Observer-based fixed-time continuous nonsingular terminal sliding mode control of quadrotor aircraft under uncertainties and disturbances for robust trajectory tracking: Theory and experiment

Omar Mechali\textsuperscript{a,} \textsuperscript{b}, Limei Xu\textsuperscript{a,} \textsuperscript{b}, Ya Huang\textsuperscript{c}, Mengji Shi\textsuperscript{a,} \textsuperscript{b}, Xiaomei Xie\textsuperscript{a,} \textsuperscript{b}, *\textsuperscript{*}\\\textsuperscript{a}School of Aeronautics and Astronautics, University of Electronic Science and Technology of China, Chengdu, 611731, China\textsuperscript{b}Aircraft Swarm Intelligent Sensing and Cooperative Control Key Laboratory of Sichuan Province, University of Electronic Science and Technology of China, Chengdu, 611731, China\textsuperscript{c}School of Mechanical and Design Engineering, University of Portsmouth PO1 3DJ Portsmouth, UK

Abstract

This paper solves an accurate fixed-time attitude and position control problems of a quadrotor UAV system. The aircraft system is subject to nonlinearities, parameter uncertainties, unmodeled dynamics, and external time-varying disturbances. To deal with the under-actuation problem of the quadrotor’s dynamics, a hierarchical control structure with an inner-outer loop framework is adopted for the flight control system design. Robust nonlinear control strategies for attitude and position control are innovatively proposed based on a new continuous nonsingular terminal sliding mode control (CNTSMC) scheme. A full-order homogeneous terminal sliding surface is designed for the attitude and position states in such a way that the sliding motion is fixed-time stable independently of the system’s initial condition. Hence, this contributes to enhancing the control system robustness. A disturbance observer-based control (DOB) approach is developed to stabilize the inner rotational subsystem (attitude-loop). This compounded control structure integrates a finite-time observer (FTO) and the CNTSMC scheme. The FTO observer is incorporated into the control framework to cope with the strong perturbations. An output-feedback control approach is adopted for the outer translational subsystem (position-loop) to ensure a velocity-free control. In this context, the CNTSMC scheme is combined with a fixed-time extended state observer (FXESO) to achieve an active disturbance rejection control (ADRC) by estimating and canceling the lumped disturbances. Therefore, within the developed control approach including the robust CNTSMC scheme, DOB, and ADRC strategies, robust and accurate trajectory tracking control can be achieved despite uncertainties and disturbances. Stability analysis of the closed-loop system is rigorously investigated by using the Lyapunov theorem, bi-limit homogeneous theory, and the notion of input-to-state stability (ISS). Extensive experimental tests under the influence of various disturbances are conducted to corroborate the theoretical findings. To this end, an effective model-based design (MBD) framework is established to implement the developed control algorithms in real autopilot hardware. Furthermore, processor-in-the-loop (PIL) experiments are also carried out within the MBD framework. A comparative study is made involving our control algorithms and other control strategies. Overall, the obtained results show that the synthesized control system yields performance improvement regarding fixed-time tracking stability featuring fast transient, strong robustness, and high steady-state precision. Besides, the chattering effect of regular linear sliding mode control (LSMC) is significantly alleviated. Moreover, unlike conventional TSMC, the control input shows no singularity.

Keywords: Quadrotor UAV; Fixed-time stability; Trajectory tracking control; Continuous nonsingular terminal sliding mode control; Disturbance observer-based control; Active disturbance rejection control; Output-feedback control

Nomenclature

<table>
<thead>
<tr>
<th>Nomenclature</th>
<th>Preliminaries part</th>
</tr>
</thead>
<tbody>
<tr>
<td>t, t, t, t</td>
<td>time variables</td>
</tr>
<tr>
<td>f(x), \phi(x)</td>
<td>nonlinear functions (or vector fields)</td>
</tr>
<tr>
<td>x_0</td>
<td>initial state of system ( \dot{x} = f(x) )</td>
</tr>
<tr>
<td>T_x(x_0)</td>
<td>convergence-time function of system ( \dot{x} = f(x) )</td>
</tr>
<tr>
<td>T_{\text{max}}</td>
<td>upper bound of the settling-time ( T_x(x_0) )</td>
</tr>
<tr>
<td>a(x)</td>
<td>nonlinear function</td>
</tr>
<tr>
<td>r, r, r, r</td>
<td>weights in the homogeneous property</td>
</tr>
<tr>
<td>\kappa, \kappa</td>
<td>degree of homogeneity</td>
</tr>
<tr>
<td>b(x)</td>
<td>vector field</td>
</tr>
<tr>
<td>\eta</td>
<td>vector approximating ( b(x) ) in the p-limit</td>
</tr>
<tr>
<td>p = 0 or ( p = \infty )</td>
<td>p-limit</td>
</tr>
<tr>
<td>\alpha, \gamma, \alpha, \mu</td>
<td>positive constants</td>
</tr>
<tr>
<td>\tau_{\text{set}}</td>
<td>settling-time</td>
</tr>
<tr>
<td>T_{\text{set}}(\alpha, \gamma, \alpha, \mu)</td>
<td>upper bound of the settling-time ( T_{\text{set}} )</td>
</tr>
<tr>
<td>\dot{V}(x(t))</td>
<td>positive-definite Lyapunov function</td>
</tr>
<tr>
<td>f(x, 0)</td>
<td>vector field</td>
</tr>
<tr>
<td>f_o(x, 0), f_o(x, 0)</td>
<td>approximating functions for ( f(x, 0) ) in 0-limit and ( \infty )-limit, respectively</td>
</tr>
<tr>
<td>L_\omega, \rho_0, \bar{a}, \bar{b}</td>
<td>positive constants</td>
</tr>
<tr>
<td>d</td>
<td>exogenous disturbances</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nomenclature</th>
<th>Rest of the paper</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_x, A_y</td>
<td>square matrices</td>
</tr>
<tr>
<td>E = (O_x, x_0, y_0, z_0)</td>
<td>earth-fixed frame ‘E-frame’ (inertial frame)</td>
</tr>
<tr>
<td>p = [x, y, z]_E</td>
<td>position of the aircraft in the E-frame</td>
</tr>
<tr>
<td>\Phi = (O_x, x_0, y_0, z_0)</td>
<td>body-fixed frame ‘B-frame’</td>
</tr>
<tr>
<td>\eta = (\Phi, \theta, \psi)_E</td>
<td>orientation, i.e., Euler angles, of the aircraft in the B-frame (roll, pitch, and the heading angles)</td>
</tr>
<tr>
<td>\dot{\theta} = \left[ \tau_x, \tau_y, \tau_z \right]_E^T</td>
<td>velocity of the aircraft in the E-frame</td>
</tr>
<tr>
<td>\zeta = \left[ p, q, r \right]_B</td>
<td>angular velocities of the aircraft in the B-frame</td>
</tr>
<tr>
<td>\dot{R}<em>{\text{e}} = R</em>{\text{e}}</td>
<td>rotation matrix from B-frame to E-frame</td>
</tr>
<tr>
<td>\eta = \left[ \theta, \phi, \psi \right]_B</td>
<td>matrix describing the relationship between ( \zeta^B ) &amp; ( \eta^B )</td>
</tr>
<tr>
<td>m</td>
<td>total mass of the quadrotor</td>
</tr>
<tr>
<td>\tau_d = \left[ d_p, F_d \right] = \left[ d_p, \tau_d \right]</td>
<td>total disturbances acting on the quadrotor</td>
</tr>
<tr>
<td>\tau</td>
<td>thrust control (total lift force)</td>
</tr>
<tr>
<td>\alpha</td>
<td>forces produced by each rotor</td>
</tr>
<tr>
<td>\beta</td>
<td>unmodeled dynamics</td>
</tr>
<tr>
<td>\gamma</td>
<td>external disturbances</td>
</tr>
<tr>
<td>\delta</td>
<td>motors torque, gyroscopic torque, and aerodynamic friction torque, respectively</td>
</tr>
<tr>
<td>\epsilon</td>
<td>lift force, gravity force, and aerodynamic drag force</td>
</tr>
<tr>
<td>\zeta</td>
<td>gravitational acceleration constant</td>
</tr>
<tr>
<td>\eta</td>
<td>identity matrix</td>
</tr>
<tr>
<td>\iota</td>
<td>control input for the attitude subsystem</td>
</tr>
<tr>
<td>\chi</td>
<td>inertia of the rotor</td>
</tr>
<tr>
<td>\lambda</td>
<td>moments of inertia matrix</td>
</tr>
<tr>
<td>\mu</td>
<td>angular speed of the rotors</td>
</tr>
<tr>
<td>\nu</td>
<td>aerodynamic friction coefficients</td>
</tr>
<tr>
<td>\sigma</td>
<td>drag coefficient and thrust coefficient</td>
</tr>
</tbody>
</table>
length of the quadrotor’s arm

\( l \)

upper bound of \(|d_\ell|\) and \(|d_\rho|\), respectively

\( L_{d_\ell}, L_{d_\rho} \)

desired signals, actual states, and initial conditions

\( (\eta_0, P_0), (q, P), (\eta_f, P_f) \)

state vectors

\( x, \dot{x}, X, \dot{X} \)

smooth vector fields

\( g, \dot{g}, F_r, F_i \)

vectors of the controlled outputs

\( g_1 = [\dot{\theta}, \dot{\psi}]^T, \quad g_2 = [\dot{x}, \dot{y}, \dot{z}]^T \)

convergence time of attitude and position tracking errors, respectively

\( \rho^n, \rho^n, \rho^n, \rho^n, \rho^n, \rho^n, \rho^n, \rho^n \)

constants for the FTO observer design

\( \rho^n, \rho^n, \rho^n, \rho^n, \rho^n, \rho^n, \rho^n, \rho^n \)

constants for the FXESO observer design

\( \Delta_0, \Delta_i \)

intermediate variables for the FTO observer design

\( \xi, \eta, \xi, \eta, \xi, \eta, \xi, \eta \)

estimates of \( \zeta, \delta_\rho \) and \( d_\rho \), respectively

\( \xi, \eta, \xi, \eta, \xi, \eta, \xi, \eta \)

estimates of \( P, Y \) and \( d_p \), respectively

\( \xi, \eta, \xi, \eta, \xi, \eta, \xi, \eta \)

observation errors of the FTO observer

\( \xi, \eta, \xi, \eta, \xi, \eta, \xi, \eta \)

