, i.e., the construction applies separately to each component of the bipolar semicopula $B = (S_1,S_2,S_3,S_4)$. This depends from extending well known construction methods from semicopulas to bipolar semicopulas. However, other possible operation to construct bipolar semicopulas are possible, e.g. operations of the type $(S_1,S_2,S_3,S_4) \mapsto$

(T_1, T_2, T_3, T_4), where $T_i = f((S_1, S_2, S_3, S_4))$. We demand to future works the investigations of possible applications of these constructions.

Observe that the transformation B_η preserves the symmetry of bipolar semicopulas, but not the associativity, in general. However, if η is continuous, then also the associativity is preserved[5], and then for any bipolar triangular norm B_T also its transform $(B_T)_\eta = B_{T_\varphi}$ is a bipolar triangular norm.

Another construction method for semicopula is based on the idea of ordinal sums of posets, of semigroups, of triangular norms, etc.. We recall that, for any system $(]a_h, b_h[)_{h\in\mathcal{H}}$ of disjoint sub-intervals of $[0,1]$, and any system $(S_{(h)})_{h\in\mathcal{H}}$ of semicopulas, also the mapping $S : [0,1]^2 \to [0,1]$ given by

$$S(x,y) = \begin{cases} a_h + (b_h - a_h)S_{(h)}\left(\frac{x-a_h}{b_h-a_h}, \frac{y-a_h}{b_h-a_h}\right) & \text{if } (x,y) \in]a_h, b_h[^2 \text{ for some } h \in \mathcal{H}, \\ \min(x,y) & \text{else.} \end{cases} \tag{5}$$

is a semicopula. S is called an ordinal sum of summands $\langle a_h, b_h, S_{(h)}\rangle$ with notation $S = \left(\langle a_h, b_h, S_{(h)}\rangle \mid h \in \mathcal{H}\right)$. For more details see [5].

Proposition 4 *Let $(]a_h, b_h[)_{h\in\mathcal{H}}$ be a disjoint system of open sub-intervals of $[0,1]$, and let $(B_{(h)})_{h\in\mathcal{H}}$ be a system of bipolar semicopulas. Then the mapping $B : [-1,1]^2 \to [-1,1]$ given by*

$$B(x,y) = \begin{cases} (sign(xy))a_h + (b_h - a_h)B_{(h)}\left(\frac{x-(sign(x))a_h}{b_h-a_h}, \frac{y-(sign(y))a_h}{b_h-a_h}\right) & \textit{if} \\ (|x|,|y|) \in]a_h, b_h[^2 \textit{ for some } h \in \mathcal{H}, & \\ & \\ B_M(x,y) & \textit{else.} \end{cases}$$

is a bipolar semicopula. B is called an ordinal sum of bipolar semicopulas, with notation $B = \left(\langle a_h, b_h, B_{(h)}\rangle \mid h \in \mathcal{H}\right)$.

Proof. It is a matter of a trivial processing to show that if $B_h = (S_1^{(h)}, S_2^{(h)}, S_3^{(h)}, S_4^{(h)})$, $h \in \mathcal{H}$, then $B|_{[0,1]^2} = S_1 = \left(\left\langle a_h, b_h, S_1^{(h)}\right\rangle \mid h \in \mathcal{H}\right)$. Similarly, one can define ordinal sums $S_i = \left(\left\langle a_h, b_h, S_i^{(h)}\right\rangle \mid h \in \mathcal{H}\right)$, $i = 2,3,4$, and one can show that $B(x,y) = -S_2(-x,y)$ if $x \in [-1,0]$ and $y \in [0,1]$, $B(x,y) = S_3(-x,-y)$ if $(x,y) \in [-1,0]^2$, and $B(x,y) = -S_4(x,-y)$ if $x \in [0,1]$ and $y \in [-1,0]$, i.e., $B = (S_1, S_2, S_3, S_4)$ is a bipolar semicopula.

[5] this claim follows from the fact, that we have required η to be odd strictly increasing with 1 as fix point; then if eta is continuous, it is an odd automorphism of $[-1,1]$, and by isomorphism (its pseudo-inverse = inverse) associativity is guaranteed. To see that without continuity the associativity may fail, consider odd η defined on $[0, 0.5]$ by $\eta(x) = x/2$ and on $]0.5, 1]$ by $\eta(x) = (1+x)/2$; then, on [0,1], we have $\eta^{(-1)} = 2x$ on $[0, 1/4]$; 0.5 on $[1/4, 3/4]$; $2x-1$ on $[3/4, 1]$ (pseudo-inverse is continuous). Take the eta-transform of the standard product, denote it as $*$; we have then $(0.6*0.6)*0.3 = 0.5*0.3 = 0.075$ but $0.6*(0.6*0.3) = 0.6*0.24 = 0.192$.

