Observed galaxy number counts on the lightcone up to second order: II. Derivation

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We present a detailed derivation of the observed galaxy number over-density on cosmological scales up to second order in perturbation theory. We include all relativistic effects that arise from observing on the past lightcone. The derivation is in a general gauge, and applies to all dark energy models (including interacting dark energy) and to metric theories of modified gravity. The result will be important for accurate cosmological parameter estimation, including non-Gaussianity, since all projection effects need to be taken into account. It also offers the potential for new probes of General Relativity, dark energy and modified gravity. This paper accompanies Paper I which presents the key results for the concordance model in Poisson gauge.

I. INTRODUCTION

The Newtonian prediction for the galaxy number over-density is accurate only on small scales. On cosmological scales, relativistic effects alter the observed number over-density through projection onto our past lightcone. This gives the well-known corrections from redshift space distortions and gravitational lensing convergence, but there are further Doppler, Sachs-Wolfe, integrated SW and time-delay type terms. The full relativistic effects have been calculated to first order in perturbation theory by [1–5]. The nature, importance and possible detectability of the relativistic corrections to the Newtonian approximation at first order have been considered by [6]–[19].

We derive the formula for the observed galaxy number over-density up to second order on cosmological scales. The result in Poisson gauge and for the concordance model is presented in the companion Paper I [20]. Paper I also considers the implications of the result for cosmological observables. Here we give the detailed derivation both in a general gauge and in Poisson gauge, for a flat Roberston-Walker background which allows for general dark energy models, including those where dark energy interacts non-gravitationally with cold dark matter. These interacting models have momentum exchange between dark energy and cold dark matter, which can lead to a velocity bias between galaxies and cold dark matter. Our results allow for velocity bias.

Our results also apply to metric theories of modified gravity as an alternative to dark energy, since we do not impose any field equations (in particular, we do not assume that the gauge invariant metric perturbations Φ and Ψ are equal). The main result contains all relativistic effects up to second order that arise from observing on the past light cone, including all redshift and lensing distortions – due to convergence, shear and rotation – and all contributions from velocities, SW, ISW and time-delay terms (for example, see [21, 22]). This is related to the second-order perturbations of the cosmological distance-redshift relation [23]–[31], and to the weak lensing shear up to second order [32, 33].

The second-order effects that we derive, especially those involving integrals along the line of sight, may make a non-negligible contribution to the observed number counts. This could be important for removing bias on parameter estimation in precision cosmology with galaxy surveys. Recently, it has been shown that second-order effects on the distance-redshift relation induce a shift in the background that may have a significant effect on estimates of the Hubble constant and matter density parameter [30, 31].

We follow the "cosmic rulers" approach of [5, 7], generalizing it from first to second order. Consequently, we use only the observed redshift z in our analysis. In particular, all background quantities are evaluated at the observed, not background, redshift. Thus we do not need to identify the perturbations of redshift (these are derived in full detail up to second order by [23, 26]). We neglect magnification bias, leaving this for future work [27].

The paper is organised as follows: in Section II we briefly review and generalize the cosmic rulers from first to second order, and apply it to obtain the second-order perturbations of galaxy number counts in redshift space. We perturb a flat Robertson-Walker universe in a general gauge in Section III and specialize to Poisson Gauge in Section IV. In Section V we describe how to relate the fluctuations of galaxy number density to the underlying matter density fluctuation δ_m. Finally, Section VI is devoted to conclusions.

The derivation of the second-order solutions is careful to include all steps needed for an independent verification of the results. It is therefore lengthy and technical.

- For readers who want to understand the derivation of the general formula, see Section II and Section III. The main results are Eqs. (157) and (158).

- For readers who are happy to skip the proof and look at the main result for galaxy number count fluctuations at second order in the Poisson gauge – see Paper I for the concordance model, or Section IV for general dark energy and modified gravity models. The main results are Eqs. (236) and Eq. (237).

Conventions: units $c = G = 1$; signature is $(-,+,+,+)$; Greek indices run over $0,1,2,3$, and Latin over $1,2,3$.

II. COSMIC RULERS AND THE OBSERVED OVERDENSITY

The cosmic rulers formalism of [5, 7] provides a map between redshift-space and real-space, without introducing a metric. Here we generalize it to second order. We denote quantities in the redshift frame with a bar.

Redshift-space is the "cosmic laboratory" where we probe the observations. We perform perturbations in real-space and not in redshift-space. In redshift-space we use coordinates which effectively flatten our past lightcone so that the photon geodesic from an observed galaxy has the following conformal space-time coordinates (see Fig. 1):

$$\bar{x}^\mu = (\eta,\ \bar{\mathbf{x}}) = (\eta_0 - \bar{\chi},\ \bar{\chi}\,\mathbf{n}). \tag{1}$$

Here $\bar{\chi}(z)$ is the comoving distance to the observed redshift in redshift-space, calculated in the background , and $\mathbf{n}$ is the observed direction to the galaxy, $n^i = \bar{x}^i/\bar{\chi} = \delta^{ij}(\partial\bar{\chi}/\partial\bar{x}^j)$. Using $\bar{\chi}$ as an affine parameter, the total derivative

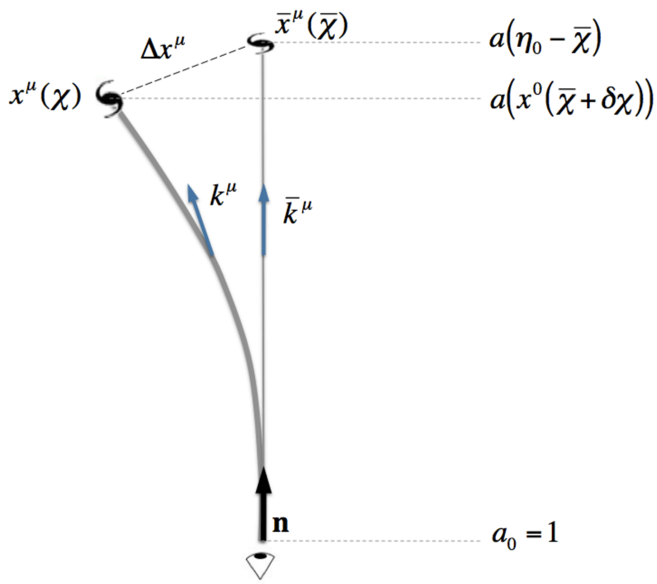

FIG. 1: The real-space and redshift-space views (adapted from [7]).

along the past light cone is

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} = -\frac{\partial}{\partial\eta} + n^i\frac{\partial}{\partial\bar{x}^i}\ . \tag{2}$$

To map from redshift-space to real-space (the "physical frame"), we introduce coordinates $x^\mu = x^\mu(\chi)$, where χ is the physical comoving distance of the source (see Fig. 1). Then, to second order

$$x^\mu(\chi) = \bar{x}^\mu(\chi) + \delta x^\mu(\chi) \quad \text{where} \quad \delta x^\mu(\chi) = \delta x^{\mu(1)}(\chi) + \frac{1}{2}\delta x^{\mu(2)}(\chi). \tag{3}$$

We map the real-space frame to the redshift frame perturbatively via

$$x^\mu(\chi) = \bar{x}^\mu(\bar{\chi}) + \Delta x^\mu(\bar{\chi}) \quad \text{where} \quad \Delta x^\mu(\bar{\chi}) = \Delta x^{\mu(1)}(\bar{\chi}) + \frac{1}{2}\Delta x^{\mu(2)}(\bar{\chi}). \tag{4}$$

The physical comoving distance to the source is a perturbation about the value in the redshift frame:

$$\chi = \bar\chi + \delta\chi \quad \text{where} \quad \delta\chi = \delta\chi^{(1)} + \frac{1}{2}\delta\chi^{(2)} \,. \tag{5}$$

Then we have

$$\begin{aligned} x^\mu(\chi) &= \bar x^\mu(\chi) + \delta x^{\mu(1)}(\chi) + \frac{1}{2}\delta x^{\mu(2)}(\chi) \\ &= \bar x^\mu(\bar\chi) + \frac{\mathrm{d}\bar x^\mu}{\mathrm{d}\bar\chi}\delta\chi^{(1)} + \delta x^{\mu(1)}(\bar\chi) + \frac{1}{2}\frac{\mathrm{d}\bar x^\mu}{\mathrm{d}\bar\chi}\delta\chi^{(2)}(\bar\chi) + \frac{1}{2}\frac{\mathrm{d}^2\bar x^\mu}{\mathrm{d}\bar\chi^2}\left(\delta\chi^{(1)}\right)^2 + \frac{\mathrm{d}\delta x^{\mu(1)}}{\mathrm{d}\bar\chi}\delta\chi^{(1)} + \frac{1}{2}\delta x^{\mu(2)}(\bar\chi) \,, \end{aligned} \tag{6}$$

which implies

$$\Delta x^{\mu(1)}(\bar\chi) = \frac{\mathrm{d}\bar x^\mu}{\mathrm{d}\bar\chi}\delta\chi^{(1)} + \delta x^{\mu(1)}(\bar\chi) \tag{7}$$

$$\Delta x^{\mu(2)}(\bar\chi) = \frac{\mathrm{d}\bar x^\mu}{\mathrm{d}\bar\chi}\delta\chi^{(2)} + \frac{\mathrm{d}^2\bar x^\mu}{\mathrm{d}\bar\chi^2}\left(\delta\chi^{(1)}\right)^2 + 2\frac{\mathrm{d}\delta x^{\mu(1)}}{\mathrm{d}\bar\chi}\delta\chi^{(1)} + \delta x^{\mu(2)}(\bar\chi) \,. \tag{8}$$

The photon 4-momentum is

$$p^\mu = \frac{\nu(a)}{a}k^\mu \tag{9}$$

where a is the scale factor, $\nu \propto 1/a$ is the frequency in a homogeneous and isotropic space-time, and k^μ is a null geodesic vector. In the redshift frame

$$\bar k^\mu = \frac{\mathrm{d}\bar x^\mu}{\mathrm{d}\bar\chi} = (-1,\ \mathbf{n}) \ , \tag{10}$$

while the physical k^μ evaluated at $\bar\chi$ is

$$k^\mu(\bar\chi) = \frac{\mathrm{d}x^\mu}{\mathrm{d}\bar\chi}(\bar\chi) = \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\bar x^\mu + \delta x^\mu\right)(\bar\chi) = \left(-1 + \delta\nu^{(1)} + \frac{1}{2}\delta\nu^{(2)},\ n^i + \delta n^{i(1)} + \frac{1}{2}\delta n^{i(2)}\right)(\bar\chi). \tag{11}$$

For $\mu = 0$ we have

$$\Delta x^{0(1)}(\bar\chi) = -\delta\chi^{(1)} + \delta x^{0(1)} \tag{12}$$

$$\Delta x^{0(2)}(\bar\chi) = -\delta\chi^{(2)} + 2\,\delta\nu^{(1)}\delta\chi^{(1)} + \delta x^{0(2)} \,, \tag{13}$$

where $\mathrm{d}\delta x^{0(n)}/\mathrm{d}\bar\chi = \delta\nu^{(n)}$, and for $\mu = i$,

$$\Delta x^{i(1)}(\bar\chi) = n^i\delta\chi^{(1)} + \delta x^{i(1)} \tag{14}$$

$$\Delta x^{i(2)}(\bar\chi) = n^i\delta\chi^{(2)} + 2\,\delta n^{i(1)}\delta\chi^{(1)} + \delta x^{i(2)} \,. \tag{15}$$

where $\mathrm{d}\delta x^{i(n)}/\mathrm{d}\bar\chi = \delta n^{i(n)}$. From Eq. (11), we obtain explicitly

$$\delta x^{0(n)}(\bar\chi) = \int_0^{\bar\chi} \mathrm{d}\tilde\chi\ \delta\nu^{(n)}(\tilde\chi) \,, \tag{16}$$

$$\delta x^{i(n)}(\bar\chi) = \int_0^{\bar\chi} \mathrm{d}\tilde\chi\ \delta n^{i(n)}(\tilde\chi) \,, \tag{17}$$

where we have imposed the boundary conditions at the observer: $\delta x_o^{0(n)} = 0$ and $\delta x_o^{i(n)} = 0$.

A. The scale factor

In real-space the scale factor is

$$a = a(x^0(\chi)) = a(\bar x^0 + \Delta x^0) = \bar a\left[1 + \mathcal{H}\Delta x^{0(1)} + \frac{1}{2}\mathcal{H}\Delta x^{0(2)} + \frac{1}{2}(\mathcal{H}' + \mathcal{H}^2)\left(\Delta x^{0(1)}\right)^2\right] \,, \tag{18}$$

where $\bar{a} = a(\bar{x}^0)$, prime is $\partial/\partial\bar{x}^0 = \partial/\partial\eta$ and $\mathcal{H} = \bar{a}'/\bar{a}$. Defining

$$\frac{a}{\bar{a}} = 1 + \Delta\ln a^{(1)} + \frac{1}{2}\Delta\ln a^{(2)} \; , \tag{19}$$

we find

$$\Delta\ln a^{(1)} \;=\; \mathcal{H}\,\Delta x^{0(1)} = \mathcal{H}\left(-\delta\chi^{(1)} + \delta x^{0(1)}\right) \tag{20}$$

$$\begin{aligned}\Delta\ln a^{(2)} \;&=\; (\mathcal{H}' + \mathcal{H}^2)\left(\Delta x^{0(1)}\right)^2 + \mathcal{H}\Delta x^{0(2)} = \frac{(\mathcal{H}' + \mathcal{H}^2)}{\mathcal{H}^2}\left(\Delta\ln a^{(1)}\right)^2 + \mathcal{H}\Delta x^{0(2)} \\ &=\; (\mathcal{H}' + \mathcal{H}^2)\left(-\delta\chi^{(1)} + \delta x^{0(1)}\right)^2 - \mathcal{H}\delta\chi^{(2)} + 2\,\mathcal{H}\delta\nu^{(1)}\delta\chi^{(1)} + \mathcal{H}\delta x^{0(2)} \; .\end{aligned} \tag{21}$$

B. Four-vectors and tetrads

The galaxy four-velocity can be given as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \frac{\mathrm{d}x^{\hat{\alpha}}}{\mathrm{d}s}\Lambda^\mu_{\hat{\alpha}} = u^{\hat{\alpha}}\Lambda^\mu_{\hat{\alpha}} \; , \tag{22}$$

where s is proper time and $\Lambda^\mu_{\hat{\alpha}}$ is an orthonormal tetrad. If we choose u^μ as the timelike basis vector, then

$$u_\mu = \Lambda_{\hat{0}\mu} = a\,E_{\hat{0}\mu} \qquad \text{and} \qquad u^\mu = \Lambda^\mu_{\hat{0}} = a^{-1}E^\mu_{\hat{0}} \; , \tag{23}$$

where $E^\mu_{\hat{\alpha}}$ is the tetrad in the comoving frame. In the background

$$E^{(0)}_{\hat{0}\mu} = (-1, \mathbf{0}) \; , \tag{24}$$

and perturbing, we obtain

$$E_{\hat{0}\mu}\left(x^\nu(\chi)\right) = E_{\hat{0}\mu}\left(\bar{x}^\nu(\bar{\chi}) + \Delta x^\nu\right) = E^{(0)}_{\hat{0}\mu}(\bar{\chi}) + E^{(1)}_{\hat{0}\mu}(\bar{\chi}) + \left(\frac{\partial E_{\hat{0}\mu}}{\partial\bar{x}^\nu}\right)^{(1)}\Delta x^{\nu(1)} + \frac{1}{2}E^{(2)}_{\hat{0}\mu}(\bar{\chi}) \; . \tag{25}$$

From k^μ, the map from redshift to real-space is given by

$$k^\mu\left(\chi\right) = \frac{\mathrm{d}x^\mu(\chi)}{\mathrm{d}\chi} = k^\mu\left(\bar{\chi} + \delta\chi\right) \;=\; k^{\mu(0)}(\bar{\chi}) + k^{\mu(1)}(\bar{\chi}) + \left(\frac{\mathrm{d}k^\mu}{\mathrm{d}\bar{\chi}}\right)^{(1)}\delta\chi^{(1)} + \frac{1}{2}k^{\mu(2)}(\bar{\chi}) \; , \tag{26}$$

where $k^{\mu(0)}(\bar{\chi}) = \bar{k}^\mu$.

C. The observed redshift

The observed redshift is given by

$$(1+z)\big|_\chi = \frac{(u_\mu p^\mu)\big|_\chi}{(u_\mu p^\mu)|_o} = \frac{\nu(\chi)}{\nu_o}\frac{(E_{\hat{0}\mu}k^\mu)\big|_\chi}{(E_{\hat{0}\mu}k^\mu)|_o} = \frac{a_o}{a(\chi)}\frac{(E_{\hat{0}\mu}k^\mu)\big|_\chi}{(E_{\hat{0}\mu}k^\mu)|_o} \; , \tag{27}$$

where we used $\nu \propto 1/a$. Quantities evaluated at the observer have a subscript o, while other quantities are assumed to be evaluated at the emitter (we suppress a subscript e for convenience). Choosing[1] $a_o = 1$ and given[2]

$$(E_{\hat{0}\mu}k^\mu)|_o = 1 \; , \tag{28}$$

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[1] Another possibility is the generalization $a_o \neq \bar{a}_o = a(\bar{x}^0) = 1$. Then $a_o = a(\eta)|_o = 1 + \Delta\ln a_o^{(1)} + \Delta\ln a_o^{(2)}/2$, and we should add $-\Delta\ln a_o^{(1)}$ to the right side of Eq. (32) and $-\Delta\ln a_o^{(2)}/2$ to the right side of Eq. (33). In order to obtain $\Delta\ln a_o^{(1)}$ and $\Delta\ln a_o^{(2)}/2$ we can follow the prescription used in [14] (which computes a_o only at first order). In our case, i.e. when we assume $a_o = 1$, by construction we automatically have $\Delta\ln a_o^{(1)} = \Delta\ln a_o^{(2)}/2 = 0$.

[2] From Eq. (9), for $\bar{\chi} = 0$ we have $p_{\hat{0}o} = (\Lambda_{\hat{0}\mu}p^\mu)|_o = \nu_o$, $p_{\hat{a}o} = (\Lambda_{\hat{a}\mu}p^\mu)|_o = n_{\hat{a}}\nu_o$.

then

$$1 + z = \frac{E_{\hat{0}\mu} k^\mu}{a} \ . \tag{29}$$

From Eq. (19), $\bar{a}$ is the scale factor in redshift-space. Then $\bar{a} = 1/(1+z)$. From Eqs. (19), (25) and (26), we get

$$1 = \frac{1 + (E_{\hat{0}\mu} k^\mu)^{(1)} + \frac{1}{2}(E_{\hat{0}\mu} k^\mu)^{(2)}}{1 + \Delta \ln a^{(1)} + \frac{1}{2}\Delta \ln a^{(2)}} \ , \tag{30}$$

where

$$(E_{\hat{0}\mu} k^\mu)^{(0)} = 1 \ . \tag{31}$$

Then from Eq. (30) we can find $\Delta \ln a^{(1)}$ and $\Delta \ln a^{(2)}$, such that[3]

$$\Delta \ln a^{(1)} = (E_{\hat{0}\mu} k^\mu)^{(1)} = E^{(1)}_{\hat{0}\mu} k^{\mu(0)} + E^{(0)}_{\hat{0}\mu} k^{\mu(1)} = -E^{(1)}_{\hat{0}0} + n^i E^{(1)}_{\hat{0}i} - \delta\nu^{(1)} \ , \tag{32}$$

$$\begin{aligned}
\Delta \ln a^{(2)} &= (E_{\hat{0}\mu} k^\mu)^{(2)} = 2E^{(1)}_{\hat{0}\mu} k^{\mu(1)} + E^{(2)}_{\hat{0}\mu} k^{\mu(0)} + E^{(0)}_{\hat{0}\mu} k^{\mu(2)} + 2\delta\chi^{(1)} E^{(0)}_{\hat{0}\mu} \left(\frac{\mathrm{d}k^\mu}{\mathrm{d}\bar{\chi}}\right)^{(1)} + 2\delta\chi^{(1)} k^{\mu(0)} \left(\frac{\mathrm{d}E_{\hat{0}\mu}}{\mathrm{d}\bar{\chi}}\right)^{(1)} \\
&+ 2k^{\mu(0)} \left(\frac{\partial E_{\hat{0}\mu}}{\partial \bar{x}^\nu}\right)^{(1)} \delta x^{\nu(1)} \\
&= 2E^{(1)}_{\hat{0}\mu} k^{\mu(1)} + E^{(2)}_{\hat{0}\mu} k^{\mu(0)} + E^{(0)}_{\hat{0}\mu} k^{\mu(2)} + 2k^{\mu(0)} \left(\frac{\partial E_{\hat{0}\mu}}{\partial \bar{x}^\nu}\right)^{(1)} \delta x^{\nu(1)} + 2\delta\chi^{(1)} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \Delta \ln a^{(1)} \\
&= 2E^{(1)}_{\hat{0}0} \delta\nu^{(1)} + 2E^{(1)}_{\hat{0}i} \delta n^{i(1)} - \delta\nu^{(2)} - E^{(2)}_{\hat{0}0} + n^i E^{(2)}_{\hat{0}i} + 2\left[-\left(\frac{\partial E_{\hat{0}0}}{\partial \bar{x}^\nu}\right)^{(1)} + n^i \left(\frac{\partial E_{\hat{0}i}}{\partial \bar{x}^\nu}\right)^{(1)}\right] \delta x^{\nu(1)} \\
&+ 2\delta\chi^{(1)} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \Delta \ln a^{(1)} \ .
\end{aligned} \tag{33}$$

Using Eqs. (20) and (32) at first order, and Eqs. (21) and (33) at second order we obtain

$$\delta\chi^{(1)} = \delta x^{0(1)} - \frac{\Delta \ln a^{(1)}}{\mathcal{H}} = \delta x^{0(1)} - \Delta x^{0(1)} \ , \tag{34}$$

$$\delta\chi^{(2)} = -\frac{1}{\mathcal{H}} \Delta \ln a^{(2)} + \frac{(\mathcal{H}' + \mathcal{H}^2)}{\mathcal{H}^3} \left(\Delta \ln a^{(1)}\right)^2 - \frac{2}{\mathcal{H}} \delta\nu^{(1)} \Delta \ln a^{(1)} + 2\delta\nu^{(1)} \delta x^{0(1)} + \delta x^{0(2)} \ . \tag{35}$$

Given $\Delta \ln a^{(1)}$ from Eq. (32), and $\Delta \ln a^{(2)}$ from Eq. (33), it is useful to rewrite Eqs. (20) and (21) as

$$\Delta x^{0(1)} = \frac{\Delta \ln a^{(1)}}{\mathcal{H}} \ , \tag{36}$$

$$\Delta x^{0(2)} = \frac{1}{\mathcal{H}} \Delta \ln a^{(2)} - \frac{(\mathcal{H}' + \mathcal{H}^2)}{\mathcal{H}^3} \left(\Delta \ln a^{(1)}\right)^2 \ . \tag{37}$$

D. Number density

The physical number density of galaxies n_g as a function of physical comoving coordinates x^μ is defined by the observed number of galaxies contained within a volume $\bar{\mathcal{V}}$:

$$\mathcal{N} = \int_{\bar{\mathcal{V}}} \sqrt{-g(x^\alpha)}\, n_g(x^\alpha)\, \varepsilon_{\mu\nu\rho\sigma} u^\mu(x^\alpha) \frac{\partial x^\nu}{\partial \bar{x}^1} \frac{\partial x^\rho}{\partial \bar{x}^2} \frac{\partial x^\sigma}{\partial \bar{x}^3} \mathrm{d}^3 \bar{\mathbf{x}} \ , \tag{38}$$

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[3] In Eq. (33) we have used $k^{\nu(0)}[\partial E_{\hat{0}\mu}/\partial \bar{x}^\nu]^{(1)} = (\mathrm{d}\bar{x}^\nu/\mathrm{d}\bar{\chi})[\partial E_{\hat{0}\mu}/\partial \bar{x}^\nu]^{(1)} = [\mathrm{d}E_{\hat{0}\mu}/\mathrm{d}\bar{\chi}]^{(1)}$.

where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor, and u^μ is the four velocity vector as a function of comoving location. In the redshift frame

$$\mathcal{N} = \int_{\bar{\mathcal{V}}} \bar{a}^3(\bar{x}^0)\, n_g\left(\bar{x}^0, \bar{\mathbf{x}}\right) \mathrm{d}^3\bar{\mathbf{x}}\ , \tag{39}$$

so that

$$a(\bar{x}^0)^3\, n_g(\bar{x}^0, \bar{\mathbf{x}}) = \sqrt{-g(x^\alpha)}\, n_g(x^\alpha)\, \varepsilon_{\mu\nu\rho\sigma} u^\mu(x^\alpha) \frac{\partial x^\nu}{\partial \bar{x}^1} \frac{\partial x^\rho}{\partial \bar{x}^2} \frac{\partial x^\sigma}{\partial \bar{x}^3}\ . \tag{40}$$

Using the cosmic rulers defined in the previous section we expand all quantities on the left side of Eq. (40) in the observed coordinates, i.e. in the redshift frame. Using Eq. (23),

$$\sqrt{-\hat{g}(x^\alpha)}\, a^3(x^\alpha) n_g(x^\alpha)\, \varepsilon_{\mu\nu\rho\sigma}\, E^\mu_{\hat{0}}(x^\alpha) \frac{\partial x^\nu}{\partial \bar{x}^1} \frac{\partial x^\rho}{\partial \bar{x}^2} \frac{\partial x^\sigma}{\partial \bar{x}^3}\ , \tag{41}$$

where $\hat{g}$ is the determinant of the comoving metric $\hat{g}_{\mu\nu} = g_{\mu\nu}/a^2$. We separate Eq. (41) into three parts:

$$(1)\ \sqrt{-\hat{g}(x^\alpha)}\ , \qquad (2)\ \varepsilon_{\mu\nu\rho\sigma}\, E^\mu_{\hat{0}}(x^\alpha) \frac{\partial x^\nu}{\partial \bar{x}^1} \frac{\partial x^\rho}{\partial \bar{x}^2} \frac{\partial x^\sigma}{\partial \bar{x}^3} \qquad \text{and} \quad (3)\ a^3(x^\alpha) n_g(x^\alpha)\ .$$

Then[4]:

(1) Splitting $\sqrt{-\hat{g}(x^\alpha)}$ as

$$\sqrt{-\hat{g}(x^\alpha)} = \sqrt{-\hat{g}(x^\alpha)}^{(0)} + \delta\sqrt{-\hat{g}(x^\alpha)}^{(1)} + \frac{1}{2}\delta\sqrt{-\hat{g}(x^\alpha)}^{(2)} \tag{42}$$

we have

$$\delta\sqrt{-\hat{g}(x^\alpha)}^{(1)} = \frac{1}{2}\sqrt{-\hat{g}(x^\alpha)}^{(0)} \hat{g}^{\mu(1)}_{\mu}(x^\alpha)\ , \tag{43}$$

$$\delta\sqrt{-\hat{g}(x^\alpha)}^{(2)} = \frac{1}{2}\sqrt{-\hat{g}(x^\alpha)}^{(0)} \left(\frac{1}{2}\hat{g}^{\mu(1)}_{\mu}\hat{g}^{\nu(1)}_{\nu} + \hat{g}^{\mu(2)}_{\mu} - \hat{g}^{\nu(1)}_{\mu}\hat{g}^{\mu(1)}_{\nu}\right)(x^\alpha)\ . \tag{44}$$

Here $\hat{g}^{\mu(n)}_{\nu} = \hat{g}^{\mu\sigma(0)}\hat{g}^{(n)}_{\sigma\nu}$ and $\hat{g}_{\mu\nu} = \hat{g}^{(0)}_{\mu\nu} + \hat{g}^{(1)}_{\mu\nu} + \hat{g}^{(2)}_{\mu\nu}/2$.

Mapping all these terms from real- to redshift-space, we find

$$\sqrt{-\hat{g}(x^\alpha)}^{(0)} = 1\ , \tag{45}$$

$$\begin{aligned}\delta\sqrt{-\hat{g}(x^\alpha)}^{(1)} &= \delta\sqrt{-\hat{g}(\bar{x}^\alpha)}^{(1)} + \left(\frac{\partial}{\partial x^\nu}\delta\sqrt{-\hat{g}(\bar{x}^\alpha)}\right)^{(1)} \Delta x^{\nu(1)} \\ &= \frac{1}{2}\hat{g}^{\mu(1)}_{\mu}(\bar{x}^\alpha) + \frac{1}{2}\left(\frac{\partial \hat{g}^\mu_\mu}{\partial \bar{x}^\nu}\right)^{(1)}(\bar{x}^\alpha)\ \Delta x^{\nu(1)}\ ,\end{aligned} \tag{46}$$

$$\delta\sqrt{-\hat{g}(x^\alpha)}^{(2)} = \delta\sqrt{-\hat{g}(\bar{x}^\alpha)}^{(2)}\ . \tag{47}$$

Rewriting

$$\sqrt{-\hat{g}(x^\alpha)} = 1 + \Delta\sqrt{-\hat{g}(\bar{x}^\alpha)}^{(1)} + \frac{1}{2}\Delta\sqrt{-\hat{g}(\bar{x}^\alpha)}^{(2)}\ , \tag{48}$$

we find

$$\Delta\sqrt{-\hat{g}(\bar{x}^\alpha)}^{(1)} = \frac{1}{2}\hat{g}^{\mu(1)}_{\mu}(\bar{x}^\alpha)\ , \tag{49}$$

$$\Delta\sqrt{-\hat{g}(\bar{x}^\alpha)}^{(2)} = \frac{1}{4}\hat{g}^{\mu(1)}_{\mu}(\bar{x}^\alpha)\ \hat{g}^{\nu(1)}_{\nu}(\bar{x}^\alpha) + \frac{1}{2}\hat{g}^{\mu(2)}_{\mu}(\bar{x}^\alpha) - \frac{1}{2}\hat{g}^{\nu(1)}_{\mu}(\bar{x}^\alpha)\ \hat{g}^{\mu(1)}_{\nu}(\bar{x}^\alpha) + \left(\frac{\partial \hat{g}^\mu_\mu}{\partial \bar{x}^\nu}\right)^{(1)}(\bar{x}^\alpha)\ \Delta x^{\nu(1)}. \tag{50}$$

[4] For a rank two tensor $\mathbb{M}$,

$$\begin{aligned} M &= \det(\mathbb{M}) = M^{(0)} + M^{(1)} + M^{(2)}/2\ , \quad \text{where} \quad M^{(0)} = \det(\mathbb{M}^{(0)})\ , \qquad M^{(1)} = M^{(0)}\mathrm{Tr}[\mathbb{M}^{(0)-1}\mathbb{M}^{(1)}]\ , \\ M^{(2)} &= M^{(0)}\{[M^{(1)}/M^{(0)}]^2 - \mathrm{Tr}[(\mathbb{M}^{(0)-1}\mathbb{M}^{(1)})(\mathbb{M}^{(0)-1}\mathbb{M}^{(1)})] + \mathrm{Tr}[\mathbb{M}^{(0)-1}\mathbb{M}^{(2)}]\}\ .\end{aligned}$$

(2) We write Eq. (42) as

$$\varepsilon_{\mu\nu\rho\sigma}\, E^{\mu}_{\hat{0}}(x^{\alpha}) \frac{\partial x^{\nu}}{\partial \bar{x}^{1}} \frac{\partial x^{\rho}}{\partial \bar{x}^{2}} \frac{\partial x^{\sigma}}{\partial \bar{x}^{3}} = E^{0}_{\hat{0}}(x^{\alpha}) \left|\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}}\right| + E^{i}_{\hat{0}}(x^{\alpha})\, \Sigma_i \,, \tag{51}$$

where

$$\left|\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}}\right| = \det\left(\frac{\partial x^i}{\partial \bar{x}^j}\right) \qquad \text{and} \qquad \Sigma_i = \epsilon_{ijk}\left(-\frac{\partial x^0}{\partial \bar{x}^1}\frac{\partial x^j}{\partial \bar{x}^2}\frac{\partial x^k}{\partial \bar{x}^3} + \frac{\partial x^j}{\partial \bar{x}^1}\frac{\partial x^0}{\partial \bar{x}^2}\frac{\partial x^k}{\partial \bar{x}^3} - \frac{\partial x^j}{\partial \bar{x}^1}\frac{\partial x^k}{\partial \bar{x}^2}\frac{\partial x^0}{\partial \bar{x}^3}\right) . \tag{52}$$

The first term on the right of Eq. (51) is

$$E^{0}_{\hat{0}}(x^{\alpha}) \left|\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}}\right| = \left[E^{0(0)}_{\hat{0}}(x^{\alpha}) + E^{0(1)}_{\hat{0}}(x^{\alpha}) + \frac{1}{2}E^{0(2)}_{\hat{0}}(x^{\alpha})\right]\left[\left|\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}}\right|^{(0)} + \left|\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}}\right|^{(1)} + \frac{1}{2}\left|\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}}\right|^{(2)}\right] , \tag{53}$$

where

$$E^{0(0)}_{\hat{0}} = 1 \qquad E^{0(1)}_{\hat{0}}(x^{\alpha}) = E^{0(1)}_{\hat{0}}(\bar{x}^{\alpha}) + \left(\frac{\partial E^{0}_{\hat{0}}}{\partial \bar{x}^{\mu}}\right)^{(1)} (\bar{x}^{\alpha})\, \Delta x^{\mu(1)} \,, \tag{54}$$

$$\left|\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}}\right|^{(0)} = 1 \,, \qquad \left|\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}}\right|^{(1)} = \left(\frac{\partial \Delta x^i}{\partial \bar{x}^i}\right)^{(1)} , \tag{55}$$

$$\left|\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}}\right|^{(2)} = \left(\frac{\partial \Delta x^i}{\partial \bar{x}^i}\right)^{(1)}\left(\frac{\partial \Delta x^j}{\partial \bar{x}^j}\right)^{(1)} - \left(\frac{\partial \Delta x^i}{\partial \bar{x}^j}\right)^{(1)}\left(\frac{\partial \Delta x^j}{\partial \bar{x}^i}\right)^{(1)} + \left(\frac{\partial \Delta x^i}{\partial \bar{x}^i}\right)^{(2)} . \tag{56}$$

The second term on the right of Eq. (51) is

$$E^{i}_{\hat{0}}(x^{\alpha})\, \Sigma_i = \left[E^{i(0)}_{\hat{0}}(x^{\alpha}) + E^{i(1)}_{\hat{0}}(x^{\alpha}) + \frac{1}{2}E^{i(2)}_{\hat{0}}(x^{\alpha})\right]\left[\Sigma^{(0)}_i + \Sigma^{(1)}_i + \frac{1}{2}\Sigma^{(2)}_i\right] , \tag{57}$$

where

$$E^{i(0)}_{\hat{0}} = 0 \qquad E^{i(1)}_{\hat{0}}(x^{\alpha}) = E^{i(1)}_{\hat{0}}(\bar{x}^{\alpha}) + \left(\frac{\partial E^{i}_{\hat{0}}}{\partial \bar{x}^{\mu}}\right)^{(1)} (\bar{x}^{\alpha})\, \Delta x^{\mu(1)} \,, \tag{58}$$

$$\Sigma^{(0)}_i = n_i \,, \qquad \Sigma^{(1)}_i = -\left(\frac{\partial \Delta x^0}{\partial \bar{x}^i}\right)^{(1)} + \epsilon_{ijr}\epsilon^{pqr} n_p \left(\frac{\partial \Delta x^j}{\partial \bar{x}^q}\right)^{(1)} . \tag{59}$$

Here we do not compute $\Sigma^{(2)}_i$ because $E^{i(0)}_{\hat{0}} = 0$.

(3) From Eq. (19),

$$a^3 = \bar{a}^3\left[1 + 3\,\Delta\ln a^{(1)} + 3\left(\Delta\ln a^{(1)}\right)^2 + \frac{3}{2}\Delta\ln a^{(2)}\right] . \tag{60}$$

Writing $n_g(x^{\alpha}) = n^{(0)}_g(x^0) + n^{(1)}_g(x^{\alpha}) + n^{(2)}_g(x^{\alpha})/2$, we have

$$n^{(0)}_g(x^0) = n^{(0)}_g(\bar{x}^0 + \Delta x^0) = \bar{n}_g(\bar{x}^0) + \frac{\partial \bar{n}_g}{\partial \bar{x}^0}\,\Delta x^{0(1)} + \frac{1}{2}\frac{\partial^2 \bar{n}_g}{\partial \bar{x}^{0\,2}}\left(\Delta x^{0(1)}\right)^2 + \frac{1}{2}\frac{\partial \bar{n}_g}{\partial \bar{x}^0}\,\Delta x^{0(2)} \,, \tag{61}$$

where $n^{(0)}_g(\bar{x}^0) = \bar{n}_g(\bar{x}^0)$ and

$$n^{(1)}_g(x^{\alpha}) = n^{(1)}_g(\bar{x}^{\alpha} + \Delta x^{\alpha(1)}) = n^{(1)}_g(\bar{x}^{\alpha}) + \left(\frac{\partial n_g}{\partial \bar{x}^{\alpha}}\right)^{(1)} \Delta x^{\alpha(1)} \,, \qquad \frac{1}{2}n^{(2)}_g(x^{\alpha}) = \frac{1}{2}n^{(2)}_g(\bar{x}^{\alpha}) \,. \tag{62}$$

Defining $\delta^{(n)}_g = n^{(n)}_g(\bar{x}^{\alpha})/\bar{n}_g(\bar{x}^0)$ and considering Eqs. (36) and (37), we have

$$\begin{aligned} n_g(x^{\alpha}) \;=\; \bar{n}_g &\Bigg\{1 + \frac{\mathrm{d}\ln \bar{n}_g}{\mathrm{d}\ln \bar{a}}\,\Delta\ln a^{(1)} + \delta^{(1)}_g + \left(\frac{\partial \delta_g}{\partial \bar{x}^{\mu}}\right)^{(1)} \Delta x^{\mu(1)} + \frac{\mathrm{d}\ln \bar{n}_g}{\mathrm{d}\ln \bar{a}}\,\Delta\ln a^{(1)}\,\delta^{(1)}_g \\ &+ \frac{1}{2}\left[-\frac{\mathrm{d}\ln \bar{n}_g}{\mathrm{d}\ln \bar{a}} + \left(\frac{\mathrm{d}\ln \bar{n}_g}{\mathrm{d}\ln \bar{a}}\right)^2 + \frac{\mathrm{d}^2\ln \bar{n}_g}{\mathrm{d}\ln \bar{a}^2}\right]\left(\Delta\ln a^{(1)}\right)^2 + \frac{1}{2}\frac{\mathrm{d}\ln \bar{n}_g}{\mathrm{d}\ln \bar{a}}\,\Delta\ln a^{(2)} + \frac{1}{2}\delta^{(2)}_g\Bigg\} . \end{aligned} \tag{63}$$

At this point, it is useful to define the parallel and perpendicular projection operators to the observed line-of-sight direction (see also [5, 7]). For any spatial vectors and tensors:

$$A_{\|} = n^i n^j A_{ij}\ , \quad B^i_{\perp} = \mathcal{P}^{ij} B_j = B^i - n^i B_{\|}\ , \quad A_{\perp} = \mathcal{P}^{ij} A_{ij}\ , \quad \mathcal{P}^i_j = \delta^i_j - n^i n_j\ . \tag{64}$$

The directional derivatives are defined as

$$\partial_{\|} = n^i \partial_i\ , \qquad \partial^2_{\|} = \partial_{\|}\partial_{\|}, \qquad \partial_{\perp i} = \mathcal{P}^j_i \partial_j = \partial_i - n_i \partial_{\|}\ , \tag{65}$$

and we have

$$\begin{aligned}
\partial_i n^j &= \frac{1}{\bar\chi}\mathcal{P}^j_i\ , \qquad\qquad n^i B_{\perp i} = 0\ , \qquad\qquad n^i \partial_{\perp i} = 0\ , \\
\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial^i_{\perp} &= \partial^i_{\perp}\frac{\mathrm{d}}{\mathrm{d}\bar\chi} - \frac{1}{\bar\chi}\partial^i_{\perp}\ , \qquad \partial_{\perp i}\left(n^k \mathcal{P}^i_j\right) = \frac{1}{\bar\chi}\left(\delta^k_j - 3 n^k n_j\right)\ , \\
\partial_j B^i &= n^i n_j \partial_{\|} B_{\|} + n^i \partial_{\perp j} B_{\|} + \partial_{\perp j} B^i_{\perp} + n_j \partial_{\|} B^i_{\perp} + \frac{1}{\bar\chi}\mathcal{P}^i_j B_{\|}\ , \\
\text{and} \qquad & \nabla^2_{\perp} = \partial_{\perp i}\partial^i_{\perp} = \nabla^2 - \partial^2_{\|} - \frac{2}{\bar\chi}\partial_{\|}\ .
\end{aligned} \tag{66}$$

We also have

$$\mathrm{Tr}[\partial B] = \partial_i B^i = \partial_{\perp i} B^i_{\perp} + \partial_{\|} B_{\|} + \frac{2}{\bar\chi} B_{\|} \tag{67}$$

$$\begin{aligned}
\mathrm{Tr}[(\partial B)^2] &= (\partial_j B^i)(\partial_i B^j) = (\partial_{\|} B_{\|})(\partial_{\|} B_{\|}) + (\partial_{\perp j} B^i_{\perp})(\partial_{\perp i} B^j_{\perp}) - \frac{2}{\bar\chi} B_{\perp i}\partial_{\|} B^i_{\perp} + \frac{2}{\bar\chi} B_{\|}\partial_{\perp i} B^i_{\perp} \\
&+ 2(\partial_{\|} B^i_{\perp})(\partial_{\perp i} B_{\|}) + \frac{2}{\bar\chi^2} B_{\|} B_{\|}\ .
\end{aligned} \tag{68}$$

Using Eq. (64), we find $\Delta x^{i(1)} = n^i \Delta x^{(1)}_{\|} + \mathcal{P}^i_j \Delta x^{j(1)} = n^i \Delta x^{(1)}_{\|} + \Delta x^{i(1)}_{\perp}$ where

$$\Delta x^{(1)}_{\|} = \delta\chi^{(1)} + \delta x^{(1)}_{\|} = \delta x^{0(1)} - \frac{\Delta \ln a^{(1)}}{\mathcal{H}} + \delta x^{(1)}_{\|} = \delta x^{0(1)} + \delta x^{(1)}_{\|} - \Delta x^{0(1)}\ , \tag{69}$$

$$\Delta x^{i(1)}_{\perp} = \delta x^{i(1)}_{\perp}\ . \tag{70}$$

For $\Delta x^{i(2)} = n^i \Delta x^{(2)}_{\|} + \mathcal{P}^i_j \Delta x^{j(2)} = n^i \Delta x^{(2)}_{\|} + \Delta x^{i(2)}_{\perp}$ we have

$$\begin{aligned}
\Delta x^{(2)}_{\|} &= \delta\chi^{(2)} + 2\delta n^{(1)}_{\|}\delta\chi^{(1)} + \delta x^{(2)}_{\|} = -\frac{1}{\mathcal{H}}\Delta \ln a^{(2)} + \frac{(\mathcal{H}' + \mathcal{H}^2)}{\mathcal{H}^3}\left(\Delta \ln a^{(1)}\right)^2 - \frac{2}{\mathcal{H}}\delta\nu^{(1)}\Delta \ln a^{(1)} \\
&\quad + 2\delta\nu^{(1)}\delta x^{0(1)} + \delta x^{0(2)} + 2\delta n^{(1)}_{\|}\delta x^{0(1)} - \frac{2}{\mathcal{H}}\delta n^{(1)}_{\|}\Delta \ln a^{(1)} + \delta x^{(2)}_{\|} \\
&= \delta x^{0(2)} + \delta x^{(2)}_{\|} - \Delta x^{0(2)} + 2\left(\delta\nu^{(1)} + \delta n^{(1)}_{\|}\right)\delta\chi^{(1)}\ ,
\end{aligned} \tag{71}$$

$$\Delta x^{i(2)}_{\perp} = 2\delta n^{i(1)}_{\perp}\delta\chi^{(1)} + \delta x^{i(2)}_{\perp} = 2\delta n^{i(1)}_{\perp}\delta x^{0(1)} - \frac{2}{\mathcal{H}}\delta n^{i(1)}_{\perp}\Delta \ln a^{(1)} + \delta x^{i(2)}_{\perp}\ . \tag{72}$$

We define the parallel and perpendicular parts of the tetrad in the comoving frame[5]:

$$E^i_{\hat\alpha} = n^i E^{\|}_{\hat\alpha} + E^{\perp i}_{\hat\alpha}\ , \qquad \text{where} \qquad E^{\|}_{\hat\alpha} = n_i E^i_{\hat\alpha} \qquad \text{and} \qquad E^{\perp i}_{\hat\alpha} = \mathcal{P}^i_j E^j_{\hat\alpha}\ , \tag{73}$$

$$E_{\hat\alpha i} = n_i E_{\hat\alpha \|} + E_{\hat\alpha \perp i}\ , \qquad \text{where} \qquad E_{\hat\alpha \|} = n^i E_{\hat\alpha i} \qquad \text{and} \qquad E_{\hat\alpha \perp i} = \mathcal{P}^j_i E_{\hat\alpha j}\ . \tag{74}$$

Taking into account Eqs. (64), (65), (66) and (67) and observing that $\epsilon_{ijr}\epsilon^{pqr} = (\delta^p_i\delta^q_j - \delta^p_j\delta^q_i)$, we obtain the observed fractional number over-density

$$\Delta_g = \frac{n_g(\bar x^0, \bar{\mathbf{x}}) - \bar n_g(\bar x^0)}{\bar n_g(\bar x^0)} = \Delta^{(1)}_g + \frac{1}{2}\Delta^{(2)}_g\ , \tag{75}$$

[5] Note that in general $E^i_{\hat\alpha}$ is not a 3-space tensor in the index i, so that $E^{\|}_{\hat 0} \neq E_{\hat 0 \|}$ and $E^{\perp i}_{\hat 0} \neq \delta^{ij} E_{\hat 0 \perp j}$.

where

$$\begin{aligned}
\Delta_g^{(1)} &= \delta_g^{(1)} + \frac{1}{2}\hat{g}_\mu^{\mu(1)} + b_e\,\Delta\ln a^{(1)} + \partial_\parallel \Delta x_\parallel^{(1)} + \frac{2}{\bar\chi}\Delta x_\parallel^{(1)} - 2\kappa^{(1)} + E_{\hat 0}^{0(1)} + E_{\hat 0}^{\parallel(1)}\,, \qquad (76)\\
\Delta_g^{(2)} &= \delta_g^{(2)} + \frac{1}{2}\hat{g}_\mu^{\mu(2)} + b_e\,\Delta\ln a^{(2)} + \partial_\parallel \Delta x_\parallel^{(2)} + \frac{2}{\bar\chi}\Delta x_\parallel^{(2)} - 2\kappa^{(2)} + E_{\hat 0}^{0(2)} + E_{\hat 0}^{\parallel(2)}\\
&+ \frac{1}{4}\hat{g}_\mu^{\mu(1)}\,\hat{g}_\nu^{\nu(1)} - \frac{1}{2}\hat{g}_\mu^{\nu(1)}\,\hat{g}_\nu^{\mu(1)} + \frac{1}{\mathcal{H}}\hat{g}_\mu^{\mu(1)\prime}\Delta\ln a^{(1)} + \left(\partial_\parallel \hat{g}_\mu^{\mu(1)}\right)\Delta x_\parallel^{(1)} + \left(\partial_{\perp i}\hat{g}_\mu^{\mu(1)}\right)\Delta x_\perp^{i(1)}\\
&+ \left(-b_e + b_e^2 + \frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln\bar a}\right)\left(\Delta\ln a^{(1)}\right)^2 + 2b_e\,\Delta\ln a^{(1)}\delta_g^{(1)} + \frac{2}{\mathcal{H}}\delta_g^{(1)\prime}\Delta\ln a^{(1)} + 2\partial_\parallel\delta_g^{(1)}\Delta x_\parallel^{(1)}\\
&+ 2\partial_\perp^i\delta_g^{(1)}\Delta x_{\perp i}^{(1)} + 4\left(\kappa^{(1)}\right)^2 + \frac{2}{\bar\chi^2}\left(\Delta x_\parallel^{(1)}\right)^2 - 4\kappa^{(1)}\partial_\parallel\Delta x_\parallel^{(1)} - \frac{4}{\bar\chi}\kappa^{(1)}\Delta x_\parallel^{(1)} + \frac{4}{\bar\chi}\Delta x_\parallel^{(1)}\partial_\parallel\Delta x_\parallel^{(1)}\\
&- \left(\partial_{\perp j}\Delta x_\perp^{i(1)}\right)\left(\partial_{\perp i}\Delta x_\perp^{j(1)}\right) + \frac{2}{\bar\chi}\Delta x_{\perp i}^{(1)}\partial_\parallel\Delta x_\perp^{i(1)} - 2\left(\partial_\parallel\Delta x_\perp^{i(1)}\right)\left(\partial_{\perp i}\Delta x_\parallel^{(1)}\right) + \frac{2}{\mathcal{H}}E_{\hat 0}^{0(1)\prime}\Delta\ln a^{(1)}\\
&+ 2\partial_\parallel E_{\hat 0}^{0(1)}\Delta x_\parallel^{(1)} + 2\partial_{\perp i}E_{\hat 0}^{0(1)}\Delta x_\perp^{i(1)} + 2E_{\hat 0}^{0(1)}\partial_\parallel\Delta x_\parallel^{(1)} - 4E_{\hat 0}^{0(1)}\kappa^{(1)} + \frac{4}{\bar\chi}E_{\hat 0}^{0(1)}\Delta x_\parallel^{(1)} + \frac{2}{\mathcal{H}}E_{\hat 0}^{\parallel(1)\prime}\Delta\ln a^{(1)}\\
&+ 2\partial_\parallel E_{\hat 0}^{\parallel(1)}\Delta x_\parallel^{(1)} + 2\partial_{\perp i}E_{\hat 0}^{\parallel(1)}\Delta x_\perp^{i(1)} - 2E_{\hat 0}^{\parallel(1)}\partial_\parallel\Delta x^{0(1)} - 2E_{\hat 0}^{\perp i(1)}\partial_{\perp i}\Delta x^{0(1)} - 4E_{\hat 0}^{\parallel(1)}\kappa^{(1)} + \frac{4}{\bar\chi}E_{\hat 0}^{\parallel(1)}\Delta x_\parallel^{(1)}\\
&- 2E_{\hat 0}^{\perp i(1)}\partial_{\perp i}\Delta x_\parallel^{(1)} + b_e\,\hat{g}_\mu^{\mu(1)}\Delta\ln a^{(1)} + \hat{g}_\mu^{\mu(1)}\delta_g^{(1)} + 2\left(\frac{1}{2}\hat{g}_\mu^{\mu(1)} + b_e\Delta\ln a^{(1)} + \delta_g^{(1)}\right)\\
&\times \left(\partial_\parallel\Delta x_\parallel^{(1)} - 2\kappa^{(1)} + \frac{2}{\bar\chi}\Delta x_\parallel^{(1)} + E_{\hat 0}^{0(1)} + E_{\hat 0}^{\parallel(1)}\right)\\
&= \delta_g^{(2)} + \frac{1}{2}\hat{g}_\mu^{\mu(2)} + b_e\,\Delta\ln a^{(2)} + \partial_\parallel\Delta x_\parallel^{(2)} + \frac{2}{\bar\chi}\Delta x_\parallel^{(2)} - 2\kappa^{(2)} + E_{\hat 0}^{0(2)} + E_{\hat 0}^{\parallel(2)} + \left(\Delta_g^{(1)}\right)^2\\
&- \frac{1}{2}\hat{g}_\mu^{\nu(1)}\,\hat{g}_\nu^{\mu(1)} + \frac{1}{\mathcal{H}}\hat{g}_\mu^{\mu(1)\prime}\Delta\ln a^{(1)} + \left(\partial_\parallel\hat{g}_\mu^{\mu(1)}\right)\Delta x_\parallel^{(1)} + \left(\partial_{\perp i}\hat{g}_\mu^{\mu(1)}\right)\Delta x_\perp^{i(1)} + \frac{2}{\mathcal{H}}\delta_g^{(1)\prime}\Delta\ln a^{(1)}\\
&+ 2\partial_\parallel\delta_g^{(1)}\Delta x_\parallel^{(1)} + 2\partial_\perp^i\delta_g^{(1)}\Delta x_{\perp i}^{(1)} + \left(-b_e + \frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln\bar a}\right)\left(\Delta\ln a^{(1)}\right)^2 - \frac{2}{\bar\chi^2}\left(\Delta x_\parallel^{(1)}\right)^2 + \frac{4}{\bar\chi}\kappa^{(1)}\Delta x_\parallel^{(1)}\\
&- \left(\partial_{\perp j}\Delta x_\perp^{i(1)}\right)\left(\partial_{\perp i}\Delta x_\perp^{j(1)}\right) + \frac{2}{\bar\chi}\Delta x_{\perp i}^{(1)}\partial_\parallel\Delta x_\perp^{i(1)} - 2\left(\partial_\parallel\Delta x_\perp^{i(1)}\right)\left(\partial_{\perp i}\Delta x_\parallel^{(1)}\right) + \frac{2}{\mathcal{H}}E_{\hat 0}^{0(1)\prime}\Delta\ln a^{(1)}\\
&+ 2\partial_\parallel E_{\hat 0}^{0(1)}\Delta x_\parallel^{(1)} + \frac{2}{\mathcal{H}}E_{\hat 0}^{\parallel(1)\prime}\Delta\ln a^{(1)} + 2\partial_\parallel E_{\hat 0}^{\parallel(1)}\Delta x_\parallel^{(1)} + 2\partial_{\perp i}\left(E_{\hat 0}^{0(1)} + E_{\hat 0}^{\parallel(1)}\right)\Delta x_\perp^{i(1)} - 2\left(\delta n_\parallel^{(1)} + \delta\nu^{(1)}\right)E_{\hat 0}^{\parallel(1)}\\
&- 2E_{\hat 0}^{\perp i(1)}\partial_{\perp i}\left(\Delta x^{0(1)} + \Delta x_\parallel^{(1)}\right) - \left(\delta_g^{(1)}\right)^2 - \left(\partial_\parallel\Delta x_\parallel^{(1)}\right)^2 - \left(E_{\hat 0}^{0(1)} + E_{\hat 0}^{\parallel(1)}\right)^2\,. \qquad (77)
\end{aligned}$$

Here

$$b_e = \frac{\mathrm{d}\ln\left(\bar a^3\bar n_g\right)}{\mathrm{d}\ln\bar a} \qquad (78)$$

is the evolution bias term related to the comoving number density [5]. Note that

$$\partial_\parallel\Delta x^{\mu(n)}(\bar\chi,\mathbf{n}) = \partial_{\bar\chi}\Delta x^{\mu(n)}\,, \qquad (79)$$

where $\partial_{\bar\chi}$ is applied to all terms that are functions of $\bar x^0 = \eta(\bar\chi)$ and/ or $\bar x^i = \bar x^i(\bar\chi)$.