observation errors of the FXESO observer

\( \eta, \delta_\rho, \delta_\rho, \delta_\rho \)

nonsingular, symmetric, and positive-definite matrices

\( \psi, \phi \)

attitude tracking error and its first-time derivative

\( \psi, \phi \)

position tracking error and its first-time derivative

\( \Delta_0, S_p \)

attitude and position sliding surfaces

\( \lambda_\phi, \mu_\phi, \lambda_\rho, \mu_\rho \)

positive constants

\( \text{const} \)

positive constants

\( a, b \)

positive exponents

\( N \)

positive constant

\( u, u \)

equivalent control and reaching control

\( \delta_\rho, \delta_\rho \)

estimates of \( \delta_\rho \) and \( d_\rho \), respectively

\( V_0 \)

positive-definite Lyapunov function

\( N, N, N \)

radially unbounded positive-definite Lyapunov functions

\( T_r, T^* \)

reaching time during the sliding motion and its upper bound

\( T_{f_2}, T_{f_2} \)

convergence time of the FTO and FXESO

\( F_0(X_0, d_0) \)

nonlinear function

\( \Delta_0 \)

input of the closed-loop system for the ISS stability

LSMC

linear sliding mode control

TSMC

terminal sliding mode control

CNTSMC

continuous nonsingular terminal sliding mode control

FTO

finite-time observer

FXESO

fixed-time extended state observer

DOB

disturbance observer-based control

ADRC

active disturbance rejection control

MBD

model-based design

ISS

input-to-state stability

GAS

global asymptotic stability

1 Introduction

1.1 Context and motivations

The flight mission of the quadrotor aircraft is one of the most useful and practical aerospace-related applications [1] [2]. The research on quadrotor’s automatic flight control is a major area of interest within the field of control science, robotics, and aeronautical engineering. Notably, the robust trajectory tracking control is a persistent control problem that has become an important topic for a wide range of industrially related academics and researchers in the control community. The autonomous flight of these unmanned aerial vehicles (UAVs) requires an effective and reliable flight control algorithm. Substantially, fast time response, strong robustness, and accuracy appear to be principal determining factors for a flight control system. Therefore, an advanced control strategy is required to achieve high performance for meeting mission requirements for the quadrotor. Position and attitude control are key elements of the flight control system, whose performance has a direct effect on the safety and stability of the flight mission [3]. On this control problem, there are usually two factors in position and attitude dynamics that bring great difficulty and challenge for the flight control design. These factors are the high nonlinearity with a strong coupling of the translational and rotational states as well as the influence of unknown disturbances encountered frequently in practical applications (including both internal system uncertainties and external environmental disturbances) [4] [5].

In this note, a new flight control system based on the CNTSMC control is designed to deal with the robust fixed-time cartesian trajectory tracking control problem for the quadrotor system affected by multiple disturbances. Within the developed FTO-CNTSMC and FXESO-CNTSMC control structures, position and attitude tracking errors can be stabilized to the origin in fixed-time independently of the system’s initial conditions despite the disturbances.

1.2 Literature review

The sliding mode control (SMC) technique is known to be one of the most efficient and robust control methods [6] [7] [8]. Such controllers are insensitive to model errors and system parameter variations [9] [10]. The design procedure of the SMC control systems mainly consists of two steps: the choice of a sliding surface with desirable dynamic characteristics, and the design of the SMC controller. The controller is designed such that the system’s states reach and remain on the sliding surface and consequently converge to the origin. More recently, many works have been devoted to the robust control of the quadrotor system subjected to disturbances using SMC theory. Among these works, a dual-loop integral sliding mode control (DLISMC) based on the linear extended state observer (LESO) is proposed in [11] to deal with the robust trajectory tracking of the quadrotor system. In [12], an integral backstepping sliding mode control (IBSMC) method has been presented to robustly track the desired attitude trajectory. To reduce the chattering effect, the authors of [12] have used a boundary layer method. The technical note in [13] comes up with a regular SMC controller for the attitude system of a quadrotor. In reference [14], a robust backstepping sliding mode controller has been developed considering the incorporation of the disturbances in the quadrotor model. However, all these methods are based on the LSMC. The most serious disadvantage of this control approach is that the switching manifold is linear; hence, only asymptotic convergence of the system’s states to the origin is ensured [15]. In addition, LSMC inevitably suffers from the undesirable chattering phenomena. The chattering impact is reflected by the presence of disrupting high switching frequencies in the control input of the system [16]. Such a control signal will cause low control accuracy, degrade the control performance, and excite unmodeled dynamics which significantly amplify measurement noise. Furthermore, it can even damage the system’s mechanical parts (the brushless motors in the case of quadrotor). To achieve finite-time
convergence for perturbed nonlinear dynamical systems, the TSMC has been introduced in which the sliding surface is designed nonlinear [17] [18]. Although TSMC exhibits faster convergence than LSMC, its control law has a singular point and suffers from the chattering. Thus, to overcome the singularity issue, nonsingular terminal sliding mode control (NTSMC) [19] has been developed. However, the control signal is still not continuous which causes the chattering effect. On the other hand, the so-called continuous-SMC techniques are known for their continuous control signal providing an effective solution to eliminating the chattering problem [20] [21]. They are also classified by the researchers as the fifth generation of SMC [22]. CNTSMMC control is a kind of continuous-SMC techniques that can achieve finite-time convergence and improve precision. Besides, its control law is singularity-free and chattering-free. The work in [23] investigates the robust control of two-link flexible manipulator systems in uncertain conditions by using a CNTSMMC controller. Authors of [24] offer a successful implementation of a CNTSMMC with time-delay estimation for shape memory alloy actuators. In the brief [25], a robust output voltage regulation problem of the DC-DC boost converter system is addressed by a CNTSMMC technique. The article [26] provides a continuous full-order NTSMC for systems under matched and mismatched disturbances. To our best knowledge, few works in the literature [27] [28] are devoted to investigating the design of the CNTSMMC control laws for the quadrotor system. The technical notes [27] present a comprehensive experimental study for a set of continuous-SMC algorithms for a quadrotor robust tracking, including those developed in [20] and [22], but these controllers are finite-time stable and do not show the fixed-time tracking stability property. Similar work is presented in [28]. Although finite-time stable systems provide better performance than asymptotically stable systems, finite-time control design suffers from an inevitable drawback. It has been shown that the convergence time of finite-time controllers grows unboundedly along with the deviation of initial conditions from the equilibrium point. To deal with this issue, fixed-time stabilization has been introduced [29]. Fixed-time stability aims to predefine and adjust a uniformly bounded settling-time. Such a method allows for stabilizing the system’s states in a fixed-time independently of initial conditions. This excellent property is very useful in practical scenarios of the quadrotor since it enhances the system’s robustness. An up-to-date publication [30] develops a continuous-TSMC control for servo motor systems. Therein, the authors propose a novel full-order TSM surface based on the bi-limit homogeneous property, which inspires us in this paper.

In terms of enhancing the robustness of a control system, the DOBC and ADRC control methods are considered as active anti-disturbance control approaches that have drawn much attention. These methods have been considered as a promising solution to cope with the strong disturbances affecting the system. Moreover, they show their effectiveness and superiority over classical passive anti-disturbance control that fails in dealing with strong disturbances leading to the degradation of nominal control performance which may compromise the system’s stability [31] [32]. For instance, the work in [33] provides an FTO observer with a smooth second-order SMC for missile guidance application. The same observer has been used in [34] within the control of a permanent magnet synchronous motor (PMSM) subjected to mismatched disturbances. Similarly, in [35] unknown disturbances affecting a marine vehicle are estimated and rejected. In the context of the ADRC control, the work in [36] comes up with a feedback trajectory tracking control for a marine surface vessel based on a new fixed-time extended state observer (FXESO).

A key observation from the above studies is that owing to the advantages of continuous-SMC techniques, continued efforts are needed in the research of the advanced theory and practice of this new and young fifth-generation of sliding mode techniques. There is, therefore, a definite need for improving and applying these control methods to a wide range of physical dynamical systems such as the quadrotor aircraft which is intended to operate in challenging flight conditions. Furthermore, more real-time implementations on dedicated hardware and embedded systems should be made available to validate the newly designed flight control algorithms. Such experiments are strongly recommended to bridge the gap between the theoretical findings and the practice, which is in the scope of the present note.

1.3 Contribution

The main contributions of this paper can be summarized from both theoretical and practical aspects as follows:

- Two robust nonlinear controllers are designed for the attitude and position subsystems of the quadrotor UAV to deal with the trajectory tracking control problem under various disturbances. These disturbances include internal unmodeled dynamics, plant parameters variation, and external time-varying wind gusts. To reject disturbances and enhance the control system robustness, the DOBC and ADRC approaches are adopted by contrast to passive anti-disturbance control methods which are not robust enough when encountering strong disturbances. Thus, passive methods do not provide satisfactory results in disturbance rejection. An FTO is used within the DOBC while an FXESO is adopted within the ADRC control framework to establish an output-feedback control (velocity-free control). Our control strategy is based on a fixed-time CNTSMMC control scheme combined with observers. A homogeneous full-order TSM surface is designed to develop a tailored CNTSMMC control algorithm for quadrotor system control. The choice of such a sliding surface together with an appropriate reaching law in control design allows that the sliding motion is fixed-time stable independently of the system’s initial conditions. Which contributes to enhancing the system’s robustness against disturbances. Unlike the works [23], [24], [25], [26], [27], and [28], we provide a clear estimation of the settling-time during the reaching phase of the sliding motion. It is shown that the sliding surface admits a fixed settling-time uniform with respect to the initial conditions. Moreover, within the synthesized control framework, the control law is continuous, hence it overcomes the chattering problem of LSMC, TSMC, and NTSMC. In addition, the singularity problem encountered in traditional TSMC is avoided. Thus, the designed control
strategy is effective and more applicable to the quadrotor system in practice. Besides, fixed-time convergence of both sliding surface and tracking error can be ensured to achieve fast convergence by contrast to asymptotic methods [11] [12] [13]. Also, the closed-loop system stability is strictly proved by using the Lyapunov theorem and bi-limit homogeneous theory. Meanwhile, based on the ISS notion, it is shown that the closed-loop system is robust to nonlinear disturbance inputs.