□

Note that each element $x \in [0,1] \setminus \bigcup_{h \in \mathcal{H}}]a_h, b_h[$ is a bipolar idempotent element of a bipolar ordinal sum $B = \left(\left\langle a_h, b_h, B_{(h)} \right\rangle \mid h \in \mathcal{H}\right)$.

For example consider $B = \left(\left\langle 0, \frac{1}{2}, \Pi \right\rangle, \left\langle \frac{1}{2}, 1, \Pi \right\rangle\right)$. Then B is given by the table 1 (note that $x = 1/2$ is a bipolar idempotent element of B).

$y(\downarrow),\ x(\rightarrow)$	$[-1, -\frac{1}{2}]$	$[-\frac{1}{2}, \frac{1}{2}]$	$[\frac{1}{2}, 1]$
$[-1, -\frac{1}{2}]$	$2xy + x + y + 1$	$-x$	$2xy + x - y - 1$
$[-\frac{1}{2}, \frac{1}{2}]$	$-y$	$2xy$	y
$[\frac{1}{2}, 1]$	$2xy - x + y - 1$	x	$2xy - x - y + 1$

Table 1. Example

Observe also that an ordinal sum of bipolar triangular norms is a bipolar triangular norm,

$$B_{\left(\left\langle a_h, b_h, T_{(h)} \right\rangle \mid h \in \mathcal{H}\right)} = \left(\left\langle a_h, b_h, B_{T_{(h)}} \right\rangle \mid h \in \mathcal{H}\right).$$

Thus the ordinal sums of associative bipolar semicopulas can be seen as the classical ordinal sums of semigroups in the sense of Clifford [3, 20], see also [16, Theorem 3.42], applying the same arguments as in the ordinal sum of semigroups representation of triangular norms, see [16, Theorem 3.43].

Evidently, the class $\mathcal{B}$ of bipolar semicopulas is convex, i.e., for any $B_1, \ldots, B_n \in \mathcal{B}$, $\lambda_1, \ldots, \lambda_n \in [0,1]$, $\sum_{i=1}^{n} \lambda_i = 1$, also $B = \sum_{i=1}^{n} \lambda_i B_i$ is a bipolar semicopula.

An interesting construction method genuine for bipolar semicopulas is related to mapping $\tau : \{1,2,3,4\} \rightarrow \{1,2,3,4\}$, which can be seen as variations of indices (with repetition).

Definition 7 *Let us consider a mapping* $\tau : \{1,2,3,4\} \rightarrow \{1,2,3,4\}$*. Then for any* $B \in \mathcal{B}$*,* $B = (S_1, S_2, S_3, S_4)$*, the* $\tau-$*variation* $\tau B \in \mathcal{B}$ *is given by*

$$\tau B = (S_{\tau(1)}, S_{\tau(2)}, S_{\tau(3)}, S_{\tau(4)}).$$

Obviously, simple bipolar semicopulas are invariant under any $\tau-$variation, $\tau B_S = B_S$. Deeper study of $\tau-$variations of bipolar semicopulas is given in the next section.

5 Dualities in the class $\mathcal{B}$

We have introduced several construction methods assigning to a considered bipolar semicopula $B \in \mathcal{B}$ a (possibly different) bipolar semicopula denoted as $c(B)$. Construction $c : \mathcal{B} \to \mathcal{B}$ which differs from identity is called a duality if c is involutive, i.e., if for any $B \in \mathcal{B}$, $c(c(B)) = B$. If already $c(B) = B$, then B is called $c-$invariant. As a typical example recall the construction $\tilde{}$ based on the reversing of arguments, $\tilde{B}(x,y) = B(y,x)$. Evidently, $\tilde{}$ is a duality on $\mathcal{B}$ and $\tilde{}-$invariance is just the symmetry of bipolar semicopula B, i.e., $\tilde{B} = B$ whenever $B(x,y) = B(y,x)$ for all $(x,y) \in [-1,1]^2$. Considering $\eta-$transforming of bipolar semicopulas, there is no duality up to the trivial case $\eta = id$ (the identity), in which case $B_\eta = B$ for each $B \in \mathcal{B}$. In the case of $\tau-$invariations, again the identity $\tau(i) = i$, $i \in \{1,2,3,4\}$ is trivial construction, $\tau B = B$ for each $B \in \mathcal{B}$. However, for any other τ, there are $B \in \mathcal{B}$ such that $\tau B \neq B$.