E. Weak lensing terms

In Eq. (77), we introduced the coordinate weak lensing convergence terms at order n:

$$\kappa^{(n)} = -\frac{1}{2}\partial_{\perp i}\Delta x_\perp^{i(n)}. \qquad (80)$$

Then the second-order transverse part of the volume distortion, which appears in Eq. (77), is

$$-\kappa^{(2)} + 2\left(\kappa^{(1)}\right)^2 - \frac{1}{2}\left(\partial_{\perp j}\Delta x_\perp^{i(1)}\right)\left(\partial_{\perp i}\Delta x_\perp^{j(1)}\right)\,. \qquad (81)$$

The coordinate weak lensing shear γ and rotation ϑ do not contribute to the observed number counts at first order but quadratic products do contribute at second order, via the third term above. They are defined by splitting $\partial_{\perp i}\Delta x^{(1)}_{\perp j}$ into its trace, tracefree and antisymmetric parts:

$$\partial_{\perp i}\Delta x^{(1)}_{\perp j} = -\gamma^{(1)}_{ij} - \mathcal{P}_{ij}\kappa^{(1)} - \vartheta^{(1)}_{ij} \ , \tag{82}$$

where

$$\gamma^{(1)}_{ij} = -\partial_{\perp(i}\Delta x^{(1)}_{\perp j)} - \mathcal{P}_{ij}\kappa^{(1)}, \quad \vartheta^{(1)}_{ij} = -\partial_{\perp[i}\Delta x^{(1)}_{\perp j]}. \tag{83}$$

Then

$$\left(\partial_{\perp j}\Delta x^{i(1)}_{\perp}\right)\left(\partial_{\perp i}\Delta x^{j(1)}_{\perp}\right) = 2\left(\kappa^{(1)}\right)^2 + 2\left|\gamma^{(1)}\right|^2 - \vartheta^{(1)}_{ij}\vartheta^{ij(1)} \ , \tag{84}$$

where $2|\gamma^{(1)}|^2 = \gamma^{(1)}_{ij}\gamma^{ij(1)}$. Explicit expressions for $\gamma^{(1)}_{ij}$ and $\vartheta^{(1)}_{ij}\vartheta^{ij(1)}$ are given in Eqs. (A15) and (A16) in a general gauge.

III. PERTURBED FLAT ROBERTSON-WALKER BACKGROUND IN A GENERAL GAUGE

The results obtained in the previous section have not yet used the specific form of the metric. Here we assume a spatially flat RW background, perturbed in a general gauge to second order:

$$\mathrm{d}s^2 = a(\eta)^2\left[-\left(1+2A^{(1)}+A^{(2)}\right)\mathrm{d}\eta^2 - 2\left(B^{(1)}_i + \frac{1}{2}B^{(2)}_i\right)\mathrm{d}\eta\mathrm{d}x^i + \left(\delta_{ij} + h^{(1)}_{ij} + \frac{1}{2}h^{(2)}_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j\right] \ , \tag{85}$$

where $B^{i(n)} = \partial_i B^{(n)} + \hat{B}^{(n)}_i$ and $\hat{B}^{(n)}_i$ is a solenoidal vector, i.e. $\partial^i\hat{B}^{(n)}_i = 0$. The 3-tensor is $h^{(n)}_{ij} = 2D^{(n)}\delta_{ij} + F^{(n)}_{ij}$, where $F^{(n)}_{ij} = (\partial_i\partial_j - \delta_{ij}\nabla^2/3)F^{(n)} + \partial_i\hat{F}^{(n)}_j + \partial_j\hat{F}^{(n)}_i + \hat{h}^{(n)}_{ij}$. Here $D^{(n)}$ and $F^{(n)}$ are scalars and $\hat{F}^{(n)}_i$ is a solenoidal vector field, $\partial^i\hat{h}^{(n)}_{ij} = \hat{h}^{i(n)}_i = 0$.

The geodesic equation for k^μ is

$$\frac{\mathrm{d}k^\mu}{\mathrm{d}\bar{\chi}} + \hat{\Gamma}^\mu_{\alpha\beta}k^\alpha k^\beta = 0 \tag{86}$$

where $\hat{\Gamma}^\mu_{\alpha\beta}$ are the Christoffel symbols defined using the comoving metric $\hat{g}_{\mu\nu}$. At zeroth order, we obtain Eq. (10). At first order, Eq. (86) yields

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(\delta\nu^{(1)} - 2A^{(1)} + B^{(1)}_\parallel\right) = A^{(1)\prime} - B^{(1)\prime}_\parallel - \frac{1}{2}h^{(1)\prime}_\parallel \ , \tag{87}$$

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(\delta n^{i(1)} + B^{i(1)} + h^{i(1)}_j n^j\right) = -\partial^i A^{(1)} + \partial^i B^{(1)}_\parallel - \frac{1}{\bar{\chi}}B^{i(1)}_\perp + \frac{1}{2}\partial^i h^{(1)}_\parallel - \frac{1}{\bar{\chi}}\mathcal{P}^{ij}h^{(1)}_{jk}n^k \ , \tag{88}$$

in agreement with [7]. At second order, we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left[\delta\nu^{(2)} - 2A^{(2)} + B^{(2)}_\parallel + 4A^{(1)}\delta\nu^{(1)} + 2B^{(1)}_i\delta n^{i(1)}\right] = A^{(2)\prime} - B^{(2)\prime}_\parallel - \frac{1}{2}h^{(2)\prime}_\parallel \\ &+ 2\delta n^{i(1)}\left[\partial_i\left(2A^{(1)} - B^{(1)}_\parallel\right) - \left(B^{(1)\prime}_i + n^j h^{(1)\prime}_{ij}\right) + \frac{1}{\bar{\chi}}B^{(1)}_{\perp i}\right] \ ,\end{aligned} \tag{89}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left[\delta n^{i(2)} + B^{i(2)} + h^{i(2)}_j n^j - 2\delta\nu^{(1)}B^{i(1)} + 2\delta n^{j(1)}h^{i(1)}_j\right] = -\partial^i A^{(2)} + \partial^i B^{(2)}_\parallel - \frac{1}{\bar{\chi}}B^{i(2)}_\perp + \frac{1}{2}\partial^i h^{(2)}_\parallel - \frac{1}{\bar{\chi}}\mathcal{P}^{ij}h^{(2)}_{jk}n^k \\ &+ 2\delta\nu^{(1)}\left[2\partial^i A^{(1)} - B^{i(1)\prime} - \partial^i B^{(1)}_\parallel + \frac{1}{\bar{\chi}}B^{i(1)}_\perp - n^j h^{i(1)\prime}_j\right] \\ &+ 2\delta n^{j(1)}\left[\partial^i B^{(1)}_j - \partial_j B^{i(1)} + n^k\partial^i h^{(1)}_{jk} - n^k\partial_j h^{i(1)}_k\right].\end{aligned} \tag{90}$$

To solve Eqs. (87), (88), (89) and (90) we require the values of $\delta\nu^{(1)}$, $\delta\nu^{(2)}$, $\delta n^{i(1)}$ and $\delta n^{i(2)}$ today. In this case we need all the components of the tetrads $\Lambda_\mu^{\hat{\alpha}}$ and $E_\mu^{\hat{\alpha}}$ (see Appendix A, Eq. (A10)) which are defined through the following relations

$$\begin{aligned} g^{\mu\nu}\Lambda_\mu^{\hat{\alpha}}\Lambda_\nu^{\hat{\beta}} &= \eta^{\hat{\alpha}\hat{\beta}} , \qquad \eta_{\hat{\alpha}\hat{\beta}}\Lambda_\mu^{\hat{\alpha}}\Lambda_\nu^{\hat{\beta}} = g_{\mu\nu} , \qquad g^{\mu\nu}\Lambda_\nu^{\hat{\beta}} = \Lambda^{\hat{\beta}\mu} , \qquad \eta_{\hat{\alpha}\hat{\beta}}\Lambda_\nu^{\hat{\beta}} = \Lambda_{\hat{\beta}\nu} , \\ \hat{g}^{\mu\nu}E_\mu^{\hat{\alpha}}E_\nu^{\hat{\beta}} &= \eta^{\hat{\alpha}\hat{\beta}} , \qquad \eta_{\hat{\alpha}\hat{\beta}}E_\mu^{\hat{\alpha}}E_\nu^{\hat{\beta}} = \hat{g}_{\mu\nu} , \qquad \hat{g}^{\mu\nu}E_\nu^{\hat{\beta}} = E^{\hat{\beta}\mu} , \qquad \eta_{\hat{\alpha}\hat{\beta}}E_\nu^{\hat{\beta}} = E_{\hat{\beta}\nu} , \end{aligned} \tag{91}$$

where $\eta_{\hat{\alpha}\hat{\beta}}$ the comoving Minkowski metric. Using Eq. (28) we have, at first order,

$$\delta\nu_o^{(1)} = A_o^{(1)} + v_{\|o}^{(1)} - B_{\|o}^{(1)} , \tag{92}$$

$$\delta n_o^{\hat{a}(1)} = -v_o^{\hat{a}(1)} - \frac{1}{2}n^i h_{i\,o}^{\hat{a}(1)} , \tag{93}$$

and, at second order,

$$\begin{aligned} \delta\nu_o^{(2)} &= A_o^{(2)} + v_{\|o}^{(2)} - B_{\|o}^{(2)} - 3\left(A_o^{(1)}\right)^2 - 2v_{\|o}^{(1)}A_o^{(1)} + 4B_{\|o}^{(1)}A_o^{(1)} - v_{k\,o}^{(1)}v_o^{k(1)} + n^i h_{ik\,o}^{(1)}v_o^{k(1)} \\ &+ n^i h_{ik\,o}^{(1)}B_o^{k(1)} + 2B_{k\,o}^{(1)}v_o^{k(1)} , \end{aligned} \tag{94}$$

$$\delta n_o^{\hat{a}(2)} = -v_o^{\hat{a}(2)} - \frac{1}{2}n^i h_{i\,o}^{\hat{a}(2)} + v_o^{\hat{a}(1)}v_{\|\,o}^{(1)} - v_o^{\hat{a}(1)}B_{\|\,o}^{(1)} + B_o^{\hat{a}(1)}v_{\|\,o}^{(1)} - B_o^{\hat{a}(1)}B_{\|\,o}^{(1)} + \frac{3}{4}n^i h_{i\,o}^{k(1)}h_{k\,o}^{\hat{a}(1)} . \tag{95}$$

From Eqs. (87), (88) and the constraint from Eq. (92), we obtain at first order

$$\begin{aligned} \delta\nu^{(1)} &= -\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right) + 2A^{(1)} - B_\|^{(1)} + \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right) \\ &= -\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right) + 2A^{(1)} - B_\|^{(1)} - 2I^{(1)} , \end{aligned} \tag{96}$$

$$\begin{aligned} \delta n^{i(1)} &= +B_o^{i(1)} - v_o^{i(1)} + \frac{1}{2}n^j h_{j\,o}^{i(1)} - B^{i(1)} - n^j h_j^{i(1)} \\ &\quad - \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left[\tilde{\partial}^i\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right) + \frac{1}{\tilde{\chi}}\left(B_\perp^{i(1)} + \mathcal{P}^{ij}h_{jk}^{(1)}n^k\right)\right] \\ &= n^i\delta n_\|^{(1)} + \delta n_\perp^{i(1)} , \end{aligned} \tag{97}$$

where

$$\delta n_\|^{(1)} = A_o^{(1)} - v_{\|\,o}^{(1)} - A^{(1)} - \frac{1}{2}h_\|^{(1)} + 2I^{(1)} , \tag{98}$$

$$\delta n_\perp^{i(1)} = B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i - \left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + 2S_\perp^{i(1)} . \tag{99}$$

Here we have defined

$$I^{(n)} = -\frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left(A^{(n)\prime} - B_\|^{(n)\prime} - \frac{1}{2}h_\|^{(n)\prime}\right) , \tag{100}$$

$$S^{i(n)} = -\frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left[\tilde{\partial}^i\left(A^{(n)} - B_\|^{(n)} - \frac{1}{2}h_\|^{(n)}\right) + \frac{1}{\tilde{\chi}}\left(B^{i(n)} + n^k h_k^{i(n)}\right)\right] , \tag{101}$$

where $I^{(n)}$ is the ISW term at order n, $\tilde{\partial}_i = \partial/\partial\tilde{x}^i$ and

$$S_\perp^{i(n)} = -\frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left[\tilde{\partial}_\perp^i\left(A^{(n)} - B_\|^{(n)} - \frac{1}{2}h_\|^{(n)}\right) + \frac{1}{\tilde{\chi}}\left(B_\perp^{i(n)} + n^k h_{kj}^{(n)}\mathcal{P}^{ij}\right)\right] , \tag{102}$$

$$S_\|^{(n)} = \frac{1}{2}\left(A_o^{(n)} - B_{\|\,o}^{(n)} - \frac{1}{2}h_{\|\,o}^{(n)}\right) - \frac{1}{2}\left(A^{(n)} - B_\|^{(n)} - \frac{1}{2}h_\|^{(n)}\right) + I^{(n)} - \frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\frac{1}{\tilde{\chi}}\left(B_\|^{(n)} + h_\|^{(n)}\right) . \tag{103}$$

Note the following useful relation

$$\delta n_\|^{(1)} + \delta\nu^{(1)} = A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)} . \tag{104}$$

At second order we find

$$\begin{aligned}\delta\nu^{(2)} \;=\;& -A_o^{(2)} + v_{\|\,o}^{(2)} + \left(A_o^{(1)}\right)^2 - 2A_o^{(1)}B_{\|\,o}^{(1)} + \left(B_{\|\,o}^{(1)}\right)^2 + 6A_o^{(1)}v_{\|\,o}^{(1)} - 2B_{\|\,o}^{(1)}v_{\|\,o}^{(1)} - v_{k\,o}^{(1)}v_o^{k(1)} \\
& + n^i h_{ik\,o}^{(1)}v_o^{k(1)} + 4\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right)\left(2A^{(1)} - B_\|^{(1)} - 2I^{(1)}\right) - 2\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right)B_{\perp i}^{(1)} \\
& + 2\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\tilde\partial_{\perp i}\left(2A^{(1)} - B_\|^{(1)}\right) - \left(B_{\perp i}^{(1)\prime} + n^j h_{jk}^{(1)\prime}\mathcal{P}_i^k\right) + \frac{1}{\tilde\chi}B_{\perp i}^{(1)}\right] \\
& + \left(2A^{(2)} - B_\|^{(2)}\right) - 3\left(2A^{(1)} - B_\|^{(1)}\right)^2 - 2B_\|^{(1)}\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right) + 2B_{\perp i}^{(1)}\left(B_\perp^{i(1)} + \mathcal{P}_j^i h_k^{j(1)}n^k\right) \\
& + 8\left(2A^{(1)} - B_\|^{(1)}\right)I^{(1)} - 4B_{\perp i}^{(1)}S_\perp^{i(1)} - 2I^{(2)} + 2\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left\{\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(2A^{(1)} - B_\|^{(1)}\right)\right. \\
& + \left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\left[-\left(B_\|^{(1)} + h_\|^{(1)}\right) + 4I^{(1)}\right] \\
& \left. + \left[-\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right) + 2S_\perp^{i(1)}\right]\left[\tilde\partial_{\perp i}\left(2A^{(1)} - B_\|^{(1)}\right) - \left(B_{\perp i}^{(1)\prime} + n^j h_{jk}^{(1)\prime}\mathcal{P}_i^k\right) + \frac{1}{\tilde\chi}B_{\perp i}^{(1)}\right]\right\}. \qquad (105)\end{aligned}$$

Splitting $\delta n^{i(2)} = n^i\delta n_\|^{(2)} + \delta n_\perp^{i(2)}$, we obtain

$$\begin{aligned}\delta n_\|^{(2)} \;=\;& +A_o^{(2)} - v_{\|\,o}^{(2)} + \left(v_{\|\,o}^{(1)}\right)^2 - \frac{1}{4}\left(h_{\|\,o}^{(1)}\right)^2 - \frac{1}{4}n^i h_{ij\,o}^{(1)}\mathcal{P}_k^j\,h_{p\,o}^{k(1)}n^p - 4A_o^{(1)}v_{\|\,o}^{(1)} - 2v_{\|\,o}^{(1)}h_{\|\,o}^{(1)} - 2n^i h_{ik\,o}^{(1)}\mathcal{P}_j^k v_o^{j(1)} \\
& - 2\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right)\left(2A^{(1)} + h_\|^{(1)}\right) - 2\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right)h_i^{p(1)}n_p + 8\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right)I^{(1)} \\
& + 2\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\partial_\|\left(B_{\perp i}^{(1)} + n^j h_{jk}^{(1)}\mathcal{P}_i^k\right) - \partial_{\perp i}\left(B_\|^{(1)} + h_\|^{(1)}\right)\right. \\
& \left. + \frac{1}{\tilde\chi}\left(B_{\perp i}^{(1)} + 2n^j h_{kj}^{(1)}\mathcal{P}_i^k\right)\right] - A^{(2)} - \frac{1}{2}h_\|^{(2)} - 2h_\|^{(1)}\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right) + \left(2A^{(1)} - B_\|^{(1)}\right)^2 \\
& + 2\left(2A^{(1)} - B_\|^{(1)}\right)\left(B_\|^{(1)} + h_\|^{(1)}\right) + 2\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right)h_i^{p(1)}n_p - 4\left(2A^{(1)} + h_\|^{(1)}\right)I^{(1)} - 4n_j h_i^{j(1)}S_\perp^{i(1)} \\
& + 2I^{(2)} + 2\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left\{\left(2A^{(1)} - B_\|^{(1)} - 4I^{(1)}\right)\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right) + \left[-\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right) + 2S_\perp^{i(1)}\right]\right. \\
& \left. \times\left[\tilde\partial_\|\left(B_{\perp i}^{(1)} + n^j h_{jk}^{(1)}\mathcal{P}_i^k\right) - \tilde\partial_{\perp i}\left(B_\|^{(1)} + h_\|^{(1)}\right) + \frac{1}{\tilde\chi}\left(B_{\perp i}^{(1)} + 2n^j h_j^{k(1)}\mathcal{P}_{ik}\right)\right]\right\}\;, \qquad (106)\end{aligned}$$

and[6]

$$
\begin{aligned}
\delta n_\perp^{i(2)} \;=\;& +B_{\perp o}^{i(2)} - v_{\perp o}^{i(2)} + \frac{1}{2} n^j h_{jk\,o}^{(2)} \mathcal{P}^{ki} + v_{\| o}^{(1)} v_{\perp o}^{i(1)} - 3 v_{\| o}^{(1)} B_{\perp o}^{i(1)} + B_{\| o}^{(1)} B_{\perp o}^{i(1)} - B_{\| o}^{(1)} v_{\perp o}^{i(1)} + 2A_o^{(1)} n^j h_{jk\,o}^{(1)} \mathcal{P}^{ki} \\
& -4 v_{\| o}^{(1)} n^j h_{jk\,o}^{(1)} \mathcal{P}^{ki} - 2 v_{\perp o}^{j(1)} \mathcal{P}^l_j\, h^{k(1)}_{l\,o} \mathcal{P}^i_k - \frac{1}{4} h_{\| o}^{(1)} n^j h_{jk\,o}^{(1)} \mathcal{P}^{ki} - \frac{1}{4} n^j h^{k(1)}_{j\,o} \mathcal{P}^l_k\, h^{p(1)}_{l\,o} \mathcal{P}^i_p \\
& -4 \left(A_o^{(1)} - v_{\| o}^{(1)}\right) \left(B_\perp^{i(1)} + n^k h_k^{j(1)} \mathcal{P}^i_j - 2 S_\perp^{i(1)}\right) + 2 \left(B_{\perp o}^{j(1)} - v_{\perp o}^{j(1)} + \frac{1}{2} n^k h^{p(1)}_{k\,o} \mathcal{P}^j_p\right) \Bigg\{ - \mathcal{P}^l_j\, h^{k(1)}_l \mathcal{P}^i_k \\
& + 2 \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\tilde\partial_\perp^{[i} B_\perp^{j](1)} + \tilde\partial_\perp^{[i} \left(\mathcal{P}^{j]}_m h^{m(1)}_q n^q\right) - \frac{1}{\tilde\chi} \left(n^{[i} B_\perp^{j](1)} + n^{[i} \mathcal{P}^{j]}_m h^{m(1)}_q n^q\right)\right] \Bigg\} - B_\perp^{i(2)} - n^j h^{(2)}_{jk} \mathcal{P}^{ki} \\
& +4 A^{(1)} B_\perp^{i(1)} - 2 B_\|^{(1)} B_\perp^{i(1)} + 2 A^{(1)} n^j h^{(1)}_{jk} \mathcal{P}^{ki} + h_\|^{(1)} n^j h^{(1)}_{jk} \mathcal{P}^{ki} + 2 \left(B_\perp^{j(1)} + n^k h_k^{p(1)} \mathcal{P}^j_p\right) \mathcal{P}^l_j\, h^{k(1)}_l \mathcal{P}^i_k \\
& -8 \left(B_\perp^{i(1)} + n^j h^{(1)}_{jk} \mathcal{P}^{ik}\right) I^{(1)} - 4\, \mathcal{P}^l_j\, h^{k(1)}_l \mathcal{P}^i_k S_\perp^{j(1)} + 2 S_\perp^{i(2)} + 2 \int_0^{\bar\chi} \mathrm{d}\tilde\chi \Bigg\{ - \left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2} h_\|^{(1)\prime}\right) \\
& \times \left(B_\perp^{i(1)} + n^j h^{(1)}_{jk} \mathcal{P}^{ik}\right) + \left(2A^{(1)} - B_\|^{(1)}\right) \frac{\mathrm{d}}{\mathrm{d}\tilde\chi} \left(B_\perp^{i(1)} + n^j h^{(1)}_{jk} \mathcal{P}^{ik}\right) + 2 \left(2A^{(1)} - B_\|^{(1)} - 2 I^{(1)}\right) \\
& \times \left[\tilde\partial_\perp^i \left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right) + \frac{1}{\tilde\chi} \left(B_\perp^{i(1)} + n^k h^{(1)}_{kj} \mathcal{P}^{ij}\right)\right] + 2 \left[- \left(B_{\perp j}^{(1)} + n^p h^{(1)}_{pk} \mathcal{P}^k_j\right) + 2 \delta_{jp} S_\perp^{p(1)}\right] \\
& \times \left[\tilde\partial_\perp^{[i} B_\perp^{j](1)} + \tilde\partial_\perp^{[i} \left(\mathcal{P}^{j]}_m h^{m(1)}_q n^q\right) - \frac{1}{\tilde\chi} \left(n^{[i} B_\perp^{j](1)} + n^{[i} \mathcal{P}^{j]}_m h^{m(1)}_q n^q\right)\right] + \left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right) \\
& \times \left[\tilde\partial_\perp^i \left(B_\|^{(1)} + h_\|^{(1)}\right) - \partial_\| \left(B_\perp^{i(1)} + n^p h^{(1)}_{pq} \mathcal{P}^{iq}\right) - \frac{1}{\tilde\chi} \left(B_\perp^{i(1)} + 2 n^p h^{(1)}_{pq} \mathcal{P}^{iq}\right)\right] \Bigg\} \,.
\end{aligned}
\tag{107}
$$

Combining Eqs. (89) and (106) we obtain the useful relation

$$
\begin{aligned}
\delta\nu^{(2)} + \delta n_\|^{(2)} \;=\;& + \left(A_o^{(1)}\right)^2 - 2 A_o^{(1)} B_{\| o}^{(1)} + \left(B_{\| o}^{(1)}\right)^2 + 2 A_o^{(1)} v_{\| o}^{(1)} - 2 B_{\| o}^{(1)} v_{\| o}^{(1)} - v_{\| o}^{(1)} h_{\| o}^{(1)} - \frac{1}{4} \left(h_{\| o}^{(1)}\right)^2 - v_{\perp k\,o}^{(1)} v_{\perp o}^{k(1)} \\
& + n^i h^{(1)}_{ik\,o} \mathcal{P}^k_j v_o^{j(1)} - n^i h^{(1)}_{ik\,o} \mathcal{P}^k_j B_o^{j(1)} - \frac{1}{4} n^i h^{(1)}_{ij\,o} \mathcal{P}^j_k\, h^{k(1)}_{p\,o} n^p - B_{\perp o}^{i(1)} B_{\perp i\,o}^{(1)} + 2 v_{\perp o}^{i(1)} B_{\perp i\,o}^{(1)} \\
& +4 \left(A_o^{(1)} - v_{\| o}^{(1)}\right) \left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right) - 8 \left(B_{\perp o}^{i(1)} - v_{\perp o}^{i(1)} + \frac{1}{2} n^k h^{j(1)}_{k\,o} \mathcal{P}^i_j\right) \delta_{il} S_\perp^{l(1)} \\
& + A^{(2)} - B_\|^{(2)} - \frac{1}{2} h_\|^{(2)} - 2 \left(2A^{(1)} - B_\|^{(1)}\right)^2 + \left(B_\|^{(1)} + h_\|^{(1)}\right) \left(2A^{(1)} + h_\|^{(1)}\right) \\
& + \left(B_\perp^{i(1)} + n^k h_k^{j(1)} \mathcal{P}^i_j\right) \left(B_{\perp i}^{(1)} + n^p h^{(1)}_{pm} \mathcal{P}^m_i\right) + 8 \left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right) I^{(1)} - 8 S_\perp^{i(1)} S_\perp^{j(1)} \delta_{ij} \\
& +2 \int_0^{\bar\chi} \mathrm{d}\tilde\chi \Bigg\{ \left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right) \left[2 \left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2} h_\|^{(1)\prime}\right) + \frac{\mathrm{d}}{\mathrm{d}\tilde\chi} \left(2A^{(1)} - B_\|^{(1)}\right)\right] \\
& - \left(B_\perp^{i(1)} + n^k h_k^{j(1)} \mathcal{P}^i_j\right) \left[\tilde\partial_{\perp i} \left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right) + \frac{1}{\tilde\chi} \left(B_{\perp i}^{(1)} + n^m h^{(1)}_{mp} \mathcal{P}^p_i\right)\right] \Bigg\} \,.
\end{aligned}
\tag{108}
$$

---

[6] From Eq. (66),

$$
\mathcal{P}^i_k \mathcal{P}^{jl} [\partial^k B_l^{(1)} - \partial_l B^{k(1)} + n^m \partial^k h^{(1)}_{lm} - n^m \partial_l h^{k(1)}_m] = 2 [\partial_\perp^{[i} B_\perp^{j](1)} + \partial_\perp^{[i} (\mathcal{P}^{j]}_m h^{m(1)}_q n^q) - \tilde\chi^{-1} (n^{[i} B_\perp^{j](1)} + n^{[i} \mathcal{P}^{j]}_m h^{m(1)}_q n^q)].
$$

From Eqs. (16), (17) and (104), we find to first order that

$$\delta x^{0(1)} = -\bar\chi\left(A^{(1)}_o - v^{(1)}_{\parallel o}\right) + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[2A^{(1)} - B^{(1)}_\parallel + (\bar\chi-\tilde\chi)\left(A^{(1)\prime} - B^{(1)\prime}_\parallel - \frac{1}{2}h^{(1)\prime}_\parallel\right)\right] \quad (109)$$

$$\delta x^{(1)}_\parallel = \bar\chi\left(A^{(1)}_o - v^{(1)}_{\parallel o}\right) - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\left(A^{(1)} + \frac{1}{2}h^{(1)}_\parallel\right) + (\bar\chi-\tilde\chi)\left(A^{(1)\prime} - B^{(1)\prime}_\parallel - \frac{1}{2}h^{(1)\prime}_\parallel\right)\right], \quad (110)$$

$$\delta x^{i(1)}_\perp = \bar\chi\left(B^{i(1)}_{\perp o} - v^{i(1)}_{\perp o} + \frac{1}{2}n^k h^{j(1)}_{k\,o}\mathcal{P}^i_j\right) - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left\{\left(B^{i(1)}_\perp + n^k h^{j(1)}_k \mathcal{P}^i_j\right)\right.$$
$$\left. + (\bar\chi-\tilde\chi)\left[\tilde\partial^i_\perp\left(A^{(1)} - B^{(1)}_\parallel - \frac{1}{2}h^{(1)}_\parallel\right) + \frac{1}{\tilde\chi}\left(B^{i(1)}_\perp + n^k h^{(1)}_{kj}\mathcal{P}^{ij}\right)\right]\right\}, \quad (111)$$

$$\delta x^{0(1)} + \delta x^{(1)}_\parallel = \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(A^{(1)} - B^{(1)}_\parallel - \frac{1}{2}h^{(1)}_\parallel\right) = -T^{(1)}, \quad (112)$$

where we defined

$$T^{(n)} = -\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(A^{(n)} - B^{(n)}_\parallel - \frac{1}{2}h^{(n)}_\parallel\right) \quad (113)$$

which is an integrated radial displacement corresponding to a time delay term at order n, which generalizes the usual (Shapiro) time delay term $T^{(1)}$ [4].

At second order,

$$\delta x^{0(2)} = \bar\chi\left[-A^{(2)}_o + v^{(2)}_{\parallel o} + \left(A^{(1)}_o\right)^2 - 2A^{(1)}_o B^{(1)}_{\parallel o} + \left(B^{(1)}_{\parallel o}\right)^2 + 6A^{(1)}_o v^{(1)}_{\parallel o} - 2B^{(1)}_{\parallel o} v^{(1)}_{\parallel o} - v^{(1)}_{k\,o} v^{k(1)}_o\right.$$
$$\left. + n^i h^{(1)}_{ik\,o} v^{k(1)}_o\right] + 4\left(A^{(1)}_o - v^{(1)}_{\parallel o}\right)\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\left(2A^{(1)} - B^{(1)}_\parallel\right) + (\bar\chi-\tilde\chi)\left(A^{(1)\prime} - B^{(1)\prime}_\parallel - \frac{1}{2}h^{(1)\prime}_\parallel\right)\right]$$
$$+2\left(B^{i(1)}_{\perp o} - v^{i(1)}_{\perp o} + \frac{1}{2}n^k h^{j(1)}_{k\,o}\mathcal{P}^i_j\right)\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left\{-B^{(1)}_{\perp i} + (\bar\chi-\tilde\chi)\left[\tilde\partial_{\perp i}\left(2A^{(1)} - B^{(1)}_\parallel\right) - \left(B^{(1)\prime}_{\perp i} + n^j h^{(1)\prime}_{jk}\mathcal{P}^k_i\right)\right.\right.$$
$$\left.\left. + \frac{1}{\tilde\chi}B^{(1)}_{\perp i}\right]\right\} + \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[2A^{(2)} - B^{(2)}_\parallel - 12\left(A^{(1)}\right)^2 + 10A^{(1)}B^{(1)}_\parallel - \left(B^{(1)}_\parallel\right)^2 + B^{(1)}_\parallel h^{(1)}_\parallel\right.$$
$$\left. + 2B^{(1)}_{\perp i}\left(B^{i(1)}_\perp + \mathcal{P}^i_j h^{j(1)}_k n^k\right) + 8\left(2A^{(1)} - B^{(1)}_\parallel\right) I^{(1)} - 4B^{(1)}_{\perp i} S^{i(1)}_\perp\right]$$
$$+ \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, (\bar\chi-\tilde\chi)\left\{A^{(2)\prime} - B^{(2)\prime}_\parallel - \frac{1}{2}h^{(2)\prime}_\parallel + 2\left(A^{(1)} - B^{(1)}_\parallel - \frac{1}{2}h^{(1)}_\parallel\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(2A^{(1)} - B^{(1)}_\parallel\right)\right.$$
$$+2\left(A^{(1)\prime} - B^{(1)\prime}_\parallel - \frac{1}{2}h^{(1)\prime}_\parallel\right)\left[-\left(B^{(1)}_\parallel + h^{(1)}_\parallel\right) + 4I^{(1)}\right] + 2\left[-\left(B^{i(1)}_\perp + n^j h^{(1)}_{jk}\mathcal{P}^{ik}\right) + 2S^{i(1)}_\perp\right]$$
$$\left. \times\left[\tilde\partial_{\perp i}\left(2A^{(1)} - B^{(1)}_\parallel\right) - \left(B^{(1)\prime}_{\perp i} + n^j h^{(1)\prime}_{jk}\mathcal{P}^k_i\right) + \frac{1}{\tilde\chi}B^{(1)}_{\perp i}\right]\right\}, \quad (114)$$

$$
\begin{aligned}
\delta x_{\|}^{(2)} \;=\; & \bar{\chi}\left[A_o^{(2)}-v_{\|\,o}^{(2)}+\left(v_{\|\,o}^{(1)}\right)^2-\frac{1}{4}\left(h_{\|\,o}^{(1)}\right)^2-\frac{1}{4}n^i h_{ij\,o}^{(1)}\mathcal{P}_k^j\, h_{p\,o}^{k(1)}n^p-4A_o^{(1)}v_{\|\,o}^{(1)}-2v_{\|\,o}^{(1)}h_{\|\,o}^{(1)}-2n^i h_{ik\,o}^{(1)}\mathcal{P}_j^k v_o^{j(1)}\right] \\
& -2\left(A_o^{(1)}-v_{\|\,o}^{(1)}\right)\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\left(2A^{(1)}+h_{\|}^{(1)}\right)+2\left(\bar{\chi}-\tilde{\chi}\right)\left(A^{(1)\prime}-B_{\|}^{(1)\prime}-\frac{1}{2}h_{\|}^{(1)\prime}\right)\right] \\
& +2\left(B_{\perp\,o}^{i(1)}-v_{\perp\,o}^{i(1)}+\frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right)\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left\{-h_i^{p(1)}n_p+\left(\bar{\chi}-\tilde{\chi}\right)\left[\tilde{\partial}_{\|}\left(B_{\perp i}^{(1)}+n^j h_{jk}^{(1)}\mathcal{P}_i^k\right)-\tilde{\partial}_{\perp i}\left(B_{\|}^{(1)}+h_{\|}^{(1)}\right)\right.\right. \\
& \left.\left.+\;\frac{1}{\tilde{\chi}}\left(B_{\perp i}^{(1)}+2n^j h_j^{k(1)}\mathcal{P}_k^i\right)\right]\right\}+\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left\{-A^{(2)}-\frac{1}{2}h_{\|}^{(2)}-2h_{\|}^{(1)}\left(A^{(1)}-B_{\|}^{(1)}-\frac{1}{2}h_{\|}^{(1)}\right)\right. \\
& +\left(2A^{(1)}-B_{\|}^{(1)}\right)^2+2\left(2A^{(1)}-B_{\|}^{(1)}\right)\left(B_{\|}^{(1)}+h_{\|}^{(1)}\right)+2\left(B_{\perp}^{i(1)}+n^k h_k^{j(1)}\mathcal{P}_j^i\right)h_i^{p(1)}n_p-4\left(2A^{(1)}+h_{\|}^{(1)}\right)I^{(1)} \\
& \left.-4n_j h_i^{j(1)}S_{\perp}^{i(1)}\right\}+\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left(\bar{\chi}-\tilde{\chi}\right)\left\{-\left(A^{(2)\prime}-B_{\|}^{(2)\prime}-\frac{1}{2}h_{\|}^{(2)\prime}\right)+2\left(2A^{(1)}-B_{\|}^{(1)}-4I^{(1)}\right)\right. \\
& \times\left(A^{(1)\prime}-B_{\|}^{(1)\prime}-\frac{1}{2}h_{\|}^{(1)\prime}\right)+2\left[-\left(B_{\perp}^{i(1)}+n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right)+2S_{\perp}^{i(1)}\right]\left[\tilde{\partial}_{\|}\left(B_{\perp i}^{(1)}+n^j h_{jk}^{(1)}\mathcal{P}_i^k\right)\right. \\
& \left.\left.-\;\tilde{\partial}_{\perp i}\left(B_{\|}^{(1)}+h_{\|}^{(1)}\right)+\frac{1}{\tilde{\chi}}\left(B_{\perp i}^{(1)}+2n^j h_j^{k(1)}\mathcal{P}_k^i\right)\right]\right\}\;, \qquad (115)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta x_{\perp}^{i(2)} \;=\; & \bar{\chi}\left[B_{\perp\,o}^{i(2)}-v_{\perp\,o}^{i(2)}+\frac{1}{2}n^j h_{jk\,o}^{(2)}\mathcal{P}^{ki}+v_{\|\,o}^{(1)}v_{\perp\,o}^{i(1)}-3v_{\|\,o}^{(1)}B_{\perp\,o}^{i(1)}+B_{\|\,o}^{(1)}B_{\perp\,o}^{i(1)}-B_{\|\,o}^{(1)}v_{\perp\,o}^{i(1)}+2A_o^{(1)}n^j h_{jk\,o}^{(1)}\mathcal{P}^{ki}\right. \\
& \left.-4v_{\|\,o}^{(1)}n^j h_{jk\,o}^{(1)}\mathcal{P}^{ki}-2v_{\perp o}^{j(1)}\mathcal{P}_j^l\, h_{l\,o}^{k(1)}\mathcal{P}_k^i-\frac{1}{4}h_{\|\,o}^{(1)}n^j h_{jk\,o}^{(1)}\mathcal{P}^{ki}-\frac{1}{4}n^j h_{j\,o}^{k(1)}\mathcal{P}_k^l\, h_{l\,o}^{p(1)}\mathcal{P}_p^i\right] \\
& -4\left(A_o^{(1)}-v_{\|\,o}^{(1)}\right)\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left\{\left(B_{\perp}^{i(1)}+n^k h_k^{j(1)}\mathcal{P}_j^i\right)+\left(\bar{\chi}-\tilde{\chi}\right)\left[\tilde{\partial}_{\perp}^i\left(A^{(1)}-B_{\|}^{(1)}-\frac{1}{2}h_{\|}^{(1)}\right)\right.\right. \\
& \left.\left.+\;\frac{1}{\tilde{\chi}}\left(B_{\perp}^{i(1)}+n^k h_{kj}^{(1)}\mathcal{P}^{ij}\right)\right]\right\}+2\left(B_{\perp\,o}^{j(1)}-v_{\perp\,o}^{j(1)}+\frac{1}{2}n^k h_{k\,o}^{l(1)}\mathcal{P}_l^j\right)\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left\{-\mathcal{P}_j^m\, h_m^{p(1)}\mathcal{P}_p^i\right. \\
& \left.+\;2\left(\bar{\chi}-\tilde{\chi}\right)\left[\tilde{\partial}_{\perp}^{[i}B_{\perp}^{j](1)}+\tilde{\partial}_{\perp}^{[i}\left(\mathcal{P}_m^{j]}h_q^{m(1)}n^q\right)-\frac{1}{\tilde{\chi}}\left(n^{[i}B_{\perp}^{j](1)}+n^{[i}\mathcal{P}_m^{j]}h_q^{m(1)}n^q\right)\right]\right\} \\
& +\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left\{-B_{\perp}^{i(2)}-n^j h_{jk}^{(2)}\mathcal{P}^{ki}+4A^{(1)}B_{\perp}^{i(1)}-2B_{\|}^{(1)}B_{\perp}^{i(1)}+2A^{(1)}n^j h_{jk}^{(1)}\mathcal{P}^{ki}+h_{\|}^{(1)}n^j h_{jk}^{(1)}\mathcal{P}^{ki}\right. \\
& \left.+2\left(B_{\perp}^{j(1)}+n^k h_k^{p(1)}\mathcal{P}_p^j\right)\mathcal{P}_j^l\, h_l^{k(1)}\mathcal{P}_k^{i(1)}-8\left(B_{\perp}^{i(1)}+n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right)I^{(1)}-4\,\mathcal{P}_j^l\, h_l^{k(1)}\mathcal{P}_k^i S_{\perp}^{j(1)}\right\} \\
& +\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left(\bar{\chi}-\tilde{\chi}\right)\left\{-\left[\tilde{\partial}_{\perp}^i\left(A^{(2)}-B_{\|}^{(2)}-\frac{1}{2}h_{\|}^{(2)}\right)+\frac{1}{\tilde{\chi}}\left(B_{\perp}^{i(2)}+n^k h_{kj}^{(2)}\mathcal{P}^{ij}\right)\right]-2\left(A^{(1)\prime}-B_{\|}^{(1)\prime}-\frac{1}{2}h_{\|}^{(1)\prime}\right)\right. \\
& \times\left(B_{\perp}^{i(1)}+n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right)+2\left(2A^{(1)}-B_{\|}^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde{\chi}}\left(B_{\perp}^{i(1)}+n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right)+4\left(2A^{(1)}-B_{\|}^{(1)}-2I^{(1)}\right) \\
& \times\left[\tilde{\partial}_{\perp}^i\left(A^{(1)}-B_{\|}^{(1)}-\frac{1}{2}h_{\|}^{(1)}\right)+\frac{1}{\tilde{\chi}}\left(B_{\perp}^{i(1)}+n^k h_{kj}^{(1)}\mathcal{P}^{ij}\right)\right]+4\left[-\left(B_{\perp j}^{(1)}+n^p h_{pk}^{(1)}\mathcal{P}_j^k\right)+2\delta_{jp}S_{\perp}^{p(1)}\right] \\
& \times\left[\tilde{\partial}_{\perp}^{[i}B_{\perp}^{j](1)}+\tilde{\partial}_{\perp}^{[i}\left(\mathcal{P}_m^{j]}h_q^{m(1)}n^q\right)-\frac{1}{\tilde{\chi}}\left(n^{[i}B_{\perp}^{j](1)}+n^{[i}\mathcal{P}_m^{j]}h_q^{m(1)}n^q\right)\right]+2\left(A^{(1)}-B_{\|}^{(1)}-\frac{1}{2}h_{\|}^{(1)}\right) \\
& \left.\times\left[\tilde{\partial}_{\perp}^i\left(B_{\|}^{(1)}+h_{\|}^{(1)}\right)-\partial_{\|}\left(B_{\perp}^{i(1)}+n^p h_{pq}^{(1)}\mathcal{P}^{iq}\right)-\frac{1}{\tilde{\chi}}\left(B_{\perp}^{i(1)}+2n^p h_{pq}^{(1)}\mathcal{P}^{iq}\right)\right]\right\}\;. \qquad (116)
\end{aligned}
$$

Combining Eqs. (114) and (115) [or integrating Eq. (108)], we have

$$\delta x^{0(2)} + \delta x_{\|}^{(2)} = \bar{\chi}\Big[\left(A_o^{(1)}\right)^2 - 2A_o^{(1)}B_{\|\,o}^{(1)} + \left(B_{\|\,o}^{(1)}\right)^2 + 2A_o^{(1)}v_{\|\,o}^{(1)} - 2B_{\|\,o}^{(1)}v_{\|\,o}^{(1)} - v_{\|\,o}^{(1)}h_{\|\,o}^{(1)} - \frac{1}{4}\left(h_{\|\,o}^{(1)}\right)^2 - v_{\perp k\,o}^{(1)}v_{\perp\,o}^{k(1)}$$
$$+n^i h_{ik\,o}^{(1)}\mathcal{P}_j^k v_o^{j(1)} - n^i h_{ik\,o}^{(1)}\mathcal{P}_j^k B_o^{j(1)} - \frac{1}{4}n^i h_{ij\,o}^{(1)}\mathcal{P}_k^j\, h_{p\,o}^{k(1)}n^p - B_{\perp\,o}^{i(1)}B_{\perp i\,o}^{(1)} + 2v_{\perp\,o}^{i(1)}B_{\perp i\,o}^{(1)}\Big]$$
$$-4\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right)T^{(1)} + 4\left(B_{\perp i\,o}^{(1)} - v_{\perp i\,o}^{(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_{ij}\right)$$
$$\times \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\left[\tilde{\partial}_\perp^i\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right) + \frac{1}{\tilde{\chi}}\left(B_\perp^{i(1)} + n^k h_{kj}^{(1)}\mathcal{P}^{ij}\right)\right]$$
$$-T^{(2)} + 2\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\Big[-\left(2A^{(1)} - B_\|^{(1)}\right)^2 + \frac{1}{2}\left(B_\|^{(1)} + h_\|^{(1)}\right)\left(2A^{(1)} + h_\|^{(1)}\right)$$
$$+\frac{1}{2}\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right)\left(B_{\perp i}^{(1)} + n^p h_{pm}^{(1)}\mathcal{P}_i^m\right) + 4\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right)I^{(1)} - 4S_\perp^{i(1)}S_\perp^{j(1)}\delta_{ij}\Big]$$
$$+2\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\Big\{\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right)\left[2\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right) + \frac{\mathrm{d}}{\mathrm{d}\tilde{\chi}}\left(2A^{(1)} - B_\|^{(1)}\right)\right]$$
$$-\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right)\left[\tilde{\partial}_{\perp i}\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right) + \frac{1}{\tilde{\chi}}\left(B_{\perp i}^{(1)} + n^m h_{mp}^{(1)}\mathcal{P}_i^p\right)\right]\Big\}\,. \tag{117}$$

To obtain all the second order terms we require $\Delta\ln a^{(1)}$ (or $\Delta x^{0(1)}$), $\delta\chi^{(1)}$, $\Delta x^{0(1)}$, $\Delta x_\|^{(1)}$, $\Delta x_\perp^{i(1)}$ and $\Delta_g^{(1)}$. From Eqs. (32) and (34) we have

$$\Delta\ln a^{(1)} = -E_{\hat{0}0}^{(1)} + E_{\hat{0}\|}^{(1)} - \delta\nu^{(1)} = \left(A_o^{(1)} - v_{\|\,o}^{(1)}\right) - A^{(1)} + v_\|^{(1)} + 2I^{(1)}$$
$$= +\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right) - A^{(1)} + v_\|^{(1)} - \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\,, \tag{118}$$
$$\delta\chi^{(1)} = \delta x^{0(1)} - \Delta x^{0(1)} = \delta x^{0(1)} - \frac{1}{\mathcal{H}}\Delta\ln a^{(1)} = -\left(\bar{\chi} + \frac{1}{\mathcal{H}}\right)\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right) + \frac{1}{\mathcal{H}}\left(A^{(1)} - v_\|^{(1)}\right)$$
$$+\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left[2A^{(1)} - B_\|^{(1)} + (\bar{\chi}-\tilde{\chi})\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\right] - \frac{2}{\mathcal{H}}I^{(1)}\,. \tag{119}$$

Then, from Eq. (36)

$$\Delta x^{0(1)} = \frac{1}{\mathcal{H}}\left[\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right) - A^{(1)} + v_\|^{(1)} + 2I^{(1)}\right]$$
$$= \frac{1}{\mathcal{H}}\left[\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right) - A^{(1)} + v_\|^{(1)} - \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\right]\,, \tag{120}$$

and from Eqs. (69) and (112)

$$\Delta x_\|^{(1)} = -T^{(1)} - \Delta x^{0(1)} = -T^{(1)} - \frac{1}{\mathcal{H}}\Delta\ln a^{(1)} = -T^{(1)} - \frac{1}{\mathcal{H}}\left[\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right) - A^{(1)} + v_\|^{(1)} + 2I^{(1)}\right]$$
$$= \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right) - \frac{1}{\mathcal{H}}\left[\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right) - A^{(1)} + v_\|^{(1)} - \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\right]. \tag{121}$$

Using Eq. (70), we have

$$\Delta x_\perp^{i(1)} = \bar{\chi}\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right) - \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\Big\{\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right)$$
$$+(\bar{\chi}-\tilde{\chi})\left[\tilde{\partial}_\perp^i\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right) + \frac{1}{\tilde{\chi}}\left(B_\perp^{i(1)} + n^k h_{kj}^{(1)}\mathcal{P}^{ij}\right)\right]\Big\}\,. \tag{122}$$

In Eq. (120) there is an ISW contribution and in Eq. (121) we have both time-delay and ISW contributions.