- An experimental comparative analysis is carried out for our designed control algorithms and other control strategies. Extensive PIL experiments and real-time tests are conducted on a real quadrotor platform to illustrate the difference between some quantitative indexes for all the controllers regarding the error signals and the control efforts. The following well-known quantitative performance indexes are used, root-mean-square error (RMSE), integral of square error (ISE), integral of the absolute value of the input (IAU), and integral of the absolute value of the derivative of the input (IADU). This comprehensive study is made by paying more attention to the differences between some sliding mode-based controllers. This aims to highlight the attained improvements regarding chattering-free control, accuracy, and transient response. The most obvious improvement to emerge from our synthesized control strategy is that it has overall superiority than the other controllers. The control performances are enhanced concerning fast transient response, high precision, strong robustness, and disturbance rejection capability. Moreover, in addition to the finite-time SMC strategies presented in [27] and [28], the present work provides one of the few investigations on new fixed-time CNTSMC control algorithms design and real-time implementation for the quadrotor system. Finally, for the control algorithms design, validation, and real-time implementation, an effective and reliable MBD framework is used. This framework allows for time-saving and avoiding coding errors.

1.4 Structure of the manuscript

The remainder of this paper is organized as follows. Section 2 is devoted to the preliminaries and problem statement. The proposed control strategy together with the stability analysis of the closed-loop system are presented in section 3. Experimental results and discussions are illustrated in section 4 to examine the theoretical findings. The paper is concluded in section 5 with possible future research directions.

2 Preliminaries and problem statement

This section will introduce the notations used in the paper and some relevant mathematical definitions and lemmas employed in the control design and fixed-time stability proof. Our main results are exploited on these fundamental facts. Besides, the control problem is formulated in this section.

2.1 Notation

Throughout the paper, the following notations will be used. A sequence of integers \(i = 1, ..., n\) is denoted \(i \in \mathbb{N}\). The symbol \(\times\) denotes the cross product, i.e., vector product, of two three-dimensional (3D) vectors. Denote \(\mathbb{R}\) the set of real numbers. The notation \(\mathbb{R}^n\) stands for the n-dimensional state-space. Let \(\mathbb{R}_+\) denotes the set of positive real numbers, where \(\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}\). The symbol \(|x|\) denotes the absolute value of \(x\) in \(\mathbb{R}\). The function \([x]^\alpha\) is defined by \([x]^\alpha := |x|^\alpha \text{sign}(x) \quad \forall x \in \mathbb{R} \quad \alpha \in \mathbb{R}_+\). Besides, \(\text{sign}(\cdot)\) is the standard signum function. The symbol \(\|\|\) stands for the Euclidean norm on \(\mathbb{R}^n\). The notations \(\dot{x}\) and \(\ddot{x}\) symbolize the first-time and second-time derivatives of \(x\), respectively. The derivative of \([x]^\alpha+1\) with respect to \(x\) is defined by \(d[x]^\alpha+1/\,dx := (\alpha + 1)[x]^\alpha\). The symbol \(\exp(\cdot)\) stands for the exponential function \(\exp(x) : \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow e^x\). For a square matrix \(A \in \mathbb{R}^{n \times n}, A^{-1}\) represents the inverse of the matrix. The symbol \(I\) denotes the identity matrix. For a real symmetric matrix \(B \in \mathbb{R}^{n \times n}, \lambda_{\text{min}}(B)\) and \(\lambda_{\text{max}}(B)\) are its minimum and maximum eigenvalue, respectively. The matrix \(\text{diag}(a_1, a_2, ..., a_n)\) with \(a_n \in \mathbb{R}\) denotes the corresponding diagonal matrix. Denote \(sx := \sin x\) and \(cx := \cos x\).

2.2 Preliminaries

Definition 1. ( [32]). (Finite-time stability). Consider the following autonomous system:

\[
\dot{x} = f(x), \quad x(0) = x_0.
\]

where \(x \in \mathbb{R}^n\), and the nonlinear function \(f : D \rightarrow \mathbb{R}^n\) is continuous on an open neighborhood \(D \subseteq \mathbb{R}^n\) of the origin. The origin \(x = 0\) is a globally finite-time convergent equilibrium of system (1) if it is globally asymptotically stable, and there are an open neighborhood \(U \subseteq D\) of the origin and a function \(T_x : U \setminus \{0\} \rightarrow (0, \infty)\) such that every solution trajectory \(x(t, x_0)\) of system (1) starting from the initial point \(x_0 \in U \setminus \{0\}\) is well-defined for \(t \in [0, T_x(x_0))\), and \(\lim_{t \to T_x(x_0)} x(t, x_0) = 0\). Here \(T_x(x_0)\) is called the convergence-time function, i.e., settling-time function, (with respect to \(x_0\)). The origin is said to be a finite-time stable equilibrium if it is finite-time convergent and Lyapunov stable. If \(U = D = \mathbb{R}^n\), the origin is said to be a globally finite-time stable equilibrium.

Definition 2. ( [29]). (Fixed-time stability). The equilibrium point \(x = 0\) of the system (1) is said to be globally fixed-time stable if it is globally finite-time stable and the settling-time function \(T_x(x_0)\) is bounded, i.e., \(\exists T_{\text{max}} \in \mathbb{R}_+\), such that \(T_x(x_0) \leq T_{\text{max}}, \forall x_0 \in \mathbb{R}^n\), that is, the upper bound \(T_{\text{max}}\) does not depend on the initial conditions \(x_0\) of the system.

Definition 3. ( [37]). (Homogeneity property). A function \(a(x) : \mathbb{R}^n \rightarrow \mathbb{R}\) is homogeneous of degree \(\kappa\) with respect to weights \(r = [r_1, ..., r_n] \in \mathbb{R}^n\), if for any given \(\varepsilon_0 > 0, a(\varepsilon_0 x_1, ..., \varepsilon_0 x_n) = \varepsilon_0^\kappa a(x), \forall x \in \mathbb{R}^n\). A vector field \(a(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is homogeneous of degree \(\kappa\) with weights \(r\), if for all \(1 \leq i \leq n\), the \(i\)th component \(a_i\) is a homogeneous function of degree \(r_i + \kappa\).

Definition 4. ( [37]). (p (p = 0 or p = \infty)-limit homogeneity). A vector field \(b(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is said to be homogeneous in the \(p\)-limit with associated triple \((r_p, \kappa_p, b_p)\), where \(r_p = [r_{p1}, ..., r_{pn}] \in \mathbb{R}_+^n\) is the weight vector, \(\kappa_p\) is the degree and \(b_p\)
is the approximating vector, if \( b_i \) and \( b_{pi} \) are continuous and not identically zero, for each compact set \( C \subseteq \mathbb{R}^n \setminus \{0\} \), there exists \( \bar{\lambda}_o \) such that:
\[
\lim \sup_{x \to 0} \frac{b_i(x^p) - b_{pi}(x)}{\bar{\lambda}_o} = 0, \quad \forall \bar{\lambda}_o \in (0, \bar{\lambda}_o).
\]
where \( b_i \) and \( b_{pi} \) denote the \( i \)th component of \( b \) and \( b_p \), respectively.

**Definition 5.** ([38]). (Bi-limit homogeneity). A function \( \phi(x) : \mathbb{R}^n \to \mathbb{R} \) (or a vector field \( \phi(x) : \mathbb{R}^n \to \mathbb{R}^n \)) is said to be homogeneous in the bi-limit if it is homogeneous in the 0-limit and homogeneous in the \( \infty \)-limit simultaneously.

**Lemma 1.** ([33]). Consider the following system:
\[
\begin{align*}
\dot{s}_0 &= -\rho_0 a_1^{1/3} |s_0|^{1/3} \operatorname{sign}(s_0) + s_1, \\
\dot{s}_i &= -\rho_i \bar{L}_1^{1/2} |s_{i-1} - s_{i-2}|^{1/2} \operatorname{sign}(s_{i-1} - s_{i-2}) + s_{i+1}, \\
\dot{s}_{n-1} &= -\rho_{n-1} \bar{L}_n^{1/2} |s_{n-1} - s_{n-2}|^{1/2} \operatorname{sign}(s_{n-1} - s_{n-2}) + s_n, \\
\dot{s}_n &= -\rho_n \bar{L}_2 \operatorname{sign}(s_n - s_{n-1}) + [-L_d, L_d].
\end{align*}
\]
(2)

If the constants \( L_d, \rho_i \) satisfy \( L_d > 0 \) and \( \rho_i > 0, i = 0, \ldots, n \), then the above system is finite-time stable.

**Lemma 2.** ([39] [40]). Suppose that there is a continuous and positive-definite Lyapunov function \( V(x(t)) : \mathbb{R}^n \to \mathbb{R} \), and its derivative satisfies:
\[
\dot{V}(x(t)) \leq -\lambda V^\mu - \mu V^\nu,
\]
where \( \lambda, \mu, \nu > 0 \), \( \frac{\lambda}{\mu} = \frac{\nu}{\nu} \), and \( \frac{\nu}{\mu} = \frac{1}{\mu} < 1 \) are some positive constants, then the origin of system (1) is finite-time stable. The settling-time function \( T_0 \) is uniform with respect to the initial condition \( x(0) \in \mathbb{R}^n \) and bounded by \( T^* \) as follows:
\[
T_0 \leq T^*(a, \gamma, \lambda, \mu) \equiv \frac{1}{\lambda(a-1)} + \frac{1}{\mu(1-\gamma)}.
\]
(4)

**Lemma 3.** ([38]). Consider the system (1). Suppose that \( f(x) \) is a homogeneous vector field in the bi-limit with associated triples \( (f_0, \kappa_0, f_0(x)) \) and \( (f_\infty, \kappa_\infty, f_\infty(x)) \). If the origins of systems \( \dot{x} = f(x), x_\infty = f_\infty(x) \) and \( x_0 = f_0(x) \) are globally asymptotically stable (GAS) and the condition \( \kappa_0 < 0 \) holds true, then the origin of system (1) is fixed-time stable. The appropriate definitions of 0-limit and \( \infty \) -limit are given in Definition 4.

**Lemma 4.** (Corollary 2.21 [38]). (Input-to-state stability (ISS)). Consider the following system with exogenous disturbances \( d = (d_1, \ldots, d_m) \) in \( \mathbb{R}^m \):
\[
\dot{x} = f(x, d),
\]
(5)
where \( f : \mathbb{R}^n \times \mathbb{R}^m \) is a continuous vector field homogeneous in the bi-limit. If the origins of the systems:
\[
\dot{x} = f(x, 0), \quad \dot{x} = f_0(x, 0), \quad \dot{x} = f_\infty(x, 0).
\]
are globally asymptotically stable equilibria, the system (5) is ISS with respect to disturbances \( d \) as input. Where the vectors \( f_0(x, 0) \) and \( f_\infty(x, 0) \) are considered as approximating functions for \( f(x, 0) \) in 0-limit and \( \infty \)-limit, respectively.