Proposition 5 *Let $\tau : \{1,2,3,4\} \to \{1,2,3,4\}$ be fixed, $\tau \neq id$. Then the $\tau-$variation is a duality on $\mathcal{B}$ if and only if τ is involutive, i.e., $\tau \circ \tau = id$*

Proof. It is enough to observe that $\tau(\tau B) = (\tau \circ \tau)B$ for any $B \in \mathcal{B}$.

□

The enumeration of all $\tau-$variations which are dualities follows from the simple fact that if $\tau \circ \tau(i) = i$ for each $i \in \{1,2,3,4\}$, then necessarily τ is a permutation which is self-inverse $\tau = \tau^{-1}$. Up to the already mentioned $\tau = id$, one can consider exactly the next permutations τ yielding a duality on $\mathcal{B}$: $\tau_1 = (2,1,3,4)$, $\tau_2 = (3,2,1,4)$, $\tau_3 = (4,2,3,1)$, $\tau_4 = (1,3,2,4)$, $\tau_5 = (1,4,3,2)$, $\tau_6 = (1,2,4,3)$, $\tau_7 = (2,1,4,3)$, $\tau_8 = (3,4,1,2)$, and $\tau_9 = (4,3,2,1)$.
The invariance with respect to $\tau-$variations follows trivially from the representation $B = (S_1,S_2,S_3,S_4)$ and $\tau B = (S_{\tau(1)},S_{\tau(2)},S_{\tau(3)},S_{\tau(4)})$. Then $\tau B = B$ if and only if $S_i = S_{\tau(i)}$ for all $i \in \{1,2,3,4\}$. For example, $\tau_1 B = B$ if and only if $S_1 = S_2$, i.e., $B = (S_1,S_1,S_3,S_4)$.

We introduce now simple formulae for 3 distinguished $\tau-$dualities.

$$\tau_7 B(x,y) = -B(-x,y),\ \tau_7 B = B\ \to\ B = (S_1,S_1,S_2,S_2);$$
$$\tau_8 B(x,y) = B(-x,-y),\ \tau_8 B = B\ \to\ B = (S_1,S_2,S_1,S_2);$$
$$\tau_9 B(x,y) = -B(x,-y),\ \tau_9 B = B\ \to\ B = (S_1,S_2,S_2,S_1).$$

Observe that $\tau_7 \circ \tau_8 = \tau_8 \circ \tau_7 = \tau_9$, $\tau_7 \circ \tau_9 = \tau_9 \circ \tau_7 = \tau_8$ and $\tau_8 \circ \tau_9 = \tau_9 \circ \tau_8 = \tau_7$. invariance with respect to these $\tau-$variations is another characterization of simple bipolar semicopulas.

Proposition 6 *Let $B \in \mathcal{B}$, then B is a simple bipolar semicopula if and only if B is $\tau-$invariant with respect to two permutations from the set $\{\tau_7,\tau_8,\tau_9\}$.*

From the geometrical point of view, one can consider the graph

$$G_B = \{(x,y,B(x,y)) \mid (x,y) \in [-1,1]^2\}$$

of a bipolar semicopula B. Then the invariance with respect to $\tau_7$variation is just the symmetry of G_B with respect to $y-$axis. In the case of τ_8 one should consider the symmetry of G_B with respect to $z-$axis, and in τ_9 case with respect to $x-$axis. Observe also that the standard symmetry of bipolar semicopulas, i.e. $\tilde{}-$invariance, is just the symmetry of G_B with respect to the plane $x - y = 0$.