Now we can obtain $\Delta_g^{(1)}$. Using Eq. (79) for $\Delta x_\parallel^{(1)}$, we find

$$\partial_\parallel \Delta x_\parallel^{(1)} = \partial_{\bar\chi}\Delta x_\parallel^{(1)} = -\partial_{\bar\chi}\left(T^{(1)}+\Delta x^{0(1)}\right) = \left(A^{(1)}-B_\parallel^{(1)}-\frac{1}{2}h_\parallel^{(1)}\right) - \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta\ln a^{(1)} - \frac{1}{\mathcal{H}}\left(\frac{\mathrm{d}\,\Delta\ln a}{\mathrm{d}\bar\chi}\right)^{(1)}$$
$$= \left(A^{(1)}-B_\parallel^{(1)}-\frac{1}{2}h_\parallel^{(1)}\right) + \frac{1}{\mathcal{H}}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v_\parallel^{(1)}\right) + \left(A^{(1)\prime}-B_\parallel^{(1)\prime}-\frac{1}{2}h_\parallel^{(1)\prime}\right)\right] - \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta\ln a^{(1)}. \quad (123)$$

From Eqs. (118), (121) and (123), we find that Eq. (76) becomes

$$\Delta_g^{(1)} = \delta_g^{(1)} + \left(b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar\chi\mathcal{H}}\right)\Delta\ln a^{(1)} + \frac{1}{\mathcal{H}}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v_\parallel^{(1)}\right) + \left(A^{(1)\prime}-B_\parallel^{(1)\prime}-\frac{1}{2}h_\parallel^{(1)\prime}\right)\right]$$
$$-\frac{2}{\bar\chi}T^{(1)} - 2\kappa^{(1)} + A^{(1)} + v_\parallel^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)} + \frac{1}{2}h_i^{i(1)}, \quad (124)$$

in agreement with [1–5].

Making explicit at first order the coordinate convergence lensing term defined in Eq. (80) (see also [7]), we find[7]

$$\kappa^{(1)} = -\frac{1}{2}\partial_{\perp i}\Delta x_\perp^{i(1)} = \kappa_1^{(1)} + \kappa_2^{(1)} + \kappa_3^{(1)} \quad (125)$$

where, using $\partial_{\perp i}\mathcal{P}^i_j = -2n_j/\bar\chi$,

$$\kappa_1^{(1)} = \frac{1}{2}\partial_{\perp i}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi-\tilde\chi)\,\tilde\partial_\perp^i\left(A^{(1)}-B_\parallel^{(1)}-\frac{1}{2}h_\parallel^{(1)}\right) = \frac{1}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi-\tilde\chi)\,\frac{\tilde\chi}{\bar\chi}\tilde\nabla_\perp^2\left(A^{(1)}-B_\parallel^{(1)}-\frac{1}{2}h_\parallel^{(1)}\right), \quad (126)$$

$$\kappa_2^{(1)} = \frac{1}{2}\partial_{\perp i}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\frac{\bar\chi}{\tilde\chi}\left(B_\perp^{i(1)} + n^k h_{kj}^{(1)}\mathcal{P}^{ij}\right) = \frac{1}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\tilde\partial_\perp^i B_i^{(1)} - \frac{2}{\tilde\chi}B_\parallel^{(1)} + \mathcal{P}^{ij}n^k\tilde\partial_i h_{jk}^{(1)} + \frac{1}{\tilde\chi}\left(h_i^{i(1)} - 3h_\parallel^{(1)}\right)\right], \quad (127)$$

$$\kappa_3^{(1)} = -\frac{\bar\chi}{2}\left[\partial_{\perp i}B_{\perp\, o}^{i(1)} - \partial_{\perp i}v_{\perp\, o}^{i(1)} + \frac{1}{2}h_{k\, o}^{j(1)}\partial_{\perp i}\left(n^k\mathcal{P}^i_j\right)\right] = -\frac{1}{4}\left(h_{i\, o}^{i(1)} - 3h_{\parallel\, o}^{(1)}\right) + \left(B_{\parallel\, o}^{(1)} - v_{\parallel\, o}^{(1)}\right). \quad (128)$$

---

[7] To compute correctly the lensing term, we need further properties of the parallel and orthogonal derivatives. If $\tilde x^j$ is not necessarily the same as $\bar x^i$ (i.e. $\tilde\chi$ can be different from $\bar\chi$), then $\partial\bar x^i(\bar\chi)/\partial\tilde x^j = \partial\bar\chi/\partial\tilde x^j n^i + \bar\chi\partial n^i/\partial\tilde x^j = (\partial\bar\chi/\partial\tilde x^j)n^i + (\bar\chi/\tilde\chi)\mathcal{P}^i_j$. (We used $\partial n^i/\partial\tilde x^j = \mathcal{P}^i_j/\tilde\chi$.) If $\bar\chi = \tilde\chi$, then $\partial\bar\chi/\partial\tilde x^j = n_j$, and if $\bar\chi \neq \tilde\chi$, we have $\partial\bar\chi/\partial\tilde x^j = 0$. Thus $\partial\bar x^i(\bar\chi)/\partial\tilde x^j = \delta^i_j$ for $\bar\chi = \tilde\chi$, and $= (\bar\chi/\tilde\chi)\mathcal{P}^i_j$ for $\bar\chi \neq \tilde\chi$. Note that for $\bar\chi \neq \tilde\chi$, the orthogonal part survives. Then

$$\tilde\partial_{\perp j}F(\bar x^i) = \mathcal{P}^k_j(\partial\bar x^i(\bar\chi)/\partial\tilde x^k)\partial F/\partial\bar x^i = (\bar\chi/\tilde\chi)\mathcal{P}^i_j\partial F/\partial\bar x^i = (\bar\chi/\tilde\chi)\partial_{\perp j}F\ .$$

We can finally compute $\Delta\ln a^{(2)}$, $\Delta x^{0(2)}$, $\Delta x_{\|}^{(2)}$ and $\Delta x_{\perp}^{i(2)}$. From Eq. (33) we find

$$
\begin{aligned}
\Delta\ln a^{(2)} &= -\delta\nu^{(2)} + E^{(2)}_{\hat{0}\|} - E^{(2)}_{\hat{0}0} - 2E^{(1)}_{\hat{0}\|}\left(\frac{\mathrm{d}T}{\mathrm{d}\bar{\chi}}\right)^{(1)} - 2\partial_{\|}\left(E^{(1)}_{\hat{0}\|} - E^{(1)}_{\hat{0}0}\right)T^{(1)} - \frac{2}{\mathcal{H}}\left(E^{(1)}_{\hat{0}\|} - E^{(1)}_{\hat{0}0}\right)\left(\frac{\mathrm{d}\,\Delta\ln a}{\mathrm{d}\bar{\chi}}\right)^{(1)} \\
&\quad +2\left[-\left(E^{(1)}_{\hat{0}\|} - E^{(1)}_{\hat{0}0}\right) + \frac{1}{\mathcal{H}}\left(\frac{\mathrm{d}\,\Delta\ln a}{\mathrm{d}\bar{\chi}}\right)^{(1)}\right]\delta\nu^{(1)} - 2\delta x^{0(1)}\left(\frac{\mathrm{d}\delta\nu}{\mathrm{d}\bar{\chi}}\right)^{(1)} + 2E^{(1)}_{\hat{0}\perp i}\delta n_{\perp}^{i(1)} \\
&\quad +2\left[\partial_{\perp i}\left(E^{(1)}_{\hat{0}\|} - E^{(1)}_{\hat{0}0}\right) - \frac{1}{\bar{\chi}}E^{(1)}_{\hat{0}\perp i}\right]\delta x_{\perp}^{i(1)} \\
&= A_o^{(2)} - v_{\|\,o}^{(2)} - \left(A_o^{(1)}\right)^2 + 2A_o^{(1)}B_{\|\,o}^{(1)} - \left(B_{\|\,o}^{(1)}\right)^2 - 6A_o^{(1)}v_{\|\,o}^{(1)} + 2B_{\|\,o}^{(1)}v_{\|\,o}^{(1)} + v_{k\,o}^{(1)}v_o^{k(1)} \\
&\quad -n^i h^{(1)}_{ij\,o}v_o^{j(1)} + 2\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right)\Bigg\{-2\left(2A^{(1)} - B_{\|}^{(1)}\right) + \left(A^{(1)} + v_{\|}^{(1)} - B_{\|}^{(1)}\right) \\
&\quad +\left(\bar{\chi} + \frac{1}{\mathcal{H}}\right)\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(2A^{(1)} - B_{\|}^{(1)}\right) - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(A^{(1)} + v_{\|}^{(1)} - B_{\|}^{(1)}\right) + \left(\bar{\chi} + \frac{1}{\mathcal{H}}\right)\left(A^{(1)\prime} - B_{\|}^{(1)\prime} - \frac{1}{2}h_{\|}^{(1)\prime}\right) \\
&\quad +4I^{(1)}\Bigg\} + 2\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h^{j(1)}_{k\,o}\mathcal{P}^i_j\right)\Bigg\{B^{(1)}_{\perp i} + \bar{\chi}\,\partial_{\perp i}\left(A^{(1)} + v_{\|}^{(1)} - B_{\|}^{(1)}\right) \\
&\quad -\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\tilde{\partial}_{\perp i}\left(2A^{(1)} - B_{\|}^{(1)}\right) - \left(B^{(1)\prime}_{\perp i} + n^j h^{(1)\prime}_{jk}\mathcal{P}^k_i\right) + \frac{1}{\tilde{\chi}}B^{(1)}_{\perp i}\right]\Bigg\} - A^{(2)} + v_{\|}^{(2)} + 7\left(A^{(1)}\right)^2 \\
&\quad +\left(B_{\|}^{(1)}\right)^2 - 4A^{(1)}B_{\|}^{(1)} + \left(v_{\|}^{(1)}\right)^2 + v^{(1)}_{\perp i}v_{\perp}^{i(1)} + v_{\|}^{(1)}h_{\|}^{(1)} - 2A^{(1)}v_{\|}^{(1)} - 2v^{(1)}_{\perp i}B_{\perp}^{i(1)} \\
&\quad -\frac{2}{\mathcal{H}}\left(A^{(1)} - v_{\|}^{(1)}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(A^{(1)} - v_{\|}^{(1)}\right) + \left(A^{(1)\prime} - B_{\|}^{(1)\prime} - \frac{1}{2}h_{\|}^{(1)\prime}\right)\right] - 4\Bigg[2\left(2A^{(1)} - B_{\|}^{(1)}\right) \\
&\quad -\left(A^{(1)} + v_{\|}^{(1)} - B_{\|}^{(1)}\right) - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(2A^{(1)} - B_{\|}^{(1)}\right) + \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(A^{(1)} + v_{\|}^{(1)} - B_{\|}^{(1)}\right) \\
&\quad -\frac{1}{\mathcal{H}}\left(A^{(1)\prime} - B_{\|}^{(1)\prime} - \frac{1}{2}h_{\|}^{(1)\prime}\right)\Bigg]I^{(1)} + 4v^{(1)}_{\perp i}S_{\perp}^{i(1)} - 2\partial_{\|}\left(A^{(1)} + v_{\|}^{(1)} - B_{\|}^{(1)}\right)T^{(1)} - 2\Bigg[\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(2A^{(1)} - B_{\|}^{(1)}\right) \\
&\quad +\left(A^{(1)\prime} - B_{\|}^{(1)\prime} - \frac{1}{2}h_{\|}^{(1)\prime}\right)\Bigg]\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[2A^{(1)} - B_{\|}^{(1)} + (\bar{\chi} - \tilde{\chi})\left(A^{(1)\prime} - B_{\|}^{(1)\prime} - \frac{1}{2}h_{\|}^{(1)\prime}\right)\right] \\
&\quad -2\left[\partial_{\perp i}\left(A^{(1)} + v_{\|}^{(1)} - B_{\|}^{(1)}\right) - \frac{1}{\bar{\chi}}\left(v^{(1)}_{\perp i} - B^{(1)}_{\perp i}\right)\right]\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\Bigg\{\left(B_{\perp}^{i(1)} + n^k h^{j(1)}_{k}\mathcal{P}^i_j\right) \\
&\quad +(\bar{\chi} - \tilde{\chi})\left[\tilde{\partial}^i_{\perp}\left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2}h_{\|}^{(1)}\right) + \frac{1}{\tilde{\chi}}\left(B_{\perp}^{i(1)} + n^k h^{(1)}_{kj}\mathcal{P}^{ij}\right)\right]\Bigg\} + 2I^{(2)} \\
&\quad +2\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\Bigg\{\left(B_{\|}^{(1)} + h_{\|}^{(1)} - 4I^{(1)}\right)\left(A^{(1)\prime} - B_{\|}^{(1)\prime} - \frac{1}{2}h_{\|}^{(1)\prime}\right) - \left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2}h_{\|}^{(1)}\right) \\
&\quad \times\frac{\mathrm{d}}{\mathrm{d}\tilde{\chi}}\left(2A^{(1)} - B_{\|}^{(1)}\right) + \left[\left(B_{\perp}^{i(1)} + n^j h^{(1)}_{jk}\mathcal{P}^{ik}\right) - 2S_{\perp}^{i(1)}\right]\Bigg[\tilde{\partial}_{\perp i}\left(2A^{(1)} - B_{\|}^{(1)}\right) \\
&\quad -\left(B^{(1)\prime}_{\perp i} + n^j h^{(1)\prime}_{jk}\mathcal{P}^k_i\right) + \frac{1}{\tilde{\chi}}B^{(1)}_{\perp i}\Bigg]\Bigg\}\;.
\end{aligned}
\tag{129}
$$

Using Eqs (118) and (164), Eq. (37) yields

$$
\begin{aligned}
\Delta x^{0(2)} &= \frac{1}{\mathcal{H}}\Delta\ln a^{(2)} - \frac{(\mathcal{H}'+\mathcal{H}^2)}{\mathcal{H}^3}\left(\Delta\ln a^{(1)}\right)^2 \\
&= +\frac{1}{\mathcal{H}}A_o^{(2)} - \frac{1}{\mathcal{H}}v_{\parallel\,o}^{(2)} - \left(\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{2}{\mathcal{H}}\right)\left(A_o^{(1)}\right)^2 - \frac{1}{\mathcal{H}}\left(B_{\parallel\,o}^{(1)}\right)^2 + \frac{2}{\mathcal{H}}A_o^{(1)}B_{\parallel\,o}^{(1)} + 2\left(\frac{\mathcal{H}'}{\mathcal{H}^3}-\frac{2}{\mathcal{H}}\right)A_o^{(1)}v_{\parallel\,o}^{(1)} \\
&\quad +\frac{2}{\mathcal{H}}B_{\parallel\,o}^{(1)}v_{\parallel\,o}^{(1)} - \frac{\mathcal{H}'}{\mathcal{H}^3}\left(v_{\parallel\,o}^{(1)}\right)^2 + \frac{1}{\mathcal{H}}v_{\perp i\,o}^{(1)}v_{\perp\,o}^{i(1)} - \frac{1}{\mathcal{H}}n^ih_{ij\,o}^{(1)}v_o^{j(1)} + 2\left(A_o^{(1)}-v_{\parallel\,o}^{(1)}\right) \\
&\quad \times\Bigg\{\left(\frac{\mathcal{H}'}{\mathcal{H}^3}-\frac{1}{\mathcal{H}}\right)\left(2A^{(1)}-B_\parallel^{(1)}\right) - \frac{\mathcal{H}'}{\mathcal{H}^3}\left(A^{(1)}+v_\parallel^{(1)}-B_\parallel^{(1)}\right) + \left(\frac{\bar\chi}{\mathcal{H}}+\frac{1}{\mathcal{H}^2}\right)\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2A^{(1)}-B_\parallel^{(1)}\right) \\
&\quad -\frac{1}{\mathcal{H}^2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}+v_\parallel^{(1)}-B_\parallel^{(1)}\right) + \left(\frac{\bar\chi}{\mathcal{H}}+\frac{1}{\mathcal{H}^2}\right)\left(A^{(1)\prime}-B_\parallel^{(1)\prime}-\frac{1}{2}h_\parallel^{(1)\prime}\right) - 2\left(\frac{\mathcal{H}'}{\mathcal{H}^3}-\frac{1}{\mathcal{H}}\right)I^{(1)}\Bigg\} \\
&\quad +2\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^kh_{k\,o}^{j(1)}\mathcal{P}_j^i\right)\Bigg\{\frac{1}{\mathcal{H}}B_{\perp i}^{(1)} + \frac{\bar\chi}{\mathcal{H}}\partial_{\perp i}\left(A^{(1)}+v_\parallel^{(1)}-B_\parallel^{(1)}\right) \\
&\quad -\frac{1}{\mathcal{H}}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\tilde\partial_{\perp i}\left(2A^{(1)}-B_\parallel^{(1)}\right) - \left(B_{\perp i}^{(1)\prime}+n^jh_{jk}^{(1)\prime}\mathcal{P}_i^k\right) + \frac{1}{\tilde\chi}B_{\perp i}^{(1)}\right]\Bigg\} - \frac{1}{\mathcal{H}}A^{(2)} + \frac{1}{\mathcal{H}}v_\parallel^{(2)} \\
&\quad +\left(-\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{6}{\mathcal{H}}\right)\left(A^{(1)}\right)^2 + \frac{1}{\mathcal{H}}\left(B_\parallel^{(1)}\right)^2 - \frac{4}{\mathcal{H}}A^{(1)}B_\parallel^{(1)} - \frac{\mathcal{H}'}{\mathcal{H}^3}\left(v_\parallel^{(1)}\right)^2 + \frac{1}{\mathcal{H}}v_{\perp i}^{(1)}v_\perp^{i(1)} \\
&\quad +\frac{1}{\mathcal{H}}v_\parallel^{(1)}h_\parallel^{(1)} + 2\frac{\mathcal{H}'}{\mathcal{H}^3}A^{(1)}v_\parallel^{(1)} - \frac{2}{\mathcal{H}}v_{\perp i}^{(1)}B_\perp^{i(1)} - \frac{2}{\mathcal{H}^2}\left(A^{(1)}-v_\parallel^{(1)}\right)\Bigg[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v_\parallel^{(1)}\right) \\
&\quad +\left(A^{(1)\prime}-B_\parallel^{(1)\prime}-\frac{1}{2}h_\parallel^{(1)\prime}\right)\Bigg] - 4\Bigg[-\left(\frac{\mathcal{H}'}{\mathcal{H}^3}-\frac{1}{\mathcal{H}}\right)\left(2A^{(1)}-B_\parallel^{(1)}\right) + \frac{\mathcal{H}'}{\mathcal{H}^3}\left(A^{(1)}+v_\parallel^{(1)}-B_\parallel^{(1)}\right) \\
&\quad -\frac{1}{\mathcal{H}^2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v_\parallel^{(1)}\right) - \frac{1}{\mathcal{H}^2}\left(A^{(1)\prime}-B_\parallel^{(1)\prime}-\frac{1}{2}h_\parallel^{(1)\prime}\right) + \left(\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{1}{\mathcal{H}}\right)I^{(1)}\Bigg]I^{(1)} \\
&\quad +\frac{4}{\mathcal{H}}v_{\perp i}^{(1)}S_\perp^{i(1)} - \frac{2}{\mathcal{H}}\partial_\parallel\left(A^{(1)}+v_\parallel^{(1)}-B_\parallel^{(1)}\right)T^{(1)} - \frac{2}{\mathcal{H}}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2A^{(1)}-B_\parallel^{(1)}\right) + \left(A^{(1)\prime}-B_\parallel^{(1)\prime}-\frac{1}{2}h_\parallel^{(1)\prime}\right)\right] \\
&\quad \times\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2A^{(1)}-B_\parallel^{(1)} + (\bar\chi-\tilde\chi)\left(A^{(1)\prime}-B_\parallel^{(1)\prime}-\frac{1}{2}h_\parallel^{(1)\prime}\right)\right] - \frac{2}{\mathcal{H}}\Bigg[\partial_{\perp i}\left(A^{(1)}+v_\parallel^{(1)}-B_\parallel^{(1)}\right) \\
&\quad -\frac{1}{\bar\chi}\left(v_{\perp i}^{(1)}-B_{\perp i}^{(1)}\right)\Bigg]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\left(B_\perp^{i(1)}+n^kh_k^{j(1)}\mathcal{P}_j^i\right) + (\bar\chi-\tilde\chi)\Bigg[\tilde\partial_\perp^i\left(A^{(1)}-B_\parallel^{(1)}-\frac{1}{2}h_\parallel^{(1)}\right) \\
&\quad +\frac{1}{\tilde\chi}\left(B_\perp^{i(1)}+n^kh_{kj}^{(1)}\mathcal{P}^{ij}\right)\Bigg]\Bigg\} + \frac{2}{\mathcal{H}}I^{(2)} + \frac{2}{\mathcal{H}}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\left(B_\parallel^{(1)}+h_\parallel^{(1)}-4I^{(1)}\right)\left(A^{(1)\prime}-B_\parallel^{(1)\prime}-\frac{1}{2}h_\parallel^{(1)\prime}\right) \\
&\quad -\left(A^{(1)}-B_\parallel^{(1)}-\frac{1}{2}h_\parallel^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(2A^{(1)}-B_\parallel^{(1)}\right) + \left[\left(B_\perp^{i(1)}+n^jh_{jk}^{(1)}\mathcal{P}^{ik}\right)-2S_\perp^{i(1)}\right] \\
&\quad \times\left[\tilde\partial_{\perp i}\left(2A^{(1)}-B_\parallel^{(1)}\right) - \left(B_{\perp i}^{(1)\prime}+n^jh_{jk}^{(1)\prime}\mathcal{P}_i^k\right) + \frac{1}{\tilde\chi}B_{\perp i}^{(1)}\right]\Bigg\}\ .
\end{aligned}
\tag{130}
$$

From Eqs. (71) and (117) we deduce

$$
\begin{aligned}
\Delta x_{\|}^{(2)} &= \delta x^{0(2)} + \delta x_{\|}^{(2)} - \frac{1}{\mathcal{H}}\Delta \ln a^{(2)} + \left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left(\Delta \ln a^{(1)}\right)^2 - 2\left(\frac{\mathrm{d}T}{\mathrm{d}\bar{\chi}}\right)^{(1)} \delta\chi^{(1)} \\
&= \bar{\chi}\Bigg[\left(A_o^{(1)}\right)^2 - 2A_o^{(1)}B_{\|\,o}^{(1)} + \left(B_{\|\,o}^{(1)}\right)^2 + 2A_o^{(1)}v_{\|\,o}^{(1)} - 2B_{\|\,o}^{(1)}v_{\|\,o}^{(1)} - v_{\|\,o}^{(1)}h_{\|\,o}^{(1)} - \frac{1}{4}\left(h_{\|\,o}^{(1)}\right)^2 - v_{\perp k\,o}^{(1)}v_{\perp\,o}^{k(1)} \\
&\quad + n^i h_{ik\,o}^{(1)}\mathcal{P}_j^k v_o^{j(1)} - n^i h_{ik\,o}^{(1)}\mathcal{P}_j^k B_o^{j(1)} - \frac{1}{4}n^i h_{ij\,o}^{(1)}\mathcal{P}_k^j\, h_{p\,o}^{k(1)} n^p - B_{\perp\,o}^{i(1)}B_{\perp i\,o}^{(1)} + 2v_{\perp\,o}^{i(1)}B_{\perp i\,o}^{(1)}\Bigg] \\
&\quad - 2\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right)\left[\bar{\chi}\left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2}h_{\|}^{(1)}\right) + 2T^{(1)}\right] + 4\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right) \\
&\quad \times \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi} - \tilde{\chi})\left[\tilde{\partial}_\perp^i\left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2}h_{\|}^{(1)}\right) + \frac{1}{\tilde{\chi}}\left(B_\perp^{i(1)} + n^k h_{kj}^{(1)}\mathcal{P}^{ij}\right)\right] \\
&\quad + 2\left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2}h_{\|}^{(1)}\right)\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left[2A^{(1)} - B_{\|}^{(1)} + (\bar{\chi} - \tilde{\chi})\left(A^{(1)\prime} - B_{\|}^{(1)\prime} - \frac{1}{2}h_{\|}^{(1)\prime}\right)\right] \\
&\quad - T^{(2)} + 2\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\Bigg[-\left(2A^{(1)} - B_{\|}^{(1)}\right)^2 + \frac{1}{2}\left(B_{\|}^{(1)} + h_{\|}^{(1)}\right)\left(2A^{(1)} + h_{\|}^{(1)}\right) \\
&\quad + \frac{1}{2}\left(B_\perp^{i(1)} + \mathcal{P}_j^i h_k^{j(1)} n^k\right)\left(B_{i\perp}^{(1)} + n^p h_{pm}^{(1)}\mathcal{P}_i^m\right) + 4\left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2}h_{\|}^{(1)}\right) I^{(1)} - 4S_\perp^{i(1)}S_\perp^{j(1)}\delta_{ij}\Bigg] \\
&\quad + 2\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi} - \tilde{\chi})\Bigg\{\left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2}h_{\|}^{(1)}\right)\left[2\left(A^{(1)\prime} - B_{\|}^{(1)\prime} - \frac{1}{2}h_{\|}^{(1)\prime}\right) + \frac{\mathrm{d}}{\mathrm{d}\tilde{\chi}}\left(2A^{(1)} - B_{\|}^{(1)}\right)\right] \\
&\quad - \left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right)\left[\tilde{\partial}_{\perp i}\left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2}h_{\|}^{(1)}\right) + \frac{1}{\tilde{\chi}}\left(B_{\perp i}^{(1)} + n^m h_{mp}^{(1)}\mathcal{P}_i^p\right)\right]\Bigg\} \\
&\quad - \frac{2}{\mathcal{H}}\left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2}h_{\|}^{(1)}\right)\Delta \ln a^{(1)} - \frac{1}{\mathcal{H}}\Delta \ln a^{(2)} + \left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left(\Delta \ln a^{(1)}\right)^2 \,, \qquad (131)
\end{aligned}
$$

and from Eqs. (72) and (116) we find

$$
\begin{aligned}
\Delta x_\perp^{i(2)} &= \delta x_\perp^{i(2)} + 2\delta n_\perp^{i(1)}\delta x^{0(1)} - \frac{2}{\mathcal{H}}\delta n_\perp^{i(1)}\Delta\ln a^{(1)} \\
&= \bar\chi\bigg[B_{\perp\,o}^{i(2)} - v_{\perp\,o}^{i(2)} + \frac{1}{2}n^j h_{jk\,o}^{(2)}\mathcal{P}^{ki} - 2A_o^{(1)}B_{\perp\,o}^{i(1)} + 2A_o^{(1)}v_{\perp\,o}^{i(1)} + A_o^{(1)}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i - v_{\|\,o}^{(1)}v_{\perp\,o}^{i(1)} \\
&\quad - v_{\|\,o}^{(1)}B_{\perp\,o}^{i(1)} + B_{\|\,o}^{(1)}B_{\perp\,o}^{i(1)} - B_{\|\,o}^{(1)}v_{\perp\,o}^{i(1)} - 3v_{\|\,o}^{(1)}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i - 2v_{\perp o}^{j(1)}\mathcal{P}_j^l\,h_{l\,o}^{k(1)}\mathcal{P}_k^i - \frac{1}{4}h_{\|\,o}^{(1)}n^j h_{jk\,o}^{(1)}\mathcal{P}^{ki} \\
&\quad - \frac{1}{4}n^j h_{j\,o}^{k(1)}\mathcal{P}_k^l\,h_{l\,o}^{p(1)}\mathcal{P}_p^{i(1)}\bigg] + 2\bar\chi\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right)\left[\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) - 2S_\perp^{i(1)}\right] \\
&\quad - 4\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\bigg\{\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + (\bar\chi-\tilde\chi)\bigg[\tilde\partial_\perp^i\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right) \\
&\quad + \frac{1}{\tilde\chi}\left(B_\perp^{i(1)} + n^k h_{kj}^{(1)}\mathcal{P}^{ij}\right)\bigg]\bigg\} + 2\left(B_{\perp j\,o}^{(1)} - v_{\perp j\,o}^{(1)} + \frac{1}{2}n^k h_{k\,o}^{l(1)}\mathcal{P}_{jl}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\bigg\{-\mathcal{P}^{jm}\,h_m^{p(1)}\mathcal{P}_p^i \\
&\quad + 2(\bar\chi-\tilde\chi)\left[\tilde\partial_\perp^{[i}B_\perp^{j](1)} + \tilde\partial_\perp^{[i}\left(\mathcal{P}_m^{j]}h_q^{m(1)}n^q\right) - \frac{1}{\tilde\chi}\left(n^{[i}B_\perp^{j](1)} + n^{[i}\mathcal{P}_m^{j]}h_q^{m(1)}n^q\right)\right]\bigg\} \\
&\quad + 2\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right)\bigg\{\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2A^{(1)} - B_\|^{(1)} + (\bar\chi-\tilde\chi)\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\right] \\
&\quad - \frac{1}{\mathcal{H}}\Delta\ln a^{(1)}\bigg\} + \frac{2}{\mathcal{H}}\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i - 2S_\perp^{i(1)}\right)\Delta\ln a^{(1)} \\
&\quad - 2\left[\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) - 2S_\perp^{i(1)}\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2A^{(1)} - B_\|^{(1)} + (\bar\chi-\tilde\chi)\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\right] \\
&\quad + \int_0^{\bar\chi}\mathrm{d}\tilde\chi\bigg\{-B_\perp^{i(2)} - n^j h_{jk}^{(2)}\mathcal{P}^{ki} + 4A^{(1)}B_\perp^{i(1)} - 2B_\|^{(1)}B_\perp^{i(1)} + 2A^{(1)}n^j h_{jk}^{(1)}\mathcal{P}^{ki} + h_\|^{(1)}n^j h_{jk}^{(1)}\mathcal{P}^{ki} \\
&\quad + 2\left(B_\perp^{j(1)} + n^k h_k^{p(1)}\mathcal{P}_p^j\right)\mathcal{P}_j^l\,h_l^{k(1)}\mathcal{P}_k^{i(1)} - 8\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right)I^{(1)} - 4\,\mathcal{P}_j^l\,h_l^{k(1)}\mathcal{P}_k^i S_\perp^{j(1)}\bigg\} \\
&\quad + \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi-\tilde\chi)\bigg\{-\left[\tilde\partial_\perp^i\left(A^{(2)} - B_\|^{(2)} - \frac{1}{2}h_\|^{(2)}\right) + \frac{1}{\tilde\chi}\left(B_\perp^{i(2)} + n^k h_{kj}^{(2)}\mathcal{P}^{ij}\right)\right] \\
&\quad - 2\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right) + 2\left(2A^{(1)} - B_\|^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right) \\
&\quad + 4\left(2A^{(1)} - B_\|^{(1)} - 2I^{(1)}\right)\left[\tilde\partial_\perp^i\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right) + \frac{1}{\tilde\chi}\left(B_\perp^{i(1)} + n^k h_{kj}^{(1)}\mathcal{P}^{ij}\right)\right] \\
&\quad - 4\left[\left(B_{\perp j}^{(1)} + n^p h_p^{k(1)}\mathcal{P}_{jk}\right) - 2\delta_{jp}S_\perp^{p(1)}\right]\left[\tilde\partial_\perp^{[i}B_\perp^{j](1)} + \tilde\partial_\perp^{[i}\left(\mathcal{P}_m^{j]}h_q^{m(1)}n^q\right) - \frac{1}{\tilde\chi}\left(n^{[i}B_\perp^{j](1)} + n^{[i}\mathcal{P}_m^{j]}h_q^{m(1)}n^q\right)\right] \\
&\quad + 2\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right)\left[\tilde\partial_\perp^i\left(B_\|^{(1)} + h_\|^{(1)}\right) - \partial_\|\left(B_\perp^{i(1)} + n^p h_{pq}^{(1)}\mathcal{P}^{iq}\right) - \frac{1}{\tilde\chi}\left(B_\perp^{i(1)} + 2n^p h_{pq}^{(1)}\mathcal{P}^{iq}\right)\right]\bigg\}\,.
\end{aligned}
\tag{132}
$$

The next step is to compute $\partial_\|\Delta x_\|^{(2)}$ and $\kappa^{(2)}$. From Eq. (79) we have $\partial_\|\Delta x_\|^{(2)}(\eta,\bar{\mathbf{x}}) = \partial_{\bar\chi}\Delta x_\|^{(2)}(\bar\chi,\mathbf{n})$, so that

$$
\begin{aligned}
\partial_\|\Delta x_\|^{(2)} &= \delta\nu^{(2)} + \delta n_\|^{(2)} - \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(2)}\right) + \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left(\Delta\ln a^{(1)}\right)^2\right] - 2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\left(\frac{\mathrm{d}T}{\mathrm{d}\bar\chi}\right)^{(1)}\delta\chi^{(1)}\right] \\
&= -\frac{\mathcal{H}'}{\mathcal{H}^2}\Delta\ln a^{(2)} + \left[-\frac{\mathcal{H}''}{\mathcal{H}^3} + 3\left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right]\left(\Delta\ln a^{(1)}\right)^2 + 2\left[\frac{\mathcal{H}'}{\mathcal{H}^2}\left(\frac{\mathrm{d}T}{\mathrm{d}\bar\chi}\right)^{(1)} + \frac{1}{\mathcal{H}}\left(\frac{\mathrm{d}^2T}{\mathrm{d}\bar\chi^2}\right)^{(1)}\right]\Delta\ln a^{(1)} \\
&\quad + 2\left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\Delta\ln a^{(1)}\left(\frac{\mathrm{d}\,\Delta\ln a}{\mathrm{d}\bar\chi}\right)^{(1)} - \frac{1}{\mathcal{H}}\left(\frac{\mathrm{d}\,\Delta\ln a}{\mathrm{d}\bar\chi}\right)^{(2)} - 2\left(\frac{\mathrm{d}^2T}{\mathrm{d}\bar\chi^2}\right)^{(1)}\delta x^{0(1)} - 2\left(\frac{\mathrm{d}T}{\mathrm{d}\bar\chi}\right)^{(1)}\delta\nu^{(1)} \\
&\quad + \frac{2}{\mathcal{H}}\left(\frac{\mathrm{d}T}{\mathrm{d}\bar\chi}\right)^{(1)}\left(\frac{\mathrm{d}\,\Delta\ln a}{\mathrm{d}\bar\chi}\right)^{(1)} + \delta\nu^{(2)} + \delta n_\|^{(2)}\,.
\end{aligned}
\tag{133}
$$

To obtain explicitly Eq. (133) we have to determine

$$
\begin{aligned}
\left(\frac{\mathrm{d}\,\Delta\ln a}{\mathrm{d}\bar\chi}\right)^{(2)} = \; & +2\left(A^{(1)}_o - v^{(1)}_{\| o}\right)\Bigg\{\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-1\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v^{(1)}_\|\right)+\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right] \\
& +\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v^{(1)}_\|\right)+\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right]+\bar\chi\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2A^{(1)}-B^{(1)}_\|\right)\right. \\
& \left.+\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right]\Bigg\}+2\left(B^{i(1)}_{\perp\, o}-v^{i(1)}_{\perp\, o}+\frac{1}{2}n^k h^{j(1)}_{k\, o}\mathcal{P}^i_j\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}B^{(1)}_{\perp i}-\partial_{\perp i}\left(A^{(1)}-v^{(1)}_\|\right)\right. \\
& \left.+\bar\chi\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_{\perp i}\left(A^{(1)}+v^{(1)}_\|-B^{(1)}_\|\right)+\left(B^{(1)\prime}_{\perp i}+n^j h^{(1)\prime}_{jk}\mathcal{P}^k_i\right)-\frac{1}{\bar\chi}B^{(1)}_{\perp i}\right]+\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(-A^{(2)}+v^{(2)}_\|\right) \\
& -\left(A^{(2)\prime}-B^{(2)\prime}_\|-\frac{1}{2}h^{(2)\prime}_\|\right)+\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[7\left(A^{(1)}\right)^2+\left(B^{(1)}_\|\right)^2-4A^{(1)}B^{(1)}_\|-2A^{(1)}v^{(1)}_\|\right. \\
& \left.+\left(v^{(1)}_\|\right)^2+v^{(1)}_\| h^{(1)}_\|\right]-2\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2A^{(1)}-B^{(1)}_\|\right)+\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right]\left(2A^{(1)}-B^{(1)}_\|\right) \\
& -2\left\{\frac{\mathcal{H}'}{\mathcal{H}^2}\left(A^{(1)}-v^{(1)}_\|\right)+\frac{1}{\mathcal{H}}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v^{(1)}_\|\right)+\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right]\right\}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v^{(1)}_\|\right)\right. \\
& \left.+\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right]+2\left[\left(2A^{(1)}+h^{(1)}_\|\right)+\left(A^{(1)}-v^{(1)}_\|\right)\right]\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right) \\
& -\frac{2}{\mathcal{H}}\left(A^{(1)}-v^{(1)}_\|\right)\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v^{(1)}_\|\right)+\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right]+2\left(A^{(1)}-B^{(1)}_\|-\frac{1}{2}h^{(1)}_\|\right) \\
& \times\left[-\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2A^{(1)}-B^{(1)}_\|\right)+\partial_\|\left(A^{(1)}+v^{(1)}_\|-B^{(1)}_\|\right)\right]+2\left(B^{i(1)}_\perp+n^j h^{(1)}_{jk}\mathcal{P}^{ik}\right)\left[\partial_{\perp i}\left(A^{(1)}-v^{(1)}_\|\right)\right. \\
& \left.-\left(B^{(1)\prime}_{\perp i}+n^j h^{(1)\prime}_{jk}\mathcal{P}^k_i\right)\right]-2v^{(1)}_{\perp i}\partial^i_\perp\left(A^{(1)}-B^{(1)}_\|-\frac{1}{2}h^{(1)}_\|\right)+2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{1}{2}v^{(1)}_{\perp i}v^{i(1)}_\perp-v^{(1)}_{\perp i}B^{i(1)}_\perp\right] \\
& +4\left\{\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-1\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v^{(1)}_\|\right)+\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right]+\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v^{(1)}_\|\right)\right.\right. \\
& \left.\left.+\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right]\right\}I^{(1)}-2\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_\|\left(A^{(1)}+v^{(1)}_\|-B^{(1)}_\|\right)\right]T^{(1)}-2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2A^{(1)}-B^{(1)}_\|\right)\right. \\
& \left.+\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2A^{(1)}-B^{(1)}_\|+(\bar\chi-\tilde\chi)\left(A^{(1)\prime}-B^{(1)\prime}_\|-\frac{1}{2}h^{(1)\prime}_\|\right)\right] \\
& -4\left[\partial_{\perp i}\left(A^{(1)}-v^{(1)}_\|\right)-\left(B^{(1)\prime}_{\perp i}+n^j h^{(1)\prime}_{jk}\mathcal{P}^k_i\right)+\frac{1}{\bar\chi}v^{(1)}_{\perp i}-\frac{\mathrm{d}}{\mathrm{d}\bar\chi}v^{(1)}_{\perp i}\right]S^{i(1)}_\perp \\
& -2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\partial_{\perp i}\left(A^{(1)}+v^{(1)}_\|-B^{(1)}_\|\right)-\frac{1}{\bar\chi}\left(v^{(1)}_{\perp i}-B^{(1)}_{\perp i}\right)\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left\{\left(B^{i(1)}_\perp+n^k h^{j(1)}_k\mathcal{P}^i_j\right)\right. \\
& \left.+(\bar\chi-\tilde\chi)\left[\tilde\partial^i_\perp\left(A^{(1)}-B^{(1)}_\|-\frac{1}{2}h^{(1)}_\|\right)+\frac{1}{\tilde\chi}\left(B^{i(1)}_\perp+n^k h^{(1)}_{kj}\mathcal{P}^{ij}\right)\right]\right\}. \qquad (134)
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
&\partial_\parallel \Delta x_\parallel^{(2)} = \left(A_o^{(1)}\right)^2 - 2A_o^{(1)} B_{\parallel o}^{(1)} + \left(B_{\parallel o}^{(1)}\right)^2 + 2A_o^{(1)} v_{\parallel o}^{(1)} - 2B_{\parallel o}^{(1)} v_{\parallel o}^{(1)} - v_{\parallel o}^{(1)} h_{\parallel o}^{(1)} - \frac{1}{4}\left(h_{\parallel o}^{(1)}\right)^2 - v_{\perp k\, o}^{(1)} v_{\perp o}^{k(1)} \\
&+ n^i h_{ik\, o}^{(1)} \mathcal{P}^k_j v_o^{j(1)} - n^i h_{ik\, o}^{(1)} \mathcal{P}^k_j B_o^{j(1)} - \frac{1}{4} n^i h_{ij\, o}^{(1)} \mathcal{P}^j_k h_{p\, o}^{k(1)} n^p - B_{\perp o}^{i(1)} B_{\perp i\, o}^{(1)} + 2 v_{\perp o}^{i(1)} B_{\perp i\, o}^{(1)} + 2\left(A_o^{(1)} - v_{\parallel o}^{(1)}\right) \\
&\times \Bigg\{ \left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right) \left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\parallel^{(1)}\right) + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right] - \frac{\bar\chi}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2A^{(1)} - B_\parallel^{(1)}\right)\right. \\
&\left. + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right] + \left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) - \bar\chi \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) - \frac{1}{\mathcal{H}^2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\parallel^{(1)}\right)\right. \\
&\left. + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right]\Bigg\} + 2\left(B_{\perp o}^{i(1)} - v_{\perp o}^{i(1)} + \frac{1}{2} n^k h_{k\, o}^{j(1)} \mathcal{P}^i_j\right)\left[-\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi} B_{\perp i}^{(1)} + \frac{1}{\mathcal{H}}\partial_{\perp i}\left(A^{(1)} - v_\parallel^{(1)}\right)\right. \\
&\left. - \frac{1}{\mathcal{H}}\left(B_{\perp i}^{(1)\prime} + n^j h_{jk}^{(1)\prime} \mathcal{P}^k_i\right) - 2\frac{\bar\chi}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_{\perp i}\left(A^{(1)} + v_\parallel^{(1)} - B_\parallel^{(1)}\right) + \frac{1}{\mathcal{H}\bar\chi} B_{\perp i}^{(1)} - 4\delta_{il} S_\perp^{l(1)}\right] + A^{(2)} - B_\parallel^{(2)} - \frac{1}{2} h_\parallel^{(2)} \\
&+ \frac{1}{\mathcal{H}}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(2)} - v_\parallel^{(2)}\right) + \left(A^{(2)\prime} - B_\parallel^{(2)\prime} - \frac{1}{2} h_\parallel^{(2)\prime}\right)\right] - 2\left(2A^{(1)} + h_\parallel^{(1)}\right)\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[7\left(A^{(1)}\right)^2\right. \\
&\left. + \left(B_\parallel^{(1)}\right)^2 - 4A^{(1)} B_\parallel^{(1)} - 2A^{(1)} v_\parallel^{(1)} + \left(v_\parallel^{(1)}\right)^2 + v_\parallel^{(1)} h_\parallel^{(1)} + v_{\perp i}^{(1)} v_\perp^{i(1)} - 2 v_{\perp i}^{(1)} B_\perp^{i(1)}\right] + \frac{2}{\mathcal{H}^2}\left(A^{(1)} - v_\parallel^{(1)}\right) \\
&\times \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\parallel^{(1)}\right) + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right] - \frac{2}{\mathcal{H}}\left[\left(2A^{(1)} + h_\parallel^{(1)}\right) + \left(A^{(1)} - v_\parallel^{(1)}\right)\right] \\
&\times \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right) + \frac{2}{\mathcal{H}}\left(2A^{(1)} - B_\parallel^{(1)}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2A^{(1)} - B_\parallel^{(1)}\right) + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right] \\
&+ 2\Bigg\{\frac{1}{\mathcal{H}}\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) + \frac{\mathcal{H}'}{\mathcal{H}^3}\left(A^{(1)} - v_\parallel^{(1)}\right) + \frac{1}{\mathcal{H}^2}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\parallel^{(1)}\right) + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right]\Bigg\} \\
&\times \left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\parallel^{(1)}\right) + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right] + \frac{2}{\mathcal{H}} v_{\perp i}^{(1)} \partial_\perp^i \left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) + \frac{2}{\mathcal{H}}\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) \\
&\times \left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2A^{(1)} - B_\parallel^{(1)}\right) - \partial_\parallel\left(A^{(1)} + v_\parallel^{(1)} - B_\parallel^{(1)}\right)\right] + \frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_\parallel\left(A^{(1)} + v_\parallel^{(1)} - B_\parallel^{(1)}\right) T^{(1)} + 2\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)} \mathcal{P}^{ik}\right) \\
&\times \left[\frac{1}{2}\left(B_{i\perp}^{(1)} + n^p h_{pm}^{(1)} \mathcal{P}^m_i\right) - \frac{1}{\mathcal{H}}\partial_{\perp i}\left(A^{(1)} - v_\parallel^{(1)}\right) + \frac{1}{\mathcal{H}}\left(B_{\perp i}^{(1)\prime} + n^j h_{jk}^{(1)\prime} \mathcal{P}^k_i\right)\right] + 4\Bigg\{\left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\parallel^{(1)}\right)\right. \\
&\left. + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right] - \frac{1}{\mathcal{H}^2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\parallel^{(1)}\right) + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right] + \left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right)\Bigg\} I^{(1)} \\
&+ 2\Bigg\{\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) + \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2A^{(1)} - B_\parallel^{(1)}\right) + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right]\Bigg\}\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[2A^{(1)} - B_\parallel^{(1)}\right. \\
&\left. + (\bar\chi - \tilde\chi)\left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right] + 4\left[\frac{1}{\mathcal{H}}\partial_{\perp i}\left(A^{(1)} - v_\parallel^{(1)}\right) - \frac{1}{\mathcal{H}}\left(B_{\perp i}^{(1)\prime} + n^j h_{jk}^{(1)\prime} \mathcal{P}^k_i\right) + \frac{1}{\mathcal{H}\bar\chi} v_{\perp i}^{(1)} - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi} v_{\perp i}^{(1)}\right. \\
&\left. - 2 S_\perp^{j(1)} \delta_{ij}\right] S_\perp^{i(1)} + \frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\partial_{\perp i}\left(A^{(1)} + v_\parallel^{(1)} - B_\parallel^{(1)}\right) - \frac{1}{\bar\chi}\left(v_{\perp i}^{(1)} - B_{\perp i}^{(1)}\right)\right]\int_0^{\bar\chi} \mathrm{d}\tilde\chi \Bigg\{\left(B_\perp^{i(1)} + n^k h_k^{j(1)} \mathcal{P}^i_j\right) \\
&+ (\bar\chi - \tilde\chi)\left[\tilde\partial_\perp^i\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) + \frac{1}{\tilde\chi}\left(B_\perp^{i(1)} + n^k h_{kj}^{(1)} \mathcal{P}^{ij}\right)\right]\Bigg\} - \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta \ln a^{(2)} + 2\Bigg\{-\frac{\mathcal{H}'}{\mathcal{H}^2}\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) \\
&- \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) - \left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\parallel^{(1)}\right) + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right)\right]\Bigg\}\Delta \ln a^{(1)} \\
&+ 2\int_0^{\bar\chi} \mathrm{d}\tilde\chi \Bigg\{\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right)\left[2\left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2} h_\parallel^{(1)\prime}\right) + \frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(2A^{(1)} - B_\parallel^{(1)}\right)\right] - \left(B_\perp^{i(1)} + n^k h_k^{j(1)} \mathcal{P}^i_j\right) \\
&\times \left[\tilde\partial_{\perp i}\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2} h_\parallel^{(1)}\right) + \frac{1}{\tilde\chi}\left(B_{\perp i}^{(1)} + n^m h_{mp}^{(1)} \mathcal{P}^p_i\right)\right]\Bigg\} + \left[-\frac{\mathcal{H}''}{\mathcal{H}^3} + 3\left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right]\left(\Delta \ln a^{(1)}\right)^2 . \qquad (135)
\end{aligned}
$$