**Lemma 5.** ([36]). Consider the following system:
\[
\begin{align*}
\dot{\sigma}_1 &= \sigma_2 - \rho_1 |\sigma_1|^{a_1} - \rho_1 |\sigma_1|^{b_1}, \\
\dot{\sigma}_2 &= \sigma_3 - \rho_2 |\sigma_2|^{a_2} - \rho_2 |\sigma_2|^{b_2}, \\
&\vdots \\
\dot{\sigma}_{n-1} &= \sigma_n - \rho_{n-1} |\sigma_{n-1}|^{a_{n-1}} - \rho_{n-1} |\sigma_{n-1}|^{b_{n-1}}, \\
\dot{\sigma}_n &= -\rho_n |\sigma_n|^{a_n} - \rho_n |\sigma_n|^{b_n} + [-L_d, L_d],
\end{align*}
\]
(7)
where \( a_i \in (0, 1), \rho_i > 1, i = 1, \ldots, n \). Also, if the nonnegative constants \( \rho_i, \bar{\sigma}_i, i = 1, \ldots, n \) are assigned to ensure the following matrices are Hurwitz, i.e., every eigenvalue of the matrices has a strictly negative real part:
\[
A_1 = \begin{bmatrix} -\rho_1 & 1 & 0 \\
-\rho_2 & 0 & 1 \\
-\rho_3 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\bar{\sigma}_1 & 1 & 0 \\
-\bar{\sigma}_2 & 0 & 1 \\
-\bar{\sigma}_3 & 0 & 0 \end{bmatrix}.
\]
Then, the above system (7) is fixed-time stable.

### 2.3 Problem statement
In the following, the mathematical modeling of the quadrotor system is established and subsequently, the control problem is formulated. For convenience, the following basic assumptions are considered in quadrotor modeling [40] [41]:
- The quadrotor is a rigid body where the mass and the moments of inertia are constant.
- The propellers are rigid.
- The B-frame’s origin and the center of gravity of the vehicle coincide.
- Propellers (1,2) rotate counterclockwise and propellers (3,4) rotate clockwise.
- The lift force and moments are proportional to the square of the rotational speeds.

#### 2.3.1 Disturbed 6-DoF dynamic model of the quadrotor
The quadrotor is considered as a rigid body where its motion in space can be described by the means of two coordinate frames as shown in Fig. 1: an earth-fixed frame E-frame (inertial frame) \( E = (O_E, x_E, y_E, z_E) \) and a body-fixed frame B-frame \( B = (O_B, x_B, y_B, z_B) \) with \( z_B \) being perpendicular to the earth. The position and attitude movements of the vehicle can be achieved via an appropriate combination of the angular speeds of the rotors. The four rotors generate four lift forces denoted by \( f_i, i = 1, \ldots, 4 \). The E-frame is used to define the translational motion, i.e., the position of the center of mass (CM) of the aircraft, by \( \bar{\eta} = [x, y, z]_E^T \in \mathbb{R}^3 \) where \( x, y \) are the coordinates in the horizontal plane, \( z \) is the vertical position. The B-frame is used to define the rotational motion, i.e., Euler angles, \( \eta^B = [\Phi, \Theta, \Psi]_B^T \in \mathbb{R}^3 \) where \( \Phi, \Theta, \Psi \) are respectively, the roll angle (around x-axis), pitch angle (around y-axis), and the yaw angle (around z-axis). These angles characterize the instantaneous orientation of the quadrotor in the B-frame.

The rotation of the vehicle from B-frame to E-frame is described by using Euler angles with respect to the ground by the means of the following rotation matrix \( R_{B \rightarrow E}(\eta) \in \mathbb{R}^{3 \times 3} \):
\[
R_{B \rightarrow E}(\eta) = \begin{bmatrix}
\cos \Theta \cos \Psi & \cos \Theta \sin \Psi & -\sin \Theta \\
\sin \Phi \sin \Theta \cos \Psi - \cos \Phi \sin \Psi & \sin \Phi \sin \Theta \sin \Psi + \cos \Phi \cos \Psi & \sin \Phi \cos \Theta \\
\cos \Phi \sin \Theta \cos \Psi + \sin \Phi \sin \Psi & \cos \Phi \sin \Theta \sin \Psi - \sin \Phi \cos \Psi & \cos \Phi \cos \Theta
\end{bmatrix}.
\]
where $\tau_m := [u_\phi, u_\theta, u_\psi]_B^T \in \mathbb{R}^3$ with $u_\phi, u_\theta, u_\psi$ are the control inputs for the roll, pitch, and yaw orientations, respectively. $J_i \in \mathbb{R}_+$, $\Omega_i \in \mathbb{R}_+$, ($i = \overline{1,4}$) and $c_\phi, c_\theta, c_\psi \in \mathbb{R}_+$ symbolize the inertia of the rotors (motor and propeller), the angular speed of the rotors, and the aerodynamic friction coefficients, respectively. $J := \text{diag}(J_{xx}, J_{yy}, J_{zz}) \in \mathbb{R}^{3 \times 3}$ represents a diagonal symmetric positive definite matrix with $J_{xx}, J_{yy}, J_{zz}$ being the moments of inertia along $O_x, O_y, O_z$ axes, respectively. The terms $F_d, \tau_d \in \mathbb{R}^3$ in (11) denote the total lumped disturbances acting on the quadrotor’s translational and rotational accelerations, respectively. These disturbances include unmodeled dynamics and parameters’ uncertainties $d_{p\text{unc}}, d_{\eta\text{unc}} \in \mathbb{R}^3$ and external disturbances $d_p^\text{ext}, d_{\eta}^\text{ext} \in \mathbb{R}^3$, i.e., wind gusts. The total lumped disturbances are defined by $F_d(t) := d_p(t) = d_{p\text{unc}} + d_p^\text{ext} = [d_x, d_y, d_z]^T \in \mathbb{R}^3$, $\tau_d(t) := d_{\eta\text{unc}} + d_{\eta}^\text{ext} = [d_\phi, d_\theta, d_\psi]^T \in \mathbb{R}^3$. The external wind disturbances $d_p^\text{ext}, d_{\eta}^\text{ext}$ are modeled as sinusoidal signals with different frequencies. We take $d_{\eta}^\text{ext}$ as an example, where its expression is given as follows [27] [47]:

$$d_{\eta}^\text{ext}(t) := d_{\eta,0}^\text{ext} + \sum_{j=1}^{n} a_j \sin(\omega_j + \varphi_j).$$

where $a_j$ is the amplitude of the sinusoid, $d_{\eta,0}^\text{ext}$ is the static wind disturbance, and $\omega_j, \varphi_j$ are the frequency and phase shift, respectively. These wind gusts are considered Lipschitz continuous matched disturbances with bounded derivative [28]. Thus, the unknown disturbances $d_{\eta}(t)$ are sufficiently smooth uncertain function. Where its first-time derivative satisfies $\|\dot{d}_{\eta}\| \leq L_{d_{\eta}}$ for a bounded known Lipschitz constant $0 < L_{d_{\eta}} < \infty$. Similarly, $\|\dot{d}_p\| \leq L_{d_p}$ [48].

On the other hand, since the flight of the quadcopter is driven by four propellers, the angular speeds of the propellers $\Omega_i$, ($i = \overline{1,4}$), will determine the total lift force (thrust control input) $F_p^E$ and the torques $u_{\phi}$, $u_{\theta}$, $u_{\psi}$. The quadrotor’s actuators (rotors) produce a total lift force defined as follows [40] [46]:

$$F_p^E := u_{\phi} \Omega_1^4 + u_{\theta} \Omega_2^4 + u_{\psi} \Omega_3^4,$$

where $c_\phi, c_\theta, c_\psi \in \mathbb{R}_+$ is the thrust coefficient and $f_i \in \mathbb{R}_+$ is the thrust force of the $i$th rotor. The control torques $u_{\phi}$, $u_{\theta}$, $u_{\psi}$ developed by the quadrotor’s actuators are defined as [49]:

$$u_{\phi} := \frac{u_\phi}{\Omega_i^4}, u_{\theta} := \frac{u_\theta}{\Omega_i^4}, u_{\psi} := \frac{u_\psi}{\Omega_i^4},$$

where the parameters $c_d$ and $f_i$ represent the drag coefficient and the distance between the CM and the rotors, i.e., arm length. Besides, the rotating velocities of the four propellers, i.e. $\Omega_i, i = \overline{1,4}$, are related to $u_{\phi}$, $u_{\theta}$, and $u_{\psi}$ (the torques) and $\Omega_i$ (the total lift force) by the means of a constant invertible matrix as follows [40]:

$$u := \begin{bmatrix} u_\phi \\ u_\theta \\ u_\psi \end{bmatrix} = \begin{bmatrix} c_t & c_t & c_t & c_t \\ -lc_i & lc_i & lc_i & -lc_i \\ lc_i & -lc_i & lc_i & -lc_i \end{bmatrix} \begin{bmatrix} \Omega_1^4 \\ \Omega_2^4 \\ \Omega_3^4 \end{bmatrix}.$$

From (9) and (11), it yields:
\[ \hat{\eta}^B = R^{-1}(\eta) \zeta^B, \]
\[ \hat{J}^B = -(\zeta^B)^T \times J \zeta^B + \tau^B + \tau^B + \tau_d. \]  
(18)