6 Concluding remarks

We have introduced a new class $\mathcal{B}$ of binary operations on a bipolar scale $[-1, 1]$, generalizing the standard multiplication, symmetric minimum and the simple bipolar semicopulas. We have studied several construction methods for bipolar semicopulas, including some dualities. We expect applications of our results in the framework of integrals on bipolar scales. Moreover, our approach can contribute to the extension of the quasi-copulas theory to the bipolar scale $[-1, 1]$, requiring the 1-Lipschitzianity of the considered bipolar semicopulas. Of course, in such case all semicopulas in the representation (S_1, S_2, S_3, S_4) should be quasi-copulas.

Acknowledgment

The authors sincerely thank the anonymous referees for their very helpful comments and suggestions.
The work on this contribution was partially supported by the grants APVV-0073-10 and by the European Regional Development Fund in the IT4Innovations Centre for Excellence project (CZ.1.05/1.1.00/02.0070).

References

1. B. Bassan and F. Spizzichino. Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. *Journal of Multivariate Analysis*, 93(2):313–339, 2005.
2. G. Choquet. Theory of Capacities. *Annales de l'Institute Fourier (Grenoble)*, 5:131–295, 1953/54.
3. A. H. Clifford. Naturally totally ordered commutative semigroups. *American Journal of Mathematics*, 76:631–646, 1954.
4. D. Dubois and H. Prade. An introduction to bipolar representations of information and preference. *International Journal of Intelligent Systems*, 23(8):866–877, 2008.
5. F. Durante and C. Sempi. Semicopulæ. *Kybernetika*, 41(3):315–328, 2005.
6. M. Grabisch. The symmetric Sugeno integral. *Fuzzy Sets and Systems*, 139(3):473–490, 2003.
7. M. Grabisch. The Möbius transform on symmetric ordered structures and its application to capacities on finite sets. *Discrete Mathematics*, 287(1-3):17–34, 2004.
8. M. Grabisch, S. Greco, and M. Pirlot. Bipolar and bivariate models in multicriteria decision analysis: Descriptive and constructive approaches. *Int. J. Intell. Syst.*, 23(9):930–969, 2008.
9. M. Grabisch and C. Labreuche. Bi-capacities–II: the Choquet integral. *Fuzzy Sets and Systems*, 151(2):237–259, 2005.
10. M. Grabisch and C. Labreuche. A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *4OR*, 6(1):1–44, 2008.

11. S. Greco, B. Matarazzo, and R. Slowinski. Bipolar Sugeno and Choquet integrals. In B. De Baets, J. Fodor, and G. Pasi, editors, *EUROWorking Group on Fuzzy Sets*, pages 191–196. Workshop on Informations Systems (EUROFUSE 2002), Varenna, Italy, 2002.
12. S. Greco, R. Mesiar, and F. Rindone. The Bipolar Universal Integral. In *IPMU 2012, Part III, CCIS 300*, pages 360–369. Greco et al, 2012.
13. S. Greco and F. Rindone. Bipolar fuzzy integrals. *Fuzzy Sets and Systems*, 220:21–33, 2013.
14. S. Greco and F. Rindone. The bipolar Choquet integral representation. *Theory and Decision*, pages 1–29, 2013.
15. D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. *Econometrica: Journal of the Econometric Society*, 47(2):263–291, 1979.
16. E. P. Klement and E. Mesiar, R. Pap. *Triangular norms*. Dordrecht, The Netherlands: Kluwer, 2000.
17. E. P. Klement, R. Mesiar, and E. Pap. Quasi-and pseudo-inverses of monotone functions, and the construction of t-norms. *Fuzzy Sets and Systems*, 104(1):3–13, 1999.
18. E.P. Klement, R. Mesiar, and E. Pap. A universal integral as common frame for Choquet and Sugeno integral. *IEEE Transactions on Fuzzy Systems*, 18(1):178–187, 2010.
19. R. B. Nelsen. *An introduction to copulas*. Springer, New York, 2006.
20. A. H. Preston and G.B. Clifford. The algebraic theory of semigroups. In *American Mathematical Society, Providence*, pages 148–159, 1961.
21. B. Schweizer and A. Sklar. Statistical metric spaces. *Pacific J. Math.*, 10:313–334, 1960.
22. N. Shilkret. Maxitive measure and integration. In *Indagationes Mathematicae (Proceedings)*, volume 74, pages 109–116. Elsevier, 1971.
23. M. Sugeno. *Theory of fuzzy integrals and its applications*. Ph.D. Thesis, Tokyo Institute of Technology, 1974.
24. A. Tversky and D. Kahneman. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and uncertainty*, 5(4):297–323, 1992.