From Eq. (132) and using Eqs. (126) (127) and (128) we obtain the coordinate convergence lensing term at second order

$$\kappa^{(2)} = -\frac{1}{2}\partial_{\perp i}\Delta x_{\perp}^{i(2)} = \kappa_1^{(2)} + \kappa_2^{(2)} + \kappa_3^{(2)} + \kappa_4^{(2)} \tag{136}$$

where

$$\kappa_1^{(2)} = \frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\,\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\nabla}_{\perp}^2\left(A^{(2)} - B_{\|}^{(2)} - \frac{1}{2}h_{\|}^{(2)}\right)\,, \tag{137}$$

$$\kappa_2^{(2)} = \frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left[\tilde{\partial}_{\perp}^{i}B_i^{(2)} - \frac{2}{\tilde{\chi}}B_{\|}^{(2)} + \mathcal{P}^{ij}n^k\tilde{\partial}_i h_{jk}^{(2)} + \frac{1}{\tilde{\chi}}\left(h_i^{i(2)} - 3h_{\|}^{(2)}\right)\right]\,, \tag{138}$$

$$
\begin{aligned}
\kappa_3^{(2)} \;=\; & \Bigg\{ -\frac{2}{\bar\chi}B_\parallel^{(1)} + \partial_\perp^i B_i^{(1)} + \frac{1}{\bar\chi}\left(h_i^{i(1)} - 3h_\parallel^{(1)}\right) + \mathcal{P}^{ij}n^k\partial_i h_{jk}^{(1)} + \frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Big[\tilde\partial_\perp^i B_i^{(1)} + \mathcal{P}^{ij}n^k\tilde\partial_i h_{jk}^{(1)} \\
& -\frac{1}{\tilde\chi}\left(2B_\parallel^{(1)} + 3h_\parallel^{(1)} - h_i^{i(1)}\right)\Big] + \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\frac{\tilde\chi}{\bar\chi}\tilde\nabla_\perp^2\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\Bigg\} \times \Bigg\{\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Big[2A^{(1)} - B_\parallel^{(1)} \\
& + (\bar\chi - \tilde\chi)\left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2}h_\parallel^{(1)\prime}\right)\Big] - \frac{1}{\mathcal{H}}\Delta\ln a^{(1)}\Bigg\} + \left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i - 2S_\perp^{i(1)}\right) \\
& \times\Bigg\{\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\frac{\tilde\chi}{\bar\chi}\left[\tilde\partial_{\perp i}\left(2A^{(1)} - B_\parallel^{(1)}\right) + (\bar\chi - \tilde\chi)\,\tilde\partial_{\perp i}\left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2}h_\parallel^{(1)\prime}\right)\right] - \frac{1}{\mathcal{H}}\partial_{\perp i}\Delta\ln a^{(1)}\Bigg\} \\
& + \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\Bigg[\frac{4}{\tilde\chi}A^{(1)}B_\parallel^{(1)} - 4A^{(1)}\tilde\partial_\perp^i B_i^{(1)} - \frac{2}{\tilde\chi}\left(B_\parallel^{(1)}\right)^2 + B_\parallel^{(1)}\tilde\partial_\perp^i B_i^{(1)} - \frac{3}{\tilde\chi}A^{(1)}h_i^{i(1)} + \frac{9}{\tilde\chi}A^{(1)}h_\parallel^{(1)} \\
& -3A^{(1)}\mathcal{P}^{ij}n^k\tilde\partial_i h_{jk}^{(1)} + \frac{2}{\tilde\chi}B_\parallel^{(1)}h_\parallel^{(1)} - h_\parallel^{(1)}\tilde\partial_\perp^i B_i^{(1)} - \frac{3}{2\tilde\chi}h_\parallel^{(1)}h_i^{i(1)} + \frac{9}{2\tilde\chi}\left(h_\parallel^{(1)}\right)^2 - \frac{3}{2}h_\parallel^{(1)}\mathcal{P}^{ij}n^k\tilde\partial_i h_{jk}^{(1)} \\
& -4\left(\frac{2}{\tilde\chi}B_\parallel^{(1)} - \tilde\partial_\perp^i B_i^{(1)} - \frac{1}{\tilde\chi}h_i^{i(1)} + \frac{3}{\tilde\chi}h_\parallel^{(1)} - \mathcal{P}^{ij}n^k\tilde\partial_i h_{jk}^{(1)}\right)I^{(1)}\Bigg] + \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\frac{\tilde\chi}{\bar\chi}\Big[-\frac{4}{\tilde\chi}A^{(1)}B_\parallel^{(1)} + 2A^{(1)}\tilde\partial_\perp^i B_i^{(1)} \\
& +\frac{2}{\tilde\chi}A^{(1)}h_i^{i(1)} - \frac{6}{\tilde\chi}A^{(1)}h_\parallel^{(1)} + 2A^{(1)}\mathcal{P}^{ij}n^k\tilde\partial_i h_{jk}^{(1)} + \frac{1}{\tilde\chi}B_\parallel^{(1)}h_i^{i(1)} - \frac{3}{\tilde\chi}B_\parallel^{(1)}h_\parallel^{(1)} - h_\parallel^{(1)}\tilde\partial_\perp^i B_i^{(1)} - \frac{4}{\tilde\chi}\left(h_\parallel^{(1)}\right)^2 \\
& +h_\parallel^{(1)}\mathcal{P}^{ij}n^k\tilde\partial_i h_{jk}^{(1)} - 2\,\tilde\partial_{\perp i}A^{(1)}B_\perp^{i(1)} + \frac{1}{\tilde\chi}B_{\perp i}^{(1)}B_\perp^{i(1)} + n^j B_\perp^{i(1)}\tilde\partial_{\perp i}B_j^{(1)} - \tilde\partial_i A^{(1)}n^j h_{jk}^{(1)}\mathcal{P}^{ki} + \frac{2}{\tilde\chi}n^p h_{pq}^{(1)}\mathcal{P}^{qj}h_{jk}^{(1)}n^k \\
& -\frac{1}{2}n^j h_{jk}^{(1)}\mathcal{P}^{ki}n^p n^q\tilde\partial_{\perp i}h_{pq}^{(1)} - h_l^{k(1)}\mathcal{P}_q^l\,\tilde\partial_{\perp k}B^{q(1)} - \frac{1}{\tilde\chi}\mathcal{P}_k^m\mathcal{P}_p^l h_m^{p(1)}h_l^{k(1)} + \frac{3}{\tilde\chi}n^l B_{\perp k}^{(1)}h_l^{k(1)} - B_\perp^{l(1)}\tilde\partial_{\perp k}h_l^{k(1)} \\
& +\frac{2}{\tilde\chi}h_\parallel^{(1)}h_i^{i(1)} - n^m h_m^{p(1)}\mathcal{P}_p^l\tilde\partial_{\perp k}h_l^{k(1)} + 4\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right)\tilde\partial_{\perp i}I^{(1)} + 2\left(-\frac{3}{\tilde\chi}n^l\mathcal{P}_{kj}h_l^{k(1)} + \mathcal{P}_j^l\tilde\partial_{\perp k}h_l^{k(1)}\right)S_\perp^{j(1)} \\
& -\frac{2}{\tilde\chi}\mathcal{P}_k^l h_l^{k(1)}S_\parallel^{(1)} + 2\,\mathcal{P}_m^l h_l^{k(1)}\tilde\partial_{\perp k}S^{m(1)}\Big] + \int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi - \tilde\chi)\,\frac{\tilde\chi}{\bar\chi}\Bigg\{\tilde\partial_{\perp i}\left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2}h_\parallel^{(1)\prime}\right) \\
& \times\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right) + \left(A^{(1)\prime} - B_\parallel^{(1)\prime} - \frac{1}{2}h_\parallel^{(1)\prime}\right)\left(-\frac{2}{\tilde\chi}B_\parallel^{(1)} + \tilde\partial_\perp^i B_i^{(1)} + \frac{1}{\tilde\chi}h_i^{i(1)} - \frac{3}{\tilde\chi}h_\parallel^{(1)} + n^k\tilde\partial_\perp^{j(1)}h_{jk}^{(1)}\right) \\
& -\left(2\partial_{\perp i}A^{(1)} - \frac{1}{\tilde\chi}B_{\perp i}^{(1)} - n_k\tilde\partial_{\perp i}B_k^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right) - \left(2A^{(1)} - B_\parallel^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(-\frac{2}{\tilde\chi}B_\parallel^{(1)} + \tilde\partial_\perp^i B_i^{(1)}\right. \\
& \left.+\frac{1}{\tilde\chi}h_i^{i(1)} - \frac{3}{\tilde\chi}h_\parallel^{(1)} + n^k\tilde\partial_\perp^{j(1)}h_{jk}^{(1)}\right) - 2\left(-\mathcal{P}_{jq}\tilde\partial_{\perp i}B^{q(1)} - n^k\mathcal{P}_{jq}\tilde\partial_{\perp i}h_k^{q(1)} + 2\mathcal{P}_{jm}\tilde\partial_{\perp j}S^{m(1)}\right) \\
& \times\left(\mathcal{P}^{l[j}\,\partial_\perp^{i]}B_l^{(1)} + n^m\mathcal{P}^{l[j}\,\partial_\perp^{i]}h_{lm}^{(1)}\right) + \left(B_{\perp j}^{(1)} + n^p h_p^{k(1)}\mathcal{P}_{jk} - 2\delta_{jp}S_\perp^{p(1)}\right)\left(\mathcal{P}^{jl}\tilde\nabla_\perp^2 B_l^{(1)} - \tilde\partial_\perp^j\tilde\partial_\perp^k B_k^{(1)}\right. \\
& \left.+\mathcal{P}^{jl}n^m\tilde\nabla_\perp^2 h_{lm}^{(1)} - n^m\,\tilde\partial_\perp^j\tilde\partial_\perp^k h_{km}^{(1)} + \frac{1}{\tilde\chi}n^m n^k\tilde\partial_\perp^j h_{km}^{(1)} + \frac{1}{\tilde\chi}\mathcal{P}^{jl}\tilde\partial_\perp^m h_{lm}^{(1)} - \frac{1}{\tilde\chi}\tilde\partial_\perp^j h_i^{i(1)}\right) \\
& -\tilde\partial_{\perp i}\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\left[\tilde\partial_\perp^i\left(B_\parallel^{(1)} + h_\parallel^{(1)}\right) - \partial_\parallel\left(B_\perp^{i(1)} + n^p h_{pq}^{(1)}\mathcal{P}^{iq}\right) - \frac{1}{\tilde\chi}\left(B_\perp^{i(1)} + 2n^p h_{pq}^{(1)}\mathcal{P}^{iq}\right)\right] \\
& -\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\left[\tilde\nabla_\perp^2\left(B_\parallel^{(1)} + h_\parallel^{(1)}\right) - \tilde\partial_\parallel\left(-\frac{2}{\tilde\chi}B_\parallel^{(1)} + \tilde\partial_\perp^i B_i^{(1)} + \frac{1}{\tilde\chi}h_i^{i(1)} - \frac{3}{\tilde\chi}h_\parallel^{(1)} + n^k\tilde\partial_\perp^{j(1)}h_{jk}^{(1)}\right)\right] \\
& -2\left(2\tilde\partial_\perp^i A^{(1)} - \frac{1}{\tilde\chi}B_{\perp j}^{(1)} - n_k\tilde\partial_{\perp i}B_k^{(1)} - 2\tilde\partial_\perp^i I^{(1)}\right)\left[\tilde\partial_\perp^i\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right) + \frac{1}{\tilde\chi}\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)}\mathcal{P}^{ki}\right)\right] \\
& -2\left(2A^{(1)} - B_\parallel^{(1)} - 2I^{(1)}\right)\tilde\nabla_\perp^2\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\Bigg\}\,, \qquad (139)
\end{aligned}
$$

$$
\begin{aligned}
\kappa_4^{(2)} &= B^{(2)}_{\|\,o} - v^{(2)}_{\|\,o} + \frac{3}{4} h^{(2)}_{\|\,o} - \frac{1}{4} h^{i(2)}_{i\,o} - 2A^{(1)}_o B^{(1)}_{\|\,o} + 2A^{(1)}_o v^{(1)}_{\|\,o} - \frac{1}{2} A^{(1)}_o h^{i(1)}_{i\,o} + \frac{3}{2} A^{(1)}_o h^{(1)}_{\|\,o} + \frac{1}{2} v^{(1)}_{\perp i\,o} v^{i(1)}_{\perp\,o} - \left(v^{(1)}_{\|\,o}\right)^2 \\
&+ v^{(1)}_{\perp i\,o} B^{i(1)}_{\perp\,o} - 2 v^{(1)}_{\|\,o} B^{(1)}_{\|\,o} - \frac{1}{2} B^{(1)}_{\perp i\,o} B^{i(1)}_{\perp\,o} + \left(B^{(1)}_{\|\,o}\right)^2 - \frac{7}{2} v^{(1)}_{\|\,o} h^{(1)}_{\|\,o} + \frac{1}{2} v^{(1)}_{\|\,o} h^{i(1)}_{i\,o} - \frac{3}{2} v^{(1)}_{i\,o} n^k h^{j(1)}_{k\,o} \mathcal{P}^i_j \\
&+ \frac{1}{8} h^{j(1)}_{k\,o} h^{k(1)}_{j\,o} - \frac{3}{8} n^i h^{j(1)}_{i\,o} h^{k(1)}_{j\,o} n_k + v^{(1)}_{\perp i\,o} \Bigg\{ \left(B^{i(1)}_\perp + n^k h^{(1)}_{kj} \mathcal{P}^{ij}\right) - 2 S^{i(1)}_\perp - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \Bigg[\frac{2}{\tilde\chi} \left(B^{i(1)}_\perp + n^k h^{j(1)}_k \mathcal{P}^i_j\right) \\
&+ 2\left(1 - \frac{\tilde\chi}{\bar\chi}\right) \tilde\partial^i_\perp \left(A^{(1)} - B^{(1)}_\| - \frac{1}{2} h^{(1)}_\|\right) \Bigg] \Bigg\} + \left(A^{(1)}_o - v^{(1)}_{\|\,o}\right) \Bigg\{ 2B^{(1)}_\| - \bar\chi \partial_{\perp i} B^{i(1)} + 3h^{(1)}_\| - h^{i(1)}_i \\
&- \bar\chi n^k \partial_{\perp j} h^{j(1)}_k - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\tilde\partial_{\perp i} B^{i(1)} + n^k \tilde\partial_{\perp j} h^{j(1)}_k - \frac{2}{\tilde\chi} B^{(1)}_\| + \frac{1}{\tilde\chi}\left(h^{i(1)}_i - 3h^{(1)}_\|\right) + \tilde\chi \tilde\nabla^2_\perp \left(A^{(1)} - B^{(1)}_\| - \frac{1}{2} h^{(1)}_\|\right) \right] \\
&+ 4\kappa^{(1)}_2 + 4\kappa^{(1)}_1 \Bigg\} - \frac{1}{\bar\chi} \left(B^{(1)}_{\|\,o} - v^{(1)}_{\|\,o} + \frac{1}{2} h^{(1)}_{\|\,o}\right) \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(h^{i(1)}_i - h^{(1)}_\|\right) + \frac{1}{2\bar\chi} \mathcal{P}^k_i\, h^{l(1)}_{k\,o} \mathcal{P}_{jl} \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(\mathcal{P}^{jm}\, h^{p(1)}_m \mathcal{P}^{i(1)}_p\right) \\
&+ \left(B^{(1)}_{\perp j\,o} - v^{(1)}_{\perp j\,o} + \frac{1}{2} n^k\, h^{l(1)}_{k\,o} \mathcal{P}_{jl}\right) \int_0^{\bar\chi} \mathrm{d}\tilde\chi \Bigg\{ - \mathcal{P}^{jl} \tilde\partial^m_\perp h^{(1)}_{lm} + \frac{\tilde\chi}{\bar\chi} \Bigg[- \frac{3}{\tilde\chi} n_p\, h^{p(1)}_m \mathcal{P}^{mj} + 2\mathcal{P}^{jm} \partial_{\perp p} h^{p(1)}_m \\
&- (\bar\chi - \tilde\chi) \left(\mathcal{P}^{jl} \tilde\nabla^2_\perp B^{(1)}_l - \tilde\partial^j_\perp \tilde\partial^k_\perp B^{(1)}_k + \mathcal{P}^{jl} n^m \tilde\nabla^2_\perp h^{(1)}_{lm} - n^m\, \tilde\partial^j_\perp \tilde\partial^k_\perp h^{(1)}_{km}\right) \\
&- \left(\frac{\bar\chi}{\tilde\chi} - 1\right) \left(n^m n^k \tilde\partial^j_\perp h^{(1)}_{km} - \tilde\partial^j_\perp h^{k(1)}_k\right) \Bigg] \Bigg\} + \left[\frac{2}{\bar\chi} \left(B^{(1)}_{\|\,o} - v^{(1)}_{\|\,o}\right) - \frac{1}{2\bar\chi} h^{i(1)}_{i\,o} + \frac{3}{2\bar\chi} h^{(1)}_{\|\,o} \right] \\
&\times \Bigg\{ \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[2A^{(1)} - B^{(1)}_\| + (\bar\chi - \tilde\chi) \left(A^{(1)\prime} - B^{(1)\prime}_\| - \frac{1}{2} h^{(1)\prime}_\|\right) \right] - \frac{1}{\mathcal{H}} \Delta \ln a^{(1)} \Bigg\} \\
&- \left(B^{i(1)}_{\perp\,o} - v^{i(1)}_{\perp\,o} + \frac{1}{2} n^k h^{j(1)}_{k\,o} \mathcal{P}^i_j\right) \Bigg\{ \int_0^{\bar\chi} \mathrm{d}\tilde\chi \frac{\tilde\chi}{\bar\chi} \left[\tilde\partial_{\perp i} \left(2A^{(1)} - B^{(1)}_\|\right) + (\bar\chi - \tilde\chi)\, \tilde\partial_{\perp i} \left(A^{(1)\prime} - B^{(1)\prime}_\| - \frac{1}{2} h^{(1)\prime}_\|\right) \right] \\
&- \frac{1}{\mathcal{H}} \partial_{\perp i} \Delta \ln a^{(1)} \Bigg\} \,. \qquad (140)
\end{aligned}
$$

Finally, in order to obtain $\Delta^{(2)}_g$ [see Eq. (77)], we need the following

$$
\hat g^{\mu(2)}_\mu - \hat g^{\nu(1)}_\mu\, \hat g^{\mu(1)}_\nu = 2A^{(2)} + h^{i(2)}_i - 4\left(A^{(1)}\right)^2 + 2B^{(1)}_i B^{i(1)} - h^{k(1)}_i h^{i(1)}_k \,, \qquad (141)
$$

$$
E^{0(2)}_{\hat 0} + E^{\|(2)}_{\hat 0} = -A^{(2)} + v^{(2)}_\| + 3\left(A^{(1)}\right)^2 + v^{(1)}_i v^{i(1)} - 2v^{(1)}_i B^{i(1)} \,, \qquad (142)
$$

$$
\begin{aligned}
\frac{1}{\mathcal{H}} \hat g^{\mu(1)\prime}_\mu \Delta \ln a^{(1)} + \left(\partial_\| \hat g^{\mu(1)}_\mu\right) \Delta x^{(1)}_\| &= -\frac{1}{\mathcal{H}} \Delta \ln a^{(1)} \frac{\mathrm{d}}{\mathrm{d}\bar\chi} \left(\hat g^{\mu(1)}_\mu\right) - \left(\partial_\| \hat g^{\mu(1)}_\mu\right) T^{(1)} \\
&= -\frac{1}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar\chi} \left(2A^{(1)} + h^{i(1)}_i\right) \Delta \ln a^{(1)} - \partial_\| \left(2A^{(1)} + h^{i(1)}_i\right) T^{(1)} \,, \qquad (143)
\end{aligned}
$$

$$
\begin{aligned}
\left(\partial_{\perp i} \hat g^{\mu(1)}_\mu\right) \Delta x^{i(1)}_\perp &= \partial_{\perp i} \left(2A^{(1)} + h^{i(1)}_i\right) \delta x^{i(1)}_\perp = \bar\chi \left(B^{i(1)}_{\perp\,o} - v^{i(1)}_{\perp\,o} + \frac{1}{2} n^k h^{j(1)}_{k\,o} \mathcal{P}^i_j\right) \partial_{\perp i} \left(2A^{(1)} + h^{i(1)}_i\right) \\
&- \partial_{\perp i} \left(2A^{(1)} + h^{i(1)}_i\right) \int_0^{\bar\chi} \mathrm{d}\tilde\chi \Bigg\{ \left(B^{i(1)}_\perp + n^k h^{j(1)}_k \mathcal{P}^i_j\right) + (\bar\chi - \tilde\chi) \\
&\times \left[\tilde\partial^i_\perp \left(A^{(1)} - B^{(1)}_\| - \frac{1}{2} h^{(1)}_\|\right) + \frac{1}{\tilde\chi} \left(B^{i(1)}_\perp + n^k h^{(1)}_{kj} \mathcal{P}^{ij}\right) \right] \Bigg\} \,, \qquad (144)
\end{aligned}
$$

$$
\frac{1}{\mathcal{H}} \delta^{(1)\prime}_g \Delta \ln a^{(1)} + \left(\partial_\| \delta^{(1)}_g\right) \Delta x^{(1)}_\| = -\frac{1}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar\chi} \left(\delta^{(1)}_g\right) \Delta \ln a^{(1)} - \left(\partial_\| \delta^{(1)}_g\right) T^{(1)} \,, \qquad (145)
$$

$$\Delta x^{(1)}_{\perp i}\,\partial^i_\perp\delta^{(1)}_g \;=\; \delta x^{(1)}_{\perp i}\,\partial^i_\perp\delta^{(1)}_g=\bar\chi\left(B^{i(1)}_{\perp o}-v^{i(1)}_{\perp o}+\frac{1}{2}n^kh^{j(1)}_{k\,o}\mathcal{P}^i_j\right)\partial_{\perp i}\delta^{(1)}_g-\left(\partial_{\perp i}\delta^{(1)}_g\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\left(B^{i(1)}_\perp+n^kh^{j(1)}_k\mathcal{P}^i_j\right)$$
$$+\;(\bar\chi-\tilde\chi)\left[\tilde\partial^i_\perp\left(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\right)+\frac{1}{\tilde\chi}\left(B^{i(1)}_\perp+n^kh^{(1)}_{kj}\mathcal{P}^{ij}\right)\right]\Bigg\}\,, \tag{146}$$

$$-\frac{1}{\bar\chi^2}\left(\Delta x^{(1)}_\parallel\right)^2=-\frac{1}{\bar\chi^2}\left(T^{(1)}\right)^2-\frac{1}{\bar\chi^2\mathcal{H}^2}\left(\Delta\ln a^{(1)}\right)^2-\frac{2}{\bar\chi^2\mathcal{H}}\Delta\ln a^{(1)}T^{(1)}\,, \tag{147}$$

$$\frac{2}{\bar\chi}\Delta x^{(1)}_\parallel\kappa^{(1)}=-\frac{2}{\bar\chi\mathcal{H}}\Delta\ln a^{(1)}\kappa^{(1)}-\frac{2}{\bar\chi}T^{(1)}\kappa^{(1)}\,, \tag{148}$$

$$-\;\frac{1}{2}\left(\partial_{\perp j}\Delta x^{i(1)}_\perp\right)\left(\partial_{\perp i}\Delta x^{j(1)}_\perp\right)=-\left(v^{(1)}_{\parallel o}\right)^2-\left(A^{(1)}_o\right)^2-\frac{1}{2}v^{(1)}_{\parallel o}h^{i(1)}_{i\,o}+\frac{1}{2}v^{(1)}_{\parallel o}h^{(1)}_{\parallel o}-\frac{1}{8}\mathcal{P}^m_p\mathcal{P}^k_nh^{n(1)}_{m\,o}h^{p(1)}_{k\,o}$$
$$+\;2A^{(1)}_ov^{(1)}_{\parallel o}+\frac{1}{2}A^{(1)}_oh^{i(1)}_{i\,o}-\frac{1}{2}A^{(1)}_oh^{(1)}_{\parallel o}+\left(2A^{(1)}_o-2v^{(1)}_{\parallel o}-\frac{1}{2}h^{i(1)}_{i\,o}+\frac{1}{2}h^{(1)}_{\parallel o}\right)\left(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel-2I^{(1)}\right)$$
$$+\;\left(\mathcal{P}^i_jA^{(1)}_o-\mathcal{P}^i_jv^{(1)}_{\parallel o}-\frac{1}{2}\mathcal{P}^k_j\mathcal{P}^i_ph^{p(1)}_{k\,o}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\frac{1}{\tilde\chi}\mathcal{P}^j_iB^{(1)}_\parallel-\mathcal{P}^j_m\tilde\partial_{\perp i}B^{m(1)}-\frac{1}{\tilde\chi}\mathcal{P}^n_i\mathcal{P}^j_mh^{m(1)}_n+\frac{1}{\tilde\chi}\mathcal{P}^j_ih^{(1)}_\parallel-n^m\mathcal{P}^j_n\tilde\partial_{\perp i}h^{n(1)}_m$$
$$-\;\frac{\tilde\chi}{\bar\chi}\left[\mathcal{P}^j_i\tilde\partial_\parallel+(\bar\chi-\tilde\chi)\,\mathcal{P}^n_i\mathcal{P}^{jm}\tilde\partial_m\tilde\partial_n\right]\left(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\right)\Bigg\}-\left(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel-2I^{(1)}\right)^2$$
$$-\;\left(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel-2I^{(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\frac{2}{\tilde\chi}B^{(1)}_\parallel-\tilde\partial_{\perp m}B^{m(1)}-\frac{1}{\tilde\chi}h^{i(1)}_i+\frac{3}{\tilde\chi}h^{(1)}_\parallel-n^m\tilde\partial_{\perp n}h^{n(1)}_m$$
$$-\;\frac{\tilde\chi}{\bar\chi}\left[2\tilde\partial_\parallel+(\bar\chi-\tilde\chi)\,\mathcal{P}^{mn}\tilde\partial_m\tilde\partial_n\right]\left(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\right)\Bigg\}-\frac{1}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\frac{1}{\tilde\chi}\mathcal{P}^i_jB^{(1)}_\parallel-\mathcal{P}^i_p\tilde\partial_{\perp j}B^{p(1)}-\frac{1}{\tilde\chi}\mathcal{P}^p_j\mathcal{P}^i_qh^{q(1)}_p$$
$$+\;\frac{1}{\tilde\chi}\mathcal{P}^i_jh^{(1)}_\parallel-n^p\mathcal{P}^i_q\tilde\partial_{\perp j}h^{q(1)}_p-\frac{\tilde\chi}{\bar\chi}\left[\mathcal{P}^i_j\tilde\partial_\parallel+(\bar\chi-\tilde\chi)\,\mathcal{P}^p_j\mathcal{P}^{iq}\tilde\partial_q\tilde\partial_p\right]\left(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\right)\Bigg\}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\frac{1}{\tilde\chi}\mathcal{P}^j_iB^{(1)}_\parallel$$
$$-\;\mathcal{P}^j_m\tilde\partial_{\perp i}B^{m(1)}-\frac{1}{\tilde\chi}\mathcal{P}^n_i\mathcal{P}^j_mh^{m(1)}_n+\frac{1}{\tilde\chi}\mathcal{P}^j_ih^{(1)}_\parallel-n^m\mathcal{P}^j_n\tilde\partial_{\perp i}h^{n(1)}_m-\frac{\tilde\chi}{\bar\chi}\left[\mathcal{P}^j_i\tilde\partial_\parallel+(\bar\chi-\tilde\chi)\,\mathcal{P}^n_i\mathcal{P}^{jm}\tilde\partial_m\tilde\partial_n\right]$$
$$\times\;\left(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\right)\Bigg\}\,, \tag{149}$$

$$\left(\frac{1}{\bar\chi}\Delta x^{(1)}_{\perp i}-\partial_{\perp i}\Delta x^{(1)}_\parallel\right)\partial_\parallel\Delta x^{i(1)}_\perp=\left(\frac{1}{\bar\chi}\delta x^{(1)}_{\perp i}-\partial_{\perp i}\Delta x^{(1)}_\parallel\right)\delta n^{i(1)}_\perp=\left(B^{(1)}_{\perp i\,o}-v^{(1)}_{\perp i\,o}+\frac{1}{2}n^kh^{j(1)}_{k\,o}\mathcal{P}_{ij}\right)$$
$$\times\left(B^{i(1)}_{\perp o}-v^{i(1)}_{\perp o}+\frac{1}{2}n^kh^{j(1)}_{k\,o}\mathcal{P}^i_j\right)+\left(B^{(1)}_{\perp i\,o}-v^{(1)}_{\perp i\,o}+\frac{1}{2}n^kh^{j(1)}_{k\,o}\mathcal{P}_{ij}\right)\Bigg\{-\left(B^{i(1)}_\perp+n^kh^{j(1)}_k\mathcal{P}^i_j\right)+2S^{i(1)}_\perp$$
$$+\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)}+T^{(1)}\right)-\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\bar\chi}{\tilde\chi}\left(B^{i(1)}_\perp+n^kh^{j(1)}_k\mathcal{P}^i_j\right)+(\bar\chi-\tilde\chi)\,\tilde\partial^i_\perp\left(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\right)\right]\Bigg\}$$
$$+\left[-\left(B^{i(1)}_\perp+n^kh^{j(1)}_k\mathcal{P}^i_j\right)+2S^{i(1)}_\perp\right]\Bigg\{-\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\bar\chi}{\tilde\chi}\left(B^{i(1)}_\perp+n^kh^{j(1)}_k\mathcal{P}^i_j\right)+(\bar\chi-\tilde\chi)\,\tilde\partial^i_\perp\left(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\right)\right]$$
$$+\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)}+T^{(1)}\right)\Bigg\}\,, \tag{150}$$

$$\frac{1}{\mathcal{H}}\left(E^{0(1)}_{\hat 0}+E^{\parallel(1)}_{\hat 0}\right)'\Delta\ln a^{(1)}+\partial_\parallel\left(E^{0(1)}_{\hat 0}+E^{\parallel(1)}_{\hat 0}\right)\Delta x^{(1)}_\parallel=\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}-v^{(1)}_\parallel\right)\Delta\ln a^{(1)}+\partial_\parallel\left(A^{(1)}-v^{(1)}_\parallel\right)T^{(1)} \tag{151}$$

$$\partial_{\perp i}\left(E^{0(1)}_{\hat{0}}+E^{\|(1)}_{\hat{0}}\right)\Delta x^{i(1)}_{\perp}=-\bar{\chi}\left(B^{i(1)}_{\perp\,o}-v^{i(1)}_{\perp\,o}+\frac{1}{2}n^k h^{j(1)}_{k\,o}\mathcal{P}^i_j\right)\partial_{\perp i}\left(A^{(1)}-v^{(1)}_{\|}\right)+\partial_{\perp i}\left(A^{(1)}-v^{(1)}_{\|}\right)$$
$$\times\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left\{\left(B^{i(1)}_{\perp}+n^k h^{j(1)}_{k}\mathcal{P}^i_j\right)+(\bar{\chi}-\tilde{\chi})\left[\tilde{\partial}^i_{\perp}\left(A^{(1)}-B^{(1)}_{\|}-\frac{1}{2}h^{(1)}_{\|}\right)+\frac{1}{\tilde{\chi}}\left(B^{i(1)}_{\perp}+n^k h^{(1)}_{kj}\mathcal{P}^{ij}\right)\right]\right\}\;, \tag{152}$$

$$-\left(\delta n^{(1)}_{\|}+\delta\nu^{(1)}\right)E^{\|(1)}_{\hat{0}}=v^{(1)}_{\|}\left(\frac{\mathrm{d}T}{\mathrm{d}\bar{\chi}}\right)^{(1)}=-v^{(1)}_{\|}\left(A^{(1)}-B^{(1)}_{\|}-\frac{1}{2}h^{(1)}_{\|}\right)\;, \tag{153}$$

$$-E^{\perp i(1)}_{\hat{0}}\partial_{\perp i}\left(\Delta x^{0(1)}+\Delta x^{(1)}_{\|}\right)=-E^{\perp i(1)}_{\hat{0}}\partial_{\perp i}\left(\delta x^{0(1)}+\delta x^{(1)}_{\|}\right)=v^{i(1)}_{\perp}\partial_{\perp i}T^{(1)}\;, \tag{154}$$

$$-\frac{1}{2}\left(\partial_{\|}\Delta x^{(1)}_{\|}\right)^2=-\frac{1}{2}\left(A^{(1)}-B^{(1)}_{\|}-\frac{1}{2}h^{(1)}_{\|}\right)^2-\frac{1}{2\mathcal{H}^2}\left[\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(A^{(1)}-v^{(1)}_{\|}\right)+\left(A^{(1)\prime}-B^{(1)\prime}_{\|}-\frac{1}{2}h^{(1)\prime}_{\|}\right)\right]^2$$
$$-\frac{1}{\mathcal{H}}\left(A^{(1)}-B^{(1)}_{\|}-\frac{1}{2}h^{(1)}_{\|}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(A^{(1)}-v^{(1)}_{\|}\right)+\left(A^{(1)\prime}-B^{(1)\prime}_{\|}-\frac{1}{2}h^{(1)\prime}_{\|}\right)\right]+\frac{\mathcal{H}'}{\mathcal{H}^2}\left(A^{(1)}-B^{(1)}_{\|}-\frac{1}{2}h^{(1)}_{\|}\right)\Delta\ln a^{(1)}$$
$$+\frac{\mathcal{H}'}{\mathcal{H}^3}\left[\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(A^{(1)}-v^{(1)}_{\|}\right)+\left(A^{(1)\prime}-B^{(1)\prime}_{\|}-\frac{1}{2}h^{(1)\prime}_{\|}\right)\right]\Delta\ln a^{(1)}-\frac{1}{2}\left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2\left(\Delta\ln a^{(1)}\right)^2\;, \tag{155}$$

$$-\frac{1}{2}\left(E^{0(1)}_{\hat{0}}+E^{\|(1)}_{\hat{0}}\right)^2=-\frac{1}{2}\left(A^{(1)}-v^{(1)}_{\|}\right)^2\;. \tag{156}$$

Putting together the above relations we obtain finally

$$
\begin{aligned}
\Delta_g^{(2)} &= \delta_g^{(2)} + b_e \, \Delta \ln a^{(2)} + \partial_\| \Delta x_\|^{(2)} + \frac{2}{\bar\chi} \Delta x_\|^{(2)} - 2\kappa^{(2)} + A^{(2)} + v_\|^{(2)} + \frac{1}{2} h_i^{i(2)} + \left(\Delta_g^{(1)}\right)^2 - 3\left(A^{(1)}\right)^2 + 6 A^{(1)} B_\|^{(1)} \\
&+ 3A^{(1)} h_\|^{(1)} - 3B_\|^{(1)} h_\|^{(1)} + v_{\perp i}^{(1)} v_\perp^{i(1)} - 2 v_{\perp i}^{(1)} B_\perp^{i(1)} - 2\left(B_\|^{(1)}\right)^2 + B_{\perp i}^{(1)} B_\perp^{i(1)} - \frac{1}{2} h_i^{k(1)} h_k^{i(1)} - \frac{3}{4}\left(h_\|^{(1)}\right)^2 - 8\left(I^{(1)}\right)^2 \\
&+ 8A^{(1)} I^{(1)} - 8B_\|^{(1)} I^{(1)} - 4h_\|^{(1)} I^{(1)} - \left(\delta_g^{(1)}\right)^2 + v_\|^{(1)} h_\|^{(1)} - \frac{1}{\mathcal{H}^2}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\|^{(1)}\right) + \left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2} h_\|^{(1)\prime}\right)\right]^2 \\
&- \frac{2}{\mathcal{H}}\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\|^{(1)}\right) + \left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2} h_\|^{(1)\prime}\right)\right] - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2v_\|^{(1)} + h_i^{i(1)} + 2\delta_g^{(1)}\right)\Delta \ln a^{(1)} \\
&- \partial_\|\left(2v_\|^{(1)} + h_i^{i(1)} + 2\delta_g^{(1)}\right) T^{(1)} - \frac{4}{\bar\chi^2 \mathcal{H}} \Delta \ln a^{(1)} T^{(1)} + 2 v_\perp^{i(1)} \partial_{\perp i} T^{(1)} - \frac{4}{\bar\chi \mathcal{H}} \Delta \ln a^{(1)} \kappa^{(1)} - \frac{4}{\bar\chi} T^{(1)} \kappa^{(1)} \\
&+ \left[-b_e + \frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln \bar a} - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 - \frac{2}{\bar\chi^2 \mathcal{H}^2}\right]\left(\Delta \ln a^{(1)}\right)^2 - \frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2 + 2\frac{\mathcal{H}'}{\mathcal{H}^2}\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right)\Delta \ln a^{(1)} \\
&+ 2\frac{\mathcal{H}'}{\mathcal{H}^3}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\|^{(1)}\right) + \left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2} h_\|^{(1)\prime}\right)\right]\Delta \ln a^{(1)} + 2\left[-\left(B_\perp^{i(1)} + n^k h_k^{j(1)} \mathcal{P}_j^i\right) + 2 S_\perp^{i(1)}\right] \\
&\times \partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta \ln a^{(1)} + T^{(1)}\right) - \left[-\frac{2}{\bar\chi}\left(B_\perp^{i(1)} + n^k h_k^{j(1)} \mathcal{P}_j^i\right) + \frac{4}{\bar\chi} S_\perp^{i(1)} + \partial_\perp^i\left(2v_\|^{(1)} + h_l^{l(1)} + 2\delta_g^{(1)}\right)\right] \\
&\times \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\frac{\bar\chi}{\tilde\chi}\left(B_{\perp i}^{(1)} + n^k h_k^{j(1)} \mathcal{P}_{ij}\right) + \left(\bar\chi - \tilde\chi\right)\tilde\partial_{\perp i}\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right)\right] \\
&- 2\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)} - 2I^{(1)}\right)\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left\{\frac{2}{\tilde\chi} B_\|^{(1)} - \tilde\partial_{\perp m} B^{m(1)} - \frac{1}{\tilde\chi} h_i^{i(1)} + \frac{3}{\tilde\chi} h_\|^{(1)} - n^m \tilde\partial_{\perp n} h_m^{n(1)}\right. \\
&\left. - \frac{\tilde\chi}{\bar\chi}\left[2\tilde\partial_\| + \left(\bar\chi - \tilde\chi\right)\mathcal{P}^{mn}\tilde\partial_m \tilde\partial_n\right]\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right)\right\} - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left\{\frac{1}{\tilde\chi}\mathcal{P}_j^i B_\|^{(1)} - \mathcal{P}_p^i \tilde\partial_{\perp j} B^{p(1)} - \frac{1}{\tilde\chi}\mathcal{P}_j^p \mathcal{P}_q^i h_p^{q(1)} + \frac{1}{\tilde\chi}\mathcal{P}_j^i h_\|^{(1)}\right. \\
&\left. - n^p \mathcal{P}_q^i \tilde\partial_{\perp j} h_p^{q(1)} - \frac{\tilde\chi}{\bar\chi}\left[\mathcal{P}_j^i \tilde\partial_\| + \left(\bar\chi - \tilde\chi\right)\mathcal{P}_j^p \mathcal{P}^{iq}\tilde\partial_q \tilde\partial_p\right]\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right)\right\} \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left\{\frac{1}{\tilde\chi}\mathcal{P}_i^j B_\|^{(1)} - \mathcal{P}_m^j \tilde\partial_{\perp i} B^{m(1)}\right. \\
&\left. + \frac{1}{\tilde\chi}\mathcal{P}_i^j h_\|^{(1)} - \frac{1}{\tilde\chi}\mathcal{P}_i^n \mathcal{P}_m^j h_n^{m(1)} - n^m \mathcal{P}_n^j \tilde\partial_{\perp i} h_m^{n(1)} - \frac{\tilde\chi}{\bar\chi}\left[\mathcal{P}_i^j \tilde\partial_\| + \left(\bar\chi - \tilde\chi\right)\mathcal{P}_i^n \mathcal{P}^{jm}\tilde\partial_m \tilde\partial_n\right]\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right)\right\} - 2\left(v_{\| o}^{(1)}\right)^2 \\
&+ v_{\| o}^{(1)} h_{\| o}^{(1)} + 4A_o^{(1)} v_{\| o}^{(1)} + A_o^{(1)} h_{i\, o}^{i(1)} - A_o^{(1)} h_{\| o}^{(1)} + 2\left(B_{\perp i\, o}^{(1)} - v_{\perp i\, o}^{(1)} + \frac{1}{2} n^k h_{k\, o}^{j(1)} \mathcal{P}_{ij}\right)\left(B_{\perp\, o}^{i(1)} - v_{\perp\, o}^{i(1)} + \frac{1}{2} n^k h_{k\, o}^{j(1)} \mathcal{P}_j^i\right) \\
&- 2\left(A_o^{(1)}\right)^2 - v_{\| o}^{(1)} h_{i\, o}^{i(1)} - \frac{1}{4}\mathcal{P}_p^m \mathcal{P}_n^k h_{m\, o}^{n(1)} h_{k\, o}^{p(1)} + 2\left(2A_o^{(1)} - 2v_{\| o}^{(1)} - \frac{1}{2} h_{i\, o}^{i(1)} + \frac{1}{2} h_{\| o}^{(1)}\right)\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)} - 2I^{(1)}\right) \\
&+ 2\left(\mathcal{P}_j^i A_o^{(1)} - \mathcal{P}_j^i v_{\| o}^{(1)} - \frac{1}{2}\mathcal{P}_j^k \mathcal{P}_p^i h_{k\, o}^{p(1)}\right)\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left\{\frac{1}{\tilde\chi}\mathcal{P}_i^j B_\|^{(1)} - \mathcal{P}_m^j \tilde\partial_{\perp i} B^{m(1)} - \frac{1}{\tilde\chi}\mathcal{P}_i^n \mathcal{P}_m^j h_n^{m(1)} + \frac{1}{\tilde\chi}\mathcal{P}_i^j h_\|^{(1)}\right. \\
&\left. - n^m \mathcal{P}_n^j \tilde\partial_{\perp i} h_m^{n(1)} - \frac{\tilde\chi}{\bar\chi}\left[\mathcal{P}_i^j \tilde\partial_\| + \left(\bar\chi - \tilde\chi\right)\mathcal{P}_i^n \mathcal{P}^{jm}\tilde\partial_m \tilde\partial_n\right]\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right)\right\} \\
&+ \left(B_{\perp i\, o}^{(1)} - v_{\perp i\, o}^{(1)} + \frac{1}{2} n^k h_{k\, o}^{j(1)} \mathcal{P}_{ij}\right)\left\{-2\left(B_\perp^{i(1)} + n^m h_m^{l(1)} \mathcal{P}_l^i\right) + 4S_\perp^{i(1)} + \bar\chi \partial_\perp^i\left(2v_\|^{(1)} + h_l^{l(1)} + 2\delta_g^{(1)}\right)\right. \\
&\left. + 2\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta \ln a^{(1)} + T^{(1)}\right) - 2\frac{1}{\bar\chi}\int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[\frac{\bar\chi}{\tilde\chi}\left(B_\perp^{i(1)} + n^k h_k^{j(1)} \mathcal{P}_j^i\right) + \left(\bar\chi - \tilde\chi\right)\tilde\partial_\perp^i\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2} h_\|^{(1)}\right)\right]\right\}, \qquad (157)
\end{aligned}
$$

where for $\Delta \ln a^{(2)}/2$ see Eq. (164), for $\Delta x_\|^{(2)}$ see Eq. (133), for $\partial_\| \Delta x_\|^{(2)}/2$ see Eq. (135) and for $\kappa^{(2)}$ see Eq. (136).

Equation (157) is the main result of this paper – giving the number counts at second order in a general gauge, valid for any dark energy model and also in metric theories of modified gravity.

We can simplify the result by explicitly introducing the weak lensing shear and rotation, defined in Eqs. (82)–(83).

This leads to

$$
\begin{aligned}
&\Delta_g^{(2)} = \delta_g^{(2)} + b_e\,\Delta\ln a^{(2)} + \partial_\|\Delta x_\|^{(2)} + \frac{2}{\bar\chi}\Delta x_\|^{(2)} - 2\kappa^{(2)} + A^{(2)} + v_\|^{(2)} + \frac{1}{2}h_i^{i(2)} + \left(\Delta_g^{(1)}\right)^2 - \left(A^{(1)}\right)^2 + 2A^{(1)}B_\|^{(1)}\\
&+A^{(1)}h_\|^{(1)} - B_\|^{(1)}h_\|^{(1)} + v_{\perp i}^{(1)}v_\perp^{i(1)} - 2v_{\perp i}^{(1)}B_\perp^{i(1)} + B_{\perp i}^{(1)}B_\perp^{i(1)} - \frac{1}{2}h_i^{k(1)}h_k^{i(1)} - \frac{1}{4}\left(h_\|^{(1)}\right)^2 - \left(\delta_g^{(1)}\right)^2 + v_\|^{(1)}h_\|^{(1)}\\
&-2|\gamma^{(1)}|^2 - 2\left(\kappa^{(1)}\right)^2 + \vartheta_{ij}^{(1)}\vartheta^{ij(1)} - \frac{1}{\mathcal{H}^2}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\|^{(1)}\right) + \left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\right]^2\\
&-\frac{2}{\mathcal{H}}\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\|^{(1)}\right) + \left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\right] - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2v_\|^{(1)} + h_i^{i(1)} + 2\delta_g^{(1)}\right)\Delta\ln a^{(1)}\\
&-\partial_\|\left(2v_\|^{(1)} + h_i^{i(1)} + 2\delta_g^{(1)}\right)T^{(1)} - \frac{4}{\bar\chi^2\mathcal{H}}\Delta\ln a^{(1)}T^{(1)} + 2v_\perp^{i(1)}\partial_{\perp i}T^{(1)} - \frac{4}{\bar\chi\mathcal{H}}\Delta\ln a^{(1)}\kappa^{(1)} - \frac{4}{\bar\chi}T^{(1)}\kappa^{(1)}\\
&+\left[-b_e + \frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln\bar a} - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 - \frac{2}{\bar\chi^2\mathcal{H}^2}\right]\left(\Delta\ln a^{(1)}\right)^2 - \frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2 + 2\frac{\mathcal{H}'}{\mathcal{H}^2}\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right)\Delta\ln a^{(1)}\\
&+2\frac{\mathcal{H}'}{\mathcal{H}^3}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)} - v_\|^{(1)}\right) + \left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\right]\Delta\ln a^{(1)} + 2\left[-\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + 2S_\perp^{i(1)}\right]\\
&\times\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) - \left[-\frac{2}{\bar\chi}\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + \frac{4}{\bar\chi}S_\perp^{i(1)} + \partial_\perp^i\left(2v_\|^{(1)} + h_l^{l(1)} + 2\delta_g^{(1)}\right)\right]\\
&\times\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\bar\chi}{\tilde\chi}\left(B_{\perp i}^{(1)} + n^k h_k^{j(1)}\mathcal{P}_{ij}\right) + (\bar\chi - \tilde\chi)\,\tilde\partial_{\perp i}\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right)\right]\\
&+2\left(B_{\perp i\,o}^{(1)} - v_{\perp i\,o}^{(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_{ij}\right)\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right)\\
&+\left(B_{\perp i\,o}^{(1)} - v_{\perp i\,o}^{(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_{ij}\right)\Bigg\{-2\left(B_\perp^{i(1)} + n^m h_m^{l(1)}\mathcal{P}_l^i\right) + 4S_\perp^{i(1)} + \bar\chi\partial_\perp^i\left(2v_\|^{(1)} + h_l^{l(1)} + 2\delta_g^{(1)}\right)\\
&+2\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) - 2\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\bar\chi}{\tilde\chi}\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + (\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right)\right]\Bigg\}\,. \qquad (158)
\end{aligned}
$$

See Eqs. (A15) and (A16) for $\gamma_{ij}^{(1)}$ and $\vartheta_{ij}^{(1)}\vartheta^{ij(1)}$.

Assuming no velocity bias

Up to now, we have made no assumption about the CDM velocity u_m^μ or about the conservation equations. If we define

$$\mathcal{E}_m^\mu = \nabla_\nu T_m^{\mu\nu}, \quad T_m^{\mu\nu} = \rho_m u_m^\mu u_m^\nu, \qquad (159)$$

then $\mathcal{E}_m^\mu = 0$ in General Relativity, in the absence of interactions between CDM and dark energy. However, to include interacting CDM and some modified gravity models, we allow for nonzero $\mathcal{E}_m^\mu$.

We now assume that galaxy velocities follow the matter velocity field (at first and second order), i.e. $u_m^\mu = u^\mu$. Then the expressions for $\mathcal{E}_m^{\mu(n)}$ are given in Eqs. (A11)–(A13).

At first order

$$\partial_\|\Delta x_\|^{(1)} \;=\; A^{(1)} - v_\|^{(1)} - \frac{1}{2}h_\|^{(1)} - \frac{1}{\mathcal{H}}\partial_\| v_\|^{(1)} - \frac{1}{2\mathcal{H}}h_\|^{(1)\prime} + \frac{\bar a^2}{\mathcal{H}\bar\rho_m}\left(\mathcal{E}_m^{\|(1)} - \mathcal{E}_m^{0(0)}v_\|^{(1)}\right) - \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta\ln a^{(1)}\,, \qquad (160)$$

where

$$
\begin{aligned}
\mathcal{E}_m^{\|(1)} \;&=\; n_i\mathcal{E}_m^{i(1)} = \frac{\bar\rho_m}{a^2}\left(v_\|^{(1)\prime} - B_\|^{(1)\prime} + \mathcal{H}v_\|^{(1)} - \mathcal{H}B_\|^{(1)} + \partial_\| A^{(1)}\right) + \mathcal{E}_m^{0(0)}v_\|^{(1)}, \qquad (161)\\
\mathcal{E}_m^{0(0)} \;&=\; \frac{1}{\bar a^2}\left(\bar\rho_m' + 3\mathcal{H}\bar\rho_m\right) = \frac{\mathcal{H}\bar\rho_m}{\bar a^2}b_m. \qquad (162)
\end{aligned}
$$

Here $b_m = \mathrm{d}(a^3\bar\rho_m)/\mathrm{d}\ln\bar a$. Then

$$
\begin{aligned}
\Delta_g^{(1)} &= \delta_g^{(1)} + \left(b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar\chi\mathcal{H}}\right)\Delta\ln a^{(1)} - \frac{1}{\mathcal{H}}\partial_\| v_\|^{(1)} - \frac{1}{2\mathcal{H}}h_\|^{(1)\prime} + A^{(1)} - \frac{1}{2}h_\|^{(1)} + \frac{1}{2}h_i^{i(1)} - b_m v_\|^{(1)} \\
&\quad + \frac{a^2}{\mathcal{H}\bar\rho_m}\mathcal{E}_m^{\|(1)} - \frac{2}{\bar\chi}T^{(1)} - 2\kappa^{(1)}\,. \qquad (163)
\end{aligned}
$$

This relation generalizes for any gauge the results previously obtained in Refs.[1–5] and can be apply to general dark energy models, including those where dark energy interacts non-gravitationally with dark matter, and to metric theories of modified gravity as an alternative to dark energy.