It should be noted that for small Euler angles \( \eta^B \) in [rad], i.e., in the limit where the angles approach zero, it results that
\[ \tan \theta \approx \theta, \sin \Phi \approx \Phi, \cos \theta \approx 1 - \frac{\theta^2}{2} \approx 1, \cos \Phi \approx 1 - \frac{\Phi^2}{2} \approx 1 \]  
(40) [50]. The small-angle approximation is frequently used in quadrotor control design like in [5] [44] [46] [51], to mention a few. Moreover, in practice and real flight missions without flips such as inspection, mapping, and coverage, etc., [52] the quadrotor aircraft is intended to fly along a feasible, continuous, differentiable, and sufficiently smooth cartesian trajectory to ensure a safe flight mission. Therefore, abrupt reference changes leading to aggressive maneuvers soliciting large attitude angles can be avoided [53] [54]. Hence, from (18) and based on small-angle approximation, one can get that \( R^{-1}(\eta) \approx I \), where \( I \) is the identity matrix. Thus, \( \hat{\eta}^B \equiv \zeta^B \), that is \( \hat{\eta}^B \equiv \zeta^B \). Consequently, the following expression is obtained [44]:
\[ \dot{\eta}^B = [\dot{\Phi}, \dot{\theta}, \dot{\psi}]^T \equiv [\dot{\theta}, \dot{\phi}, \dot{r}]^T, \]  
(19)

Then, the accelerations’ dynamics for the rotational and translational movements of the quadrotor in the presence of disturbances is given as follows [5] [51]:
\[ \ddot{\phi} = J_{\phi 2}^{-1} \left( J_{\phi 2} J_{\phi 2}^T \theta \dot{\psi} - c \dot{\phi}^2 \theta - J_1 \ddot{\phi} + u_\phi \right), \]
\[ \ddot{\theta} = J_{\theta 2}^{-1} \left( J_{\theta 2} J_{\theta 2}^T \phi \dot{\psi} - c \dot{\theta}^2 \phi - J_1 \ddot{\theta} + u_\theta \right), \]
\[ \ddot{\psi} = J_{\psi 2}^{-1} \left( J_{\psi 2} J_{\psi 2}^T \theta \dot{\phi} - c \dot{\psi}^2 \theta + u_\psi \right), \]  
(20)

and
\[ \ddot{x} = -m^{-1} \left[ (c \Phi s \theta c \psi + s \Phi s \theta) u_x - k_x \dot{x} + d_x^\text{ext} \right], \]
\[ \ddot{y} = -m^{-1} \left[ (c \Phi s \theta s \psi - s \Phi c \psi) u_y - k_y \dot{y} + d_y^\text{ext} \right], \]
\[ \ddot{z} = -m^{-1} \left[ (c \Phi c \theta) u_z - k_z \dot{z} + d_z^\text{ext} \right] + g, \]  
(21)

where \( \ddot{\omega} := \Omega_1 - \Omega_2 + \Omega_3 - \Omega_4 \) is the overall residual angular velocity of the rotor. The following assumption is introduced for the quadrotor’s nonlinear dynamic model given by (20) and (21).

**Assumption 1.** We assume in our study that the model uncertainties, i.e., internal unmodeled dynamics, is the aerodynamical and gyroscopic effect moments denoted by \( \tau^B \) and \( \tau^B \), that is:
\[ d^\text{unc}_{\eta} = \begin{bmatrix} d^\text{unc}_{\phi} \\ d^\text{unc}_{\theta} \\ d^\text{unc}_{\psi} \end{bmatrix} = \begin{bmatrix} -c \Phi \dot{\phi}^2 - J_1 \ddot{\phi} \\ -c \dot{\theta}^2 \phi + J_1 \ddot{\theta} \\ -c \Phi \dot{\psi}^2 \end{bmatrix}. \]  
(22)

Besides, since it is difficult to identify the aerodynamic coefficients \( K_a = \text{diag}(k_x, k_y, k_z) \) in practice, the unmodeled dynamics for the translational subsystem is the drag force \( F^o_a \). These uncertainties are defined as follows:
\[ d^\text{unc}_{\eta} = \begin{bmatrix} d^\text{unc}_{x} \\ d^\text{unc}_{y} \\ d^\text{unc}_{z} \end{bmatrix} = \begin{bmatrix} -k_x v_x/m \\ -k_y v_y/m \\ -k_z v_z/m \end{bmatrix}. \]  
(23)

Thus, the unmodeled dynamics for the quadrotor system will be considered as disturbances for the control law.

**Remark 1.** This assumption will not affect the system stability and control performance, because the internal unmodeled dynamics and uncertainties are considered as a part of total the disturbances \( d^\text{unc}_\eta(t) \) and \( d^\text{unc}_\phi(t) \). Therefore, the model uncertainties \( d^\text{unc}_\eta \), \( d^\text{unc}_\phi \) can be dealt with by the FTO and the FXE0 observers, respectively. Thus, the simplifications of the mathematical model adopted in Assumption 1 are reasonable and can be accepted within the DOBC and ADRC strategies.

Considering Assumption 1, the mathematical model given by (20) and (21) describing the quadrotor’s rotational and translational movements can be rewritten in a state-space form as follows:
\[ \begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= f_1(x_2, t) + g_\eta(t) u_\eta(t) + d_\eta(t), \\
\dot{y}_1 &= x_2, \\
\dot{y}_2 &= f_2(x_2, t) + F_p(x_1, t) + d_p(t), \\
\dot{y}_3 &= x_2,
\end{aligned} \]  
(24)

where \( x := [\Phi, \theta, \psi, \dot{x}, y, \dot{y}, z] \in \mathbb{R}^{12} \) is the state vector. The model given by (24) can be generalized for the attitude and position subsystems in which the rotational and translational dynamics are considered as disturbed nonlinear second-order systems. Therefore, the following two models will be used in the design of the observers and the control algorithms:
\[ \begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= f_1(x_2, t) + g_\eta(t) u_\eta(t) + d_\eta(t), \\
\dot{y}_1(t) &= x_2(t),
\end{aligned} \]  
(25)

and
\[ \begin{aligned}
\dot{x}_1(t) &= x_4(t), \\
\dot{x}_2(t) &= f_2(x_4, t) + F_p(x_1, t) + d_p(t), \\
\dot{y}_2(t) &= x_4(t).
\end{aligned} \]  
(26)
\[
\mathbf{f}_\eta(\chi_2, t) = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} J_{xx}(\mathbf{I}_{yy} - J_{zz})\theta \psi \\ J_{yy}(J_{xx} - J_{zz})\theta \phi \\ J_{zz}(J_{yy} - J_{xx})\theta \phi \end{bmatrix}. \tag{27}
\]

\[
g_p(t) = \begin{bmatrix} g_{\omega} & g_{\theta} & g_{\psi} \end{bmatrix}^T = \begin{bmatrix} f_{xx} & f_{yy} & f_{zz} \end{bmatrix}^T. \tag{28}
\]

Also, \(X_0 = [x, y, z]^T \in \mathbb{R}^{3 \times 1}\) is the vector of states (only the position \(x_3\) is assumed to be measurable), \(X_3 = P = [x, y, z]^T \in \mathbb{R}^{3 \times 1}\), and \(Y_2 = [x, y, z]^T \in \mathbb{R}^{3 \times 1}\) is the vector of the controlled outputs, and the uncertain function \(d_p(t) \in \mathbb{R}^{3 \times 1}\) presents the lumped disturbances. The functions \(f_p(\chi_4, t)\) and \(F_p(\chi_4, t)\) are defined by:

\[
f_p(\chi_4, t) = F_a = -K_a \chi_4. \tag{29}
\]

\[
F_p(\chi_4, t) = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} -u_x m^{-1}(c \phi s \theta \psi + s \phi s \psi) \\ -u_y m^{-1}(c \phi s \theta \psi - s \phi c \psi) \\ -u_z m^{-1}(\phi \psi \theta + g) \end{bmatrix}. \tag{30}
\]

**Definition 6.** (Robust trajectory tracking control problem). The considered control problem of our study consists of designing robust fixed-time TSM control laws \(\mathbf{u}_d(t) = [u_{\phi}, u_{\theta}, u_{\psi}]^T\) and \(\mathbf{u}_a(t)\) for both attitude and position subsystems affected by perturbations in (25) and (26), such that:

- The attitude and position tracking errors tend to the origin in a fixed-time, i.e., for \(\forall e_p(t) = \eta(t) - \eta_d(t), \forall e_p(t) = P(t) - P_d(t), \) there exist two constants \(T_\eta, T_P\), such that:

\[
\begin{align*}
\lim_{t \to T_\eta} e_p(t) &= 0, \\
\lim_{t \to T_P} e_p(t) &= 0,
\end{align*}
\]

(31)

where \(\eta_d(t)\) and \(P_d(t)\) are the desired reference signals for the attitude and position subsystems, respectively.

- The controller must ensure strong robustness against uncertainties and disturbances.

- The control signal is nonsingular, continuous, and chattering-free.

The following part of this note moves on to describe in detail the proposed fixed-time attitude and position control structures to achieve the foregoing objective.

## 3 Main results

The control objective is to realize a robust cartesian trajectory tracking for the quadrotor system in the presence of multiple disturbances. This can be attained through a robust tracking of the position and attitude references.

The quadrotor’s dynamics (24) are nonlinear, underactuated, and strongly coupled. This aircraft system has six output variables \((x, y, z, \phi, \theta, \psi)\) but only four control inputs available \((u_x, u_y, u_z, u\psi)\). Notably, the translational movements are directly achieved by the rotational motions. To deal with this problem, the hierarchical control scheme is adopted as shown in Fig. 2. In the context of the hierarchical control scheme, the flight control of the quadrotor system is divided into two control loops: an inner-loop for the rotational subsystem control and an outer-loop for the translational subsystem control. The inner-loop is based on the robust nonlinear FTO-CNTSMC controller that is designed to ensure the stabilization of the quadrotor’s attitude by controlling the angular variables. The input of this control loop is the reference angles \((\phi_d, \theta_d, \psi_d)\) and the output is the appropriate rolling, pitching, and yawing torques \((u_x, u_y, u_z)\). While a robust FXESO-CNTSMC controller is synthesized for the outer-loop to achieve a robust and accurate position tracking. This loop has as input the desired cartesian trajectory signals \((x_d, y_d, z_d)\) and as output, it generates the reference angles \((\phi_d, \theta_d)\) for the inner-loop and the total thrust force control \(u_z\).

In the following, the synthesis of the cartesian trajectory tracking control system (flight control system) is divided into two steps. First, attitude control is addressed. Second, the position control scheme is developed.

### 3.1 Attitude control design and stability analysis

In this section, the proposed nonlinear control strategy is addressed to track the attitude reference trajectory under model uncertainties, parameters variation, and external wind disturbances. In the following, the synthesis of the attitude control system is divided into two steps. In the first step, the disturbance observer is addressed. In the second step, the continuous-SMC controller is developed. Consequently, the stability analysis will be provided for the observer-controller closed-loop system. Notably, the proof of fixed-time convergence during the sliding phase of the motioning and fixed-time stability of the tracking errors.

#### 3.1.1 Finite-time observer design

In this subsection, the FTO observer is designed to identify and cancel the lumped disturbances affecting the attitude subsystem of the quadrotor. This aims to enhance the robustness and achieve better control performance within the DOBC control strategy.

Since \(\chi_1 = \eta\) and \(\chi_2 = \zeta\), the system (25) can be written as:

\[
\begin{align*}
\dot{\eta} &= \zeta, \\
\dot{\zeta} &= f_\eta + g_\eta \eta + d_\eta. \tag{32}
\end{align*}
\]

**Theorem 1.** Given the disturbed attitude dynamic model described in (32), an FTO observer is designed as follows:

\[
\begin{align*}
\dot{\eta}^0 &= \zeta^0 + f_\eta + g_\eta \eta, \\
\dot{\zeta}^0 &= -\rho_1^0 \zeta^0 + \frac{1}{3} |I_1^\eta - \zeta^0|^{2/3} \text{sign}(I_1^\eta - \zeta^0) + I_1^\eta, \\
\dot{I}_1^\eta &= \dot{\eta}^0, \\
\dot{\zeta}^1 &= -\rho_2^0 \dot{\zeta}^0 - \hat{\alpha}_0 \left( I_1^\eta - \dot{\alpha}_0 \right)^{1/2} \text{sign}(I_1^\eta - \dot{\alpha}_0) + I_2^\eta, \\
\dot{I}_2^\eta &= -\rho_3^0 \dot{\zeta}^0 \text{sign}(I_2^\eta - \dot{\alpha}_1).
\end{align*}
\]

(33)

here \(\rho_i^0, i = 1, 3\) and \(L_{d_\eta}\) are constants, where \(L_{d_\eta} > 0\) and \(\rho_i^0 > 0\). Besides, \(I_1^\eta, I_2^\eta\) and \(I_3^\eta\) are the estimates of \(\zeta,\dot{d}_\eta,\) and \(\ddot{d}_\eta\), respectively. Then, the disturbance \(d_\eta\) can be exactly identified within a finite-time \(T_0\), that is, \(d_\eta \equiv \ddot{d}_\eta\).
Proof. Let us define the observation errors as follows:
\[
\begin{align*}
\dot{e}_1 &= \eta - \dot{\eta}_d, \\
\dot{e}_2 &= \theta - \dot{\theta}_d, \\
\dot{e}_3 &= \psi - \dot{\psi}_d.
\end{align*}
\] (34)

By differentiating these errors with respect to time and substituting \(\dot{\eta}, \dot{\theta}, \dot{\psi}\) by their expressions, the corresponding error dynamics can be written as follows:
\[
\begin{align*}
\dot{e}_1 &= -p_1 L \dot{d}_n [\dot{\eta}^n - \dot{\eta}] + \dot{\eta}_d, \\
\dot{e}_2 &= -p_2 L \dot{d}_n [\dot{\eta}^n - \dot{\eta}_d], \\
\dot{e}_3 &= -p_3 L \dot{d}_n [\dot{\eta}^n - \dot{\eta}_d].
\end{align*}
\] (35)

Based on Lemma 1, the observation errors \(\{\dot{e}_1, \dot{e}_2, \dot{e}_3\}\) can be stabilized to zero in finite-time \(T_0\) by selecting appropriate parameters \(p_1, p_2, p_3, L\), hence \(\dot{\eta}^n \equiv \hat{\eta}, \dot{\eta}_d^2 \equiv \hat{\eta}_d\). □

3.1.2 Continuous nonsingular terminal sliding mode control design

In the following, the FTO-based CNTSMC control scheme will be synthesized in detail.

The tracking error vector for the orientations is defined by:
\[
\begin{align*}
\dot{e}_\eta &= \eta - \eta_d, \\
\dot{e}_\phi &= \phi - \phi_d,
\end{align*}
\] (37)

where \(\eta_d = [\phi_d, \theta_d, \psi_d]^T \in \mathbb{R}^3\) and \(\eta = [\phi, \theta, \psi]^T\) are the desired attitude trajectory and the corresponding orientation, respectively, and \(e_{\eta} = [e_{\eta}, e_{\phi}, e_{\psi}]^T\), \(e_{\eta} = \dot{e}_\eta = [e_{\eta}, e_{\phi}, e_{\psi}]^T\). By differentiating (37), the tracking error dynamics is given by the following expression:
\[
\begin{align*}
\dot{\dot{e}}_\eta &= e_{\dot{\eta}}, \\
\dot{e}_\eta &= \dot{\eta} - \dot{\eta}_d.
\end{align*}
\] (38)

The control objective is that the quadrotor’s orientations \(\eta(t)\) track robustly the desired signals \(\eta_d(t)\), i.e. to render \(e_{\eta}(t) = 0\) satisfying: \(\forall t \geq T_\eta\) in a fixed-time. This by designing a continuous-SMC control law \(u_{\eta} = [u_\phi, u_\theta, u_\psi]^T\).

Thus, to ensure accurate attitude control, the following key sliding mode control is considered in the rotational system (25):
\[
\begin{align*}
\dot{s}_\eta &= \ddot{e}_\eta + [c_{\eta_1} \dot{e}_\eta]^{\alpha_1} + [c_{\eta_2} \dot{e}_\eta]^{\alpha_2} + [c_{\eta_3} \dot{e}_\eta]^{\alpha_3} + k_{\eta_1} |e_{\eta}|^{\beta_1} + k_{\eta_2} |e_{\eta}|^{\beta_2} + k_{\eta_3} |e_{\eta}|^{\beta_3},
\end{align*}
\] (39)

where \(c_{\eta_j} = [c_{\eta_1}, c_{\eta_2}, c_{\eta_3}]^T\), \(k_{\eta_j} = [k_{\eta_1}, k_{\eta_2}, k_{\eta_3}]^T \in \mathbb{R}^3\), \((j = 1, 3)\) are constants to be designed, and \(\alpha_i, \beta_i \in \mathbb{R}_+\) are positive exponents.

The input control needed for the establishment of the reaching phase of the sliding surface \(s_\eta\) and sliding motion on \(s_\eta = 0\) has the following form:
\[
u_{\eta} := g_{\eta}^{-1} \left(u_{eq\eta} + u_n\right).
\] (40)

This control structure consists of two parts. The term \(u_{eq\eta}\) is the equivalent control part which maintains the variables on the sliding surface, and \(u_{n\eta}\) is the reaching control part, which exploits to ensure faster convergence; hence, leading to improve the robustness. On the one hand, the \(u_{eq\eta}\) control can be obtained from the sliding motion \(s_\eta = 0\). Thus, when the closed-loop system approaches the sliding surface \(s_\eta = 0\), the sliding surface dynamics has the following form:
\[
\begin{align*}
\ddot{e}_\eta + [c_{\eta_1} \dot{e}_\eta]^{\alpha_1} + [c_{\eta_2} \dot{e}_\eta]^{\alpha_2} + [c_{\eta_3} \dot{e}_\eta]^{\alpha_3} + k_{\eta_1} |e_{\eta}|^{\beta_1} + k_{\eta_2} |e_{\eta}|^{\beta_2} + k_{\eta_3} |e_{\eta}|^{\beta_3} = 0,
\end{align*}
\] (41)

where \(\ddot{e}_\eta := \ddot{\eta} - \ddot{\eta}_d\). By substituting \(\ddot{e}_\eta\) by its expression from (25) into (41), and after some manipulations, the equivalent control can be obtained as follows:
\[
u_{eq\eta} := g_{\eta}^{-1} \left(-f_{\eta} - c_{\eta_1} [\dot{e}_\eta]^{\alpha_1} - c_{\eta_2} [\dot{e}_\eta]^{\alpha_2} - k_{\eta_1} |e_{\eta}|^{\beta_1} - k_{\eta_2} |e_{\eta}|^{\beta_2} - k_{\eta_3} |e_{\eta}|^{\beta_3} - \ddot{\eta}_d + \ddot{\eta}_d\right).
\] (42)
here $\hat{d}_\eta$ is estimated by the FTO observer given by (33) to cancel the lumped disturbances $d_\eta$. On the other hand, the $u_{n\eta}$ control is chosen to ensure the property of fixed-time reaching of the sliding surface, it is proposed as follows:

$$\dot{u}_{n\eta} = - \lambda_\eta [s_\eta]^\xi - \mu_\eta [s_\eta]^\xi,$$

where $\lambda_\eta, \mu_\eta, \xi, \epsilon \in \mathbb{R}^+$ are positive constants.

**Remark 2.** The control law (42) has the form of a standard fixed-time stabilization structure that generally consists of two classes of feedback terms: one with exponents greater than 1

$$[\dot{\hat{\varepsilon}}_\eta]^\alpha, [\varepsilon_\eta]^\beta$$

and the other with fractional exponents $[\dot{\hat{\varepsilon}}_\eta]^\gamma, [\varepsilon_\eta]^\delta$. On the one hand, the exponents $a_i, b_i, \gamma_i (i = 1,3)$ are chosen according to the bi-limit homogeneity reasoning to ensure fixed-time stability. They are chosen as follows [30]: $a_1 = \nu, a_2 = 1, a_3 = 1 + \frac{1 - \nu}{3 - 2\nu}$. $b_1 = \frac{\nu}{2 - \nu}, b_2 = 1, b_3 = 2 - \frac{\nu}{2 - \nu}$ where $0 < \nu < 1$. Also, they should satisfy $\alpha_i < \alpha_i = 1 < \alpha_3$, and $\beta_i < \beta_2 = 1 < \beta_3$. Similarly to the fixed-time stabilizing feedback controller designed in [56], the high-order terms $[\dot{\hat{\varepsilon}}_\eta]^\alpha, [\varepsilon_\eta]^\beta$ are responsible for driving the system states into a compact set in finite-time uniform with respect to the initial condition, whereas the low-order terms $[\dot{\hat{\varepsilon}}_\eta]^\gamma, [\varepsilon_\eta]^\delta$ are used to ensure the finite-time stabilization of the system (38) with its initial state being in any compact set. On the other hand, the positive constants $c_{\eta_i}, k_{\eta_i} > 0, (i = 1,3)$ should be selected to make the n-order polynomials $s^n + k_{\eta_1}s^{n-1} + \cdots + k_{\eta_3} + k_1$ be Hurwitz in terms of the Laplace operator $s$. Besides, the positive constants $\lambda_\eta, \mu_\eta > 0, \xi > 1$ and $\epsilon < 1$ are chosen to preadjust a settling-time for the reaching phase of the sliding mode as it is shown in Lemma 2. In addition, the positive parameters of the observer $\rho_{\eta_i}^n > 0, (i = 1,3)$ should be selected to satisfy the following condition $\rho_{\eta_2}^n > \rho_{\eta_1}^n > \rho_\eta^1$. The parameter $L_{d\eta}$ is the upper bound of the total disturbances. It is a bounded constant $0 < L_{d\eta} < \infty$.