At second order, using Eqs. (A13) we have

$$
\begin{aligned}
\Delta\ln a^{(2)} &= +A_o^{(2)} - v_{\|\,o}^{(2)} - \left(A_o^{(1)}\right)^2 + 2A_o^{(1)}B_{\|\,o}^{(1)} - \left(B_{\|\,o}^{(1)}\right)^2 - 6A_o^{(1)}v_{\|\,o}^{(1)} + 2B_{\|\,o}^{(1)}v_{\|\,o}^{(1)} + v_{k\,o}^{(1)}v_o^{k(1)} \\
&\quad - n^i h_{ij\,o}^{(1)}v_o^{j(1)} + 2\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right)\Bigg[-3A^{(1)} + 2B_\|^{(1)} - \frac{1}{\mathcal{H}}\partial_\| v_\|^{(1)} - \frac{1}{2\mathcal{H}}h_\|^{(1)\prime} - b_m v_\|^{(1)} \\
&\quad + \bar\chi\partial_\|\left(2A^{(1)} - B_\|^{(1)}\right) - \bar\chi\left(A^{(1)\prime} + \frac{1}{2}h_\|^{(1)\prime}\right) + \frac{a^2}{\mathcal{H}\bar\rho_m}\mathcal{E}_m^{\|(1)} + 4I^{(1)}\Bigg] + 2\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right) \\
&\quad \times\Bigg\{B_{\perp i}^{(1)} + \bar\chi\,\partial_{\perp i}\left(A^{(1)} + v_\|^{(1)} - B_\|^{(1)}\right) - \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\tilde\partial_{\perp i}\left(2A^{(1)} - B_\|^{(1)}\right) - \left(B_{\perp i}^{(1)\prime} + n^j h_{jk}^{(1)\prime}\mathcal{P}_i^k\right) + \frac{1}{\tilde\chi}B_{\perp i}^{(1)}\right]\Bigg\} \\
&\quad - A^{(2)} + v_\|^{(2)} + 7\left(A^{(1)}\right)^2 + \left(B_\|^{(1)}\right)^2 - 6A^{(1)}B_\|^{(1)} - 2\left(\frac{1}{2} + b_m\right)\left(v_\|^{(1)}\right)^2 + v_{\perp i}^{(1)}v_\perp^{i(1)} + v_\|^{(1)}h_\|^{(1)} \\
&\quad + 2b_m A^{(1)}v_\|^{(1)} + 2v_\|^{(1)}B_\|^{(1)} - 2v_{\perp i}^{(1)}B_\perp^{i(1)} + \frac{2}{\mathcal{H}}\left(A^{(1)} - v_\|^{(1)}\right)\left(\frac{1}{2}h_\|^{(1)\prime} + \partial_\| v_\|^{(1)} - \frac{a^2}{\bar\rho_m}\mathcal{E}_m^{\|(1)}\right) \\
&\quad + 4\Bigg[-3A^{(1)} + 2B_\|^{(1)} - \frac{1}{\mathcal{H}}\partial_\| v_\|^{(1)} - \frac{1}{2\mathcal{H}}h_\|^{(1)\prime} - b_m v_\|^{(1)} + \frac{\bar a^2}{\mathcal{H}\bar\rho_m}\mathcal{E}_m^{\|(1)}\Bigg]I^{(1)} + 4v_{\perp i}^{(1)}S_\perp^{i(1)} \\
&\quad - 2\partial_\|\left(A^{(1)} + v_\|^{(1)} - B_\|^{(1)}\right)T^{(1)} - 2\Bigg[\partial_\|\left(2A^{(1)} - B_\|^{(1)}\right) - \left(A^{(1)\prime} + \frac{1}{2}h_\|^{(1)\prime}\right)\Bigg]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg[2A^{(1)} - B_\|^{(1)} \\
&\quad + (\bar\chi - \tilde\chi)\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right)\Bigg] - 2\Bigg[\partial_{\perp i}\left(A^{(1)} + v_\|^{(1)} - B_\|^{(1)}\right) - \frac{1}{\bar\chi}\left(v_{\perp i}^{(1)} - B_{\perp i}^{(1)}\right)\Bigg] \\
&\quad \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left\{\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + (\bar\chi - \tilde\chi)\left[\tilde\partial_\perp^i\left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right) + \frac{1}{\tilde\chi}\left(B_\perp^{i(1)} + n^k h_{kj}^{(1)}\mathcal{P}^{ij}\right)\right]\right\} + 2I^{(2)} \\
&\quad + 2\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\left(B_\|^{(1)} + h_\|^{(1)} - 4I^{(1)}\right)\left(A^{(1)\prime} - B_\|^{(1)\prime} - \frac{1}{2}h_\|^{(1)\prime}\right) - \left(A^{(1)} - B_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\left(2A^{(1)} - B_\|^{(1)}\right) \\
&\quad + \left[\left(B_\perp^{i(1)} + n^j h_{jk}^{(1)}\mathcal{P}^{ik}\right) - 2S_\perp^{i(1)}\right]\left[\tilde\partial_{\perp i}\left(2A^{(1)} - B_\|^{(1)}\right) - \left(B_{\perp i}^{(1)\prime} + n^j h_{jk}^{(1)\prime}\mathcal{P}_i^k\right) + \frac{1}{\tilde\chi}B_{\perp i}^{(1)}\right]\Bigg\}\,, \qquad (164)
\end{aligned}
$$

$$
\begin{aligned}
\Delta x^{0(2)} \; = \; & +\frac{1}{\mathcal{H}} A_o^{(2)} - \frac{1}{\mathcal{H}} v_{\|\,o}^{(2)} - \left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{2}{\mathcal{H}}\right) \left(A_o^{(1)}\right)^2 - \frac{1}{\mathcal{H}} \left(B_{\|\,o}^{(1)}\right)^2 + \frac{2}{\mathcal{H}} A_o^{(1)} B_{\|\,o}^{(1)} + 2\left(\frac{\mathcal{H}'}{\mathcal{H}^3} - \frac{2}{\mathcal{H}}\right) A_o^{(1)} v_{\|\,o}^{(1)} \\
& + \frac{2}{\mathcal{H}} B_{\|\,o}^{(1)} v_{\|\,o}^{(1)} - \frac{\mathcal{H}'}{\mathcal{H}^3} \left(v_{\|\,o}^{(1)}\right)^2 + \frac{1}{\mathcal{H}} v_{\perp i\,o}^{(1)} v_{\perp\,o}^{i(1)} - \frac{1}{\mathcal{H}} n^i h_{ij\,o}^{(1)} v_o^{j(1)} + 2\left(A_o^{(1)} - v_{\|\,o}^{(1)}\right) \\
& \times \Bigg\{ \left(\frac{\mathcal{H}'}{\mathcal{H}^3} - \frac{2}{\mathcal{H}}\right) A^{(1)} - \left[\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\left(1 + b_m\right)\right] v_{\|}^{(1)} + \frac{2}{\mathcal{H}} B_{\|}^{(1)} - \frac{\bar{\chi}}{\mathcal{H}} A^{(1)\prime} - \frac{1}{\mathcal{H}^2}\left(\partial_{\|} v_{\|}^{(1)} - \frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}^{\|(1)}\right) \\
& - \frac{1}{2}\left(\frac{\bar{\chi}}{\mathcal{H}} + \frac{1}{\mathcal{H}^2}\right) h_{\|}^{(1)\prime} + \frac{\bar{\chi}}{\mathcal{H}} \partial_{\|}\left(2A^{(1)} - B_{\|}^{(1)}\right) - 2\left(\frac{\mathcal{H}'}{\mathcal{H}^3} - \frac{1}{\mathcal{H}}\right) I^{(1)} \Bigg\} + 2\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2} n^k h_{k\,o}^{j(1)} \mathcal{P}_j^i\right) \\
& \times \Bigg\{ \frac{1}{\mathcal{H}} B_{\perp i}^{(1)} + \frac{\bar{\chi}}{\mathcal{H}} \partial_{\perp i}\left(A^{(1)} + v_{\|}^{(1)} - B_{\|}^{(1)}\right) - \frac{1}{\mathcal{H}} \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi} \left[\tilde{\partial}_{\perp i}\left(2A^{(1)} - B_{\|}^{(1)}\right) - \left(B_{\perp i}^{(1)\prime} + n^j h_{jk}^{(1)\prime} \mathcal{P}_i^k\right) \right. \\
& \left. + \frac{1}{\tilde{\chi}} B_{\perp i}^{(1)}\right]\Bigg\} - \frac{1}{\mathcal{H}} A^{(2)} + \frac{1}{\mathcal{H}} v_{\|}^{(2)} + \left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{6}{\mathcal{H}}\right)\left(A^{(1)}\right)^2 + \frac{1}{\mathcal{H}}\left(B_{\|}^{(1)}\right)^2 - \frac{6}{\mathcal{H}} A^{(1)} B_{\|}^{(1)} + \frac{2}{\mathcal{H}} v_{\|}^{(1)} B_{\|}^{(1)} \\
& - \left[\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{2}{\mathcal{H}}\left(1 + b_m\right)\right]\left(v_{\|}^{(1)}\right)^2 + \frac{1}{\mathcal{H}} v_{\perp i}^{(1)} v_{\perp}^{i(1)} + \frac{1}{\mathcal{H}} v_{\|}^{(1)} h_{\|}^{(1)} + 2\left[\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\left(1 + b_m\right)\right] A^{(1)} v_{\|}^{(1)} - \frac{2}{\mathcal{H}} v_{\perp i}^{(1)} B_{\perp}^{i(1)} \\
& + \frac{2}{\mathcal{H}^2}\left(A^{(1)} - v_{\|}^{(1)}\right)\left(\frac{1}{2} h_{\|}^{(1)\prime} + \partial_{\|} v_{\|}^{(1)} - \frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}_m^{\|(1)}\right) + 4\Bigg\{\left(\frac{\mathcal{H}'}{\mathcal{H}^3} - \frac{2}{\mathcal{H}}\right) A^{(1)} - \left[\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\left(1 + b_m\right)\right] v_{\|}^{(1)} \\
& + \frac{2}{\mathcal{H}} B_{\|}^{(1)} - \frac{1}{\mathcal{H}^2}\left(\frac{1}{2} h_{\|}^{(1)\prime} + \partial_{\|} v_{\|}^{(1)} - \frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}_m^{\|(1)}\right) - \left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right) I^{(1)}\Bigg\} I^{(1)} + \frac{4}{\mathcal{H}} v_{\perp i}^{(1)} S_{\perp}^{i(1)} \\
& - \frac{2}{\mathcal{H}} \partial_{\|}\left(A^{(1)} + v_{\|}^{(1)} - B_{\|}^{(1)}\right) T^{(1)} - \frac{2}{\mathcal{H}}\left[\partial_{\|}\left(2A^{(1)} - B_{\|}^{(1)}\right) - \left(A^{(1)\prime} + \frac{1}{2} h_{\|}^{(1)\prime}\right)\right] \\
& \times \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi} \left[2A^{(1)} - B_{\|}^{(1)} + \left(\bar{\chi} - \tilde{\chi}\right)\left(A^{(1)\prime} - B_{\|}^{(1)\prime} - \frac{1}{2} h_{\|}^{(1)\prime}\right)\right] - \frac{2}{\mathcal{H}}\left[\partial_{\perp i}\left(A^{(1)} + v_{\|}^{(1)} - B_{\|}^{(1)}\right)\right. \\
& \left. - \frac{1}{\bar{\chi}}\left(v_{\perp i}^{(1)} - B_{\perp i}^{(1)}\right)\right] \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi} \Bigg\{\left(B_{\perp}^{i(1)} + n^k h_k^{j(1)} \mathcal{P}_j^i\right) + \left(\bar{\chi} - \tilde{\chi}\right)\left[\tilde{\partial}_{\perp}^i\left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2} h_{\|}^{(1)}\right)\right. \\
& \left. + \frac{1}{\tilde{\chi}}\left(B_{\perp}^{i(1)} + n^k h_{kj}^{(1)} \mathcal{P}^{ij}\right)\right]\Bigg\} + \frac{2}{\mathcal{H}} I^{(2)} + \frac{2}{\mathcal{H}} \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi} \Bigg\{\left(B_{\|}^{(1)} + h_{\|}^{(1)} - 4I^{(1)}\right)\left(A^{(1)\prime} - B_{\|}^{(1)\prime} - \frac{1}{2} h_{\|}^{(1)\prime}\right) \\
& - \left(A^{(1)} - B_{\|}^{(1)} - \frac{1}{2} h_{\|}^{(1)}\right) \frac{\mathrm{d}}{\mathrm{d}\tilde{\chi}}\left(2A^{(1)} - B_{\|}^{(1)}\right) + \left[\left(B_{\perp}^{i(1)} + n^j h_{jk}^{(1)} \mathcal{P}^{ik}\right) - 2S_{\perp}^{i(1)}\right] \\
& \times \left[\tilde{\partial}_{\perp i}\left(2A^{(1)} - B_{\|}^{(1)}\right) - \left(B_{\perp i}^{(1)\prime} + n^j h_{jk}^{(1)\prime} \mathcal{P}_i^k\right) + \frac{1}{\tilde{\chi}} B_{\perp i}^{(1)}\right]\Bigg\} \,, \qquad (165)
\end{aligned}
$$

$$
\begin{aligned}
&\partial_\parallel \Delta x^{(2)}_\parallel = \Big(A^{(1)}_o\Big)^2 - 2A^{(1)}_o B^{(1)}_{\parallel o} + \Big(B^{(1)}_{\parallel o}\Big)^2 + 2A^{(1)}_o v^{(1)}_{\parallel o} - 2B^{(1)}_{\parallel o} v^{(1)}_{\parallel o} - v^{(1)}_{\parallel o} h^{(1)}_{\parallel o} - \frac{1}{4} \Big(h^{(1)}_{\parallel o}\Big)^2 - v^{(1)}_{\perp k\, o} v^{k(1)}_{\perp o} \\
&+ n^i h^{(1)}_{ik\,o} \mathcal{P}^k_j v^{j(1)}_o - n^i h^{(1)}_{ik\,o}\mathcal{P}^k_j B^{j(1)}_o - \frac{1}{4} n^i h^{(1)}_{ij\,o}\mathcal{P}^j_k h^{k(1)}_{p\,o} n^p - B^{i(1)}_{\perp o} B^{(1)}_{\perp i\,o} + 2 v^{i(1)}_{\perp o} B^{(1)}_{\perp i\,o} + 2\Big(A^{(1)}_o - v^{(1)}_{\parallel o}\Big) \\
&\times\bigg\{ \Big(-\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{1}{\mathcal{H}}\Big)\bigg[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel - \mathcal{H}(b_m+1)v^{(1)}_\parallel + \mathcal{H} B^{(1)}_\parallel - \frac{1}{2} h^{(1)\prime}_\parallel\bigg] - \frac{\bar\chi}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\bigg[\partial_\parallel\Big(2A^{(1)}-B^{(1)}_\parallel\Big) \\
&-\Big(A^{(1)\prime}+\frac{1}{2} h^{(1)\prime}_\parallel\Big)\bigg] + \Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2} h^{(1)}_\parallel\Big) - \bar\chi\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\Big) - \frac{1}{\mathcal{H}^2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\bigg[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel \\
&-\mathcal{H}(b_m+1)v^{(1)}_\parallel + \mathcal{H} B^{(1)}_\parallel - \frac{1}{2} h^{(1)\prime}_\parallel\bigg]\bigg\} + 2\Big(B^{i(1)}_{\perp o} - v^{i(1)}_{\perp o} + \frac{1}{2} n^k h^{j(1)}_{k\,o}\mathcal{P}^i_j\Big)\bigg[-\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}B^{(1)}_{\perp i} + \frac{1}{\mathcal{H}}\partial_{\perp i}\Big(A^{(1)}-v^{(1)}_\parallel\Big) \\
&-\frac{1}{\mathcal{H}}\Big(B^{(1)\prime}_{\perp i} + n^j h^{(1)\prime}_{jk}\mathcal{P}^k_i\Big) - \frac{\bar\chi}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_{\perp i}\Big(A^{(1)}+v^{(1)}_\parallel - B^{(1)}_\parallel\Big) + \frac{1}{\mathcal{H}\bar\chi}B^{(1)}_{\perp i} - 4\delta_{il}S^{l(1)}_\perp\bigg] + A^{(2)} - B^{(2)}_\parallel - \frac{1}{2} h^{(2)}_\parallel - \frac{1}{2\mathcal{H}}h^{(2)\prime}_\parallel \\
&+2\frac{\bar a^2}{\mathcal{H}\bar\rho_m}\bigg[\frac{1}{2}\mathcal{E}^{\parallel(2)}_m - \mathcal{E}^{0(1)}_m v^{(1)}_\parallel - \mathcal{E}^{\parallel(1)}_m\Big(\delta^{(1)}_m - A^{(1)}\Big)\bigg] - 2b_m\Big(\frac{1}{2} v^{(2)}_\parallel - \delta^{(1)}_m v^{(1)}_\parallel + 2A^{(1)}v^{(1)}_\parallel\Big) - \frac{2}{\mathcal{H}}\bigg[\frac{1}{2}\partial_\parallel v^{(2)}_\parallel - v^{(1)}_\parallel\partial_\parallel v^{(1)}_\parallel \\
&+\mathcal{H}\Big(\frac{1}{2} v^{(2)}_\parallel - \frac{1}{2} B^{(2)}_\parallel\Big) - \frac{2}{\bar\chi}\Big(v^{(1)}_\parallel\Big)^2 - v^{(1)}_\parallel\partial_{\perp j}v^{j(1)}_\perp + A^{(1)}B^{(1)\prime}_\parallel + A^{(1)\prime}B^{(1)}_\parallel + \mathcal{H} A^{(1)}B^{(1)}_\parallel + v^{(1)}_\parallel h^{(1)\prime}_\parallel + \mathcal{H} B^{(1)}_\parallel h^{(1)}_\parallel \\
&+B^{(1)\prime}_\parallel h^{(1)}_\parallel - A^{(1)}\partial_\parallel A^{(1)} - \partial_\parallel A^{(1)} h^{(1)}_\parallel + v^{(1)}_{\perp k}\partial_\parallel B^{k(1)}_\perp - v^{j(1)}_\perp\partial_{\perp j}B^{(1)}_\parallel + \frac{1}{\bar\chi}v^{j(1)}_\perp B^{(1)}_{\perp j} + \mathcal{H} B^{(1)}_k\mathcal{P}^k_j h^{ij(1)} n_i \\
&+v^{(1)}_k\mathcal{P}^k_j h^{ij(1)\prime}n_i + B^{(1)\prime}_k \mathcal{P}^k_j h^{ij(1)}n_i - \partial_k A^{(1)}\mathcal{P}^k_j h^{ij(1)}n_i\bigg] - 2\Big(2A^{(1)}+h^{(1)}_\parallel\Big)\Big(A^{(1)}-B^{(1)}_\parallel - \frac{1}{2} h^{(1)}_\parallel\Big) \\
&-\frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\bigg[\frac{7}{2}\Big(A^{(1)}\Big)^2 + \frac{1}{2}\Big(B^{(1)}_\parallel\Big)^2 - 2A^{(1)}B^{(1)}_\parallel - A^{(1)}v^{(1)}_\parallel + \frac{1}{2}\Big(v^{(1)}_\parallel\Big)^2 + \frac{1}{2} v^{(1)}_\parallel h^{(1)}_\parallel + \frac{1}{2} v^{(1)}_{\perp i}v^{i(1)}_\perp - v^{(1)}_{\perp i}B^{i(1)}_\perp\bigg] \\
&+\frac{2}{\mathcal{H}^2}\Big(A^{(1)}-v^{(1)}_\parallel\Big)\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\bigg[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel - \mathcal{H}(b_m+1)v^{(1)}_\parallel + \mathcal{H} B^{(1)}_\parallel - \frac{1}{2} h^{(1)\prime}_\parallel\bigg] - \frac{2}{\mathcal{H}}\Big(3A^{(1)}+h^{(1)}_\parallel - v^{(1)}_\parallel\Big) \\
&\times\Big(A^{(1)\prime}-B^{(1)\prime}_\parallel - \frac{1}{2} h^{(1)\prime}_\parallel\Big) + \frac{2}{\mathcal{H}}\Big(2A^{(1)}-B^{(1)}_\parallel\Big)\bigg[\partial_\parallel\Big(2A^{(1)}-B^{(1)}_\parallel\Big) - \Big(A^{(1)\prime}+\frac{1}{2}h^{(1)\prime}_\parallel\Big)\bigg] + \bigg\{2\frac{\mathcal{H}'}{\mathcal{H}^3}\Big(A^{(1)}-v^{(1)}_\parallel\Big) \\
&+\frac{2}{\mathcal{H}}\Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\Big) + \frac{2}{\mathcal{H}^2}\bigg[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel - \mathcal{H}(b_m+1)v^{(1)}_\parallel + \mathcal{H}B^{(1)}_\parallel - \frac{1}{2}h^{(1)\prime}_\parallel\bigg]\bigg\}\bigg[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel \\
&-\mathcal{H}(b_m+1)v^{(1)}_\parallel + \mathcal{H}B^{(1)}_\parallel - \frac{1}{2}h^{(1)\prime}_\parallel\bigg] + \frac{2}{\mathcal{H}}\bigg[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Big(2A^{(1)}-B^{(1)}_\parallel\Big) - \partial_\parallel\Big(A^{(1)}+v^{(1)}_\parallel - B^{(1)}_\parallel\Big)\bigg]\Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\Big) \\
&+2\Big(B^{i(1)}_\perp + n^j h^{(1)}_{jk}\mathcal{P}^{ik}\Big)\bigg[\frac{1}{2}\Big(B^{(1)}_{i\perp} + n^p h^{(1)}_{pm}\mathcal{P}^m_i\Big) - \frac{1}{\mathcal{H}}\partial_{\perp i}\Big(A^{(1)}-v^{(1)}_\parallel\Big) + \frac{1}{\mathcal{H}}\Big(B^{(1)\prime}_{\perp i} + n^j h^{(1)\prime}_{jk}\mathcal{P}^k_i\Big)\bigg] + \frac{2}{\mathcal{H}}v^{(1)}_{\perp i} \\
&\times\partial^i_\perp\Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\Big) + \frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_\parallel\Big(A^{(1)}+v^{(1)}_\parallel-B^{(1)}_\parallel\Big)T^{(1)} + 4\bigg\{\Big(-\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{1}{\mathcal{H}}\Big)\bigg[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel + \mathcal{H}B^{(1)}_\parallel \\
&-\mathcal{H}(b_m+1)v^{(1)}_\parallel - \frac{1}{2}h^{(1)\prime}_\parallel\bigg] - \frac{1}{\mathcal{H}^2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\bigg[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel - \mathcal{H}(b_m+1)v^{(1)}_\parallel + \mathcal{H}B^{(1)}_\parallel - \frac{1}{2}h^{(1)\prime}_\parallel\bigg] + \Big(A^{(1)}-B^{(1)}_\parallel \\
&-\frac{1}{2}h^{(1)}_\parallel\Big)\bigg\}I^{(1)} + 2\bigg\{\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\Big) + \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\bigg[\partial_\parallel\Big(2A^{(1)}-B^{(1)}_\parallel\Big) - \Big(A^{(1)\prime}+\frac{1}{2}h^{(1)\prime}_\parallel\Big)\bigg]\bigg\}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\bigg[2A^{(1)} \\
&-B^{(1)}_\parallel + (\bar\chi-\tilde\chi)\Big(A^{(1)\prime}-B^{(1)\prime}_\parallel-\frac{1}{2}h^{(1)\prime}_\parallel\Big)\bigg] + 4\bigg[\frac{1}{\mathcal{H}}\partial_{\perp i}\Big(A^{(1)}-v^{(1)}_\parallel\Big) - \frac{1}{\mathcal{H}}\Big(B^{(1)\prime}_{\perp i}+n^jh^{(1)\prime}_{jk}\mathcal{P}^k_i\Big) + \frac{1}{\mathcal{H}\bar\chi}v^{(1)}_{\perp i} \\
&-\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}v^{(1)}_{\perp i} - 2S^{j(1)}_\perp\delta_{ij}\bigg]S^{i(1)}_\perp + \frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\bigg[\partial_{\perp i}\Big(A^{(1)}+v^{(1)}_\parallel-B^{(1)}_\parallel\Big) - \frac{1}{\bar\chi}\Big(v^{(1)}_{\perp i}-B^{(1)}_{\perp i}\Big)\bigg]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\bigg\{\Big(B^{i(1)}_\perp + n^kh^{j(1)}_k\mathcal{P}^i_j\Big) \\
&+(\bar\chi-\tilde\chi)\bigg[\tilde\partial^i_\perp\Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\Big) + \frac{1}{\tilde\chi}\Big(B^{i(1)}_\perp + n^kh^{(1)}_{kj}\mathcal{P}^{ij}\Big)\bigg]\bigg\} - \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta\ln a^{(2)} + \bigg\{-2\frac{\mathcal{H}'}{\mathcal{H}^2}\Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\Big) \\
&-\frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\Big) - 2\Big(\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{1}{\mathcal{H}}\Big)\bigg[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel + \mathcal{H}B^{(1)}_\parallel - \mathcal{H}(b_m+1)v^{(1)}_\parallel - \frac{1}{2}h^{(1)\prime}_\parallel\bigg]\bigg\}\Delta\ln a^{(1)} \\
&+\bigg[-\frac{\mathcal{H}''}{\mathcal{H}^3} + 3\Big(\frac{\mathcal{H}'}{\mathcal{H}^2}\Big)^2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\bigg]\Big(\Delta\ln a^{(1)}\Big)^2 + 2\int_0^{\bar\chi}\mathrm{d}\tilde\chi\bigg\{\Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\Big)\bigg[2\Big(A^{(1)\prime}-B^{(1)\prime}_\parallel-\frac{1}{2}h^{(1)\prime}_\parallel\Big) \\
&+\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\Big(2A^{(1)}-B^{(1)}_\parallel\Big)\bigg] - 2\Big(B^{i(1)}_\perp + n^kh^{j(1)}_k\mathcal{P}^i_j\Big)\bigg[\tilde\partial_{\perp i}\Big(A^{(1)}-B^{(1)}_\parallel-\frac{1}{2}h^{(1)}_\parallel\Big) + \frac{1}{\tilde\chi}\Big(B^{(1)}_{\perp i}+n^mh^{(1)}_{mp}\mathcal{P}^p_i\Big)\bigg]\bigg\}\,. \qquad (166)
\end{aligned}
$$

We can see immediately that $\partial_\| \Delta x_\|^{(2)}$ can be further expanded. We apply again Eq. (A11) in the following terms

$$-\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}v_{\perp i}^{(1)} = \frac{1}{\mathcal{H}}\frac{\bar a^2}{\bar\rho_m}\mathcal{P}_{ij}\mathcal{E}_m^{j(1)} - \frac{1}{\mathcal{H}}\partial_\| v_{\perp i}^{(1)} - \left(b_m+1\right)v_{\perp i}^{(1)} + B_{\perp i}^{(1)} - \frac{1}{\mathcal{H}}\partial_{\perp i}A^{(1)} + \frac{1}{\mathcal{H}}B_{\perp i}^{(1)\prime}\,, \tag{167}$$

$$+\frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_\|\left(A^{(1)}+v_\|^{(1)}-B_\|^{(1)}\right)T^{(1)} = +\frac{2}{\mathcal{H}}\Bigg[-\frac{\bar a^2}{\bar\rho_m}\partial_\|\mathcal{E}_m^{\|(1)} + \partial_\|^2 v_\|^{(1)} + \mathcal{H}\left(b_m+1\right)\partial_\| v_\|^{(1)} - \mathcal{H}\partial_\| B_\|^{(1)} - \partial_\| A^{(1)\prime}$$
$$+2\partial_\|^2 A^{(1)} - \partial_\|^2 B_\|^{(1)}\Bigg]T^{(1)}\,, \tag{168}$$

$$-\frac{\bar\chi}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_{\perp i}\left(A^{(1)}+v_\|^{(1)}-B_\|^{(1)}\right) = -\frac{\bar\chi}{\mathcal{H}}\partial_{\perp i}\Bigg[-A^{(1)\prime} - \frac{\bar a^2}{\bar\rho_m}\mathcal{E}_m^{\|(1)} + \partial_\| v_\|^{(1)} + \mathcal{H}\left(b_m+1\right)v_\|^{(1)} - \mathcal{H}B_\|^{(1)} + 2\partial_\| A^{(1)}$$
$$-\partial_\| B_\|^{(1)}\Bigg] + \frac{1}{\mathcal{H}}\partial_{\perp i}\left(A^{(1)}+v_\|^{(1)}-B_\|^{(1)}\right)\,, \tag{169}$$

$$+\frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\partial_{\perp i}\left(A^{(1)}+v_\|^{(1)}-B_\|^{(1)}\right) - \frac{1}{\bar\chi}\left(v_{\perp i}^{(1)}-B_{\perp i}^{(1)}\right)\right] = \frac{2}{\mathcal{H}}\Bigg[\partial_{\perp i}\Bigg(-A^{(1)\prime} - \frac{\bar a^2}{\bar\rho_m}\mathcal{E}_m^{\|(1)} + \partial_\| v_\|^{(1)} + \mathcal{H}\left(b_m+1\right)v_\|^{(1)}$$
$$-\mathcal{H}B_\|^{(1)} + 2\partial_\| A^{(1)} - \partial_\| B_\|^{(1)}\Bigg) + \frac{1}{\bar\chi}\Bigg(-2\partial_{\perp i}A^{(1)} - \partial_{\perp i}v_\|^{(1)} - \partial_\| v_{\perp i}^{(1)} + \partial_{\perp i}B_\|^{(1)} + \frac{\bar a^2}{\bar\rho_m}\mathcal{P}_{ij}\mathcal{E}_m^{j(1)} - \mathcal{H}\left(b_m+1\right)v_{\perp i}^{(1)}$$
$$+\mathcal{H}B_{\perp i}^{(1)} + \partial_\| B_{\perp i}^{(1)}\Bigg) + \frac{1}{\bar\chi^2}\left(v_{\perp i}^{(1)}-B_{\perp i}^{(1)}\right)\Bigg]\,, \tag{170}$$

$$\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}_m^{\|(1)} - \partial_\| v_\|^{(1)} - \mathcal{H}\left(b_m+1\right)v_\|^{(1)} + \mathcal{H}B_\|^{(1)} - \frac{1}{2}h_\|^{(1)\prime}\right] = -\left(\frac{\bar a^2}{\bar\rho_m}\mathcal{E}_m^{\|(1)}\right)' + \frac{\bar a^2\mathcal{H}}{\bar\rho_m}\left(b_m+1\right)\mathcal{E}_m^{\|(1)}$$
$$+\mathcal{H}^2\frac{\mathrm{d}b_m}{\mathrm{d}\ln a}v_\|^{(1)} + b_m\Bigg[-\mathcal{H}^2\left(b_m+1\right)v_\|^{(1)} + \mathcal{H}B_\|^{(1)\prime} - 2\mathcal{H}\partial_\| v_\|^{(1)} + \left(\mathcal{H}'-\mathcal{H}^2\right)v_\|^{(1)} + \mathcal{H}^2 B_\|^{(1)} - \mathcal{H}\partial_\| A^{(1)}\Bigg]$$
$$-2\mathcal{H}\partial_\| v_\|^{(1)} - \partial_\|^2 A^{(1)} + \partial_\| B_\|^{(1)\prime} - \partial_\|^2 v_\|^{(1)} + 2\mathcal{H}\partial_\| B_\|^{(1)} + \left(\mathcal{H}'-\mathcal{H}^2\right)\left(v_\|^{(1)}-B_\|^{(1)}\right) - \mathcal{H}\partial_\| A^{(1)} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}h_\|^{(1)\prime}\,, \tag{171}$$

$$\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{7}{2}\left(A^{(1)}\right)^2 + \frac{1}{2}\left(B_\|^{(1)}\right)^2 - 2A^{(1)}B_\|^{(1)} - A^{(1)}v_\|^{(1)} + \frac{1}{2}\left(v_\|^{(1)}\right)^2 + \frac{1}{2}v_\|^{(1)}h_\|^{(1)} + \frac{1}{2}v_{\perp i}^{(1)}v_\perp^{i(1)} - v_{\perp i}^{(1)}B_\perp^{i(1)}\right]$$
$$= 6A^{(1)}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}A^{(1)} - A^{(1)}A^{(1)\prime} + B_\|^{(1)}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}B_\|^{(1)} - 2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(A^{(1)}B_\|^{(1)}\right) + A^{(1)}B_\|^{(1)\prime} + v_\|^{(1)}A^{(1)\prime} + \frac{1}{2}v_\|^{(1)}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}h_\|^{(1)}$$
$$-A^{(1)}\partial_\| v_\|^{(1)} - \mathcal{H}A^{(1)}v_\|^{(1)} + \mathcal{H}A^{(1)}B_\|^{(1)} - v_\|^{(1)}B_\|^{(1)\prime} + v_\|^{(1)}\partial_\| v_\|^{(1)} + \mathcal{H}\left(v_\|^{(1)}\right)^2 - \mathcal{H}v_\|^{(1)}B_\|^{(1)} + \frac{1}{2}h_\|^{(1)}\partial_\| A^{(1)} - \frac{1}{2}h_\|^{(1)}B_\|^{(1)\prime}$$
$$+\frac{1}{2}h_\|^{(1)}\partial_\| v_\|^{(1)} + \frac{\mathcal{H}}{2}v_\|^{(1)}h_\|^{(1)} - \frac{\mathcal{H}}{2}B_\|^{(1)}h_\|^{(1)} + \left(B_{\perp i}^{(1)}-v_{\perp i}^{(1)}\right)\left(-\partial_\| v_\perp^{i(1)} + B_\perp^{i(1)\prime} - \mathcal{H}v_\perp^{i(1)} + \mathcal{H}B_\perp^{i(1)} - \partial_\perp^i A^{(1)}\right)$$
$$-\left(A^{(1)} - v_\|^{(1)} - \frac{1}{2}h_\|^{(1)}\right)\left(\frac{\bar a^2}{\bar\rho_m}\mathcal{E}_m^{\|(1)} - \mathcal{H}b_m v_\|^{(1)}\right) + \left(B_{\perp i}^{(1)} - v_{\perp i}^{(1)}\right)\left(\frac{\bar a^2}{\bar\rho_m}\mathcal{P}_j^i\mathcal{E}_m^{j(1)} - \mathcal{H}b_m v_\perp^{i(1)}\right)\,. \tag{172}$$

Finally, we get

$$
\begin{aligned}
&\Delta_g^{(2)} = \delta_g^{(2)} + b_e\,\Delta\ln a^{(2)} + \partial_\parallel \Delta x_\parallel^{(2)} + \frac{2}{\bar\chi}\Delta x_\parallel^{(2)} - 2\kappa^{(2)} + A^{(2)} + v_\parallel^{(2)} + \frac{1}{2}h_i^{i(2)} + \left(\Delta_g^{(1)}\right)^2 - 3\left(A^{(1)}\right)^2 + 4A^{(1)}B_\parallel^{(1)} \\
&+3A^{(1)}h_\parallel^{(1)} - 2B_\parallel^{(1)}h_\parallel^{(1)} - \left(v_\parallel^{(1)}\right)^2 - \left(B_\parallel^{(1)}\right)^2 - \frac{1}{2}h_i^{k(1)}h_k^{i(1)} - \frac{3}{4}\left(h_\parallel^{(1)}\right)^2 + 2A^{(1)}v_\parallel^{(1)} - \frac{1}{\mathcal{H}^2}\left(\partial_\parallel v_\parallel^{(1)}\right)^2 + \frac{1}{\mathcal{H}}A^{(1)}h_\parallel^{(1)\prime} \\
&-\frac{1}{4\mathcal{H}^2}\left(h_\parallel^{(1)\prime}\right)^2 - \frac{1}{\mathcal{H}}v_\parallel^{(1)}h_\parallel^{(1)\prime} - \frac{1}{2\mathcal{H}}h_\parallel^{(1)}h_\parallel^{(1)\prime} + \frac{2}{\mathcal{H}}A^{(1)}\partial_\parallel v_\parallel^{(1)} - \frac{2}{\mathcal{H}}v_\parallel^{(1)}\partial_\parallel v_\parallel^{(1)} - \frac{1}{\mathcal{H}}h_\parallel^{(1)}\partial_\parallel v_\parallel^{(1)} - \frac{1}{\mathcal{H}^2}h_\parallel^{(1)\prime}\partial_\parallel v_\parallel^{(1)} \\
&+B_{\perp i}^{(1)}B_\perp^{i(1)} + v_{\perp i}^{(1)}v_\perp^{i(1)} - 2v_{\perp i}^{(1)}B_\perp^{i(1)} - 8\left(I^{(1)}\right)^2 + 8A^{(1)}I^{(1)} - 8B_\parallel^{(1)}I^{(1)} - 4h_\parallel^{(1)}I^{(1)} - \left(\delta_g^{(1)}\right)^2 + 2v_\perp^{i(1)}\partial_{\perp i}T^{(1)} \\
&+\frac{2}{\mathcal{H}}\left(-\partial_\parallel A^{(1)} + B_\parallel^{(1)\prime} - \partial_\parallel v_\parallel^{(1)} - \mathcal{H}v_\parallel^{(1)} + \mathcal{H}B_\parallel^{(1)}\right)\Delta\ln a^{(1)} - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(h_i^{i(1)} + 2\delta_g^{(1)}\right)\Delta\ln a^{(1)} - \frac{4}{\bar\chi^2\mathcal{H}}\Delta\ln a^{(1)}T^{(1)} \\
&+2\frac{\mathcal{H}'}{\mathcal{H}^2}\left(A^{(1)} - v_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel v_\parallel^{(1)} - \frac{1}{2\mathcal{H}}h_\parallel^{(1)\prime}\right)\Delta\ln a^{(1)} - \partial_\parallel\left(2v_\parallel^{(1)} + h_i^{i(1)} + 2\delta_g^{(1)}\right)T^{(1)} - \frac{4}{\bar\chi\mathcal{H}}\Delta\ln a^{(1)}\kappa^{(1)} \\
&-\frac{4}{\bar\chi}T^{(1)}\kappa^{(1)} + \left[-b_e + \frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln\bar a} - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 - \frac{2}{\bar\chi^2\mathcal{H}^2}\right]\left(\Delta\ln a^{(1)}\right)^2 - \frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2 + 2\left[-\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + 2S_\perp^{i(1)}\right] \\
&\times\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) - \left[-\frac{2}{\bar\chi}\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + \frac{4}{\bar\chi}S_\perp^{i(1)} + \partial_\perp^i\left(2v_\parallel^{(1)} + h_l^{l(1)} + 2\delta_g^{(1)}\right)\right] \\
&\times\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\bar\chi}{\tilde\chi}\left(B_{\perp i}^{(1)} + n^k h_k^{j(1)}\mathcal{P}_{ij}\right) + \left(\bar\chi - \tilde\chi\right)\tilde\partial_{\perp i}\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\right] \\
&-2\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)} - 2I^{(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\frac{2}{\tilde\chi}B_\parallel^{(1)} - \tilde\partial_{\perp m}B^{m(1)} - \frac{1}{\tilde\chi}h_i^{i(1)} + \frac{3}{\tilde\chi}h_\parallel^{(1)} - n^m\tilde\partial_{\perp n}h_m^{n(1)} \\
&-\frac{\bar\chi}{\tilde\chi}\left[2\tilde\partial_\parallel + \left(\bar\chi - \tilde\chi\right)\mathcal{P}^{mn}\tilde\partial_m\tilde\partial_n\right]\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\Bigg\} - \int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\frac{1}{\tilde\chi}\mathcal{P}_j^i B_\parallel^{(1)} - \mathcal{P}_p^i\tilde\partial_{\perp j}B^{p(1)} - \frac{1}{\tilde\chi}\mathcal{P}_j^p\mathcal{P}_q^i h_p^{q(1)} \\
&+\frac{1}{\tilde\chi}\mathcal{P}_j^i h_\parallel^{(1)} - n^p\mathcal{P}_q^i\tilde\partial_{\perp j}h_p^{q(1)} - \frac{\tilde\chi}{\bar\chi}\left[\mathcal{P}_j^i\tilde\partial_\parallel + \left(\bar\chi - \tilde\chi\right)\mathcal{P}_j^p\mathcal{P}^{iq}\tilde\partial_q\tilde\partial_p\right]\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\Bigg\} \\
&\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\frac{1}{\tilde\chi}\mathcal{P}_i^j B_\parallel^{(1)} - \mathcal{P}_m^j\tilde\partial_{\perp i}B^{m(1)} - \frac{1}{\tilde\chi}\mathcal{P}_i^n\mathcal{P}_m^j h_n^{m(1)} + \frac{1}{\tilde\chi}\mathcal{P}_i^j h_\parallel^{(1)} - n^m\mathcal{P}_n^j\tilde\partial_{\perp i}h_m^{n(1)} \\
&-\frac{\tilde\chi}{\bar\chi}\left[\mathcal{P}_i^j\tilde\partial_\parallel + \left(\bar\chi - \tilde\chi\right)\mathcal{P}_i^n\mathcal{P}^{jm}\tilde\partial_m\tilde\partial_n\right]\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\Bigg\} - 2\left(v_{\parallel o}^{(1)}\right)^2 - 2\left(A_o^{(1)}\right)^2 - v_{\parallel o}^{(1)}h_{i\,o}^{i(1)} + v_{\parallel o}^{(1)}h_{\parallel o}^{(1)} \\
&+4A_o^{(1)}v_{\parallel o}^{(1)} + A_o^{(1)}h_{i\,o}^{i(1)} - A_o^{(1)}h_{\parallel o}^{(1)} + 2\left(B_{\perp i\,o}^{(1)} - v_{\perp i\,o}^{(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_{ij}\right)\left(B_{\perp o}^{i(1)} - v_{\perp o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right) \\
&-\frac{1}{4}\mathcal{P}_p^m\mathcal{P}_n^k h_{m\,o}^{n(1)}h_{k\,o}^{p(1)} + 2\left(2A_o^{(1)} - 2v_{\parallel o}^{(1)} - \frac{1}{2}h_{i\,o}^{i(1)} + \frac{1}{2}h_{\parallel o}^{(1)}\right)\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)} - 2I^{(1)}\right) \\
&+2\left(\mathcal{P}_j^i A_o^{(1)} - \mathcal{P}_j^i v_{\parallel o}^{(1)} - \frac{1}{2}\mathcal{P}_j^k\mathcal{P}_p^i h_{k\,o}^{p(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\frac{1}{\tilde\chi}\mathcal{P}_i^j B_\parallel^{(1)} - \mathcal{P}_m^j\tilde\partial_{\perp i}B^{m(1)} - \frac{1}{\tilde\chi}\mathcal{P}_i^n\mathcal{P}_m^j h_n^{m(1)} + \frac{1}{\tilde\chi}\mathcal{P}_i^j h_\parallel^{(1)} \\
&-n^m\mathcal{P}_n^j\tilde\partial_{\perp i}h_m^{n(1)} - \frac{\tilde\chi}{\bar\chi}\left[\mathcal{P}_i^j\tilde\partial_\parallel + \left(\bar\chi - \tilde\chi\right)\mathcal{P}_i^n\mathcal{P}^{jm}\tilde\partial_m\tilde\partial_n\right]\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\Bigg\} \\
&+\left(B_{\perp i\,o}^{(1)} - v_{\perp i\,o}^{(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_{ij}\right)\Bigg\{-2\left(B_\perp^{i(1)} + n^m h_m^{l(1)}\mathcal{P}_l^i\right) + 4S_\perp^{i(1)} + \bar\chi\partial_\perp^i\left(2v_\parallel^{(1)} + h_l^{l(1)} + 2\delta_g^{(1)}\right) \\
&+2\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) - 2\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\bar\chi}{\tilde\chi}\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + \left(\bar\chi - \tilde\chi\right)\tilde\partial_\perp^i\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\right]\Bigg\} \\
&-\left(\frac{\bar a^2}{\bar\rho_m\mathcal{H}}\mathcal{E}_m^{\parallel(1)} - b_m v_\parallel^{(1)}\right)^2 - 2\left[A^{(1)} - v_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel v_\parallel^{(1)} - \frac{1}{2\mathcal{H}}h_\parallel^{(1)\prime} - \left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\Delta\ln a^{(1)}\right] \\
&\times\left(\frac{\bar a^2}{\bar\rho_m\mathcal{H}}\mathcal{E}_m^{\parallel(1)} - b_m v_\parallel^{(1)}\right)\,. \qquad (173)
\end{aligned}
$$

This is the main result in a general gauge with no velocity bias. If we explicitly identify the weak lensing shear and rotation contributions, we arrive at Eq. (A17).

IV. PERTURBATIONS IN THE POISSON GAUGE

The Poisson gauge [34] is defined by $\partial^i {B_i}^{(n)} = \partial^j {h_{ij}}^{(n)} = 0$. In this case, one scalar degree of freedom is eliminated from g_{0i} and one scalar and two vector degrees of freedom from g_{ij}. In addition, here we neglect vector and tensor perturbations at first order. First-order vector perturbations have a decreasing amplitude and are not generated by inflation. The first-order tensor perturbations give a negligible contribution to second-order perturbations. Then the metric is

$$\mathrm{d}s^2 = a(\eta)^2 \left\{ -\left(1 + 2\Phi^{(1)} + \Phi^{(2)}\right) \mathrm{d}\eta^2 + 2\omega_i^{(2)} \mathrm{d}\eta\, \mathrm{d}x^i + \left[\delta_{ij}\left(1 - 2\Psi^{(1)} - \Psi^{(2)}\right) + \frac{1}{2}\hat{h}_{ij}^{(2)}\right] \mathrm{d}x^i \mathrm{d}x^j \right\} , \tag{174}$$

where $A^{(n)} = \Phi^{(n)}$, $B_i^{(1)} = 0$, $B^{(2)} = 0$, $\hat{B}_i^{(2)} = -2\omega_i^{(2)}$, $D^{(n)} = -\Psi^{(n)}$, $F^{(n)} = 0$, $\hat{F}_j^{(n)} = 0$ (i.e. $h_{ij}^{(1)} = -2\delta_{ij}\Psi^{(1)}$ and $h_{ij}^{(2)} = -2\delta_{ij}\Psi^{(2)} + \hat{h}_{ij}^{(2)}$) .

For the geodesic equation we obtain, at first order,

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(\delta\nu^{(1)} - 2\Phi^{(1)}\right) = \Phi^{(1)\prime} + \Psi^{(1)\prime} , \qquad \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(\delta n^{i(1)} - 2\Psi^{(1)} n^i\right) = -\partial^i\left(\Phi^{(1)} + \Psi^{(1)}\right) , \tag{175}$$

and, at second order,

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(\delta\nu^{(2)} - 2\Phi^{(2)} - 2\omega_\parallel^{(2)} + 4\Phi^{(1)}\delta\nu^{(1)}\right) = \Phi^{(2)\prime} + 2\omega_\parallel^{(2)\prime} + \Psi^{(2)\prime} - \frac{1}{2}\hat{h}_\parallel^{(2)\prime} + 4\delta n^{i(1)}\partial_i\Phi^{(1)} + 4\delta n_\parallel^{(1)}\Psi^{(1)\prime} \tag{176}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}&\left(\delta n^{i(2)} - 2\omega^{i(2)} - 2\Psi^{(2)} n^i + \hat{h}_j^{i(2)} n^j - 4\delta n^{i(1)}\Psi^{(1)}\right) = -\partial^i\left(\Phi^{(2)} + \Psi^{(2)}\right) - 2\partial^i\omega_\parallel^{(2)} + \frac{2}{\bar{\chi}}\omega_\perp^{i(2)} + \frac{1}{2}\partial^i\hat{h}_\parallel^{(2)} \\ &- \frac{1}{\bar{\chi}}\mathcal{P}^{ij}\hat{h}_{jk}^{(2)} n^k + 4\delta\nu^{(1)}\left(\partial^i\Phi^{(1)} + n^i\Psi^{(1)\prime}\right) - 4\delta n_\parallel^{(1)}\partial^i\Psi^{(1)} + 4\delta n^{j(1)} n^i\partial_j\Psi^{(1)}. \end{aligned} \tag{177}$$

Using the constraints

$$\begin{aligned} \delta\nu_o^{(1)} &= \Phi_o^{(1)} + v_{\parallel o}^{(1)} , \qquad \delta n_o^{\hat{a}(1)} = -v_o^{\hat{a}(1)} + n^{\hat{a}}\Psi_o^{(1)} , \\ \delta\nu_o^{(2)} &= \Phi_o^{(2)} + v_{\parallel o}^{(2)} + 2\omega_{\parallel o}^{(2)} - 3\left(\Phi_o^{(1)}\right)^2 - 2v_{\parallel o}^{(1)}\Phi_o^{(1)} - v_{k\,o}^{(1)} v_o^{k(1)} - 2\Psi_o^{(1)} v_{\parallel o}^{(1)} \\ \delta n_o^{\hat{a}(2)} &= -v_o^{\hat{a}(2)} + n^{\hat{a}}\Psi_o^{(2)} - \frac{1}{2}n^i\hat{h}_{i\,o}^{\hat{a}(2)} + v_o^{\hat{a}(1)} v_{\parallel o}^{(1)} + 3\, n^{\hat{a}}\left(\Psi_o^{(1)}\right)^2 , \end{aligned} \tag{178}$$

we obtain at first order

$$\delta\nu^{(1)} = -\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) + 2\Phi^{(1)} + \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right) = -\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) + 2\Phi^{(1)} - 2I^{(1)} , \tag{179}$$

$$\delta n^{i(1)} = -v_o^{i(1)} - n^i\Psi_o^{(1)} + 2n^i\Psi^{(1)} - \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,\tilde{\partial}^i\left(\Phi^{(1)} + \Psi^{(1)}\right) = n^i\delta n_\parallel^{(1)} + \delta n_\perp^{i(1)} , \tag{180}$$

where

$$\delta n_\parallel^{(1)} = \Phi_o^{(1)} - v_{\parallel o}^{(1)} - \Phi^{(1)} + \Psi^{(1)} + 2I^{(1)} , \qquad \delta n_\perp^{i(1)} = -v_{\perp o}^{i(1)} + 2S_\perp^{i(1)} , \tag{181}$$

$$I^{(1)} = -\frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right) , \tag{182}$$

$$S^{i(1)} = -\frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\left[\tilde{\partial}^i\left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{2}{\tilde{\chi}}n^i\Psi^{(1)}\right] , \tag{183}$$

$$S_\perp^{i(1)} = -\frac{1}{2}\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,\tilde{\partial}_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right) , \tag{184}$$

$$S_\parallel^{(1)} = n_i S^{i(1)} = \frac{1}{2}\left(\Phi_o^{(1)} + \Psi_{\parallel o}^{(1)}\right) - \frac{1}{2}\left(\Phi^{(1)} + \Psi^{(1)}\right) + I^{(1)} + \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\frac{\Psi^{(1)}}{\tilde{\chi}}. \tag{185}$$

Note the following useful relation

$$\delta n_\parallel^{(1)} + \delta\nu^{(1)} = \Phi^{(1)} + \Psi^{(1)} . \tag{186}$$

At second order we find

$$\delta\nu^{(2)} = -\Phi_o^{(2)} + v_{\parallel o}^{(2)} + \left(\Phi_o^{(1)}\right)^2 + 6\Phi_o^{(1)} v_{\parallel o}^{(1)} - v_{k\,o}^{(1)} v_o^{k(1)} - 2\Psi_o^{(1)} v_{\parallel o}^{(1)} + 4\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right)\left(2\Phi^{(1)} - 2I^{(1)}\right)$$
$$-4\, v_{\perp o}^{i(1)} \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(\tilde\partial_{\perp i}\Phi^{(1)}\right) + 2\Phi^{(2)} + 2\omega_\parallel^{(2)} - 12\left(\Phi^{(1)}\right)^2 + 16\,\Phi^{(1)} I^{(1)} - 2I^{(2)}$$
$$+4\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left\{\left(\Phi^{(1)} + \Psi^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\Phi^{(1)} + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\left(\Psi^{(1)} + 2I^{(1)}\right) + 2\tilde\partial_{\perp i}\left(\Phi^{(1)}\right) S_\perp^{i(1)}\right\}, \qquad (187)$$

and, splitting $\delta n^{i(2)} = n^i \delta n_\parallel^{(2)} + \delta n_\perp^{i(2)}$, we obtain

$$\delta n_\parallel^{(2)} = \Phi_o^{(2)} - v_{\parallel o}^{(2)} + \left(v_{\parallel o}^{(1)}\right)^2 - \left(\Psi_o^{(1)}\right)^2 - 4\Phi_o^{(1)} v_{\parallel o}^{(1)} + 4 v_{\parallel o}^{(1)}\Psi_o^{(1)} - 4\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right)\left(\Phi^{(1)} - \Psi^{(1)}\right) + 8\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) I^{(1)}$$
$$-\ 4 v_{\perp o}^{i(1)} \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(\partial_{\perp i}\Psi^{(1)}\right) - \Phi^{(2)} + \Psi^{(2)} - \frac{1}{2}\hat h_\parallel^{(2)} + 4\left(\Psi^{(1)}\right)^2 + 4\left(\Phi^{(1)}\right)^2 - 4\Phi^{(1)}\Psi^{(1)} - 8\left(\Phi^{(1)} - \Psi^{(1)}\right) I^{(1)}$$
$$+\ 2I^{(2)} + 4\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\left(\Phi^{(1)} - 2I^{(1)}\right)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right) + 2\,\tilde\partial_{\perp i}\Psi^{(1)} S_\perp^{i(1)}\right], \qquad (188)$$

and

$$\delta n_\perp^{i(2)} = -2\omega_{\perp o}^{i(2)} - v_{\perp o}^{i(2)} + \frac{1}{2} n^j \hat h_{jk\,o}^{(2)}\mathcal{P}^{ki} + v_{\parallel o}^{(1)} v_{\perp o}^{i(1)} + 4 v_{\perp o}^{i(1)}\Psi_o^{(1)} + 8\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) S_\perp^{i(1)} - 4 v_{\perp o}^{i(1)}\Psi^{(1)} + 2\omega_\perp^{i(2)}$$
$$-n^j \hat h_{jk}^{(2)}\mathcal{P}^{ki} + 8\,\Psi^{(1)} S_\perp^{i(1)} + 2S_\perp^{i(2)} + 4\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left\{+2\left(\Phi^{(1)} - I^{(1)}\right)\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right) - \left(\Phi^{(1)} + \Psi^{(1)}\right)\tilde\partial_\perp^i\Psi^{(1)}\right\}, \qquad (189)$$

where

$$I^{(2)} = -\frac{1}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(\Phi^{(2)\prime} + 2\omega_\parallel^{(2)\prime} + \Psi^{(2)\prime} - \frac{1}{2}\hat h_\parallel^{(2)\prime}\right), \qquad (190)$$
$$S_\perp^{i(2)} = -\frac{1}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\tilde\partial_\perp^i\left(\Phi^{(2)} + 2\omega_\parallel^{(2)} + \Psi^{(2)} - \frac{1}{2}\hat h_\parallel^{(2)}\right) + \frac{1}{\tilde\chi}\left(-2\omega_\perp^{i(2)} + n^k \hat h_{kj}^{(2)}\mathcal{P}^{ij}\right)\right]. \qquad (191)$$

Combining Eqs. (89) and (106), we obtain

$$\delta\nu^{(2)} + \delta n_\parallel^{(2)} = +\left(\Phi_o^{(1)}\right)^2 + 2\Phi_o^{(1)} v_{\parallel o}^{(1)} + 2 v_{\parallel o}^{(1)}\Psi_o^{(1)} - \left(\Psi_o^{(1)}\right)^2 - v_{\perp k\,o}^{(1)} v_{\perp o}^{k(1)} + 4\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)$$
$$+8\, v_{\perp i\,o}^{(1)} S_\perp^{i(1)} + \Phi^{(2)} + 2\omega_\parallel^{(2)} + \Psi^{(2)} - \frac{1}{2}\hat h_\parallel^{(2)} - 8\left(\Phi^{(1)}\right)^2 - 4\Psi^{(1)}\left(\Phi^{(1)} - \Psi^{(1)}\right) + 8\left(\Phi^{(1)} + \Psi^{(1)}\right) I^{(1)}$$
$$-8 S_\perp^{i(1)} S_\perp^{j(1)}\delta_{ij} + 4\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left\{\left(\Phi^{(1)} + \Psi^{(1)}\right)\left[\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right) + \frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\Phi^{(1)}\right]\right\}. \qquad (192)$$

From Eqs (16) and (17), we find

$$\delta x^{0(1)} = -\bar\chi\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) + \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2\Phi^{(1)} + (\bar\chi - \tilde\chi)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \qquad (193)$$
$$\delta x_\parallel^{(1)} = \bar\chi\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) - \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\left(\Phi^{(1)} - \Psi^{(1)}\right) + (\bar\chi - \tilde\chi)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right], \qquad (194)$$
$$\delta x_\perp^{i(1)} = -\bar\chi\, v_{\perp o}^{i(1)} - \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right], \qquad (195)$$

to first order. Using Eq. (104), we note that

$$T^{(1)} = -\left(\delta x^{0(1)} + \delta x_\parallel^{(1)}\right) = -\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(\Phi^{(1)} + \Psi^{(1)}\right). \qquad (196)$$

At second order,

$$
\begin{aligned}
\delta x^{0(2)} &= \bar{\chi}\left[-\Phi_o^{(2)}+v_{\| o}^{(2)}+\left(\Phi_o^{(1)}\right)^2+6\Phi_o^{(1)}v_{\| o}^{(1)}-v_{k\,o}^{(1)}v_o^{k(1)}-2\Psi_o^{(1)}v_{\| o}^{(1)}\right]\\
&+ 4\left(\Phi_o^{(1)}-v_{\| o}^{(1)}\right)\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[2\Phi^{(1)}+(\bar{\chi}-\tilde{\chi})\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]\\
&- 4v_{\perp o}^{i(1)}\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\,\tilde{\partial}_{\perp i}\Phi^{(1)}+2\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\Phi^{(2)}+\omega_{\|}^{(2)}-6\left(\Phi^{(1)}\right)^2+8\Phi^{(1)}I^{(1)}\right]\\
&+ \int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\left\{\Phi^{(2)\prime}+2\omega_{\|}^{(2)\prime}+\Psi^{(2)}-\frac{1}{2}\hat{h}_{\|}^{(2)\prime}+4\left(\Phi^{(1)}+\Psi^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde{\chi}}\Phi^{(1)}\right.\\
&+ \left.4\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\left(\Psi^{(1)}+2I^{(1)}\right)+8S_{\perp}^{i(1)}\tilde{\partial}_{\perp i}\Phi^{(1)}\right\}\ ,
\end{aligned}
\tag{197}
$$

$$
\begin{aligned}
\delta x_{\|}^{(2)} &= \bar{\chi}\left[\Phi_o^{(2)}-v_{\| o}^{(2)}+\left(v_{\| o}^{(1)}\right)^2-\left(\Psi_o^{(1)}\right)^2-4\Phi_o^{(1)}v_{\| o}^{(1)}+4v_{\| o}^{(1)}\Psi_o^{(1)}\right]\\
&- 4\left(\Phi_o^{(1)}-v_{\| o}^{(1)}\right)\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\left(\Phi^{(1)}-\Psi^{(1)}\right)+(\bar{\chi}-\tilde{\chi})\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]-4v_{\perp o}^{i(1)}\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\,\tilde{\partial}_{\perp i}\Psi^{(1)}\\
&+ \int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left\{-\Phi^{(2)}+\Psi^{(2)}-\frac{1}{2}\hat{h}_{\|}^{(2)}-4\Psi^{(1)}\Phi^{(1)}+4\left(\Psi^{(1)}\right)^2+4\left(\Phi^{(1)}\right)^2-8\left(\Phi^{(1)}-\Psi^{(1)}\right)I^{(1)}\right\}\\
&+ \int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\left\{-\left(\Phi^{(2)\prime}+2\omega_{\|}^{(2)\prime}+\Psi^{(2)\prime}-\frac{1}{2}\hat{h}_{\|}^{(2)\prime}\right)+4\left(\Phi^{(1)}-2I^{(1)}\right)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)+8S_{\perp}^{i(1)}\tilde{\partial}_{\perp i}\Psi^{(1)}\right\}\ ,
\end{aligned}
\tag{198}
$$

and

$$
\begin{aligned}
\delta x_{\perp}^{i(2)} &= \bar{\chi}\left[-2\omega_{\perp o}^{i(2)}-v_{\perp o}^{i(2)}+\frac{1}{2}n^j\hat{h}_{jk\,o}^{(2)}\mathcal{P}^{ki}+v_{\| o}^{(1)}v_{\perp o}^{i(1)}+4v_{\perp o}^{i(1)}\Psi_o^{(1)}\right]\\
&- 4\left(\Phi_o^{(1)}-v_{\| o}^{(1)}\right)\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\,\tilde{\partial}_{\perp}^i\left(\Phi^{(1)}+\Psi^{(1)}\right)-4v_{\perp o}^{i(1)}\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\Psi^{(1)}\\
&+ \int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left(2\omega_{\perp}^{i(2)}-n^j\hat{h}_{jk}^{(2)}\mathcal{P}^{ki}+8\,\Psi^{(1)}S_{\perp}^{i(1)}\right)+\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\left\{-\left[\tilde{\partial}_{\perp}^i\left(\Phi^{(2)}+2\omega_{\|}^{(2)}+\Psi^{(2)}-\frac{1}{2}\hat{h}_{\|}^{(2)}\right)\right.\right.\\
&+ \left.\left.\frac{1}{\tilde{\chi}}\left(-2\omega_{\perp}^{i(2)}+n^k\hat{h}_{kj}^{(2)}\mathcal{P}^{ij}\right)\right]+8\left(\Phi^{(1)}-I^{(1)}\right)\tilde{\partial}_{\perp}^i\left(\Phi^{(1)}+\Psi^{(1)}\right)-4\left(\Phi^{(1)}+\Psi^{(1)}\right)\tilde{\partial}_{\perp}^i\Psi^{(1)}\right]\right\}.
\end{aligned}
\tag{199}
$$

Combining Eqs. (114) and (115) we have

$$
\begin{aligned}
\delta x^{0(2)}+\delta x_{\|}^{(2)} &= \bar{\chi}\left[\left(\Phi_o^{(1)}\right)^2+2\Phi_o^{(1)}v_{\| o}^{(1)}+2v_{\| o}^{(1)}\Psi_o^{(1)}-\left(\Psi_o^{(1)}\right)^2-v_{\perp k\,o}^{(1)}v_{\perp o}^{k(1)}\right]-4\left(\Phi_o^{(1)}-v_{\| o}^{(1)}\right)T^{(1)}\\
&-4\,v_{\perp o}^{i(1)}\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\left[\tilde{\partial}_{\perp}^i\left(\Phi^{(1)}+\Psi^{(1)}\right)\right]-T^{(2)}+4\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[-2\left(\Phi^{(1)}\right)^2\right.\\
&\left.-\Psi^{(1)}\left(\Phi^{(1)}-\Psi^{(1)}\right)+2\left(\Phi^{(1)}+\Psi^{(1)}\right)I^{(1)}-2S_{\perp}^{i(1)}S_{\perp}^{j(1)}\delta_{ij}\right]\\
&+4\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\left\{\left(\Phi^{(1)}+\Psi^{(1)}\right)\left[\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)+\frac{\mathrm{d}}{\mathrm{d}\tilde{\chi}}\Phi^{(1)}\right]\right\}.
\end{aligned}
\tag{200}
$$

To obtain all the second order terms we require $\Delta\ln a^{(1)}$ (or $\Delta x^{0(1)}$), $\delta\chi^{(1)}$, $\Delta x^{0(1)}$, $\Delta x_{\|}^{(1)}$, $\Delta x_{\perp}^{i(1)}$ and $\Delta_g^{(1)}$. From Eqs. (32) and (34) we have

$$
\Delta\ln a^{(1)} = \left(\Phi_o^{(1)}-v_{\| o}^{(1)}\right)-\Phi^{(1)}+v_{\|}^{(1)}+2I^{(1)}=\left(\Phi_o^{(1)}-v_{\| o}^{(1)}\right)-\Phi^{(1)}+v_{\|}^{(1)}-\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\ , \tag{201}
$$

$$
\delta\chi^{(1)} = -\left(\bar{\chi}+\frac{1}{\mathcal{H}}\right)\left(\Phi_o^{(1)}-v_{\| o}^{(1)}\right)+\frac{1}{\mathcal{H}}\left(\Phi^{(1)}-v_{\|}^{(1)}\right)+\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[2\Phi^{(1)}+(\bar{\chi}-\tilde{\chi})\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]-\frac{2}{\mathcal{H}}I^{(1)}\ . \tag{202}
$$

Then, from Eq. (36)

$$\Delta x^{0(1)} = \frac{1}{\mathcal{H}}\left[\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) - \Phi^{(1)} + v_\parallel^{(1)} + 2I^{(1)}\right] = \frac{1}{\mathcal{H}}\left[\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) - \Phi^{(1)} + v_\parallel^{(1)} - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] , \tag{203}$$

and from Eqs. (69) and (112)

$$\begin{aligned}\Delta x_\parallel^{(1)} &= -T^{(1)} - \frac{1}{\mathcal{H}}\left[\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) - \Phi^{(1)} + v_\parallel^{(1)} + 2I^{(1)}\right] \\ &= \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{1}{\mathcal{H}}\left[\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) - \Phi^{(1)} + v_\parallel^{(1)} - \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] . \end{aligned}\tag{204}$$

Using Eq. (70), we have

$$\Delta x_\perp^{i(1)} = -\bar\chi\, v_{\perp o}^{i(1)} - \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, (\bar\chi - \tilde\chi)\, \tilde\partial_\perp^i \left(\Phi^{(1)} + \Psi^{(1)}\right) . \tag{205}$$

In Eq. (120) there is an ISW contribution and in Eq. (121) we have both time-delay and ISW contributions.

Now we can obtain $\Delta_g^{(1)}$. Using Eq. (79) for $\Delta x_\parallel^{(1)}$, we find

$$\begin{aligned}\partial_\parallel \Delta x_\parallel^{(1)} &= \left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta \ln a^{(1)} - \frac{1}{\mathcal{H}}\left(\frac{\mathrm{d}\,\Delta \ln a}{\mathrm{d}\bar\chi}\right)^{(1)} \\ &= \left(\Phi^{(1)} + \Psi^{(1)}\right) + \frac{1}{\mathcal{H}}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\parallel^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] - \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta \ln a^{(1)} . \end{aligned}\tag{206}$$

With Eqs. (118), (121) and (123), Eq. (76) becomes

$$\begin{aligned}\Delta_g^{(1)} &= \delta_g^{(1)} + \left(b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar\chi\mathcal{H}}\right)\Delta \ln a^{(1)} + \frac{1}{\mathcal{H}}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\parallel^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] - \frac{2}{\bar\chi}T^{(1)} - 2\kappa^{(1)} \\ &\quad + \Phi^{(1)} + v_\parallel^{(1)} - 2\Psi^{(1)} , \end{aligned}\tag{207}$$

in agreement with [3, 4]. At first order, the coordinate convergence lensing term defined in Eq. (80) yields

$$\kappa^{(1)} = -\frac{1}{2}\partial_{\perp i}\Delta x_\perp^{i(1)} = \frac{1}{2}\int_0^{\bar\chi} \mathrm{d}\tilde\chi\, (\bar\chi - \tilde\chi)\, \frac{\tilde\chi}{\bar\chi}\, \tilde\nabla_\perp^2 \left(\Phi^{(1)} + \Psi^{(1)}\right) - v_{\parallel o}^{(1)} . \tag{208}$$

At this point, we can finally compute $\Delta \ln a^{(2)}$, $\Delta x^{0(2)}$, $\Delta x_\parallel^{(2)}$ and $\Delta x_\perp^{i(2)}$. From Eq. (33) we find

$$\begin{aligned}\Delta \ln a^{(2)} &= \Phi_o^{(2)} - v_{\parallel o}^{(2)} - \left(\Phi_o^{(1)}\right)^2 - 6\Phi_o^{(1)} v_{\parallel o}^{(1)} + v_{k\,o}^{(1)} v_o^{k(1)} + 2\Psi_o^{(1)} v_{\parallel o}^{(1)} + 2\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right)\Bigg[- 3\Phi^{(1)} + v_\parallel^{(1)} \\ &\quad + \left(2\bar\chi + \frac{1}{\mathcal{H}}\right)\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)} - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}v_\parallel^{(1)} + \left(\bar\chi + \frac{1}{\mathcal{H}}\right)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right) + 4I^{(1)}\Bigg] \\ &\quad - 2\, v_{\perp o}^{i(1)}\left[\bar\chi\, \partial_{\perp i}\left(\Phi^{(1)} + v_\parallel^{(1)}\right) - 2\int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \tilde\partial_{\perp i}\Phi^{(1)}\right] - \Phi^{(2)} + v_\parallel^{(2)} + 7\left(\Phi^{(1)}\right)^2 + v_i^{(1)} v^{i(1)} \\ &\quad - 2v_\parallel^{(1)}\left(\Psi^{(1)} + \Phi^{(1)}\right) - \frac{2}{\mathcal{H}}\left(\Phi^{(1)} - v_\parallel^{(1)}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\parallel^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \\ &\quad - 4\left[3\Phi^{(1)} - v_\parallel^{(1)} - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\parallel^{(1)}\right) - \frac{1}{\mathcal{H}}\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] I^{(1)} + 4 v_{\perp i}^{(1)} S_\perp^{i(1)} \\ &\quad - 2\partial_\parallel\left(\Phi^{(1)} + v_\parallel^{(1)}\right) T^{(1)} - 2\left[2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)} + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right]\int_0^{\bar\chi} \mathrm{d}\tilde\chi\left[2\Phi^{(1)} + (\bar\chi - \tilde\chi)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \\ &\quad - 2\left[\partial_{\perp i}\left(\Phi^{(1)} + v_\parallel^{(1)}\right) - \frac{1}{\bar\chi}v_{\perp i}^{(1)}\right]\int_0^{\bar\chi} \mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\, \tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] + 2I^{(2)} \\ &\quad - 4\int_0^{\bar\chi} \mathrm{d}\tilde\chi\left\{\left(\Psi^{(1)} + 2I^{(1)}\right)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right) + \left(\Phi^{(1)} + \Psi^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\Phi^{(1)} + 2S_\perp^{i(1)}\tilde\partial_{\perp i}\Phi^{(1)}\right\} , \end{aligned}\tag{209}$$

Using Eqs (118) and (164), Eq. (37) yields

$$
\begin{aligned}
\Delta x^{0(2)} = & +\frac{1}{\mathcal{H}}\Phi^{(2)}_o - \frac{1}{\mathcal{H}}v^{(2)}_{\|o} - \left(\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{2}{\mathcal{H}}\right)\left(\Phi^{(1)}_o\right)^2 + 2\left(\frac{\mathcal{H}'}{\mathcal{H}^3}-\frac{2}{\mathcal{H}}\right)\Phi^{(1)}_o v^{(1)}_{\|o} - \frac{\mathcal{H}'}{\mathcal{H}^3}\left(v^{(1)}_{\|o}\right)^2 + \frac{2}{\mathcal{H}}\Psi^{(1)}_o v^{(1)}_{\|o} \\
& +\frac{1}{\mathcal{H}}v^{(1)}_{\perp i\,o}v^{i(1)}_{\perp o} + 2\left(\Phi^{(1)}_o - v^{(1)}_{\|o}\right)\left\{\left(\frac{\mathcal{H}'}{\mathcal{H}^3}-\frac{2}{\mathcal{H}}\right)\Phi^{(1)} - \frac{\mathcal{H}'}{\mathcal{H}^3}v^{(1)}_{\|} + \left(2\frac{\bar\chi}{\mathcal{H}}+\frac{1}{\mathcal{H}^2}\right)\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)} - \frac{1}{\mathcal{H}^2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}v^{(1)}_{\|}\right. \\
& \left.+\left(\frac{\bar\chi}{\mathcal{H}}+\frac{1}{\mathcal{H}^2}\right)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right) - 2\left(\frac{\mathcal{H}'}{\mathcal{H}^3}-\frac{1}{\mathcal{H}}\right)I^{(1)}\right\} - 2v^{i(1)}_{\perp o}\left[+\frac{\bar\chi}{\mathcal{H}}\partial_{\perp i}\left(\Phi^{(1)}+v^{(1)}_{\|}\right) - \frac{2}{\mathcal{H}}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\tilde\partial_{\perp i}\Phi^{(1)}\right] \\
& -\frac{1}{\mathcal{H}}\Phi^{(2)} + \frac{1}{\mathcal{H}}v^{(2)}_{\|} + \left(-\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{6}{\mathcal{H}}\right)\left(\Phi^{(1)}\right)^2 - \frac{\mathcal{H}'}{\mathcal{H}^3}\left(v^{(1)}_{\|}\right)^2 + \frac{1}{\mathcal{H}}v^{(1)}_{\perp i}v^{i(1)}_{\perp} - \frac{2}{\mathcal{H}}v^{(1)}_{\|}\Psi^{(1)} + 2\frac{\mathcal{H}'}{\mathcal{H}^3}\Phi^{(1)}v^{(1)}_{\|} \\
& -\frac{2}{\mathcal{H}^2}\left(\Phi^{(1)}-v^{(1)}_{\|}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)}-v^{(1)}_{\|}\right) + \left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right] - 4\left[\left(\frac{2}{\mathcal{H}}-\frac{\mathcal{H}'}{\mathcal{H}^3}\right)\Phi^{(1)} + \frac{\mathcal{H}'}{\mathcal{H}^3}v^{(1)}_{\|}\right. \\
& \left.-\frac{1}{\mathcal{H}^2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)}-v^{(1)}_{\|}\right) - \frac{1}{\mathcal{H}^2}\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right) + \left(\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{1}{\mathcal{H}}\right)I^{(1)}\right]I^{(1)} + \frac{4}{\mathcal{H}}v^{(1)}_{\perp i}S^{i(1)}_{\perp} - \frac{2}{\mathcal{H}}\partial_{\|}\left(\Phi^{(1)}+v^{(1)}_{\|}\right)T^{(1)} \\
& -\frac{2}{\mathcal{H}}\left[2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)} + \left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2\Phi^{(1)} + (\bar\chi-\tilde\chi)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right] - \frac{2}{\mathcal{H}}\left[\partial_{\perp i}\left(\Phi^{(1)}+v^{(1)}_{\|}\right)\right. \\
& \left.-\frac{1}{\bar\chi}v^{(1)}_{\perp i}\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi-\tilde\chi)\,\tilde\partial^i_{\perp}\left(\Phi^{(1)}+\Psi^{(1)}\right)\right] + \frac{2}{\mathcal{H}}I^{(2)} - \frac{4}{\mathcal{H}}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\left(\Psi^{(1)}+2I^{(1)}\right)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right. \\
& \left.+\left(\Phi^{(1)}+\Psi^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\Phi^{(1)} + 2S^{i(1)}_{\perp}\tilde\partial_{\perp i}\Phi^{(1)}\right]. \qquad (210)
\end{aligned}
$$

From Eqs. (71) and (117) we deduce

$$
\begin{aligned}
\Delta x^{(2)}_{\|} = & \;\bar\chi\left[\left(\Phi^{(1)}_o\right)^2 + 2\Phi^{(1)}_o v^{(1)}_{\|o} + 2v^{(1)}_{\|o}\Psi^{(1)}_o - \left(\Psi^{(1)}_o\right)^2 - v^{(1)}_{\perp k\,o}v^{k(1)}_{\perp o}\right] - 2\left(\Phi^{(1)}_o - v^{(1)}_{\|o}\right)\left[\bar\chi\left(\Phi^{(1)}+\Psi^{(1)}\right) + 2T^{(1)}\right] \\
& -4v^{i(1)}_{\perp o}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi-\tilde\chi)\left[\tilde\partial^i_{\perp}\left(\Phi^{(1)}+\Psi^{(1)}\right)\right] + 2\left(\Phi^{(1)}+\Psi^{(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2\Phi^{(1)} + (\bar\chi-\tilde\chi)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right] \\
& -T^{(2)} + 4\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[-2\left(\Phi^{(1)}\right)^2 - \Psi^{(1)}\left(\Phi^{(1)}-\Psi^{(1)}\right) + 2\left(\Phi^{(1)}+\Psi^{(1)}\right)I^{(1)} - 2S^{i(1)}_{\perp}S^{j(1)}_{\perp}\delta_{ij}\right] \\
& +4\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left\{(\bar\chi-\tilde\chi)\left(\Phi^{(1)}+\Psi^{(1)}\right)\left[\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right) + \frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\Phi^{(1)}\right]\right\} - \frac{2}{\mathcal{H}}\left(\Phi^{(1)}+\Psi^{(1)}\right)\Delta\ln a^{(1)} \\
& -\frac{1}{\mathcal{H}}\Delta\ln a^{(2)} + \left(\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{1}{\mathcal{H}}\right)\left(\Delta\ln a^{(1)}\right)^2\,, \qquad (211)
\end{aligned}
$$

and from Eqs. (72) and (116) we find

$$
\begin{aligned}
\Delta x^{i(2)}_{\perp} = & \;\bar\chi\left[-2\omega^{i(2)}_{\perp o} - v^{i(2)}_{\perp o} + \frac{1}{2}n^j\hat h^{(2)}_{jk\,o}\mathcal{P}^{ki} + 2\Phi^{(1)}_o v^{i(1)}_{\perp o} - v^{(1)}_{\|o}v^{i(1)}_{\perp o} + 4v^{i(1)}_{\perp o}\Psi^{(1)}_o\right] - 4\bar\chi\left(\Phi^{(1)}_o - v^{(1)}_{\|o}\right)S^{i(1)}_{\perp} \\
& -4\left(\Phi^{(1)}_o - v^{(1)}_{\|o}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi-\tilde\chi)\,\tilde\partial^i_{\perp}\left(\Phi^{(1)}+\Psi^{(1)}\right)\right] - 2\,v^{i(1)}_{\perp o}\left\{\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2\left(\Phi^{(1)}+\Psi^{(1)}\right)\right.\right. \\
& \left.\left.+(\bar\chi-\tilde\chi)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right] - \frac{1}{\mathcal{H}}\Delta\ln a^{(1)}\right\} - \frac{4}{\mathcal{H}}S^{i(1)}_{\perp}\Delta\ln a^{(1)} + 4\,S^{i(1)}_{\perp}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2\Phi^{(1)}\right. \\
& \left.+(\bar\chi-\tilde\chi)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right] + \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left\{2\omega^{i(2)}_{\perp} - n^j\hat h^{(2)}_{jk}\mathcal{P}^{ki} + 8\,\Psi^{(1)}S^{i(1)}_{\perp}\right\} \\
& +\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi-\tilde\chi)\left\{-\left[\tilde\partial^i_{\perp}\left(\Phi^{(2)} + 2\omega^{(2)}_{\|} + \Psi^{(2)} - \frac{1}{2}\hat h^{(2)}_{\|}\right) + \frac{1}{\tilde\chi}\left(-2\,\omega^{i(2)}_{\perp} + n^k\hat h^{(2)}_{kj}\mathcal{P}^{ij}\right)\right]\right. \\
& \left.+8\left(\Phi^{(1)}-I^{(1)}\right)\tilde\partial^i_{\perp}\left(\Phi^{(1)}+\Psi^{(1)}\right) - 4\left(\Phi^{(1)}+\Psi^{(1)}\right)\tilde\partial^i_{\perp}\Psi^{(1)}\right\}. \qquad (212)
\end{aligned}
$$

To obtain explicitly Eq. (133) we need

$$
\begin{aligned}
&\left(\frac{\mathrm{d}\,\Delta\ln a}{\mathrm{d}\bar\chi}\right)^{(2)} = +2\left(\Phi^{(1)}_o - v^{(1)}_{\parallel\,o}\right)\left\{\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-1\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)+\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]+\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)\right.\right.\\
&\left.\left.+\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]+\bar\chi\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)}+\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]\right\}-2\,v^{i(1)}_{\perp\,o}\left[-\partial_{\perp i}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)+\bar\chi\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_{\perp i}\left(\Phi^{(1)}+v^{(1)}_\parallel\right)\right]\\
&+\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(-\Phi^{(2)}+v^{(2)}_\parallel\right)-\left(\Phi^{(2)\prime}+2\,\omega^{(2)\prime}_\parallel+\Psi^{(2)\prime}-\frac{1}{2}\hat h^{(2)\prime}_\parallel\right)+\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[7\left(\Phi^{(1)}\right)^2-2\Phi^{(1)}v^{(1)}_\parallel+\left(v^{(1)}_\parallel\right)^2-2v^{(1)}_\parallel\Psi^{(1)}\right]\\
&-4\left[2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)}+\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]\Phi^{(1)}-2\left\{\frac{\mathcal{H}'}{\mathcal{H}^2}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)+\frac{1}{\mathcal{H}}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)+\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]\right\}\\
&\times\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)+\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]+2\left[3\Phi^{(1)}-2\Psi^{(1)}-v^{(1)}_\parallel\right]\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\\
&-\frac{2}{\mathcal{H}}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)+\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]+2\left(\Phi^{(1)}+\Psi^{(1)}\right)\left[-2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)}+\partial_\parallel\left(\Phi^{(1)}+v^{(1)}_\parallel\right)\right]\\
&-2v^{(1)}_{\perp i}\partial^i_\perp\left(\Phi^{(1)}+\Psi^{(1)}\right)+2v^{(1)}_{\perp i}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}v^{i(1)}_\perp+4\left\{\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-1\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)+\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]\right.\\
&\left.+\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)+\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]\right\}I^{(1)}-2\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_\parallel\left(\Phi^{(1)}+v^{(1)}_\parallel\right)\right]T^{(1)}-2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)}\right.\\
&\left.+\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2\Phi^{(1)}+(\bar\chi-\tilde\chi)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right]-4\left[\partial_{\perp i}\left(\Phi^{(1)}-v^{(1)}_\parallel\right)+\frac{1}{\tilde\chi}v^{(1)}_{\perp i}-\frac{\mathrm{d}}{\mathrm{d}\bar\chi}v^{(1)}_{\perp i}\right]S^{i(1)}_\perp\\
&-2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\partial_{\perp i}\left(\Phi^{(1)}+v^{(1)}_\parallel\right)-\frac{1}{\bar\chi}v^{(1)}_{\perp i}\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi-\tilde\chi)\,\tilde\partial^i_\perp\left(\Phi^{(1)}+\Psi^{(1)}\right)\right]\,. \qquad (213)
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
&\partial_\| \Delta x_\|^{(2)} = \left(\Phi_o^{(1)}\right)^2 + 2\Phi_o^{(1)} v_{\| o}^{(1)} + 2 v_{\| o}^{(1)} \Psi_o^{(1)} - \left(\Psi_o^{(1)}\right)^2 - v_{\perp k\, o}^{(1)} v_{\perp\, o}^{k(1)} + 2\left(\Phi_o^{(1)} - v_{\| o}^{(1)}\right) \\
&\times \Bigg\{ \left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right) \left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] - \frac{\bar\chi}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)} + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \\
&+ \left(\Phi^{(1)} + \Psi^{(1)}\right) - \bar\chi \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{1}{\mathcal{H}^2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right]\Bigg\} - 2 v_{\perp\, o}^{i(1)} \\
&\times \left[+ \frac{1}{\mathcal{H}}\partial_{\perp i}\left(\Phi^{(1)} - v_\|^{(1)}\right) - 2\frac{\bar\chi}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_{\perp i}\left(\Phi^{(1)} + v_\|^{(1)}\right) - 4\delta_{il} S_\perp^{l(1)}\right] + \Phi^{(2)} + 2\omega_\|^{(2)} + \Psi^{(2)} - \frac{1}{2}\hat h_\|^{(2)} \\
&+ \frac{1}{\mathcal{H}}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(2)} - v_\|^{(2)}\right) + \left(\Phi^{(2)\prime} + 2\omega_\|^{(2)\prime} + \Psi^{(2)\prime} - \frac{1}{2}\hat h_\|^{(2)\prime}\right)\right] - 4\left(\Phi^{(1)} - \Psi^{(1)}\right)\left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\bigg[7\left(\Phi^{(1)}\right)^2 \\
&- 2\Phi^{(1)} v_\|^{(1)} + \left(v_\|^{(1)}\right)^2 - 2 v_\|^{(1)}\Psi^{(1)} + v_{\perp i}^{(1)} v_\perp^{i(1)}\bigg] + \frac{2}{\mathcal{H}^2}\left(\Phi^{(1)} - v_\|^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \\
&- \frac{4}{\mathcal{H}}\left[\left(\Phi^{(1)} - \Psi^{(1)}\right) + \left(\Phi^{(1)} - v_\|^{(1)}\right)\right]\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right) + \frac{4}{\mathcal{H}}\Phi^{(1)}\left[2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)} + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \\
&+ 2\Bigg\{\frac{1}{\mathcal{H}}\left(\Phi^{(1)} + \Psi^{(1)}\right) + \frac{\mathcal{H}'}{\mathcal{H}^3}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \frac{1}{\mathcal{H}^2}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right]\Bigg\} \\
&\times \left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] + \frac{2}{\mathcal{H}}\left[2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)} - \partial_\|\left(\Phi^{(1)} + v_\|^{(1)}\right)\right]\left(\Phi^{(1)} + \Psi^{(1)}\right) \\
&+ \frac{2}{\mathcal{H}} v_{\perp i}^{(1)}\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right) + \frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_\|\left(\Phi^{(1)} + v_\|^{(1)}\right) T^{(1)} + 4\Bigg\{\left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \\
&- \frac{1}{\mathcal{H}^2}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] + \left(\Phi^{(1)} + \Psi^{(1)}\right)\Bigg\} I^{(1)} + 2\Bigg\{\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} + \Psi^{(1)}\right) \\
&+ \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)} + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right]\Bigg\}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2\Phi^{(1)} + (\bar\chi - \tilde\chi)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] + 4\bigg[\frac{1}{\mathcal{H}}\partial_{\perp i}\left(\Phi^{(1)} - v_\|^{(1)}\right) \\
&+ \frac{1}{\mathcal{H}\bar\chi} v_{\perp i}^{(1)} - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi} v_{\perp i}^{(1)} - 2 S_\perp^{j(1)}\delta_{ij}\bigg] S_\perp^{i(1)} + \frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[\partial_{\perp i}\left(\Phi^{(1)} + v_\|^{(1)}\right) - \frac{1}{\bar\chi} v_{\perp i}^{(1)}\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\ \tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] \\
&- \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta\ln a^{(2)} + \left[-\frac{\mathcal{H}''}{\mathcal{H}^3} + 3\left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right]\left(\Delta\ln a^{(1)}\right)^2 + 4\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\left(\Phi^{(1)} + \Psi^{(1)}\right)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime} + \frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\Phi^{(1)}\right)\right] \\
&+ 2\Bigg\{-\frac{\mathcal{H}'}{\mathcal{H}^2}\left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} + \Psi^{(1)}\right) - \left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right]\Bigg\}\Delta\ln a^{(1)} .
\end{aligned}
\tag{214}
$$

From Eq. (132) and using Eqs. (126), (127) and (128), we obtain the coordinate convergence lensing term at second order

$$
\kappa^{(2)} = -\frac{1}{2}\partial_{\perp i}\Delta x_\perp^{i(2)} = \kappa_1^{(2)} + \kappa_2^{(2)} + \kappa_3^{(2)} + \kappa_4^{(2)} \tag{215}
$$

where

$$
\kappa_1^{(2)} = \frac{1}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi - \tilde\chi)\,\frac{\tilde\chi}{\bar\chi}\tilde\nabla_\perp^2\left(\Phi^{(2)} + 2\omega_\|^{(2)} + \Psi^{(2)} - \frac{1}{2}\hat h_\|^{(2)}\right) , \tag{216}
$$

$$
\kappa_2^{(2)} = \frac{1}{2}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(-2\tilde\partial_\perp^i\omega_i^{(2)} + \frac{4}{\tilde\chi}\omega_\|^{(2)} + \mathcal{P}^{ij}n^k\tilde\partial_i\hat h_{jk}^{(2)} - \frac{3}{\tilde\chi}\hat h_\|^{(2)}\right) , \tag{217}
$$

$$\kappa_3^{(2)} = \left[\int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \frac{\tilde\chi}{\bar\chi}\, \tilde\nabla_\perp^2 \left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\left\{ \int_0^{\bar\chi} \mathrm{d}\tilde\chi \left[2\Phi^{(1)} + (\bar\chi - \tilde\chi)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] - \frac{1}{\mathcal{H}}\Delta\ln a^{(1)}\right\}$$
$$-2S_\perp^{i(1)}\left\{ \int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \frac{\tilde\chi}{\bar\chi}\left[2\,\tilde\partial_{\perp i}\Phi^{(1)} + (\bar\chi - \tilde\chi)\,\tilde\partial_{\perp i}\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] - \frac{1}{\mathcal{H}}\partial_{\perp i}\Delta\ln a^{(1)}\right\}$$
$$+\int_0^{\bar\chi} \mathrm{d}\tilde\chi\, \frac{\tilde\chi}{\bar\chi}\left[-4\,\tilde\partial_{\perp j}\Psi^{(1)} S_\perp^{j(1)} + \frac{8}{\tilde\chi}\Psi^{(1)}S_\parallel^{(1)} - 4\,\Psi^{(1)}\tilde\partial_{\perp m}S^{m(1)}\right]$$
$$+2\int_0^{\bar\chi} \mathrm{d}\tilde\chi\left\{ (\bar\chi - \tilde\chi)\,\frac{\tilde\chi}{\bar\chi}\left[\tilde\partial_\perp^i\Psi^{(1)}\tilde\partial_{\perp i}\left(\Phi^{(1)} + \Psi^{(1)}\right) + \left(\Phi^{(1)} + \Psi^{(1)}\right)\tilde\nabla_\perp^2\Psi^{(1)}\right.\right.$$
$$\left.\left.-2\tilde\partial_{\perp i}\left(\Phi^{(1)} - I^{(1)}\right)\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right) - 2\left(\Phi^{(1)} - I^{(1)}\right)\tilde\nabla_\perp^2\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\right\}, \tag{218}$$

$$\kappa_4^{(2)} = -2\omega_{\parallel o}^{(2)} - v_{\parallel o}^{(2)} + \frac{3}{4}\hat h_{\parallel o}^{(2)} - \frac{1}{4}\hat h_{i\,o}^{i(2)} + 2\Phi_o^{(1)}v_{\parallel o}^{(1)} + \frac{1}{2}v_{\perp i\,o}^{(1)}v_{\perp o}^{i(1)} - \left(v_{\parallel o}^{(1)}\right)^2 + 4v_{\parallel o}^{(1)}\Psi_o^{(1)}$$
$$+\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right)\left\{-\int_0^{\bar\chi} \mathrm{d}\tilde\chi\left[\tilde\chi\tilde\nabla_\perp^2\left(\Phi^{(1)} + \Phi^{(1)}\right)\right] + 4\kappa_2^{(1)} + 4\kappa_1^{(1)}\right\}$$
$$-\frac{2}{\bar\chi}v_{\parallel o}^{(1)}\left\{ \int_0^{\bar\chi} \mathrm{d}\tilde\chi\left[2\left(\Phi^{(1)} + \Psi^{(1)}\right) + (\bar\chi - \tilde\chi)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] - \frac{1}{\mathcal{H}}\Delta\ln a^{(1)}\right\}$$
$$+v_{\perp i\,o}^{(1)}\left\{-2S_\perp^{i(1)} + \int_0^{\bar\chi} \mathrm{d}\tilde\chi\left[-2\left(1 - 2\frac{\tilde\chi}{\bar\chi}\right)\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right) + \frac{\tilde\chi}{\bar\chi}(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right]\right.$$
$$\left.-\frac{1}{\mathcal{H}}\partial_\perp^i\Delta\ln a^{(1)}\right\}. \tag{219}$$

Finally, in order to obtain $\Delta_g^{(2)}$ [see Eq. (77)], we use

$$\frac{1}{2}\hat g_\mu^{\mu(2)} - \frac{1}{2}\hat g_\mu^{\nu(1)}\,\hat g_\nu^{\mu(1)} = \Phi^{(2)} - 3\Psi^{(2)} - 2\left(\Phi^{(1)}\right)^2 - 6\left(\Psi^{(1)}\right)^2, \tag{220}$$

$$E_{\hat 0}^{0(2)} + E_{\hat 0}^{\parallel(2)} = -\Phi^{(2)} + v_\parallel^{(2)} + 3\left(\Phi^{(1)}\right)^2 + v_i^{(1)}v^{i(1)}, \tag{221}$$

$$\frac{1}{\mathcal{H}}\hat g_\mu^{\mu(1)\prime}\Delta\ln a^{(1)} + \left(\partial_\parallel\hat g_\mu^{\mu(1)}\right)\Delta x_\parallel^{(1)} = -\frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - 3\Psi^{(1)}\right)\Delta\ln a^{(1)} - 2\partial_\parallel\left(\Phi^{(1)} - 3\Psi^{(1)}\right)T^{(1)}, \tag{222}$$

$$\left(\partial_{\perp i}\hat g_\mu^{\mu(1)}\right)\Delta x_\perp^{i(1)} = -2\bar\chi v_{\perp o}^{i(1)}\partial_{\perp i}\left(\Phi^{(1)} - 3\Psi^{(1)}\right) - 2\partial_{\perp i}\left(\Phi^{(1)} - 3\Psi^{(1)}\right)\int_0^{\bar\chi} \mathrm{d}\tilde\chi\left\{+(\bar\chi - \tilde\chi)\left[\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\right\}, \tag{223}$$

$$\frac{2}{\mathcal{H}}\delta_g^{(1)\prime}\Delta\ln a^{(1)} + 2\left(\partial_\parallel\delta_g^{(1)}\right)\Delta x_\parallel^{(1)} = -\frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\delta_g^{(1)}\right)\Delta\ln a^{(1)} - 2\left(\partial_\parallel\delta_g^{(1)}\right)T^{(1)}, \tag{224}$$

$$\Delta x_{\perp i}^{(1)}\,\partial_\perp^i\delta_g^{(1)} = \delta x_{\perp i}^{(1)}\,\partial_\perp^i\delta_g^{(1)} = -\bar\chi v_{\perp o}^{i(1)}\partial_{\perp i}\delta_g^{(1)} - \left(\partial_{\perp i}\delta_g^{(1)}\right)\int_0^{\bar\chi} \mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right], \tag{225}$$

$$-\frac{2}{\bar\chi^2}\left(\Delta x_\parallel^{(1)}\right)^2 = -\frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2 - \frac{2}{\bar\chi^2\mathcal{H}^2}\left(\Delta\ln a^{(1)}\right)^2 - \frac{4}{\bar\chi^2\mathcal{H}}\Delta\ln a^{(1)}T^{(1)}, \tag{226}$$

$$\frac{4}{\bar\chi}\Delta x_\parallel^{(1)}\kappa^{(1)} = -\frac{4}{\bar\chi\mathcal{H}}\Delta\ln a^{(1)}\kappa^{(1)} - \frac{4}{\bar\chi}T^{(1)}\kappa^{(1)}, \tag{227}$$

$$-\left(\partial_{\perp j}\Delta x_{\perp}^{i(1)}\right)\left(\partial_{\perp i}\Delta x_{\perp}^{j(1)}\right) = -2\left(v_{\| o}^{(1)}\right)^2 - 2\left(\Phi_o^{(1)}\right)^2 + 4v_{\| o}^{(1)}\Psi_o^{(1)} - 2\left(\Psi_o^{(1)}\right)^2 + 4\Phi_o^{(1)}v_{\| o}^{(1)} - 4\Phi_o^{(1)}\Psi_o^{(1)}$$
$$+2\left(\Phi_o^{(1)} - v_{\| o}^{(1)} + \Psi_o^{(1)}\right)\left\{2\left(\Phi^{(1)} + \Psi^{(1)} - 2I^{(1)}\right) - \int_0^{\bar\chi} d\tilde\chi\left[\frac{\tilde\chi}{\bar\chi}\left(2\tilde\partial_\| + (\bar\chi - \tilde\chi)\,\mathcal{P}^{mn}\tilde\partial_m\tilde\partial_n\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\right\}$$
$$-2\left(\Phi^{(1)} + \Psi^{(1)} - 2I^{(1)}\right)^2 + 2\left(\Phi^{(1)} + \Psi^{(1)} - 2I^{(1)}\right)\int_0^{\bar\chi} d\tilde\chi\left[\frac{\tilde\chi}{\bar\chi}\left(2\tilde\partial_\| + (\bar\chi - \tilde\chi)\,\mathcal{P}^{mn}\tilde\partial_m\tilde\partial_n\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]$$
$$-\left[\int_0^{\bar\chi} d\tilde\chi\frac{\tilde\chi}{\bar\chi}\left(\mathcal{P}^i_j\tilde\partial_\| + (\bar\chi - \tilde\chi)\,\mathcal{P}^p_j\mathcal{P}^{iq}\tilde\partial_q\tilde\partial_p\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]$$
$$\times\left[\int_0^{\bar\chi} d\tilde\chi\frac{\tilde\chi}{\bar\chi}\left(\mathcal{P}^j_i\tilde\partial_\| + (\bar\chi - \tilde\chi)\,\mathcal{P}^n_i\mathcal{P}^{jm}\tilde\partial_m\tilde\partial_n\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right], \tag{228}$$

$$2\left(\frac{1}{\bar\chi}\Delta x_{\perp i}^{(1)} - \partial_{\perp i}\Delta x_\|^{(1)}\right)\partial_\|\Delta x_\perp^{i(1)} = 2v_{\perp i\, o}^{(1)}v_{\perp o}^{i(1)} - 2v_{\perp i\, o}^{(1)}\left\{ + 2S_\perp^{i(1)} + \partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right)\right.$$
$$\left. -\frac{1}{\bar\chi}\int_0^{\bar\chi} d\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\right\} + 4S_\perp^{i(1)}\left\{ - \frac{1}{\bar\chi}\int_0^{\bar\chi} d\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\right.$$
$$\left. +\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right)\right\}, \tag{229}$$

$$\frac{2}{\mathcal{H}}\left(E_{\hat 0}^{0(1)} + E_{\hat 0}^{\|(1)}\right)'\Delta\ln a^{(1)} + 2\partial_\|\left(E_{\hat 0}^{0(1)} + E_{\hat 0}^{\|(1)}\right)\Delta x_\|^{(1)} = \frac{2}{\mathcal{H}}\frac{d}{d\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right)\Delta\ln a^{(1)} + 2\partial_\|\left(\Phi^{(1)} - v_\|^{(1)}\right)T^{(1)} \tag{230}$$

$$\partial_{\perp i}\left(E_{\hat 0}^{0(1)} + E_{\hat 0}^{\|(1)}\right)\Delta x_\perp^{i(1)} = \bar\chi v_{\perp o}^{i(1)}\partial_{\perp i}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \partial_{\perp i}\left(\Phi^{(1)} - v_\|^{(1)}\right)\int_0^{\bar\chi} d\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right], \tag{231}$$

$$-2\left(\delta n_\|^{(1)} + \delta\nu^{(1)}\right)E_{\hat 0}^{\|(1)} = -2v_\|^{(1)}\left(\Phi^{(1)} + \Psi^{(1)}\right), \tag{232}$$

$$-2E_{\hat 0}^{\perp i(1)}\partial_{\perp i}\left(\Delta x^{0(1)} + \Delta x_\|^{(1)}\right) = +2v_\perp^{i(1)}\partial_{\perp i}T^{(1)}, \tag{233}$$

$$-\left(\partial_\|\Delta x_\|^{(1)}\right)^2 = -\left\{\left(\Phi^{(1)} + \Psi^{(1)}\right) + \frac{1}{\mathcal{H}}\left[\frac{d}{d\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] - \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta\ln a^{(1)}\right\}^2$$
$$= -\left(\Phi^{(1)} + \Psi^{(1)}\right)^2 - \frac{1}{\mathcal{H}^2}\left[\frac{d}{d\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right]^2 - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2\left(\Delta\ln a^{(1)}\right)^2$$
$$-\frac{2}{\mathcal{H}}\left(\Phi^{(1)} + \Psi^{(1)}\right)\left[\frac{d}{d\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] + 2\frac{\mathcal{H}'}{\mathcal{H}^2}\left(\Phi^{(1)} + \Psi^{(1)}\right)\Delta\ln a^{(1)}$$
$$+2\frac{\mathcal{H}'}{\mathcal{H}^3}\left[\frac{d}{d\bar\chi}\left(\Phi^{(1)} - v_\|^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right]\Delta\ln a^{(1)} \tag{234}$$

$$-\left(E_{\hat 0}^{0(1)} + E_{\hat 0}^{\|(1)}\right)^2 = -\left(\Phi^{(1)} - v_\|^{(1)}\right)^2. \tag{235}$$

Using $\Delta\ln a^{(2)}$ in Eq. (201) , $\Delta x_\|^{(2)}$ in Eq. (211) , $\partial_\|\Delta x_\|^{(2)}$ in Eq. (214) and $\kappa^{(2)}$ in Eq. (215), we finally obtain in Poisson gauge the number density fluctuations at second order:

$$
\begin{aligned}
\Delta_g^{(2)} \;=\;& \delta_g^{(2)} + b_e\,\Delta\ln a^{(2)} + \partial_\parallel \Delta x_\parallel^{(2)} + \frac{2}{\bar\chi}\Delta x_\parallel^{(2)} - 2\kappa^{(2)} + v_\parallel^{(2)} - 3\Psi^{(2)} + \left(\Delta_g^{(1)}\right)^2 \\
& - \left(\delta_g^{(1)}\right)^2 - 3\left(\Phi^{(1)}\right)^2 - 9\left(\Psi^{(1)}\right)^2 + v_{\perp i}^{(1)} v_\perp^{i(1)} - 2v_\parallel^{(1)}\Psi^{(1)} - 6\Phi^{(1)}\Psi^{(1)} + 8\Phi^{(1)}I^{(1)} + 8\Psi^{(1)}I^{(1)} - 8\left(I^{(1)}\right)^2 \\
& - \frac{1}{\mathcal{H}^2}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\parallel^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right]^2 - \frac{2}{\mathcal{H}}\left(\Phi^{(1)} + \Psi^{(1)}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\parallel^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \\
& + \Bigg\{ - \frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(v_\parallel^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right) + 2\frac{\mathcal{H}'}{\mathcal{H}^2}\left(\Phi^{(1)} + \Psi^{(1)}\right) + 2\frac{\mathcal{H}'}{\mathcal{H}^3}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\parallel^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \\
& - \frac{4}{\bar\chi\mathcal{H}}\kappa^{(1)}\Bigg\}\Delta\ln a^{(1)} + \left[-b_e + \frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln\bar a} - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 - \frac{2}{\bar\chi^2\mathcal{H}^2}\right]\left(\Delta\ln a^{(1)}\right)^2 - \frac{4}{\bar\chi^2\mathcal{H}}\Delta\ln a^{(1)}T^{(1)} + 2v_\perp^{i(1)}\partial_{\perp i}T^{(1)} \\
& - 2\partial_\parallel\left(v_\parallel^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right)T^{(1)} - \frac{4}{\bar\chi}T^{(1)}\kappa^{(1)} - \frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2 + 4\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right)S_\perp^{i(1)} \\
& + 2\left(\Phi^{(1)} + \Psi^{(1)} - 2I^{(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\tilde\chi}{\bar\chi}\left(2\tilde\partial_\parallel + (\bar\chi - \tilde\chi)\,\mathcal{P}^{mn}\tilde\partial_m\tilde\partial_n\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] \\
& - 2\partial_{\perp i}\left(v_\parallel^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{4}{\bar\chi}S_\perp^{i(1)}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] \\
& - \left[\int_0^{\bar\chi}\mathrm{d}\tilde\chi\frac{\tilde\chi}{\bar\chi}\left(\mathcal{P}_j^i\tilde\partial_\parallel + (\bar\chi - \tilde\chi)\,\mathcal{P}_j^p\mathcal{P}^{iq}\tilde\partial_q\tilde\partial_p\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] \\
& \times\left[\int_0^{\bar\chi}\mathrm{d}\tilde\chi\frac{\tilde\chi}{\bar\chi}\left(\mathcal{P}_i^j\tilde\partial_\parallel + (\bar\chi - \tilde\chi)\,\mathcal{P}_i^n\mathcal{P}^{jm}\tilde\partial_m\tilde\partial_n\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] \\
& - 2\left(v_{\parallel o}^{(1)}\right)^2 - 2\left(\Phi_o^{(1)}\right)^2 + 4v_{\parallel o}^{(1)}\Psi_o^{(1)} - 2\left(\Psi_o^{(1)}\right)^2 + 4\Phi_o^{(1)}v_{\parallel o}^{(1)} - 4\Phi_o^{(1)}\Psi_o^{(1)} + 2v_{\perp i\,o}^{(1)}v_{\perp o}^{i(1)} \\
& + 2\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)} + \Psi_o^{(1)}\right)\left\{2\left(\Phi^{(1)} + \Psi^{(1)} - 2I^{(1)}\right) - \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\tilde\chi}{\bar\chi}\left(2\tilde\partial_\parallel + (\bar\chi - \tilde\chi)\,\mathcal{P}^{mn}\tilde\partial_m\tilde\partial_n\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\right\} \\
& - 2v_{\perp i\,o}^{(1)}\Bigg\{\bar\chi\partial_\perp^i\left(v_\parallel^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right) + 2S_\perp^{i(1)} + \partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) \\
& - \frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\Bigg\}. \qquad (236)
\end{aligned}
$$

This is the main result in Poisson gauge. If we explicitly identify the weak lensing shear and rotation contributions, it becomes

$$\begin{aligned}
\Delta_g^{(2)} &= \delta_g^{(2)} + b_e\,\Delta\ln a^{(2)} + \partial_\parallel \Delta x_\parallel^{(2)} + \frac{2}{\bar\chi}\Delta x_\parallel^{(2)} - 2\kappa^{(2)} + v_\parallel^{(2)} - 3\Psi^{(2)} + \left(\Delta_g^{(1)}\right)^2 - \left(\delta_g^{(1)}\right)^2 - \left(\Phi^{(1)}\right)^2 - 7\left(\Psi^{(1)}\right)^2 \\
&\quad + v_{\perp i}^{(1)} v_\perp^{i(1)} - 2 v_\parallel^{(1)}\Psi^{(1)} - 2\Phi^{(1)}\Psi^{(1)} - 2\left(\kappa^{(1)}\right)^2 - 2\left|\gamma^{(1)}\right|^2 + \vartheta_{ij}^{(1)}\vartheta^{ij(1)} \\
&\quad - \frac{1}{\mathcal{H}^2}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\parallel^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right]^2 - \frac{2}{\mathcal{H}}\left(\Phi^{(1)} + \Psi^{(1)}\right)\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\parallel^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \\
&\quad + \Bigg\{ - \frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(v_\parallel^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right) + 2\frac{\mathcal{H}'}{\mathcal{H}^2}\left(\Phi^{(1)} + \Psi^{(1)}\right) + 2\frac{\mathcal{H}'}{\mathcal{H}^3}\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} - v_\parallel^{(1)}\right) + \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] \\
&\quad - \frac{4}{\bar\chi\mathcal{H}}\kappa^{(1)}\Bigg\}\Delta\ln a^{(1)} + \left[-b_e + \frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln\bar a} - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 - \frac{2}{\bar\chi^2\mathcal{H}^2}\right]\left(\Delta\ln a^{(1)}\right)^2 - \frac{4}{\bar\chi^2\mathcal{H}}\Delta\ln a^{(1)} T^{(1)} + 2 v_\perp^{i(1)}\partial_{\perp i} T^{(1)} \\
&\quad - 2\partial_\parallel\left(v_\parallel^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right) T^{(1)} - \frac{4}{\bar\chi} T^{(1)}\kappa^{(1)} - \frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2 + 4\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) S_\perp^{i(1)} \\
&\quad - 2\partial_{\perp i}\left(v_\parallel^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{4}{\bar\chi} S_\perp^{i(1)}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] \\
&\quad + 2 v_{\perp i\,o}^{(1)} v_{\perp\,o}^{i(1)} - 2 v_{\perp i\,o}^{(1)}\Bigg\{\bar\chi\partial_\perp^i\left(v_\parallel^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right) + 2 S_\perp^{i(1)} + \partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) \\
&\quad - \frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\Bigg\},
\end{aligned} \tag{237}$$

where $\gamma_{ij}^{(1)}$ and $\vartheta_{ij}^{(1)}\vartheta^{ij(1)}$ are given in Eqs. (B10) and (B11).