Finally, the control inputs of the attitude subsystem are given as follows:

$$u_{\eta} = g_{\eta}^{-1} \left(- f_\eta - c_{\eta}[\dot{\varepsilon}_\eta]^\alpha - c_{\eta}[\varepsilon_\eta]^\beta - c_{\eta}[\hat{\varepsilon}_\eta]^{\gamma} - k_{\eta}[\varepsilon_\eta]^\delta - k_{\eta}[\dot{\varepsilon}_\eta]^\alpha - k_{\eta}[\varepsilon_\eta]^\beta - k_{\eta}[\hat{\varepsilon}_\eta]^{\gamma} - d_\eta + \ddot{\eta}_d \right),$$

where $\eta = \{ \Phi, \Theta, \Psi \}$.

**Remark 3.** The tracking differentiator (TD) designed in [57] is employed to supply smooth and bounded attitude reference signals and their derivatives for the attitude controller, that is, $(\Phi_d, \Theta_d), (\Phi_d, \Theta_d)$, and $(\Phi_d, \Theta_d)$.

### 3.1.3 Stability analysis of the closed-loop control system

Now, the main results of this paper are given in the following two theorems. Theorem 2 investigates the fixed-time reaching of the proposed sliding surface and the convergence of the tracking error to the origin within a fixed-time. In Theorem 3, the input-to-state-stability (ISS) of the closed-loop control system with the FTO-CNTSMC controller dynamics under the influence of disturbances inputs is proved.

**Theorem 2.** For the nonlinear perturbed attitude system (25) and the sliding surface $s_\eta$ given by (39), if the control law $u_{\eta}$ is designed by (44), where the control parameters are chosen according to Remark 2 and employing observer (33), then the attitude system is fixed-time stable, i.e., $\eta_\eta(t) \equiv 0, \forall t > T_\eta$.

**Proof.** To prove this theorem, we should go through the following two consecutive steps. First, we prove that the system’s trajectories reach the sliding surface $s_\eta = 0$ in fixed-time, i.e., bounded reaching time independent of the system’s initial conditions. Second, we prove that once the sliding manifold is reached, the tracking error will converge to the origin $\eta_\eta \to 0$ along with the sliding manifold within a fixed-time.

**Step 1.** By substituting $\ddot{\eta}$ from (25) into the sliding surface $s_\eta$ in (39), one gets:

$$s_\eta = f_\eta + g_\eta u_{\eta} + d_\eta - \ddot{\eta}_d + c_{\eta}[\dot{\varepsilon}_\eta]^\alpha + c_{\eta}[\varepsilon_\eta]^\beta + k_{\eta}[\hat{\varepsilon}_\eta]^{\gamma} + k_{\eta}[\dot{\varepsilon}_\eta]^\alpha + k_{\eta}[\varepsilon_\eta]^\beta + k_{\eta}[\hat{\varepsilon}_\eta]^{\gamma},$$

Then, by substituting the control law $u_{\eta}$ designed by (44) into $s_\eta$ (45), the expression of the sliding law $s_\eta$ becomes as:

$$s_\eta = u_{\eta} + d_\eta - \ddot{\eta}_d.$$  

Moreover, according to Theorem 1 the disturbances $d_\eta$ can be estimated by the FTO observer, hence $d_\eta \equiv \hat{d}_\eta$, and thereby:

$$s_\eta = u_{\eta}, \text{ for } t \geq T_0.$$  

By differentiating (47) and substituting (43), one gets:

$$\ddot{\eta}_d = u_{\eta} - \ddot{\eta}_d.$$  

Subsequently, the following positive-definite Lyapunov function is chosen:

$$V_\eta(s_\eta) = \frac{s_\eta^2}{2}.$$  

By differentiating $V_\eta(s_\eta)$ and substituting (48), it yields:

$$\dot{V}_\eta(s_\eta) = s_\eta \ddot{\eta}_d = s_\eta (- \lambda_\eta s_\eta)^\xi - \mu_\eta s_\eta s_\eta^\xi,$$

$$\leq - \lambda_\eta s_\eta s_\eta^\xi - \mu_\eta s_\eta s_\eta^\xi,$$

By noting that $s_\eta = [s_\eta], [s_\eta] [s_\eta]^\xi = |s_\eta| [s_\eta]^\xi$, we have:

$$\dot{V}_\eta(s_\eta) \leq - \lambda_\eta [s_\eta]^{\xi+1} - \mu_\eta [s_\eta]^{\xi+1},$$

$$\leq - \lambda_\eta [\eta_\eta]^{(\xi+1)/2} - \mu_\eta [\eta_\eta]^{(\xi+1)/2}.$$  

By considering that $\xi > 1, \epsilon \in (0,1)$, it yields $(\xi + 1)/2 > 1$ and $(\epsilon + 1)/2 < 1$. According to Lemma 2, the fixed-time reaching of sliding surface $s_\eta = 0$ is guaranteed within the
following bounded reaching-time \( T_r \leq T^* (\xi, \varepsilon, \lambda_\eta, \mu_\eta) := \frac{1}{\lambda_\eta(1-(\xi+1)/2)} + \frac{1}{\mu_\eta(1-(\xi+1)/2)} \) that is independent of the initial conditions.

**Step 2.** The proof is accomplished in three steps.

(a). Establishment of the closed-loop control system dynamics. Once \( s_\eta = 0 \), from (39) it yields:

\[
\dot{\eta} = \eta_d - c_\eta \left[ \eta_d^2 - c_\eta \left[ \eta_d^2 - 2 \Omega_c \eta + \Omega_3 \eta^3 \right] \right] - k_\eta \left[ \eta_1 \eta_{\beta_1}^2 - k_\eta \eta_{\beta_1} \eta_{\beta_2} - k_\eta \eta_1 \eta_{\beta_3} \right].
\]

Recalling the errors’ dynamics for the angular signals from (38):

\[
\left\{ \begin{array}{l}
\dot{\eta}_1 = \eta_1 \\
\dot{\eta}_2 = \eta_2 - \eta_d \\
\end{array} \right.
\]

Substituting (52) into the errors’ dynamics, one gets:

\[
\left\{ \begin{array}{l}
\dot{\eta}_1 = \eta_1 \\
\dot{\eta}_2 = -c_{\eta_1} \left[ \eta_1^2 - c_{\eta_1} \eta_1 \eta_2 \right] - c_{\eta_1} \left[ \eta_2 - \eta_d \right] \\
\end{array} \right.
\]

(b). Definition of approximating functions (systems) in the bi-limit for the system \( \hat{x} := f(x) \) given in (55). We define the following vectors which are considered as approximating functions for the system \( \hat{x} := f(x) \) (55) in 0-limit and \( \infty \)-limit:

\[
f_0(x) = \left[ x_2 - k_{\eta_1} x_{11} \right]^T, \quad f_\infty(x) = \left[ x_2 - k_{\eta_2} x_{12} \right]^T.
\]

As in [30] [37], it can be obtained that the vector \( f_0(x) \) is \( r_0 \)-homogeneous of degree \( k_0 = -1 \), and the vector \( f_\infty(x) \) is \( r_\infty \)-homogeneous of degree \( k_\infty = 1 \), where \( r_0 = \left[ \frac{2-v}{1-v}, \frac{1}{1-v} \right]^T \in \mathbb{R}^2_+ \) and \( r_\infty = \left[ \frac{2-v}{1-v}, \frac{3-2v}{1-v} \right]^T \in \mathbb{R}^2_+ \). As a result, according to Definition 5 the closed-loop system (55) is bi-limit homogeneous with the associated triples \( \{(r_0, k_0, f_0(x)), (r_\infty, k_\infty, f_\infty(x))\} \).

(c). Proof of the global asymptotic stabilty (GAS) of systems \( \hat{x} = f(x) \), \( \hat{x}_0 = f_0(x) \) and \( \hat{x}_\infty = f_\infty(x) \).

(c.1). Proving the GAS of \( \hat{x} = f(x) \). The following radially unbounded positive-definite Lyapunov function is chosen for the system (55):

\[
V = k_{\eta_1} \sum_{i=1}^3 |x_{11}|^{\beta_1 + 1} + k_{\eta_2} \sum_{i=1}^3 |x_{12}|^{\beta_2 + 1} + k_{\eta_3} \sum_{i=1}^3 |x_{13}|^{\beta_3 + 1} + \frac{k_{\eta_1}}{2} \sum_{i=1}^3 |x_{11}|^2 + \frac{k_{\eta_2}}{2} \sum_{i=1}^3 |x_{12}|^2 + \frac{k_{\eta_3}}{2} \sum_{i=1}^3 |x_{13}|^2.
\]

where \( x_{11} \) and \( x_{12} \) denote the \( i \)th components of the vectors \( x_1 \) and \( x_2 \), respectively. By differentiating (58) along (55) and substituting \( \dot{x}_2 \) from (55) into the resulting expression, we get:

\[
\dot{V} = k_{\eta_1} x_{11}^2 |x_{11}|^{\beta_1} + k_{\eta_2} x_{12}^2 |x_{12}|^{\beta_2} + k_{\eta_3} x_{13}^2 |x_{13}|^{\beta_3} + 2 x_{12}^2 x_{13}^2 |x_{12}|^{\beta_2} |x_{13}|^{\beta_3}.
\]

From (59), we can conclude that \( \dot{V} \leq 0 \). Besides, \( \dot{V} = 0 \) implies that \( x_2 = \bar{x}_\eta \). Therefore, by LaSalle’s invariance theorem [37], we have \( x_1 = \bar{x}_\eta = 0 \), \( x_2 = \bar{x}_\eta = 0 \), and the closed-loop system (55) is globally asymptotically stable.