Assuming no velocity bias

At first order, assuming galaxy velocities follow the matter velocity field, then the comoving velocities are the same and we can write

$$\partial_\parallel \Delta x_\parallel^{(1)} = \Phi^{(1)} - v_\parallel^{(1)} + \Psi^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel v_\parallel^{(1)} + \frac{1}{\mathcal{H}}\Psi^{(1)\prime} + \frac{\bar a^2}{\mathcal{H}\bar\rho_m}\left(\mathcal{E}_m^{\parallel(1)} - \mathcal{E}_m^{0(0)} v_\parallel^{(1)}\right) - \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta\ln a^{(1)}. \tag{238}$$

Then

$$\begin{aligned}
\Delta_g^{(1)} &= \delta_g^{(1)} + \left(b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar\chi\mathcal{H}}\right)\Delta\ln a^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel v_\parallel^{(1)} + \frac{1}{\mathcal{H}}\Psi^{(1)\prime} + \Phi^{(1)} - 2\Psi^{(1)} - b_m v_\parallel^{(1)} - \frac{2}{\bar\chi}T^{(1)} - 2\kappa^{(1)} \\
&\quad + \frac{a^2}{\mathcal{H}\bar\rho_m}\mathcal{E}_m^{\parallel(1)}\,.
\end{aligned} \tag{239}$$

At second order, through Eq. (B8) we have

$$
\begin{aligned}
\Delta \ln a^{(2)} = & +\Phi_o^{(2)} - v_{\| o}^{(2)} - \left(\Phi_o^{(1)}\right)^2 - 6\Phi_o^{(1)} v_{\| o}^{(1)} + v_{k\,o}^{(1)} v_o^{k(1)} + 2\Psi_o^{(1)} v_{\| o}^{(1)} + 2\left(\Phi_o^{(1)} - v_{\| o}^{(1)}\right)\Bigg[-3\Phi^{(1)} - \frac{1}{\mathcal{H}}\partial_\| v_\|^{(1)} \\
& +\frac{1}{\mathcal{H}}\Psi^{(1)\prime} - b_m v_\|^{(1)} + 2\bar\chi\partial_\|\Phi^{(1)} - \bar\chi\left(\Phi^{(1)\prime}-\Psi^{(1)\prime}\right) + \frac{a^2}{\mathcal{H}\bar\rho_m}\mathcal{E}_m^{\|(1)} + 4I^{(1)}\Bigg] - 2v_{\perp o}^{i(1)}\Bigg[\bar\chi\,\partial_{\perp i}\left(\Phi^{(1)}+v_\|^{(1)}\right) \\
& -2\int_0^{\bar\chi} \mathrm{d}\tilde\chi\,\tilde\partial_{\perp i}\Phi^{(1)}\Bigg] - \Phi^{(2)} + v_\|^{(2)} + 7\left(\Phi^{(1)}\right)^2 - (1+2b_m)\left(v_\|^{(1)}\right)^2 + v_{\perp i}^{(1)} v_\perp^{i(1)} - 2v_\|^{(1)}\Psi^{(1)} + 2b_m\Phi^{(1)}v_\|^{(1)} \\
& -\frac{2}{\mathcal{H}}\left(\Phi^{(1)}-v_\|^{(1)}\right)\left(\Psi^{(1)\prime} - \partial_\| v_\|^{(1)} + \frac{a^2}{\bar\rho_m}\mathcal{E}_m^{\|(1)}\right) - 4\Bigg[3\Phi^{(1)} + \frac{1}{\mathcal H}\partial_\| v_\|^{(1)} - \frac{1}{\mathcal H}\Psi^{(1)\prime} + b_m v_\|^{(1)} \\
& -\frac{\bar a^2}{\mathcal H\bar\rho_m}\mathcal E_m^{\|(1)}\Bigg] I^{(1)} + 4 v_{\perp i}^{(1)} S_\perp^{i(1)} - 2\partial_\|\left(\Phi^{(1)}+v_\|^{(1)}\right)T^{(1)} - 2\Bigg[2\partial_\|\Phi^{(1)} - \left(\Phi^{(1)\prime}-\Psi^{(1)\prime}\right)\Bigg] \\
& \times\int_0^{\bar\chi} \mathrm{d}\tilde\chi\left[2\Phi^{(1)} + (\bar\chi-\tilde\chi)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right] - 2\Bigg[\partial_{\perp i}\left(\Phi^{(1)}+v_\|^{(1)}\right) - \frac{1}{\bar\chi}v_{\perp i}^{(1)}\Bigg] \\
& \times\int_0^{\bar\chi} \mathrm{d}\tilde\chi\left[(\bar\chi-\tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)}+\Psi^{(1)}\right)\right] + 2I^{(2)} - 4\int_0^{\bar\chi} \mathrm{d}\tilde\chi\Bigg[\left(\Psi^{(1)}+2I^{(1)}\right)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right) \\
& +\left(\Phi^{(1)}+\Psi^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\Phi^{(1)} + 2S_\perp^{i(1)}\tilde\partial_{\perp i}\Phi^{(1)}\Bigg] \,, \qquad (240)
\end{aligned}
$$

$$
\begin{aligned}
\Delta x^{0(2)} = & +\frac{1}{\mathcal H}\Phi_o^{(2)} - \frac{1}{\mathcal H}v_{\| o}^{(2)} - \left(\frac{\mathcal H'}{\mathcal H^3}+\frac{2}{\mathcal H}\right)\left(\Phi_o^{(1)}\right)^2 + 2\left(\frac{\mathcal H'}{\mathcal H^3}-\frac{2}{\mathcal H}\right)\Phi_o^{(1)}v_{\| o}^{(1)} - \frac{\mathcal H'}{\mathcal H^3}\left(v_{\| o}^{(1)}\right)^2 + \frac{1}{\mathcal H}v_{\perp i\, o}^{(1)}v_{\perp o}^{i(1)} \\
& +\frac{2}{\mathcal H}\Psi_o^{(1)}v_{\| o}^{(1)} + 2\left(\Phi_o^{(1)}-v_{\| o}^{(1)}\right)\Bigg\{\left(\frac{\mathcal H'}{\mathcal H^3}-\frac{2}{\mathcal H}\right)\Phi^{(1)} - \left[\frac{\mathcal H'}{\mathcal H^3}+\frac{1}{\mathcal H}(1+b_m)\right]v_\|^{(1)} - \frac{\bar\chi}{\mathcal H}\Phi^{(1)\prime} \\
& -\frac{1}{\mathcal H^2}\left(\partial_\| v_\|^{(1)} - \frac{\bar a^2}{\bar\rho_m}\mathcal E^{\|(1)}\right) + \left(\frac{\bar\chi}{\mathcal H}+\frac{1}{\mathcal H^2}\right)\Psi^{(1)\prime} + 2\frac{\bar\chi}{\mathcal H}\partial_\|\Phi^{(1)} - 2\left(\frac{\mathcal H'}{\mathcal H^3}-\frac{1}{\mathcal H}\right)I^{(1)}\Bigg\} \\
& -2v_{\perp o}^{i(1)}\left[\frac{\bar\chi}{\mathcal H}\partial_{\perp i}\left(\Phi^{(1)}+v_\|^{(1)}\right) - \frac{2}{\mathcal H}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,\tilde\partial_{\perp i}\Phi^{(1)}\right] - \frac{1}{\mathcal H}\Phi^{(2)} + \frac{1}{\mathcal H}v_\|^{(2)} + \left(-\frac{\mathcal H'}{\mathcal H^3}+\frac{6}{\mathcal H}\right)\left(\Phi^{(1)}\right)^2 \\
& -\left[\frac{\mathcal H'}{\mathcal H^3}+\frac{2}{\mathcal H}(1+b_m)\right]\left(v_\|^{(1)}\right)^2 + \frac{1}{\mathcal H}v_{\perp i}^{(1)}v_\perp^{i(1)} - \frac{2}{\mathcal H}v_\|^{(1)}\Psi^{(1)} + 2\left[\frac{\mathcal H'}{\mathcal H^3}+\frac{1}{\mathcal H}(1+b_m)\right]\Phi^{(1)}v_\|^{(1)} \\
& +\frac{2}{\mathcal H^2}\left(\Phi^{(1)}-v_\|^{(1)}\right)\left(-\Psi^{(1)\prime}+\partial_\| v_\|^{(1)} - \frac{\bar a^2}{\bar\rho_m}\mathcal E_m^{\|(1)}\right) + 4\Bigg\{\left(\frac{\mathcal H'}{\mathcal H^3}-\frac{2}{\mathcal H}\right)\Phi^{(1)} - \left[\frac{\mathcal H'}{\mathcal H^3}+\frac{1}{\mathcal H}(1+b_m)\right]v_\|^{(1)} \\
& -\frac{1}{\mathcal H^2}\left(-\Psi^{(1)\prime}+\partial_\| v_\|^{(1)} - \frac{\bar a^2}{\bar\rho_m}\mathcal E_m^{\|(1)}\right) - \left(\frac{\mathcal H'}{\mathcal H^3}+\frac{1}{\mathcal H}\right)I^{(1)}\Bigg\}I^{(1)} + \frac{4}{\mathcal H}v_{\perp i}^{(1)}S_\perp^{i(1)} - \frac{2}{\mathcal H}\partial_\|\left(\Phi^{(1)}+v_\|^{(1)}\right)T^{(1)} \\
& -\frac{2}{\mathcal H}\Bigg[2\partial_\|\Phi^{(1)} - \left(\Phi^{(1)\prime}-\Psi^{(1)\prime}\right)\Bigg]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2\Phi^{(1)} + (\bar\chi-\tilde\chi)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right)\right] - \frac{2}{\mathcal H}\Bigg[\partial_{\perp i}\left(\Phi^{(1)}+v_\|^{(1)}\right) \\
& -\frac{1}{\bar\chi}v_{\perp i}^{(1)}\Bigg]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi-\tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)}+\Psi^{(1)}\right)\right] + \frac{2}{\mathcal H}I^{(2)} - \frac{4}{\mathcal H}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg[\left(\Psi^{(1)}+2I^{(1)}\right)\left(\Phi^{(1)\prime}+\Psi^{(1)\prime}\right) \\
& +\left(\Phi^{(1)}+\Psi^{(1)}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\Phi^{(1)} + 2S_\perp^{i(1)}\tilde\partial_{\perp i}\Phi^{(1)}\Bigg] \,, \qquad (241)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\parallel \Delta x_\parallel^{(2)} =& \left(\Phi_o^{(1)}\right)^2 + 2\Phi_o^{(1)} v_{\parallel o}^{(1)} + 2 v_{\parallel o}^{(1)} \Psi_o^{(1)} - \left(\Psi_o^{(1)}\right)^2 - v_{\perp k\, o}^{(1)} v_{\perp o}^{k(1)} + 2\left(\Phi_o^{(1)} - v_{\parallel o}^{(1)}\right) \\
&\times \Bigg\{ \left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right) \left[\frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}_m^{\parallel(1)} - \partial_\parallel v_\parallel^{(1)} - \mathcal{H}\left(b_m + 1\right) v_\parallel^{(1)} + \Psi^{(1)\prime}\right] - \frac{\bar{\chi}}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \left[2\partial_\parallel \Phi^{(1)} - \left(\Phi^{(1)\prime} - \Psi^{(1)\prime}\right)\right] \\
&+ \left(\Phi^{(1)} + \Psi^{(1)}\right) - \bar{\chi} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{1}{\mathcal{H}^2} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \left[\frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}_m^{\parallel(1)} - \partial_\parallel v_\parallel^{(1)} - \mathcal{H}\left(b_m + 1\right) v_\parallel^{(1)} + \Psi^{(1)\prime}\right] \Bigg\} - 2 v_{\perp o}^{i(1)} \\
&\times \left[\frac{1}{\mathcal{H}} \partial_{\perp i} \left(\Phi^{(1)} - v_\parallel^{(1)}\right) - \frac{\bar{\chi}}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \partial_{\perp i} \left(\Phi^{(1)} + v_\parallel^{(1)}\right) - 4\delta_{il} S_\perp^{l(1)}\right] + \Phi^{(2)} + \Psi^{(2)} - \frac{1}{2}\hat{h}_\parallel^{(2)} + \frac{1}{\mathcal{H}} \Psi^{(2)\prime} - \frac{1}{2\mathcal{H}} \hat{h}_\parallel^{(2)\prime} \\
&+ 2\frac{\bar{a}^2}{\mathcal{H}\bar{\rho}_m} \left[\frac{1}{2}\mathcal{E}_m^{\parallel(2)} - \mathcal{E}_m^{0(1)} v_\parallel^{(1)} - \mathcal{E}_m^{\parallel(1)} \left(\delta_m^{(1)} - \Phi^{(1)}\right)\right] - 2b_m \left(\frac{1}{2} v_\parallel^{(2)} - \delta_m^{(1)} v_\parallel^{(1)} + 2\Phi^{(1)} v_\parallel^{(1)}\right) - \frac{2}{\mathcal{H}} \left[\frac{1}{2}\partial_\parallel v_\parallel^{(2)} - v_\parallel^{(1)} \partial_\parallel v_\parallel^{(1)}\right. \\
&\left.+ \frac{\mathcal{H}}{2} v_\parallel^{(2)} - \frac{2}{\bar{\chi}} \left(v_\parallel^{(1)}\right)^2 - v_\parallel^{(1)} \partial_{\perp j} v_\perp^{j(1)} - 2 v_\parallel^{(1)} \Psi^{(1)\prime} - \Phi^{(1)} \partial_\parallel \Phi^{(1)} + 2\Psi^{(1)} \partial_\parallel \Phi^{(1)}\right] - 4\left(\Phi^{(1)}\right)^2 + 4\left(\Psi^{(1)}\right)^2 \\
&- \frac{2}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \left[\frac{7}{2}\left(\Phi^{(1)}\right)^2 - \Phi^{(1)} v_\parallel^{(1)} + \frac{1}{2}\left(v_\parallel^{(1)}\right)^2 - v_\parallel^{(1)} \Psi^{(1)} + \frac{1}{2} v_{\perp i}^{(1)} v_\perp^{i(1)}\right] + \frac{4}{\mathcal{H}} \Phi^{(1)} \left[2\partial_\parallel \Phi^{(1)} - \left(\Phi^{(1)\prime} - \Psi^{(1)\prime}\right)\right] \\
&+ \frac{2}{\mathcal{H}^2} \left(\Phi^{(1)} - v_\parallel^{(1)}\right) \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \left[\frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}_m^{\parallel(1)} - \partial_\parallel v_\parallel^{(1)} - \mathcal{H}\left(b_m + 1\right) v_\parallel^{(1)} + \Psi^{(1)\prime}\right] - \frac{2}{\mathcal{H}} \left(3\Phi^{(1)} - 2\Psi^{(1)} - v_\parallel^{(1)}\right) \left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right) \\
&+ 2\Bigg\{ \frac{1}{\mathcal{H}} \left(\Phi^{(1)} + \Psi^{(1)}\right) + \frac{\mathcal{H}'}{\mathcal{H}^3} \left(\Phi^{(1)} - v_\parallel^{(1)}\right) + \frac{1}{\mathcal{H}^2} \left[\frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}_m^{\parallel(1)} - \partial_\parallel v_\parallel^{(1)} - \mathcal{H}\left(b_m + 1\right) v_\parallel^{(1)} + \Psi^{(1)\prime}\right] \Bigg\} \left[\frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}_m^{\parallel(1)} - \partial_\parallel v_\parallel^{(1)}\right. \\
&\left.- \mathcal{H}\left(b_m + 1\right) v_\parallel^{(1)} + \Psi^{(1)\prime}\right] + \frac{2}{\mathcal{H}} \left[2\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \Phi^{(1)} - \partial_\parallel \left(\Phi^{(1)} + v_\parallel^{(1)}\right)\right] \left(\Phi^{(1)} + \Psi^{(1)}\right) + \frac{2}{\mathcal{H}} v_{\perp i}^{(1)} \partial_\perp^i \left(\Phi^{(1)} + \Psi^{(1)}\right) \\
&+ \frac{2}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \partial_\parallel \left(\Phi^{(1)} + v_\parallel^{(1)}\right) T^{(1)} + 4\Bigg\{ \left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right) \left[\frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}_m^{\parallel(1)} - \partial_\parallel v_\parallel^{(1)} - \mathcal{H}\left(b_m + 1\right) v_\parallel^{(1)} + \Psi^{(1)\prime}\right] \\
&- \frac{1}{\mathcal{H}^2} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \left[\frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}_m^{\parallel(1)} - \partial_\parallel v_\parallel^{(1)} - \mathcal{H}\left(b_m + 1\right) v_\parallel^{(1)} + \Psi^{(1)\prime}\right] + \Phi^{(1)} + \Psi^{(1)} \Bigg\} I^{(1)} + 2\Bigg\{ \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \left(\Phi^{(1)} + \Psi^{(1)}\right) \\
&+ \frac{1}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \left[2\partial_\parallel \Phi^{(1)} - \left(\Phi^{(1)\prime} - \Psi^{(1)\prime}\right)\right] \Bigg\} \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi} \left[2\Phi^{(1)} + \left(\bar{\chi} - \tilde{\chi}\right)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] + 4\left[\frac{1}{\mathcal{H}} \partial_{\perp i} \left(\Phi^{(1)} - v_\parallel^{(1)}\right)\right. \\
&\left.+ \frac{1}{\mathcal{H}\bar{\chi}} v_{\perp i}^{(1)} - \frac{1}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} v_{\perp i}^{(1)} - 2 S_\perp^{j(1)} \delta_{ij}\right] S_\perp^{i(1)} + \frac{2}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \left[\partial_{\perp i} \left(\Phi^{(1)} + v_\parallel^{(1)}\right) - \frac{1}{\bar{\chi}} v_{\perp i}^{(1)}\right] \\
&\times \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi} \left[\left(\bar{\chi} - \tilde{\chi}\right) \tilde{\partial}_\perp^i \left(\Phi^{(1)} + \Psi^{(1)}\right)\right] - \frac{\mathcal{H}'}{\mathcal{H}^2} \Delta \ln a^{(2)} + \left[-\frac{\mathcal{H}''}{\mathcal{H}^3} + 3\left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right] \left(\Delta \ln a^{(1)}\right)^2 \\
&+ 2\Bigg\{ -\frac{\mathcal{H}'}{\mathcal{H}^2} \left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{1}{\mathcal{H}} \frac{\mathrm{d}}{\mathrm{d}\bar{\chi}} \left(\Phi^{(1)} + \Psi^{(1)}\right) - \left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right) \left[\frac{\bar{a}^2}{\bar{\rho}_m} \mathcal{E}_m^{\parallel(1)} - \partial_\parallel v_\parallel^{(1)} - \mathcal{H}\left(b_m + 1\right) v_\parallel^{(1)}\right. \\
&\left.+ \Psi^{(1)\prime}\right] \Bigg\} \Delta \ln a^{(1)} + 4\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi} \Bigg\{ \left(\Phi^{(1)} + \Psi^{(1)}\right) \left[\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right) + \frac{\mathrm{d}}{\mathrm{d}\tilde{\chi}} \Phi^{(1)}\right] \Bigg\} \,.
\end{aligned} \tag{242}
$$

The last equation can be further expanded, applying again Eq. (B8):

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}v_{\perp i}^{(1)}=\frac{1}{\mathcal{H}}\frac{\bar{a}^2}{\bar{\rho}_m}\mathcal{P}_{ij}\mathcal{E}_m^{j(1)}-\frac{1}{\mathcal{H}}\partial_\parallel v_{\perp i}^{(1)}-\left(b_m+1\right)v_{\perp i}^{(1)}-\frac{1}{\mathcal{H}}\partial_{\perp i}\Phi^{(1)}\ , \tag{243}$$

$$\frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\partial_\parallel\left(\Phi^{(1)}+v_\parallel^{(1)}\right)T^{(1)}=+\frac{2}{\mathcal{H}}\left[-\frac{\bar{a}^2}{\bar{\rho}_m}\partial_\parallel\mathcal{E}_m^{\parallel(1)}+\partial_\parallel^2v_\parallel^{(1)}+\mathcal{H}\left(b_m+1\right)\partial_\parallel v_\parallel^{(1)}-\partial_\parallel\Phi^{(1)\prime}+2\partial_\parallel^2\Phi^{(1)}\right]T^{(1)}\ , \tag{244}$$

$$\frac{\bar{\chi}}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\partial_{\perp i}\left(\Phi^{(1)}+v_\parallel^{(1)}\right)=-\frac{\bar{\chi}}{\mathcal{H}}\partial_{\perp i}\left(-\Phi^{(1)\prime}-\frac{\bar{a}^2}{\bar{\rho}_m}\mathcal{E}_m^{\parallel(1)}+\partial_\parallel v_\parallel^{(1)}+\mathcal{H}\left(b_m+1\right)v_\parallel^{(1)}+2\partial_\parallel\Phi^{(1)}\right)$$
$$+\frac{1}{\mathcal{H}}\partial_{\perp i}\left(\Phi^{(1)}+v_\parallel^{(1)}\right) \tag{245}$$

$$\frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left[\partial_{\perp i}\left(\Phi^{(1)}+v_\parallel^{(1)}\right)-\frac{1}{\bar{\chi}}v_{\perp i}^{(1)}\right]=\frac{2}{\mathcal{H}}\left[\partial_{\perp i}\left(-\Phi^{(1)\prime}-\frac{\bar{a}^2}{\bar{\rho}_m}\mathcal{E}_m^{\parallel(1)}+\partial_\parallel v_\parallel^{(1)}+\mathcal{H}\left(b_m+1\right)v_\parallel^{(1)}+2\partial_\parallel\Phi^{(1)}\right)\right.$$
$$\left.+\frac{1}{\bar{\chi}}\left(-2\partial_{\perp i}\Phi^{(1)}-\partial_\parallel v_{\perp i}^{(1)}-\partial_{\perp i}v_\parallel^{(1)}+\frac{\bar{a}^2}{\bar{\rho}_m}\mathcal{P}_{ij}\mathcal{E}_m^{j(1)}-\mathcal{H}\left(b_m+1\right)v_{\perp i}^{(1)}\right)+\frac{1}{\bar{\chi}^2}v_{\perp i}^{(1)}\right] \tag{246}$$

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left[\frac{\bar{a}^2}{\bar{\rho}_m}\mathcal{E}_m^{\parallel(1)}-\partial_\parallel v_\parallel^{(1)}-\mathcal{H}\left(b_m+1\right)v_\parallel^{(1)}+\Psi^{(1)\prime}\right]=-\left(\frac{\bar{a}^2}{\bar{\rho}_m}\mathcal{E}_m^{\parallel(1)}\right)'+\frac{\bar{a}^2\mathcal{H}}{\bar{\rho}_m}\left(b_m+1\right)\mathcal{E}_m^{\parallel(1)}+\mathcal{H}^2\frac{\mathrm{d}b_m}{\mathrm{d}\ln a}v_\parallel^{(1)}$$
$$+b_m\left[-\mathcal{H}^2\left(b_m+1\right)v_\parallel^{(1)}-2\mathcal{H}\partial_\parallel v_\parallel^{(1)}+\left(\mathcal{H}'-\mathcal{H}^2\right)v_\parallel^{(1)}-\mathcal{H}\partial_\parallel\Phi^{(1)}\right]-2\mathcal{H}\partial_\parallel v_\parallel^{(1)}-\partial_\parallel^2\Phi^{(1)}-\partial_\parallel^2v_\parallel^{(1)}$$
$$+\left(\mathcal{H}'-\mathcal{H}^2\right)v_\parallel^{(1)}-\mathcal{H}\partial_\parallel\Phi^{(1)}+\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\Psi^{(1)\prime}\ , \tag{247}$$

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left[\frac{7}{2}\left(\Phi^{(1)}\right)^2-\Phi^{(1)}v_\parallel^{(1)}+\frac{1}{2}\left(v_\parallel^{(1)}\right)^2-v_\parallel^{(1)}\Psi^{(1)}+\frac{1}{2}v_{\perp i}^{(1)}v_\perp^{i(1)}\right]=6\Phi^{(1)}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\Phi^{(1)}-\Phi^{(1)}\Phi^{(1)\prime}+v_\parallel^{(1)}\Phi^{(1)\prime}$$
$$-v_\parallel^{(1)}\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\Psi^{(1)}-\Phi^{(1)}\partial_\parallel v_\parallel^{(1)}-\mathcal{H}\Phi^{(1)}v_\parallel^{(1)}+v_\parallel^{(1)}\partial_\parallel v_\parallel^{(1)}+\mathcal{H}\left(v_\parallel^{(1)}\right)^2-\Psi^{(1)}\partial_\parallel\Phi^{(1)}-\Psi^{(1)}\partial_\parallel v_\parallel^{(1)}-\mathcal{H}v_\parallel^{(1)}\Psi^{(1)}$$
$$+v_{\perp i}^{(1)}\partial_\parallel v_\perp^{i(1)}+\mathcal{H}v_{\perp i}^{(1)}v_\perp^{i(1)}+v_{\perp i}^{(1)}\partial_\perp^i\Phi^{(1)}-\left(\Phi^{(1)}-v_\parallel^{(1)}+\Psi^{(1)}\right)\left(\frac{\bar{a}^2}{\bar{\rho}_m}\mathcal{E}_m^{\parallel(1)}-\mathcal{H}b_mv_\parallel^{(1)}\right)$$
$$-v_{\perp i}^{(1)}\left(\frac{\bar{a}^2}{\bar{\rho}_m}\mathcal{P}_j^i\mathcal{E}_m^{j(1)}-\mathcal{H}b_mv_\perp^{i(1)}\right)\ . \tag{248}$$

$$
\begin{aligned}
&\partial_\parallel \Delta x^{(2)}_\parallel = \left(\Phi^{(1)}_o\right)^2 + 2\Phi^{(1)}_o v^{(1)}_{\parallel o} + 2 v^{(1)}_{\parallel o}\Psi^{(1)}_o - \left(\Psi^{(1)}_o\right)^2 - v^{(1)}_{\perp k\, o} v^{k(1)}_{\perp o} + 2\left(\Phi^{(1)}_o - v^{(1)}_{\parallel o}\right)\Bigg\{\left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m\right.\\
&\left.-\partial_\parallel v^{(1)}_\parallel - \mathcal{H}\left(b_m+1\right)v^{(1)}_\parallel + \Psi^{(1)\prime}\right] - \frac{\bar\chi}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left[2\partial_\parallel\Phi^{(1)} - \left(\Phi^{(1)\prime} - \Psi^{(1)\prime}\right)\right] + \Phi^{(1)} + \Psi^{(1)} - \bar\chi\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} + \Psi^{(1)}\right)\\
&-\frac{1}{\mathcal{H}^2}\left[-2\mathcal{H}\partial_\parallel v^{(1)}_\parallel - \partial^2_\parallel\Phi^{(1)} - \partial^2_\parallel v^{(1)}_\parallel + \left(\mathcal{H}'-\mathcal{H}^2\right)v^{(1)}_\parallel - \mathcal{H}\partial_\parallel\Phi^{(1)} + \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Psi^{(1)\prime}\right] - \frac{1}{\mathcal{H}^2}\left[-\left(\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m\right)'\right.\\
&\left.+\frac{\bar a^2\mathcal{H}}{\bar\rho_m}\left(b_m+1\right)\mathcal{E}^{\parallel(1)}_m + \mathcal{H}^2\frac{\mathrm{d}b_m}{\mathrm{d}\ln a}v^{(1)}_\parallel - \mathcal{H}^2 b_m\left(b_m+1\right)v^{(1)}_\parallel - 2\mathcal{H}b_m\partial_\parallel v^{(1)}_\parallel + \left(\mathcal{H}'-\mathcal{H}^2\right)b_m v^{(1)}_\parallel - \mathcal{H}b_m\partial_\parallel\Phi^{(1)}\right]\Bigg\}\\
&-2v^{i(1)}_{\perp o}\left[+\frac{2}{\mathcal{H}}\partial_{\perp i}\Phi^{(1)} - \frac{\bar\chi}{\mathcal{H}}\partial_{\perp i}\left(-\Phi^{(1)\prime} - \frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m + \partial_\parallel v^{(1)}_\parallel + \mathcal{H}\left(b_m+1\right)v^{(1)}_\parallel + 2\partial_\parallel\Phi^{(1)}\right) - 4\delta_{il}S^{l(1)}_\perp\right]\\
&+\Phi^{(2)} + \Psi^{(2)} - \frac{1}{2}\hat h^{(2)}_\parallel + \frac{1}{\mathcal{H}}\Psi^{(2)\prime} - \frac{1}{2\mathcal{H}}\hat h^{(2)\prime}_\parallel - \frac{2}{\mathcal{H}}\left[\frac{1}{2}\partial_\parallel v^{(2)}_\parallel - v^{(1)}_\parallel\partial_\parallel v^{(1)}_\parallel + \frac{\mathcal{H}}{2}v^{(2)}_\parallel - \frac{2}{\bar\chi}\left(v^{(1)}_\parallel\right)^2 - v^{(1)}_\parallel\partial_{\perp j}v^{j(1)}_\perp\right.\\
&\left.-2v^{(1)}_\parallel\Psi^{(1)\prime} - \Phi^{(1)}\partial_\parallel\Phi^{(1)} + 2\Psi^{(1)}\partial_\parallel\Phi^{(1)}\right] - 4\left(\Phi^{(1)}\right)^2 + 4\left(\Psi^{(1)}\right)^2 - \frac{2}{\mathcal{H}}\left[6\Phi^{(1)}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)} - \Phi^{(1)}\Phi^{(1)\prime} + v^{(1)}_\parallel\Phi^{(1)\prime}\right.\\
&-v^{(1)}_\parallel\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Psi^{(1)} - \Phi^{(1)}\partial_\parallel v^{(1)}_\parallel - \mathcal{H}\Phi^{(1)}v^{(1)}_\parallel + v^{(1)}_\parallel\partial_\parallel v^{(1)}_\parallel + \mathcal{H}\left(v^{(1)}_\parallel\right)^2 - \Psi^{(1)}\partial_\parallel\Phi^{(1)} - \Psi^{(1)}\partial_\parallel v^{(1)}_\parallel - \mathcal{H}v^{(1)}_\parallel\Psi^{(1)} + v^{(1)}_{\perp i}\partial_\parallel v^{i(1)}_\perp\\
&\left.+\mathcal{H}v^{(1)}_{\perp i}v^{i(1)}_\perp + v^{(1)}_{\perp i}\partial^i_\perp\Phi^{(1)} - \left(\Phi^{(1)} - v^{(1)}_\parallel + \Psi^{(1)}\right)\left(\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \mathcal{H}b_m v^{(1)}_\parallel\right) - v^{(1)}_{\perp i}\left(\frac{\bar a^2}{\bar\rho_m}\mathcal{P}^i_j\mathcal{E}^{j(1)}_m - \mathcal{H}b_m v^{i(1)}_\perp\right)\right]\\
&+2\frac{\bar a^2}{\mathcal{H}\bar\rho_m}\left[\frac{1}{2}\mathcal{E}^{\parallel(2)}_m - \mathcal{E}^{0(1)}_m v^{(1)}_\parallel - \mathcal{E}^{\parallel(1)}_m\left(\delta^{(1)}_m - \Phi^{(1)}\right)\right] - 2b_m\left(\frac{1}{2}v^{(2)}_\parallel - \delta^{(1)}_m v^{(1)}_\parallel + 2\Phi^{(1)}v^{(1)}_\parallel\right)\\
&+\frac{2}{\mathcal{H}^2}\left(\Phi^{(1)} - v^{(1)}_\parallel\right)\left[-\left(\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m\right)' + \frac{\bar a^2\mathcal{H}}{\bar\rho_m}\left(b_m+1\right)\mathcal{E}^{\parallel(1)}_m + \mathcal{H}^2\frac{\mathrm{d}b_m}{\mathrm{d}\ln a}v^{(1)}_\parallel - \mathcal{H}^2 b_m\left(b_m+1\right)v^{(1)}_\parallel - 2\mathcal{H}b_m\partial_\parallel v^{(1)}_\parallel\right.\\
&\left.+\left(\mathcal{H}'-\mathcal{H}^2\right)b_m v^{(1)}_\parallel - \mathcal{H}b_m\partial_\parallel\Phi^{(1)} - 2\mathcal{H}\partial_\parallel v^{(1)}_\parallel - \partial^2_\parallel\Phi^{(1)} - \partial^2_\parallel v^{(1)}_\parallel + \left(\mathcal{H}'-\mathcal{H}^2\right)v^{(1)}_\parallel - \mathcal{H}\partial_\parallel\Phi^{(1)} + \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Psi^{(1)\prime}\right]\\
&+\frac{4}{\mathcal{H}}\Phi^{(1)}\left[2\partial_\parallel\Phi^{(1)} - \left(\Phi^{(1)\prime} - \Psi^{(1)\prime}\right)\right] - \frac{2}{\mathcal{H}}\left(3\Phi^{(1)} - 2\Psi^{(1)} - v^{(1)}_\parallel\right)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right) + 2\Bigg\{\frac{1}{\mathcal{H}}\left(\Phi^{(1)} + \Psi^{(1)}\right)\\
&+\frac{\mathcal{H}'}{\mathcal{H}^3}\left(\Phi^{(1)} - v^{(1)}_\parallel\right) + \frac{1}{\mathcal{H}^2}\left[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel - \mathcal{H}\left(b_m+1\right)v^{(1)}_\parallel + \Psi^{(1)\prime}\right]\Bigg\}\left[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel - \mathcal{H}\left(b_m+1\right)v^{(1)}_\parallel\right.\\
&\left.+\Psi^{(1)\prime}\right] + \frac{2}{\mathcal{H}}\left[2\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Phi^{(1)} - \partial_\parallel\left(\Phi^{(1)} + v^{(1)}_\parallel\right)\right]\left(\Phi^{(1)} + \Psi^{(1)}\right) + \frac{2}{\mathcal{H}}v^{(1)}_{\perp i}\partial^i_\perp\left(\Phi^{(1)} + \Psi^{(1)}\right) + \frac{2}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\partial_\parallel\left(\Phi^{(1)} + v^{(1)}_\parallel\right)T^{(1)}\\
&+4\Bigg\{\left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel - \mathcal{H}\left(b_m+1\right)v^{(1)}_\parallel + \Psi^{(1)\prime}\right] - \frac{1}{\mathcal{H}^2}\left[-\left(\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m\right)' + \frac{\bar a^2\mathcal{H}}{\bar\rho_m}\left(b_m+1\right)\mathcal{E}^{\parallel(1)}_m\right.\\
&+\mathcal{H}^2\frac{\mathrm{d}b_m}{\mathrm{d}\ln a}v^{(1)}_\parallel + b_m\left[-\mathcal{H}^2\left(b_m+1\right)v^{(1)}_\parallel - 2\mathcal{H}\partial_\parallel v^{(1)}_\parallel + \left(\mathcal{H}'-\mathcal{H}^2\right)v^{(1)}_\parallel - \mathcal{H}\partial_\parallel\Phi^{(1)}\right] - 2\mathcal{H}\partial_\parallel v^{(1)}_\parallel - \partial^2_\parallel\Phi^{(1)} - \partial^2_\parallel v^{(1)}_\parallel\\
&\left.+\left(\mathcal{H}'-\mathcal{H}^2\right)v^{(1)}_\parallel - \mathcal{H}\partial_\parallel\Phi^{(1)} + \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Psi^{(1)\prime}\right] + \Phi^{(1)} + \Psi^{(1)}\Bigg\}I^{(1)} + 2\left[\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} + \Psi^{(1)}\right) + \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(2\partial_\parallel\Phi^{(1)} - \Phi^{(1)\prime}\right.\right.\\
&\left.\left.+\Psi^{(1)\prime}\right)\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[2\Phi^{(1)} + \left(\bar\chi - \tilde\chi\right)\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right] + 4\left[\frac{1}{\mathcal{H}}\partial_{\perp i}\left(\Phi^{(1)} - v^{(1)}_\parallel\right) + \frac{1}{\mathcal{H}\bar\chi}v^{(1)}_{\perp i} + \frac{1}{\mathcal{H}}\frac{\bar a^2}{\bar\rho_m}\mathcal{P}_{ij}\mathcal{E}^{j(1)}_m - \frac{1}{\mathcal{H}}\partial_\parallel v^{(1)}_{\perp i}\right.\\
&\left.-\left(b_m+1\right)v^{(1)}_{\perp i} - \frac{1}{\mathcal{H}}\partial_{\perp i}\Phi^{(1)} - 2S^{j(1)}_\perp\delta_{ij}\right]S^{i(1)}_\perp + \frac{2}{\mathcal{H}}\left[\partial_{\perp i}\left(-\Phi^{(1)\prime} - \frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m + \partial_\parallel v^{(1)}_\parallel + \mathcal{H}\left(b_m+1\right)v^{(1)}_\parallel + 2\partial_\parallel\Phi^{(1)}\right)\right.\\
&\left.+\frac{1}{\bar\chi}\left(-2\partial_{\perp i}\Phi^{(1)} - \partial_{\perp i}v^{(1)}_\parallel - \partial_\parallel v^{(1)}_{\perp i} + \frac{\bar a^2}{\bar\rho_m}\mathcal{P}_{ij}\mathcal{E}^{j(1)}_m - \mathcal{H}\left(b_m+1\right)v^{(1)}_{\perp i}\right) + \frac{1}{\bar\chi^2}v^{(1)}_{\perp i}\right]\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\left(\bar\chi - \tilde\chi\right)\tilde\partial^i_\perp\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\\
&-\frac{\mathcal{H}'}{\mathcal{H}^2}\Delta\ln a^{(2)} + \left[-\frac{\mathcal{H}''}{\mathcal{H}^3} + 3\left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right]\left(\Delta\ln a^{(1)}\right)^2 + 2\Bigg\{-\frac{\mathcal{H}'}{\mathcal{H}^2}\left(\Phi^{(1)} + \Psi^{(1)}\right) - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(\Phi^{(1)} + \Psi^{(1)}\right)\\
&-\left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left[\frac{\bar a^2}{\bar\rho_m}\mathcal{E}^{\parallel(1)}_m - \partial_\parallel v^{(1)}_\parallel - \mathcal{H}\left(b_m+1\right)v^{(1)}_\parallel + \Psi^{(1)\prime}\right]\Bigg\}\Delta\ln a^{(1)} + 4\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Bigg\{\left(\Phi^{(1)} + \Psi^{(1)}\right)\left[\left(\Phi^{(1)\prime} + \Psi^{(1)\prime}\right)\right.\\
&\left.+\frac{\mathrm{d}}{\mathrm{d}\tilde\chi}\Phi^{(1)}\right]\Bigg\}\,. \qquad (249)
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
\Delta_g^{(2)} &= \delta_g^{(2)} + v_\parallel^{(2)} - 3\Psi^{(2)} + b_e\,\Delta\ln a^{(2)} + \partial_\parallel \Delta x_\parallel^{(2)} + \frac{2}{\bar\chi}\Delta x_\parallel^{(2)} - 2\kappa^{(2)} + \left(\Delta_g^{(1)}\right)^2 \\
&- \left(\delta_g^{(1)}\right)^2 - 3\left(\Phi^{(1)}\right)^2 - \left(v_\parallel^{(1)}\right)^2 + 2\Phi^{(1)} v_\parallel^{(1)} - 9\left(\Psi^{(1)}\right)^2 - \frac{1}{\mathcal{H}^2}\left(\partial_\parallel v_\parallel^{(1)}\right)^2 - \frac{1}{\mathcal{H}^2}\left(\Psi^{(1)\prime}\right)^2 - 6\Phi^{(1)}\Psi^{(1)} \\
&- \frac{2}{\mathcal{H}}\Phi^{(1)}\Psi^{(1)\prime} - \frac{2}{\mathcal{H}} v_\parallel^{(1)}\partial_\parallel v_\parallel^{(1)} + \frac{2}{\mathcal{H}} v_\parallel^{(1)}\Psi^{(1)\prime} + \frac{2}{\mathcal{H}}\Psi^{(1)}\partial_\parallel v_\parallel^{(1)} - \frac{2}{\mathcal{H}}\Psi^{(1)}\Psi^{(1)\prime} + \frac{2}{\mathcal{H}^2}\Psi^{(1)\prime}\partial_\parallel v_\parallel^{(1)} + \frac{2}{\mathcal{H}}\Phi^{(1)}\partial_\parallel v_\parallel^{(1)} \\
&+ v_{\perp i}^{(1)} v_\perp^{i(1)} - 8\left(I^{(1)}\right)^2 + 8\Phi^{(1)} I^{(1)} + 8\Psi^{(1)} I^{(1)} + 2\partial_\parallel\left(+3\Psi^{(1)} - v_\parallel^{(1)} - \delta_g^{(1)}\right) T^{(1)} - \frac{4}{\bar\chi}\kappa^{(1)} T^{(1)} - \frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2 \\
&+ \frac{2}{\mathcal{H}}\left(-\partial_\parallel v_\parallel^{(1)} - \mathcal{H} v_\parallel^{(1)} - \partial_\parallel \Phi^{(1)} + 3\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Psi^{(1)} - \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\delta_g^{(1)} - \frac{2}{\bar\chi^2} T^{(1)} - \frac{2}{\bar\chi}\kappa^{(1)}\right)\Delta\ln a^{(1)} \\
&+ 2\frac{\mathcal{H}'}{\mathcal{H}^2}\left(\Phi^{(1)} - v_\parallel^{(1)} + \Psi^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel v_\parallel^{(1)} + \frac{1}{\mathcal{H}}\Psi^{(1)\prime}\right)\Delta\ln a^{(1)} + \left(-b_e + \frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln\bar a} - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 - \frac{2}{\bar\chi^2\mathcal{H}^2}\right)\left(\Delta\ln a^{(1)}\right)^2 \\
&+ 2\left(\Phi^{(1)} + \Psi^{(1)} - 2I^{(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\tilde\chi}{\bar\chi}\left(2\tilde\partial_\parallel + (\bar\chi - \tilde\chi)\,\mathcal{P}^{mn}\tilde\partial_m\tilde\partial_n\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] \\
&- \left[\int_0^{\bar\chi}\mathrm{d}\tilde\chi\frac{\tilde\chi}{\bar\chi}\left(\mathcal{P}_j^i\tilde\partial_\parallel + (\bar\chi - \tilde\chi)\,\mathcal{P}_j^p\mathcal{P}^{iq}\tilde\partial_q\tilde\partial_p\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] \\
&\times\left[\int_0^{\bar\chi}\mathrm{d}\tilde\chi\frac{\tilde\chi}{\bar\chi}\left(\mathcal{P}_i^j\tilde\partial_\parallel + (\bar\chi - \tilde\chi)\,\mathcal{P}_i^n\mathcal{P}^{jm}\tilde\partial_m\tilde\partial_n\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] \\
&- 2\partial_{\perp i}\left(v_\parallel^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] + 2v_\perp^{i(1)}\partial_{\perp i}T^{(1)} \\
&+ 4S_\perp^{i(1)}\left\{-\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] + \partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right)\right\} \\
&+ 2v_{\perp i\,o}^{(1)} v_{\perp\,o}^{i(1)} - 2\left(v_{\parallel\,o}^{(1)}\right)^2 - 2\left(\Phi_o^{(1)}\right)^2 + 4v_{\parallel\,o}^{(1)}\Psi_o^{(1)} - 2\left(\Psi_o^{(1)}\right)^2 + 4\Phi_o^{(1)} v_{\parallel\,o}^{(1)} - 4\Phi_o^{(1)}\Psi_o^{(1)} - 2v_{\perp i\,o}^{(1)} \\
&\times\left\{\bar\chi\partial_\perp^i\left(v_\parallel^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right) + 2S_\perp^{i(1)} + \partial_\perp^i\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) - \frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[(\bar\chi - \tilde\chi)\,\tilde\partial_\perp^i\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\right\} \\
&+ 2\left(\Phi_o^{(1)} - v_{\parallel\,o}^{(1)} + \Psi_o^{(1)}\right)\left\{2\left(\Phi^{(1)} + \Psi^{(1)} - 2I^{(1)}\right) - \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\tilde\chi}{\bar\chi}\left(2\tilde\partial_\parallel + (\bar\chi - \tilde\chi)\,\mathcal{P}^{mn}\tilde\partial_m\tilde\partial_n\right)\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\right\} \\
&- 2\frac{\bar a^2}{\mathcal{H}\bar\rho_m}\left(\mathcal{E}_m^{\parallel(1)} - \frac{\mathcal{H}}{\bar a^2}\bar\rho_m b_m v_\parallel^{(1)}\right)\left[\Phi^{(1)} - v_\parallel^{(1)} + \Psi^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel v_\parallel^{(1)} + \frac{1}{\mathcal{H}}\Psi^{(1)\prime} - \left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\Delta\ln a^{(1)}\right] \\
&- \left(\frac{\bar a^2}{\mathcal{H}\bar\rho_m}\right)^2\left(\mathcal{E}_m^{\parallel(1)} - \frac{\mathcal{H}}{\bar a^2}\bar\rho_m b_m v_\parallel^{(1)}\right)^2 .
\end{aligned}
\tag{250}
$$

This is the main result in Poisson gauge with no velocity bias. If we explicitly identify the weak lensing shear and rotation contributions, we arrive at Eq. (B12) in Appendix B.

V. PRESCRIPTION FOR THE GALAXY BIAS

We need to relate the fluctuations of galaxy number density to the underlying matter density fluctuation δ_m, assuming scale-independent bias. In order to define correctly the bias we have to choose an appropriate frame where the baryon velocity perturbations vanish. If we assume that both at first and second order the baryon rest frame coincides with the rest frame of CDM, the most general gauge that meets these requirements is the comoving-time orthogonal (CO) gauge (see e.g. [36]), which becomes the usual comoving-synchronous gauge when the perturbations are dominated by pressure-free matter, for example in the ΛCDM model[8]. In this frame, galaxy and matter over-

---

[8] An analogous approach is used in [37].

densities are gauge invariant [36]. This gauge is defined by the conditions $B^{i(n)} = v^{i(n)} = 0$. Then

$$\mathrm{d}s^2 = a(\eta)^2 \left[-\left(1 + 2\varphi^{(1)} + \varphi^{(2)}\right) \mathrm{d}\eta^2 + \left(\delta_{ij} + h^{(1)}_{ij\,\mathrm{CO}} + \frac{1}{2} h^{(2)}_{ij\,\mathrm{CO}}\right) \mathrm{d}x^i \mathrm{d}x^j \right] , \tag{251}$$

where $A^{(n)}_{\mathrm{CO}} = \varphi^{(n)}$, $h^{(n)}_{ij\,\mathrm{CO}} = -2\psi^{(n)}\delta_{ij} + F^{(n)}_{ij\,\mathrm{CO}}$, with $F^{(n)}_{ij\,\mathrm{CO}} = (\partial_i\partial_j - \delta_{ij}\nabla^2/3)\xi^{(n)} + \partial_i\hat{\xi}^{(n)}_j + \partial_j\hat{\xi}^{(n)}_i + \hat{h}^{(n)}_{ij}$, $\partial_i\hat{\xi}^{i(n)} = \partial_i\hat{h}^{ij(n)} = 0$. For simplicity, we neglect vector and tensor perturbations at first order, i.e. $\hat{\xi}^{(1)}_j = \hat{h}^{(1)}_{ij} = 0$.

In order to find $\delta_{g\,\mathrm{CO}}$, we transform the metric perturbations from the Poisson gauge to the comoving-time orthogonal gauge. Using [35], we get, at first order,

$$\delta^{(1)}_{g\,\mathrm{P}} = \delta^{(1)}_{g\,\mathrm{CO}} - b_e\mathcal{H}v^{(1)}_{\mathrm{P}} + 3\mathcal{H}v^{(1)}_{\mathrm{P}}, \tag{252}$$

and, at second order,

$$\begin{aligned}
\delta^{(2)}_{g\,\mathrm{P}} \;=\;& \delta^{(2)}_{g\,\mathrm{CO}} - b_e\mathcal{H}v^{(2)}_{\mathrm{P}} + 3\mathcal{H}v^{(2)}_{\mathrm{P}} + \left(b_e\mathcal{H}' - 3\mathcal{H}' + \mathcal{H}^2\frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln\bar{a}} + b_e^2\mathcal{H}^2 - 6b_e\mathcal{H}^2 + 9\mathcal{H}^2\right)\left(v^{(1)}_{\mathrm{P}}\right)^2 \\
&+\mathcal{H}b_e v^{(1)}_{\mathrm{P}} v^{(1)\prime}_{\mathrm{P}} - 3\mathcal{H}v^{(1)}_{\mathrm{P}} v^{(1)\prime}_{\mathrm{P}} - 2\mathcal{H}b_e v^{(1)}_{\mathrm{P}}\delta^{(1)}_{g\,\mathrm{CO}} + 6\mathcal{H}v^{(1)}_{\mathrm{P}}\delta^{(1)}_{g\,\mathrm{CO}} - 2v^{(1)}_{\mathrm{P}}\delta^{(1)\,\prime}_{g\mathrm{CO}} \\
&-\frac{1}{2}\partial^i\xi^{(1)}\left(-b_e\mathcal{H}\partial_i v^{(1)}_{\mathrm{P}} + 3\mathcal{H}\partial_i v^{(1)}_{\mathrm{P}} + 2\partial_i\delta^{(1)}_{g\,\mathrm{CO}}\right) - (b_e - 3)\,\mathcal{H}\nabla^{-2}\Xi.
\end{aligned} \tag{253}$$

Here

$$\begin{aligned}
\Xi \;=\;& +v^{(1)}_{\mathrm{P}}\nabla^2 v^{(1)\prime}_{\mathrm{P}} - v^{(1)\prime}_{\mathrm{P}}\nabla^2 v^{(1)}_{\mathrm{P}} - 2\partial_i\Phi^{(1)}\partial^i v^{(1)}_{\mathrm{P}} - 2\Phi^{(1)}\nabla^2 v^{(1)}_{\mathrm{P}} - 4\Psi^{(1)}\nabla^2 v^{(1)}_{\mathrm{P}} - 4\partial_i\Psi^{(1)}\partial^i v^{(1)}_{\mathrm{P}} \\
&+\frac{1}{2}\partial_i\xi^{(1)}\partial^i\nabla^2 v^{(1)}_{\mathrm{P}} + \frac{1}{2}\partial_i v^{(1)}_{\mathrm{P}}\partial^i\nabla^2\xi^{(1)} + \partial_i\partial_j\xi^{(1)}\partial^i\partial^j v^{(1)}_{\mathrm{P}} \;.
\end{aligned} \tag{254}$$

(Another useful relation is $\xi^{(1)\prime}/2 = v^{(1)}_{\mathrm{P}}$.)

The scale-independent bias at first and second order is defined by

$$\delta^{(1)}_{g\,\mathrm{CO}} = b_1(\eta)\delta^{(1)}_{m\,\mathrm{CO}} \qquad \text{and} \qquad \delta^{(2)}_{g\,\mathrm{CO}} = b_2(\eta)\delta^{(2)}_{m\,\mathrm{CO}}. \tag{255}$$

These are substituted into Eqs. (252) and (253), and then we can replace the term $\delta^{(2)}_g$ in the expression for the observed number overdensity at second order in Poisson gauge, Eq. (236). This allows us to relate the observed number counts to the underlying matter overdensity in a gauge invariant way.

VI. CONCLUSIONS

We presented for the first time a derivation of the observed galaxy number counts to second order on cosmological scales, including all relativistic effects. Our results are given both in a general gauge and in Poisson gauge, and apply to general dark energy models, including those where dark energy interacts non-gravitationally with dark matter. Our results also apply to metric theories of modified gravity as an alternative to dark energy. The main, fully general, result for the galaxy number count fluctuations at second order is Eq. (157), which is specialized to the Poisson gauge in Eq. (236). We also gave these results in the form where the contribution from weak lensing shear and rotation is made explicit, in Eqs. (158) and (237).

We derived the expressions needed to relate the observed number over-density to the matter over-density via the bias in a gauge-invariant way, in Eqs. (252)–(255).

The second-order effects that we derive, especially those involving integrals along the line of sight, may make a non-negligible contribution to the observed number counts. This will be important for removing potential biases on parameter estimation in precision cosmology with galaxy surveys. It will also be important for an accurate analysis of the 'contamination' of primordial non-Gaussianity by relativistic projection effects. This is discussed in Paper I [20] and is the subject of ongoing work [38].

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Appendix A: Perturbation terms in general gauge

From Eq. (85) the perturbations of $g_{\mu\nu}$ and $g^{\mu\nu}$ are

$$\begin{aligned}
g_{00} &= a^2\hat{g}_{00} = -a^2\left(1+2A^{(1)}+A^{(2)}\right), & g^{00} &= a^{-2}\hat{g}^{00} = -a^{-2}\left[1-2A^{(1)}-A^{(2)}+4\left(A^{(1)}\right)^2-B_i^{(1)}B^{i(1)}\right],\\
g_{0i} &= a^2\hat{g}_{0i} = a^2\left(-B_i^{(1)}-B_i^{(2)}/2\right), & g^{0i} &= a^{-2}\hat{g}^{0i} = a^{-2}\left[-B^{i(1)}-B^{i(2)}/2+2A^{(1)}B^{i(1)}+B_k^{(1)}h^{ki(1)}\right],\\
g_{ij} &= a^2\hat{g}_{ij} = a^2\left(\delta_{ij}+h_{ij}^{(1)}+h_{ij}^{(2)}/2\right), & g^{ij} &= a^{-2}\hat{g}^{ij} = a^{-2}\left[\delta^{ij}-h^{ij(1)}-h^{ij(2)}/2+h^{ik(1)}h_k^{j(1)}-B^{i(1)}B^{j(1)}\right],
\end{aligned} \tag{A1}$$

For Christoffel symbols $\Gamma^\mu_{\rho\sigma} = \Gamma^{\mu(0)}_{\rho\sigma}+\Gamma^{\mu(1)}_{\rho\sigma}+\Gamma^{\mu(2)}_{\rho\sigma}/2$ and $\hat{\Gamma}^\mu_{\rho\sigma} = \hat{\Gamma}^{\mu(0)}_{\rho\sigma}+\hat{\Gamma}^{\mu(1)}_{\rho\sigma}+\hat{\Gamma}^{\mu(2)}_{\rho\sigma}/2$ in comoving coordinates, we obtain

$$\begin{aligned}
&\Gamma^{0(0)}_{00} = \mathcal{H}, \quad \Gamma^{0(0)}_{0i} = 0, \quad \Gamma^{i(0)}_{00} = 0,\\
&\Gamma^{i(0)}_{00} = 0, \quad \Gamma^{i(0)}_{j0} = \mathcal{H}\delta^i_j, \quad \Gamma^{i(0)}_{jk} = 0, \quad \hat{\Gamma}^{\mu(0)}_{\rho\sigma} = 0,
\end{aligned} \tag{A2}$$

$$\begin{aligned}
\Gamma^{0(1)}_{00} &= \hat{\Gamma}^{0(1)}_{00} & \Gamma^{0(1)}_{0i} &= \hat{\Gamma}^{0(1)}_{0i} - \mathcal{H}B_i^{(1)}\\
\Gamma^{0(1)}_{ij} &= \hat{\Gamma}^{0(1)}_{ij} + \mathcal{H}\left(-2A^{(1)}\delta_{ij}+h_{ij}^{(1)}\right) & \Gamma^{i(1)}_{00} &= \hat{\Gamma}^{i(1)}_{00} - \mathcal{H}B^{i(1)}\\
\Gamma^{i(1)}_{j0} &= \hat{\Gamma}^{i(1)}_{j0} & \Gamma^{i(1)}_{jk} &= \hat{\Gamma}^{i(1)}_{jk} + \mathcal{H}B^{i(1)}\delta_{jk},\\
\hat{\Gamma}^{0(1)}_{00} &= A^{(1)\prime}, & \hat{\Gamma}^{0(1)}_{0i} &= \partial_i A^{(1)},\\
\hat{\Gamma}^{0(1)}_{ij} &= \tfrac{1}{2}\partial_i B_j^{(1)} + \tfrac{1}{2}\partial_j B_i^{(1)} + \tfrac{1}{2}h_{ij}^{(1)\prime}, & \hat{\Gamma}^{i(1)}_{00} &= \partial^i A^{(1)} - B^{i(1)\prime},\\
\hat{\Gamma}^{i(1)}_{j0} &= \tfrac{1}{2}\partial^i B_j^{(1)} - \tfrac{1}{2}\partial_j B^{i(1)} + \tfrac{1}{2}h_j^{i(1)\prime}, & \hat{\Gamma}^{i(1)}_{jk} &= \tfrac{1}{2}\partial_j h_k^{i(1)} + \tfrac{1}{2}\partial_k h_j^{i(1)} - \tfrac{1}{2}\partial^i h_{jk}^{(1)},
\end{aligned} \tag{A3}$$

$$
\begin{aligned}
\frac{1}{2}\Gamma^{0(2)}_{00} &= \frac{1}{2}\hat{\Gamma}^{0(2)}_{00} + \mathcal{H}B^{(1)}_k B^{k(1)} \,, \\
\frac{1}{2}\Gamma^{0(2)}_{0i} &= \frac{1}{2}\hat{\Gamma}^{0(2)}_{0i} - \frac{\mathcal{H}}{2}B^{(2)}_i + 2\mathcal{H}A^{(1)}B^{(1)}_i \,, \\
\frac{1}{2}\Gamma^{i(2)}_{00} &= \frac{1}{2}\hat{\Gamma}^{i(2)}_{00} - \frac{\mathcal{H}}{2}B^{i(2)} + \mathcal{H}B^{(1)}_k h^{ik(1)} \,, \\
\frac{1}{2}\Gamma^{i(2)}_{j0} &= \frac{1}{2}\hat{\Gamma}^{i(2)}_{j0} - \mathcal{H}B^{i(1)}B^{(1)}_j \,, \\
\frac{1}{2}\Gamma^{0(2)}_{ij} &= \frac{1}{2}\hat{\Gamma}^{0(2)}_{ij} + \mathcal{H}\left\{\left[-A^{(2)} + 4\left(A^{(1)}\right)^2 - B^{(1)}_k B^{k(1)}\right]\delta_{ij} + \frac{1}{2}h^{(2)}_{ij} - 2A^{(1)}h^{(1)}_{ij}\right\} \,, \\
\frac{1}{2}\Gamma^{i(2)}_{jk} &= \frac{1}{2}\hat{\Gamma}^{i(2)}_{jk} + \mathcal{H}\left[\left(\frac{1}{2}B^{i(2)} - 2A^{(1)}B^{i(1)} - B^{(1)}_l h^{il(1)}\right)\delta_{jk} - B^{i(1)}h^{(1)}_{jk}\right] \,, \\
\frac{1}{2}\hat{\Gamma}^{0(2)}_{00} &= +\frac{1}{2}A^{2)\prime} - 2A^{(1)}A^{(1)\prime} + B^{(1)}_k B^{k(1)\prime} - \partial_k A^{(1)}B^{k(1)} \,, \\
\frac{1}{2}\hat{\Gamma}^{0(2)}_{0i} &= \frac{1}{2}\partial_i A^{(2)} - 2A^{(1)}\partial_i A^{(1)} - \frac{1}{2}B^{k(1)}h^{(1)\prime}_{ik} + \frac{1}{2}B^{k(1)}\left(\partial_i B^{(1)}_k - \partial_k B^{(1)}_i\right) \,, \\
\frac{1}{2}\hat{\Gamma}^{i(2)}_{00} &= \frac{1}{2}\partial^i A^{(2)} - \frac{1}{2}B^{i(2)\prime} + A^{(1)\prime}B^{i(1)} + B^{(1)\prime}_k h^{ik(1)} - \partial_k A^{(1)}h^{ik(1)} \,, \\
\frac{1}{2}\hat{\Gamma}^{i(2)}_{j0} &= \frac{1}{4}\partial^i B^{(2)}_j - \frac{1}{4}\partial_j B^{i(2)} + \frac{1}{4}h^{i(2)\prime}_j - \frac{1}{2}h^{ik(1)}h^{(1)\prime}_{jk} + \partial_j A^{(1)}B^{i(1)} - \frac{1}{2}h^{ik(1)}\left(\partial_k B^{(1)}_j - \frac{1}{2}\partial_j B^{(1)}_k\right) \,, \\
\frac{1}{2}\hat{\Gamma}^{0(2)}_{ij} &= \frac{1}{4}\partial_i B^{(2)}_j + \frac{1}{4}\partial_j B^{(2)}_i + \frac{1}{4}h^{(2)\prime}_{ij} - A^{(1)}\left(\partial_i B^{(1)}_j + \partial_j B^{(1)}_i\right) - A^{(1)}h^{(1)\prime}_{ij} - \frac{1}{2}B^{k(1)}\left(\partial_i h^{(1)}_{jk} + \partial_j h^{(1)}_{ik} - \partial_k h^{(1)}_{ij}\right) \,, \\
\frac{1}{2}\hat{\Gamma}^{i(2)}_{jk} &= \frac{1}{4}\partial_j h^{i(2)}_k + \frac{1}{4}\partial_k h^{i(2)}_j - \frac{1}{4}\partial^i h^{(2)}_{jk} + \frac{1}{2}B^{i(1)}\left(\partial_j B^{(1)}_k + \partial_k B^{(1)}_j\right) + \frac{1}{2}B^{i(1)}h^{(1)\prime}_{jk} \\
&\quad -\frac{1}{2}h^{il(1)}\left(\partial_j h^{(1)}_{kl} + \partial_k h^{(1)}_{jl} - \partial_l h^{(1)}_{jk}\right) \,. \qquad \text{(A4)}
\end{aligned}
$$

For four-velocity u^μ ($g_{\mu\nu}u^\mu u^\nu = -1$), we find

$$u_0 = -a\left[1 + A^{(1)} + \frac{1}{2}A^{(2)} - \frac{1}{2}\left(A^{(1)}\right)^2 + \frac{1}{2}v^{(1)}_k v^{k(1)}\right], \qquad \text{(A5)}$$

$$u_i = a\left[v^{(1)}_i - B^{(1)}_i + \frac{1}{2}\left(v^{(2)}_i - B^{(2)}_i\right) + A^{(1)}B^{(1)}_i + h^{(1)}_{ik}v^{k(1)}\right], \qquad \text{(A6)}$$

$$u^0 = \frac{1}{a}\left[1 - A^{(1)} - \frac{1}{2}A^{(2)} + \frac{3}{2}\left(A^{(1)}\right)^2 + \frac{1}{2}v^{(1)}_k v^{k(1)} - v^{(1)}_k B^{k(1)}\right], \qquad \text{(A7)}$$

$$u^i = \frac{1}{a}\left(v^{i(1)} + \frac{1}{2}v^{i(2)}\right) \,. \qquad \text{(A8)}$$

From Eqs. (23) and (A5) we obtain all components of $\Lambda^{(n)}_{\hat{0}\mu}$ and $E^{(n)}_{\hat{0}\mu}$. From Eq. (23), we have

$$u_\mu = \Lambda_{\hat{0}\mu} = a\,E_{\hat{0}\mu} \qquad \text{and} \qquad u^\mu = \Lambda^\mu_{\hat{0}} = E^\mu_{\hat{0}}/a \,, \qquad \text{(A9)}$$

and, using Eq. (91) we can deduce all components of $\Lambda^{(n)}_{\hat{a}\mu}$ and $E^{(n)}_{\hat{a}\mu}$. We summarize as follows:

$$\Lambda^{(1)}_{\hat{0}0} = aE^{(1)}_{\hat{0}0} = -aA^{(1)} \ , \qquad \Lambda^{(1)}_{\hat{0}i} = aE^{(1)}_{\hat{0}i} = a\left(v^{(1)}_i - B^{(1)}_i\right) \ ,$$

$$\Lambda^{(1)}_{\hat{a}0} = aE^{(1)}_{\hat{a}0} = -av^{(1)}_{\hat{a}} \ , \qquad \Lambda^{(1)}_{\hat{a}i} = aE^{(1)}_{\hat{a}i} = \tfrac{1}{2}ah^{(1)}_{\hat{a}i} \ ,$$

$$\tfrac{1}{2}\Lambda^{(2)}_{\hat{0}0} = \tfrac{1}{2}aE^{(2)}_{\hat{0}0} = a\left[-\tfrac{1}{2}A^{(2)} + \tfrac{1}{2}\left(A^{(1)}\right)^2 - \tfrac{1}{2}v^{(1)}_k v^{k(1)}\right] \ ,$$

$$\tfrac{1}{2}\Lambda^{(2)}_{\hat{0}i} = \tfrac{1}{2}aE^{(2)}_{\hat{0}i} = a\left[\tfrac{1}{2}\left(v^{(2)}_i - B^{(2)}_i\right) + A^{(1)}B^{(1)}_i + h^{(1)}_{ik}v^{k(1)}\right] \ , \tag{A10}$$

$$\tfrac{1}{2}\Lambda^{(2)}_{\hat{a}0} = \tfrac{1}{2}aE^{(2)}_{\hat{a}0} = a\left[-\tfrac{1}{2}v^{(2)}_{\hat{a}} - A^{(1)}v^{(1)}_{\hat{a}} - \tfrac{1}{2}v^{k(1)}h^{(1)}_{\hat{a}k}\right] \ ,$$

$$\tfrac{1}{2}\Lambda^{(2)}_{\hat{a}i} = \tfrac{1}{2}aE^{(2)}_{\hat{a}i} = a\left[\tfrac{1}{4}h^{(2)}_{\hat{a}j} + \tfrac{1}{2}\left(v^{(1)}_i - B^{(1)}_i\right)\left(v^{(1)}_{\hat{a}} - B^{(1)}_{\hat{a}}\right) - \tfrac{1}{8}h^{k(1)}_j h^{(1)}_{\hat{a}k}\right] \ .$$

The four-vector $\mathcal{E}^\nu_m$ defined in Eq. (159) can be nonzero. For the background, first- and second-order perturbations we obtain

$$\begin{aligned}
\mathcal{E}^{0(0)}_m &= \frac{1}{a^2}\rho^{(0)\prime}_m + \frac{3}{a^2}\mathcal{H}\rho^{(0)}_m \\
\mathcal{E}^{i(0)}_m &= 0 \ , \\
\mathcal{E}^{0(1)}_m &= \frac{1}{a^2}\rho^{(0)}_m\left(\delta^{(1)\prime}_m + \partial_i v^{i(1)} + \frac{1}{2}h^{i(1)\prime}_i\right) + \mathcal{E}^{0(0)}_m\left(-2A^{(1)} + \delta^{(1)}_m\right) \ , \\
\mathcal{E}^{i(1)}_m &= \frac{1}{a^2}\rho^{(0)}_m\left[\left(v^{i(1)} - B^{i(1)}\right)' + \mathcal{H}\left(v^{i(1)} - B^{i(1)}\right) + \partial^i A^{(1)}\right] + \mathcal{E}^{0(0)}_m v^{i(1)} \ , \\
\frac{1}{2}\mathcal{E}^{0(2)}_m &= \frac{1}{a^2}\rho^{(0)}_m\left[\frac{1}{2}\delta^{(2)\prime}_m + \frac{1}{2}\partial_i v^{i(2)} + \frac{1}{4}h^{i(2)\prime}_i - \mathcal{H}\left(v^{i(1)} - B^{i(1)}\right)\left(v^{(1)}_i - B^{(1)}_i\right) + B^{(1)}_i\partial^i A^{(1)} + \left(A^{(1)} + \delta^{(1)}_m\right)\partial_i v^{i(1)}\right. \\
&\quad \left. + v^{i(1)}\partial_i\delta^{(1)}_m + \frac{1}{2}\delta^{(1)}_m h^{i(1)\prime}_i + \frac{1}{2}v^{j(1)}\partial_j h^{i(1)}_i - \frac{1}{2}h^{ij(1)}h^{(1)\prime}_{ij}\right] + \mathcal{E}^{0(0)}_m\left(-A^{(2)} + \frac{1}{2}\delta^{(2)}_m - v^{i(1)}v^{(1)}_i\right) \\
&\quad + 2\left(v^{i(1)} - B^{i(1)}\right)\mathcal{E}^{i(1)}_m - 2A^{(1)}\mathcal{E}^{0(1)}_m \ , \\
\frac{1}{2}\mathcal{E}^{i(2)}_m &= \frac{1}{a^2}\rho^{(0)}_m\left[\left(\frac{1}{2}v^{i(2)} - \frac{1}{2}B^{i(2)}\right)' + \mathcal{H}\left(\frac{1}{2}v^{i(2)} - \frac{1}{2}B^{i(2)}\right) + \frac{1}{2}\partial^i A^{(2)} - v^{i(1)}\partial_j v^{j(1)} + A^{(1)}B^{i(1)\prime} + \mathcal{H}A^{(1)}B^{i(1)}\right. \\
&\quad + A^{(1)\prime}B^{i(1)} + \mathcal{H}B^{(1)}_k h^{ik(1)} + B^{(1)\prime}_k h^{ik(1)} + v^{k(1)}h^{i(1)\prime}_k - A^{(1)}\partial^i A^{(1)} + \partial^i B^{(1)}_k v^{k(1)} - v^{k(1)}\partial_k B^{i(1)} \\
&\quad \left. -\partial_k A^{(1)}h^{ik(1)}\right] + \mathcal{E}^{0(0)}_m\left(\frac{1}{2}v^{i(2)} - \delta^{(1)}_m v^{i(1)} + 2A^{(1)}v^{i(1)}\right) + \mathcal{E}^{0(1)}_m v^{i(1)} + \mathcal{E}^{i(1)}_m\left(\delta^{(1)}_m - A^{(1)}\right) \ .
\end{aligned} \tag{A11}$$

Here $\delta^{(1)}_m = \rho^{(1)}_m/\rho^{(0)}_m - 1$ is the CDM fractional overdensity. If $\mathcal{E}^\nu_m$ is evaluated in redshift-space, then $\rho^{(0)}_m = \bar{\rho}_m$, $a(\bar{x}^0) = \bar{a}$ and

$$\mathcal{E}^{0(0)}_m = \frac{\mathcal{H}}{\bar{a}^2}\bar{\rho}_m b_m \ , \tag{A12}$$

where $b_m = \mathrm{d}(a^3\bar{\rho}_m)/\mathrm{d}\ln\bar{a}$.

$\mathcal{E}_m^{\|(1)}$ and $\mathcal{E}_m^{\|(2)}$, evaluated at $\bar{x}^\mu$, are given by

$$\begin{aligned}
\mathcal{E}_m^{\|(1)} &= \frac{1}{\bar{a}^2}\bar{\rho}_m\left[\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(A^{(1)}-v_\|^{(1)}\right)+A^{(1)\prime}-B_\|^{(1)\prime}+\partial_\| v_\|^{(1)}+\mathcal{H}\left(v_\|^{(1)}-B_\|^{(1)}\right)\right]+\frac{\mathcal{H}}{\bar{a}^2}\bar{\rho}_m b_m v_\|^{(1)}\,,\\
\frac{1}{2}\mathcal{E}_m^{\|(2)} &= \frac{1}{\bar{a}^2}\bar{\rho}_m\left[\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(\frac{1}{2}A^{(2)}-\frac{1}{2}v_\|^{(2)}\right)+\frac{1}{2}A^{(2)\prime}-\frac{1}{2}B_\|^{(2)\prime}+\frac{1}{2}\partial_\| v_\|^{(2)}+\mathcal{H}\left(\frac{1}{2}v_\|^{(2)}-\frac{1}{2}B_\|^{(2)}\right)-v_\|^{(1)}\partial_\| v_\|^{(1)}\right.\\
&\quad -\frac{2}{\bar{\chi}}\left(v_\|^{(1)}\right)^2-v_\|^{(1)}\partial_{\perp j}v_\perp^{j(1)}+A^{(1)}B_\|^{(1)\prime}+A^{(1)\prime}B_\|^{(1)}+\mathcal{H}A^{(1)}B_\|^{(1)}+v_\|^{(1)}h_\|^{(1)\prime}+\mathcal{H}B_\|^{(1)}h_\|^{(1)}+B_\|^{(1)\prime}h_\|^{(1)}\\
&\quad -A^{(1)}\partial_\| A^{(1)}-\partial_\| A^{(1)}h_\|^{(1)}+v_{\perp k}^{(1)}\partial_\| B_\perp^{k(1)}-v_\perp^{j(1)}\partial_{\perp j}B_\|^{(1)}+\frac{1}{\chi}v_\perp^{j(1)}B_{\perp j}^{(1)}+\mathcal{H}B_k^{(1)}\mathcal{P}_j^k h^{ij(1)}n_i\\
&\quad \left.+v_k^{(1)}\mathcal{P}_j^k h^{ij(1)\prime}n_i+B_k^{(1)\prime}\mathcal{P}_j^k h^{ij(1)}n_i-\partial_k A^{(1)}\mathcal{P}_j^k h^{ij(1)}n_i\right]+\frac{\mathcal{H}}{\bar{a}^2}\bar{\rho}_m b_m\left(\frac{1}{2}v_\|^{(2)}-\delta_m^{(1)}v_\|^{(1)}+2A^{(1)}v_\|^{(1)}\right)\\
&\quad +\mathcal{E}_m^{0(1)}v_\|^{(1)}+\mathcal{E}_m^{\|(1)}\left(\delta_m^{(1)}-A^{(1)}\right)\,. \qquad \text{(A13)}
\end{aligned}$$

For the weak lensing shear and rotation:

$$\begin{aligned}
\partial_{\perp i}\Delta x_{\perp j}^{(1)} &= -\mathcal{P}_{ij}\left(B_{\|\,o}^{(1)}-v_{\|\,o}^{(1)}\right)+\frac{1}{2}\mathcal{P}_i^m\mathcal{P}_j^n h_{mn\,o}^{(1)}-\frac{1}{2}\mathcal{P}_{ij}h_{\|\,o}^{(1)}-\left(B_{\perp io}^{(1)}-v_{\perp i\,o}^{(1)}\right)n_j-\frac{1}{2}\mathcal{P}_{ip}n^k h_{k\,o}^{p(1)}n_j\\
&\quad +\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\frac{1}{\tilde{\chi}}\mathcal{P}_{ij}B_\|^{(1)}-\mathcal{P}_{jp}\tilde{\partial}_{\perp i}B^{p(1)}-\frac{1}{\tilde{\chi}}\mathcal{P}_i^p\mathcal{P}_{jq}h_p^{q(1)}+\frac{1}{\tilde{\chi}}\mathcal{P}_{ij}h_\|^{(1)}-n^p\mathcal{P}_{jq}\tilde{\partial}_{\perp i}h_p^{q(1)}\right.\\
&\quad \left.+\frac{1}{\tilde{\chi}}\left(B_{\perp i}^{(1)}+\mathcal{P}_{in}n^m h_m^{n(1)}\right)n_j\right]+\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\,\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\partial}_{\perp i}\tilde{\partial}_{\perp j}\left(A^{(1)}-B_\|^{(1)}-\frac{1}{2}h_\|^{(1)}\right)\,, \qquad \text{(A14)}\\
\gamma_{ij}^{(1)} &= \mathcal{P}_{ij}\left(B_{\|\,o}^{(1)}-v_{\|\,o}^{(1)}\right)-\frac{1}{2}\mathcal{P}_{(i}^m\mathcal{P}_{j)}^n h_{mn\,o}^{(1)}+\frac{1}{2}\mathcal{P}_{ij}h_{\|\,o}^{(1)}+n_{(j}\left(B_{\perp i)\,o}^{(1)}-v_{\perp i)\,o}^{(1)}\right)+\frac{1}{2}\mathcal{P}_{p(i}n_{j)}n^k h_{k\,o}^{p(1)}\\
&\quad -\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\frac{1}{\tilde{\chi}}\mathcal{P}_{ij}B_\|^{(1)}-\mathcal{P}_{p(j}\tilde{\partial}_{\perp i)}B^{p(1)}-\frac{1}{\tilde{\chi}}\mathcal{P}_{(i}^p\mathcal{P}_{j)q}h_p^{q(1)}+\frac{1}{\tilde{\chi}}\mathcal{P}_{ij}h_\|^{(1)}-n^p\mathcal{P}_{q(j}\tilde{\partial}_{\perp i)}h_p^{q(1)}\right.\\
&\quad \left.+\frac{1}{\tilde{\chi}}\left(B_{\perp(i}^{(1)}+n^m h_m^{n(1)}\mathcal{P}_{n(i}\right)n_{j)}\right]-\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\,\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\partial}_{\perp(i}\tilde{\partial}_{\perp j)}\left(A^{(1)}-B_\|^{(1)}-\frac{1}{2}h_\|^{(1)}\right)-\mathcal{P}_{ij}\kappa^{(1)}\,, \qquad \text{(A15)}\\
\vartheta_{ij}^{(1)}\vartheta^{ij(1)} &= \frac{1}{2}B_{\perp io}^{(1)}B_{\perp o}^{i(1)}-B_{\perp io}^{(1)}v_{\perp o}^{i(1)}+\frac{1}{2}v_{\perp io}^{(1)}v_{\perp o}^{i(1)}+\frac{1}{2}\left(B_{\perp io}^{(1)}-v_{\perp i\,o}^{(1)}\right)n^k h_{k\,o}^{i(1)}+\frac{1}{8}n^m\mathcal{P}_n^j h_m^{n(1)}n^k h_{jk}^{(1)}\\
&\quad -\left(B_{\perp io}^{(1)}-v_{\perp i\,o}^{(1)}+\frac{1}{2}\mathcal{P}_{ip}n^k h_{k\,o}^{p(1)}\right)\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\frac{1}{\tilde{\chi}}\left(B_\perp^{i(1)}+\mathcal{P}_l^i n^m h_m^{l(1)}\right)+(\bar{\chi}-\tilde{\chi})\,\frac{1}{\bar{\chi}}\tilde{\partial}_\perp^i\left(A^{(1)}-B_\|^{(1)}-\frac{1}{2}h_\|^{(1)}\right)\right]\\
&\quad +\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\mathcal{P}_{q[i}\tilde{\partial}_{\perp j]}B^{q(1)}+n^k\mathcal{P}_{q[i}\tilde{\partial}_{\perp j]}h_k^{q(1)}\right]\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\mathcal{P}_p^{[i}\tilde{\partial}_\perp^{j]}B^{p(1)}+n^m\mathcal{P}_p^{[i}\tilde{\partial}_\perp^{j]}h_m^{p(1)}\right]\\
&\quad +\frac{1}{2}\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\frac{1}{\tilde{\chi}}\left(B_{\perp i}^{(1)}+\mathcal{P}_{in}n^m h_m^{n(1)}\right)+(\bar{\chi}-\tilde{\chi})\,\frac{1}{\bar{\chi}}\tilde{\partial}_{\perp i}\left(A^{(1)}-B_\|^{(1)}-\frac{1}{2}h_\|^{(1)}\right)\right]\\
&\quad \times\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[\frac{1}{\tilde{\chi}}\left(B_\perp^{i(1)}+\mathcal{P}_p^i n^q h_q^{p(1)}\right)+(\bar{\chi}-\tilde{\chi})\,\frac{1}{\bar{\chi}}\tilde{\partial}_\perp^i\left(A^{(1)}-B_\|^{(1)}-\frac{1}{2}h_\|^{(1)}\right)\right]\,. \qquad \text{(A16)}
\end{aligned}$$

If we assume that galaxy velocities follow the matter velocity field,

$$
\begin{aligned}
\Delta_g^{(2)} &= \delta_g^{(2)} + b_e \,\Delta\ln a^{(2)} + \partial_\parallel \Delta x_\parallel^{(2)} + \frac{2}{\bar\chi}\Delta x_\parallel^{(2)} - 2\kappa^{(2)} + A^{(2)} + v_\parallel^{(2)} + \frac{1}{2}h_i^{i(2)} + \left(\Delta_g^{(1)}\right)^2 - \left(A^{(1)}\right)^2 \\
&+ A^{(1)}h_\parallel^{(1)} - \left(v_\parallel^{(1)}\right)^2 + \left(B_\parallel^{(1)}\right)^2 - \frac{1}{2}h_i^{k(1)}h_k^{i(1)} - \frac{1}{4}\left(h_\parallel^{(1)}\right)^2 + 2A^{(1)}v_\parallel^{(1)} - \frac{1}{\mathcal{H}^2}\left(\partial_\parallel v_\parallel^{(1)}\right)^2 + \frac{1}{\mathcal{H}}A^{(1)}h_\parallel^{(1)\prime} \\
&- \frac{1}{4\mathcal{H}^2}\left(h_\parallel^{(1)\prime}\right)^2 - \frac{1}{\mathcal{H}}v_\parallel^{(1)}h_\parallel^{(1)\prime} - \frac{1}{2\mathcal{H}}h_\parallel^{(1)}h_\parallel^{(1)\prime} + \frac{2}{\mathcal{H}}A^{(1)}\partial_\parallel v_\parallel^{(1)} - \frac{2}{\mathcal{H}}v_\parallel^{(1)}\partial_\parallel v_\parallel^{(1)} - \frac{1}{\mathcal{H}}h_\parallel^{(1)}\partial_\parallel v_\parallel^{(1)} - \frac{1}{\mathcal{H}^2}h_\parallel^{(1)\prime}\partial_\parallel v_\parallel^{(1)} \\
&- 2\left|\gamma^{(1)}\right|^2 - 2\left(\kappa^{(1)}\right)^2 + \vartheta_{ij}^{(1)}\vartheta^{ij(1)} + B_{\perp i}^{(1)}B_\perp^{i(1)} + v_{\perp i}^{(1)}v_\perp^{i(1)} - 2v_{\perp i}^{(1)}B_\perp^{i(1)} - \left(\delta_g^{(1)}\right)^2 + 2v_\perp^{i(1)}\partial_{\perp i}T^{(1)} \\
&+ \frac{2}{\mathcal{H}}\left(-\partial_\parallel A^{(1)} + B_\parallel^{(1)\prime} - \partial_\parallel v_\parallel^{(1)} - \mathcal{H}v_\parallel^{(1)} + \mathcal{H}B_\parallel^{(1)}\right)\Delta\ln a^{(1)} - \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\left(h_i^{i(1)} + 2\delta_g^{(1)}\right)\Delta\ln a^{(1)} - \frac{4}{\bar\chi^2\mathcal{H}}\Delta\ln a^{(1)}T^{(1)} \\
&+ 2\frac{\mathcal{H}'}{\mathcal{H}^2}\left(A^{(1)} - v_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel v_\parallel^{(1)} - \frac{1}{2\mathcal{H}}h_\parallel^{(1)\prime}\right)\Delta\ln a^{(1)} - \partial_\parallel\left(2v_\parallel^{(1)} + h_i^{i(1)} + 2\delta_g^{(1)}\right)T^{(1)} - \frac{4}{\bar\chi\mathcal{H}}\Delta\ln a^{(1)}\kappa^{(1)} \\
&- \frac{4}{\bar\chi}T^{(1)}\kappa^{(1)} + \left[-b_e + \frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln\bar a} - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 - \frac{2}{\bar\chi^2\mathcal{H}^2}\right]\left(\Delta\ln a^{(1)}\right)^2 - \frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2 + 2\left[-\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + 2S_\perp^{i(1)}\right] \\
&\times \partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) - \left[-\frac{2}{\bar\chi}\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + \frac{4}{\bar\chi}S_\perp^{i(1)} + \partial_\perp^i\left(2v_\parallel^{(1)} + h_l^{l(1)} + 2\delta_g^{(1)}\right)\right] \\
&\times \int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\bar\chi}{\tilde\chi}\left(B_{\perp i}^{(1)} + n^k h_k^{j(1)}\mathcal{P}_{ij}\right) + \left(\bar\chi - \tilde\chi\right)\tilde\partial_{\perp i}\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\right] \\
&+ 2\left(B_{\perp i\,o}^{(1)} - v_{\perp i\,o}^{(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_{ij}\right)\left(B_{\perp\,o}^{i(1)} - v_{\perp\,o}^{i(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_j^i\right) \\
&+ \left(B_{\perp i\,o}^{(1)} - v_{\perp i\,o}^{(1)} + \frac{1}{2}n^k h_{k\,o}^{j(1)}\mathcal{P}_{ij}\right)\Bigg\{-2\left(B_\perp^{i(1)} + n^m h_m^{l(1)}\mathcal{P}_l^i\right) + 4S_\perp^{i(1)} + \bar\chi\partial_\perp^i\left(2v_\parallel^{(1)} + h_l^{l(1)} + 2\delta_g^{(1)}\right) \\
&+ 2\partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) - 2\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\frac{\bar\chi}{\tilde\chi}\left(B_\perp^{i(1)} + n^k h_k^{j(1)}\mathcal{P}_j^i\right) + \left(\bar\chi - \tilde\chi\right)\tilde\partial_\perp^i\left(A^{(1)} - B_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)}\right)\right]\Bigg\} \\
&- \left(\frac{\bar a^2}{\bar\rho_m\mathcal{H}}\mathcal{E}_m^{\parallel(1)} - b_m v_\parallel^{(1)}\right)^2 - 2\left[A^{(1)} - v_\parallel^{(1)} - \frac{1}{2}h_\parallel^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel v_\parallel^{(1)} - \frac{1}{2\mathcal{H}}h_\parallel^{(1)\prime} - \left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\Delta\ln a^{(1)}\right] \\
&\times \left(\frac{\bar a^2}{\bar\rho_m\mathcal{H}}\mathcal{E}_m^{\parallel(1)} - b_m v_\parallel^{(1)}\right) \,. \qquad \text{(A17)}
\end{aligned}
$$

Appendix B: Perturbation terms in Poisson Gauge

From Eq. (174), the perturbations of $g_{\mu\nu}$ and $g^{\mu\nu}$ are

$$
\begin{aligned}
g_{00} &= -a^2\left(1 + 2\Phi^{(1)} + \Phi^{(2)}\right), & g^{00} &= -a^{-2}\left[1 - 2\Phi^{(1)} - \Phi^{(2)} + 4\left(\Phi^{(1)}\right)^2\right], \\
g_{0i} &= a^2\omega_i^{(2)}, & g^{0i} &= a^{-2}\omega^{i(2)}, \qquad \text{(B1)} \\
g_{ij} &= a^2\left(\delta_{ij} - 2\delta_{ij}\Psi^{(1)} - \delta_{ij}\Psi^{(2)} + \hat h_{ij}^{(2)}/2\right), & g^{ij} &= a^{-2}\left[\delta^{ij} + 2\delta^{ij}\Psi^{(1)} + \delta^{ij}\Psi^{(2)} - \hat h^{ij(2)}/2 + 4\delta^{ij}(\Psi^{(1)})^2\right],
\end{aligned}
$$

For four-velocity u^μ, we find

$$
u_0 = -a\left[1 + \Phi^{(1)} + \frac{1}{2}\Phi^{(2)} - \frac{1}{2}\left(\Phi^{(1)}\right)^2 + \frac{1}{2}v_k^{(1)}v^{k(1)}\right], \qquad \text{(B2)}
$$

$$
u_i = a\left[v_i^{(1)} + \frac{1}{2}\left(v_i^{(2)} + 2\omega_i^{(2)}\right) - 2\Psi^{(1)}v_i^{(1)}\right], \qquad \text{(B3)}
$$

$$
u^0 = \frac{1}{a}\left[1 - \Phi^{(1)} - \frac{1}{2}\Phi^{(2)} + \frac{3}{2}\left(\Phi^{(1)}\right)^2 + \frac{1}{2}v_k^{(1)}v^{k(1)}\right], \qquad \text{(B4)}
$$

$$
u^i = \frac{1}{a}\left(v^{i(1)} + \frac{1}{2}v^{i(2)}\right) . \qquad \text{(B5)}
$$

For the tetrad:

$$
\begin{aligned}
&\Lambda^{(1)}_{\hat{0}0} = aE^{(1)}_{\hat{0}0} = -a\Phi^{(1)}\,, && \Lambda^{(1)}_{\hat{0}i} = aE^{(1)}_{\hat{0}i} = av^{(1)}_i\,,\\
&\Lambda^{(1)}_{\hat{a}0} = aE^{(1)}_{\hat{a}0} = -av^{(1)}_{\hat{a}}\,, && \Lambda^{(1)}_{\hat{a}i} = aE^{(1)}_{\hat{a}i} = -a\delta_{\hat{a}i}\Psi^{(1)}\,,\\
&\tfrac{1}{2}\Lambda^{(2)}_{\hat{0}0} = \tfrac{1}{2}aE^{(2)}_{\hat{0}0} = a\left[-\tfrac{1}{2}\Phi^{(2)} + \tfrac{1}{2}\left(\Phi^{(1)}\right)^2 - \tfrac{1}{2}v^{(1)}_k v^{k(1)}\right]\,,\\
&\tfrac{1}{2}\Lambda^{(2)}_{\hat{0}i} = \tfrac{1}{2}aE^{(2)}_{\hat{0}i} = a\left[\tfrac{1}{2}\left(v^{(2)}_i + 2\omega^{(2)}_i\right) - 2\Psi^{(1)}v^{(1)}_i\right]\,,\\
&\tfrac{1}{2}\Lambda^{(2)}_{\hat{a}0} = \tfrac{1}{2}aE^{(2)}_{\hat{a}0} = a\left[-\tfrac{1}{2}v^{(2)}_{\hat{a}} - \Phi^{(1)}v^{(1)}_{\hat{a}} + v^{(1)}_{\hat{a}}\Psi^{(1)}\right]\,,\\
&\tfrac{1}{2}\Lambda^{(2)}_{\hat{a}i} = \tfrac{1}{2}aE^{(2)}_{\hat{a}i} = a\left[-\tfrac{1}{2}\delta_{\hat{a}j}\Psi^{(2)} + \tfrac{1}{4}\hat{h}^{(2)}_{\hat{a}j} + \tfrac{1}{2}v^{(1)}_i v^{(1)}_{\hat{a}} - \tfrac{1}{2}\delta_{\hat{a}j}\left(\Phi^{(1)}\right)^2\right]\,.
\end{aligned}
\tag{B6}
$$

For the energy-momentum exchange four-vector $\mathcal{E}^{\nu}_m$, defined in Eq. (159):

$$
\begin{aligned}
\mathcal{E}^{0(1)}_m &= \frac{1}{a^2}\rho^{(0)}_m\left(\delta^{(1)\prime}_m + \partial_i v^{i(1)} - 3\Psi^{(1)\prime}\right) + \mathcal{E}^{0(0)}_m\left(-2\Phi^{(1)} + \delta^{(1)}_m\right)\,,\\
\mathcal{E}^{i(1)}_m &= \frac{1}{a^2}\rho^{(0)}_m\left[v^{i(1)\prime} + \mathcal{H}v^{i(1)} + \partial^i\Phi^{(1)}\right] + \mathcal{E}^{0(0)}_m v^{i(1)}\,,\\
\frac{1}{2}\mathcal{E}^{0(2)}_m &= \frac{1}{a^2}\rho^{(0)}_m\left[\frac{1}{2}\delta^{(2)\prime}_m + \frac{1}{2}\partial_i v^{i(2)} - \frac{3}{2}\Psi^{(2)\prime} + \frac{1}{4}\hat{h}^{i(2)\prime}_i - \mathcal{H}v^{i(1)}v^{(1)}_i + \left(\Phi^{(1)} + \delta^{(1)}_m\right)\partial_i v^{i(1)}\right.\\
&\quad\left.+v^{i(1)}\partial_i\delta^{(1)}_m - 3\delta^{(1)}_m\Psi^{(1)\prime} - 3v^{j(1)}\partial_j\Psi^{(1)} - 6\Psi^{(1)}\Psi^{(1)\prime}\right] + \mathcal{E}^{0(0)}_m\left(-\Phi^{(2)} + \frac{1}{2}\delta^{(2)}_m - v^{i(1)}v^{(1)}_i\right)\\
&\quad+2v^{i(1)}\mathcal{E}^{i(1)}_m - 2\Phi^{(1)}\mathcal{E}^{0(1)}_m\,,\\
\frac{1}{2}\mathcal{E}^{i(2)}_m &= \frac{1}{a^2}\rho^{(0)}_m\left[\left(\frac{1}{2}v^{i(2)} + \omega^{i(2)}\right)' + \mathcal{H}\left(\frac{1}{2}v^{i(2)} + \omega^{i(2)}\right) + \frac{1}{2}\partial^i\Phi^{(2)} - v^{i(1)}\partial_j v^{j(1)} - 2v^{i(1)}\Psi^{(1)\prime} - \Phi^{(1)}\partial^i\Phi^{(1)}\right.\\
&\quad\left.+2\partial^i\Phi^{(1)}\Psi^{(1)}\right] + \mathcal{E}^{0(0)}_m\left(\frac{1}{2}v^{i(2)} - \delta^{(1)}_m v^{i(1)} + 2\Phi^{(1)}v^{i(1)}\right) + \mathcal{E}^{0(1)}_m v^{i(1)} + \mathcal{E}^{i(1)}_m\left(\delta^{(1)}_m - \Phi^{(1)}\right)\,,
\end{aligned}
\tag{B7}
$$

or

$$
\begin{aligned}
\mathcal{E}^{\|(1)}_m &= \frac{1}{\bar{a}^2}\bar{\rho}_m\left[\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(\Phi^{(1)} - v^{(1)}_\|\right) + \Phi^{(1)\prime} + \partial_\| v^{(1)}_\| + \mathcal{H}v^{(1)}_\|\right] + \frac{\mathcal{H}}{\bar{a}^2}\bar{\rho}_m b_m v^{(1)}_\|\,,\\
\frac{1}{2}\mathcal{E}^{\|(2)}_m &= \frac{1}{\bar{a}^2}\bar{\rho}_m\left[\frac{\mathrm{d}}{\mathrm{d}\bar{\chi}}\left(\frac{1}{2}\Phi^{(2)} - \frac{1}{2}v^{(2)}_\|\right) + \frac{1}{2}\Phi^{(2)\prime} + \omega^{(2)\prime}_\| + \frac{1}{2}\partial_\| v^{(2)}_\| + \mathcal{H}\left(\frac{1}{2}v^{(2)}_\| + \omega^{(2)}_\|\right) - v^{(1)}_\|\partial_\| v^{(1)}_\| - \frac{2}{\bar{\chi}}\left(v^{(1)}_\|\right)^2\right.\\
&\quad\left.-v^{(1)}_\|\partial_{\perp j}v^{j(1)}_\perp - 2v^{(1)}_\|\Psi^{(1)\prime} - \Phi^{(1)}\partial_\|\Phi^{(1)} + 2\partial_\|\Phi^{(1)}\Psi^{(1)}\right] + \frac{\mathcal{H}}{\bar{a}^2}\bar{\rho}_m b_m\left(\frac{1}{2}v^{(2)}_\| - \delta^{(1)}_m v^{(1)}_\| + 2\Phi^{(1)}v^{(1)}_\|\right)\\
&\quad+\mathcal{E}^{0(1)}_m v^{(1)}_\| + \mathcal{E}^{\|(1)}_m\left(\delta^{(1)}_m - \Phi^{(1)}\right)\,.
\end{aligned}
\tag{B8}
$$

For the weak lensing shear and rotation:

$$
\partial_{\perp i}\Delta x^{(1)}_{\perp j} = \mathcal{P}_{ij}v^{(1)}_{\|\,o} + v^{(1)}_{\perp i\,o}n_j + \int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\,\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\partial}_{\perp i}\tilde{\partial}_{\perp j}\left(\Phi^{(1)} + \Psi^{(1)}\right), \tag{B9}
$$

$$
\gamma_{ij} = -\mathcal{P}_{ij}v^{(1)}_{\|\,o} - n_{(j}v^{(1)}_{\perp i)\,o} - \int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[(\bar{\chi}-\tilde{\chi})\,\frac{\tilde{\chi}}{\bar{\chi}}\tilde{\partial}_{\perp(i}\tilde{\partial}_{\perp j)}\left(\Phi^{(1)} + \Psi^{(1)}\right)\right] - \mathcal{P}_{ij}\kappa^{(1)}, \tag{B10}
$$

$$
\begin{aligned}
\vartheta^{(1)}_{ij}\vartheta^{ij(1)} &= +\frac{1}{2}v^{(1)}_{\perp io}v^{i(1)}_{\perp o} + \frac{1}{\bar{\chi}}v^{(1)}_{\perp i\,o}\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[(\bar{\chi}-\tilde{\chi})\,\tilde{\partial}^i_\perp\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\\
&\quad+\frac{1}{2\bar{\chi}^2}\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[(\bar{\chi}-\tilde{\chi})\,\tilde{\partial}_{\perp i}\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\left[(\bar{\chi}-\tilde{\chi})\,\tilde{\partial}^i_\perp\left(\Phi^{(1)} + \Psi^{(1)}\right)\right]\,.
\end{aligned}
\tag{B11}
$$

Assuming that galaxy velocities follow the matter velocity field, we find

$$\begin{aligned}
\Delta_g^{(2)} \ =\ & \delta_g^{(2)} + v_\|^{(2)} - 3\Psi^{(2)} + b_e\,\Delta\ln a^{(2)} + \partial_\|\Delta x_\|^{(2)} + \frac{2}{\bar\chi}\Delta x_\|^{(2)} - 2\kappa^{(2)} + \left(\Delta_g^{(1)}\right)^2 - \left(\delta_g^{(1)}\right)^2 - \left(\Phi^{(1)}\right)^2 - \left(v_\|^{(1)}\right)^2 \\
&+2\Phi^{(1)}v_\|^{(1)} - 7\left(\Psi^{(1)}\right)^2 - \frac{1}{\mathcal{H}^2}\left(\partial_\| v_\|^{(1)}\right)^2 - \frac{1}{\mathcal{H}^2}\left(\Psi^{(1)\prime}\right)^2 - 2\Phi^{(1)}\Psi^{(1)} - 2\left|\gamma^{(1)}\right|^2 - 2\left(\kappa^{(1)}\right)^2 + \vartheta_{ij}^{(1)}\vartheta^{ij(1)} \\
&-\frac{2}{\mathcal{H}}\Phi^{(1)}\Psi^{(1)\prime} - \frac{2}{\mathcal{H}}v_\|^{(1)}\partial_\| v_\|^{(1)} + \frac{2}{\mathcal{H}}v_\|^{(1)}\Psi^{(1)\prime} + \frac{2}{\mathcal{H}}\Psi^{(1)}\partial_\| v_\|^{(1)} - \frac{2}{\mathcal{H}}\Psi^{(1)}\Psi^{(1)\prime} + \frac{2}{\mathcal{H}^2}\Psi^{(1)\prime}\partial_\| v_\|^{(1)} + \frac{2}{\mathcal{H}}\Phi^{(1)}\partial_\| v_\|^{(1)} \\
&+v_{\perp i}^{(1)}v_\perp^{i(1)} + 2\partial_\|\left(+3\Psi^{(1)} - v_\|^{(1)} - \delta_g^{(1)}\right)T^{(1)} - \frac{4}{\bar\chi}\kappa^{(1)}T^{(1)} - \frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2 \\
&+\frac{2}{\mathcal{H}}\left(-\partial_\| v_\|^{(1)} - \mathcal{H}v_\|^{(1)} - \partial_\|\Phi^{(1)} + 3\frac{\mathrm{d}}{\mathrm{d}\bar\chi}\Psi^{(1)} - \frac{\mathrm{d}}{\mathrm{d}\bar\chi}\delta_g^{(1)} - \frac{2}{\bar\chi^2}T^{(1)} - \frac{2}{\bar\chi}\kappa^{(1)}\right)\Delta\ln a^{(1)} \\
&+2\frac{\mathcal{H}'}{\mathcal{H}^2}\left(\Phi^{(1)} - v_\|^{(1)} + \Psi^{(1)} - \frac{1}{\mathcal{H}}\partial_\| v_\|^{(1)} + \frac{1}{\mathcal{H}}\Psi^{(1)\prime}\right)\Delta\ln a^{(1)} + \left(-b_e + \frac{\mathrm{d}\ln b_e}{\mathrm{d}\ln\bar a} - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 - \frac{2}{\bar\chi^2\mathcal{H}^2}\right)\left(\Delta\ln a^{(1)}\right)^2 \\
&-2\partial_{\perp i}\left(v_\|^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right)\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\left(\bar\chi-\tilde\chi\right)\tilde\partial_\perp^i\left(\Phi^{(1)}+\Psi^{(1)}\right)\right] + 2v_\perp^{i(1)}\partial_{\perp i}T^{(1)} \\
&+4S_\perp^{i(1)}\left\{-\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\left(\bar\chi-\tilde\chi\right)\tilde\partial_{\perp i}\left(\Phi^{(1)}+\Psi^{(1)}\right)\right] + \partial_{\perp i}\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right)\right\} + 2v_{\perp i\,o}^{(1)}v_{\perp\,o}^{i(1)} \\
&-2v_{\perp i\,o}^{(1)}\left\{\bar\chi\partial_\perp^i\left(v_\|^{(1)} - 3\Psi^{(1)} + \delta_g^{(1)}\right) + 2S_\perp^{i(1)} + \partial_\perp^i\left(\frac{1}{\mathcal{H}}\Delta\ln a^{(1)} + T^{(1)}\right) - \frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[\left(\bar\chi-\tilde\chi\right)\tilde\partial_\perp^i\left(\Phi^{(1)}+\Psi^{(1)}\right)\right]\right\} \\
&-2\frac{\bar a^2}{\mathcal{H}\bar\rho_m}\left(\mathcal{E}_m^{\|(1)} - \frac{\mathcal{H}}{\bar a^2}\bar\rho_m b_m v_\|^{(1)}\right)\left[\Phi^{(1)} - v_\|^{(1)} + \Psi^{(1)} - \frac{1}{\mathcal{H}}\partial_\| v_\|^{(1)} + \frac{1}{\mathcal{H}}\Psi^{(1)\prime} - \left(1+\frac{\mathcal{H}'}{\mathcal{H}^2}\right)\Delta\ln a^{(1)}\right] \\
&-\left(\frac{\bar a^2}{\mathcal{H}\bar\rho_m}\right)^2\left(\mathcal{E}_m^{\|(1)} - \frac{\mathcal{H}}{\bar a^2}\bar\rho_m b_m v_\|^{(1)}\right)^2 . \qquad \text{(B12)}
\end{aligned}$$

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