(c.2). Proving the GAS of \( \dot{x}_0 = f_0(x) \) and \( \dot{x}_\infty = f_\infty(x) \). For the system \( \dot{x}_0 = f_0(x) \) with \( f_0(x) \) being defined in (56), the radially unbounded positive-definite Lyapunov function is proposed as follows:

\[
V_0 := \frac{1}{2} \sum_{i=1}^3 |x_{2i}|^2 + k_\eta \sum_{i=1}^3 |x_{11}|^{\beta_1 + 1}.
\]

Taking the time derivative of (60) along with system (56) yields:

\[
\dot{V}_0 = (\beta_1 + 1) x_{21}^2 x_{22} + k_{\eta_1} \sum_{i=1}^3 |x_{11}|^{\beta_1 + 1}.
\]

Similarly, for the system \( \dot{x}_\infty = f_\infty(x) \), the radially unbounded positive-definite Lyapunov function is chosen as:

\[
V_\infty := \frac{1}{2} \sum_{i=1}^3 |x_{2i}|^2 + k_{\eta_2} \sum_{i=1}^3 |x_{11}|^{\beta_2 + 1}.
\]

Taking the derivative of (62) along with system (57), we have:

\[
\dot{V}_\infty = (\beta_3 + 1) x_{21}^2 x_{22} + k_{\eta_3} \sum_{i=1}^3 |x_{11}|^{\beta_3 + 1}.
\]

From (61) and (63), we can conclude that \( V_0 \leq 0 \) and \( V_\infty \leq 0 \). As in step (c.1), by LaSalle’s invariance theorem [37], the approximating systems \( \dot{x}_0 = f_0(x) \) and \( \dot{x}_\infty = f_\infty(x) \) in the bi-limit are guaranteed to be GAS. Finally, based on Lemma 3 the fixed-time stability of the closed-loop system (55) directly follows. Thus, the tracking error can be stabilized to zero \( \bar{e}_\eta = [\bar{e}_\varphi, \bar{e}_\varphi, 0]^T \rightarrow 0 \) along \( s_\eta = 0 \) within fixed-time for all \( t \geq T_\eta \). Hence, the attitude system is fixed-time stable. This completes the proof. \( \Box \)

**Theorem 3.** The control laws \( u_\eta = [u_\varphi, u_\varphi, u_\varphi]^T \) designed by (44) applied to the perturbed system (25), where \( \tilde{d}_\eta \) is estimated by the FTO observer given by (33) can guarantee the ISS stability of attitude closed-loop system despite the influence of disturbances input.

**Proof.** By applying the control input designed in (44) to the system (25), the attitude closed-loop system dynamics becomes:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= F_\eta(x_\eta, d_\eta), \\
\end{align*}
\]

where \( F_\eta(x_\eta, d_\eta) := [F_\varphi, F_\varphi, F_\varphi]^T \in \mathbb{R}^3 \) is defined as follows:

\[
F_\eta := \bar{\eta},
\]

\[
\begin{align*}
&= c_\eta_1 [\eta_1^{\alpha_1} - c_\eta_1 \eta_1^{\alpha_2} - c_\eta_3 \eta_1^{\alpha_3} - k_{\eta_1} \eta_1^{\beta_1} - k_{\eta_2} \eta_1^{\beta_2} - k_{\eta_3} \eta_1^{\beta_3} + u_{n_\eta} + \tilde{d}_\eta - \tilde{d}_\eta. 
\end{align*}
\]
Let us define the new input of the system (64) as follows:
\[
\delta_u := u_{n\eta} + d_q - \dot{d}_q, \tag{66}
\]
where \(u_{n\eta}\) is the control input, \(d_q\) is the disturbances input, and \(\dot{d}_q\) is the estimate of the disturbances.

As in step 2 of Theorem 2’s proof, the following closed-loop system \(\dot{x} := f(x, \delta_u)\) with disturbances input can be obtained:
\[
\begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = -c_{i7}[x_2]^a_1 - c_{i2}[x_2]^a_2 - c_{i3}[x_2]^a_3 \\
- k_{i7}[x_1]^p_1 - k_{i2}[x_1]^p_2 - k_{i3}[x_1]^p_3 + \delta_{n\eta},
\end{cases} \tag{67}
\]
We define the following vectors which are considered as approximating functions for the system \(\dot{x} = f(x, \delta_u)\) in 0-limit and co-limit:
\[
f_0(x, \delta_u) := [x_2, -k_{i7}[x_1]^p_1 - c_{i3}[x_2]^a_3]^T, \tag{68}
\]
\[
f_{\infty}(x, \delta_u) := [x_2, -k_{i7}[x_1]^p_1 - c_{i3}[x_2]^a_3]^T. \tag{69}
\]
When \(\delta_{n\eta} = 0\), we have \(\dot{x} = f(x, 0), x_0 = f_0(x, 0)\) and \(x_{\infty} = f_{\infty}(x, 0)\). These systems are equivalent to the systems (55), (56), and (57), respectively. Furthermore, it has been previously shown (in proof of Theorem 2) that the system \(\dot{x} = f(x, 0)\) is bi-limit homogeneous with associated triples \(\{(f_0, \kappa, f_{\infty}(x)), (\tau_{\infty}, \kappa_{\infty}, f_{\infty}(x))\}\). In addition, we have also proved that the systems (55), (56), and (57) are GAS. Hence, it follows that the systems (67), (68), and (69) are also GAS. According to Lemma 4, the system (67) is ISS with respect to the input \(\delta_{n\eta}\). That is, the origin of the closed-loop system (67) is robust to the nonlinear disturbances, which concludes the proof.

\[\Box\]

### 3.2 Position control design

In this section, the proposed ADRC control strategy is addressed to track the position reference trajectory under disturbances. In the following, the synthesis of the position control system is divided into two steps. The disturbance observer is presented in the first step. Then, the position controller is developed.

#### 3.2.1 Fixed-time extended state observer design for the position loop

In this subsection, the FXESO observer is designed to attenuate the disturbances affecting the position subsystem and at the same time estimates the velocities of the quadrotor.

Given the disturbed translational dynamic model described by (26), the FXESO observer is designed as follows [36]:
\[
\begin{align*}
\dot{\ell}_1^p &= \ell_2^p + \rho_p^1[\ell_1^p]^p_1 + \tilde{\rho}_1^p[\ell_1^p]^p_1, \\
\dot{\ell}_2^p &= \ell_3^p + \rho_p^2[\ell_2^p]^p_2 + \tilde{\rho}_2^p[\ell_2^p]^p_2, \\
\dot{\ell}_3^p &= \rho_p^3[\ell_3^p]^p_3 + \tilde{\rho}_3^p[\ell_3^p]^p_3 + L_{d_p}\text{sgn}(\ell_1^p),
\end{align*} \tag{70}
\]
where the exponents \(a_i, b_i\) are selected as follows: \(a_i \in (0, 1), i = 1, 3, 5\) satisfy the recurrent relations \(a_i = i\bar{a} - (i - 1), i = 2, 3, 5\), and \(a_1 = \bar{a}\) where \(\bar{a} \in (1 - \varepsilon_1, 1)\) for a sufficiently small \(\varepsilon_1 > 0\). Also, \(b_i > 1, i = 1, 3\), and \(b_1 = i\bar{b} - (i - 1), i = 2, 3, 5\), and \(b = \bar{b}\) where \(\bar{b} \in (1, 1 + \varepsilon_2)\) for a sufficiently small \(\varepsilon_2 > 0\). Besides, \(\ell_1^p := P - \dot{P}\) is the observation error and \(\ell_1^p := f_p + F_p\). Also, \(\rho_p^1, \tilde{\rho}_1^p, \rho_p^2, \tilde{\rho}_2^p, \rho_p^3, \tilde{\rho}_3^p, i = 1, 3, 5\) are positive constants. In addition, \(\ell_1^p, \ell_2^p, \ell_3^p\) are the estimates of \(P, Y, d_p\), respectively. Thus, \(\ell_1^p \equiv 0, \ell_2^p \equiv 0, \ell_3^p \equiv 0\) can be achieved in a fixed-time \(T_1\) bounded as in (74) hereafter.

In (70), the terms \(\ell_1^p\) and \(\ell_2^p\) are continuous and differentiable. However, the term \(\text{sgn}(\ell_1^p)\) is discontinuous. Thus, to avoid the chattering effect and make the observer applicable in practice, the following signum function is used to approximate the signum function [58]:
\[
\text{sgn}(\ell_1^p) := \begin{cases}
\left(\frac{2}{1 + \exp(-h \ell_1^p)}) - 1, |\ell_1^p| \leq \delta \\
\text{sign}(\ell_1^p), & |\ell_1^p| > \delta
\end{cases} \tag{71}
\]
where \(h\) is a constant that is inversely proportional to the boundary layer \(\delta\), i.e., \(\delta = 1/h, \delta \in \mathbb{R}_+\).

**Theorem 4.** The FXESO observer designed in (70) can exactly estimate the disturbance and the angular velocities within a fixed-time, that is, \(T_1 = \bar{d}_p \equiv \dot{d}_p\) and \(T_1^2 \equiv \dot{Y} \equiv Y\).

**Proof.** Let us define the observation errors as follows:
\[
\begin{align*}
\ell_1^p &:= P - \dot{P}, \\
\ell_2^p &:= Y - \dot{Y}, \\
\ell_3^p &:= d_p - \ddot{d}_p.
\end{align*} \tag{72}
\]
By differentiating these errors with respect to time, it yields:
\[
\begin{align*}
\dot{\ell}_1^p &= \ell_2^p - \rho_p^1[\ell_1^p]^p_1 - \tilde{\rho}_1^p[\ell_1^p]^p_1, \\
\dot{\ell}_2^p &= \ell_3^p - \rho_p^2[\ell_2^p]^p_2 - \tilde{\rho}_2^p[\ell_2^p]^p_2, \\
\dot{\ell}_3^p &= \dot{d}_p - \rho_p^3[\ell_3^p]^p_3 - \tilde{\rho}_3^p[\ell_3^p]^p_3 - L_{d_p}\text{sgn}(\ell_1^p).
\end{align*} \tag{73}
\]
Based on Lemma 5, the observation errors \((\ell_1^p, \ell_2^p, \ell_3^p)\) are guaranteed to converge to the origin in fixed-time bounded as:
\[
T_1 \leq \frac{2\max(\rho_p)}{\varepsilon_1} + \frac{1}{\varepsilon_2\sigma\varepsilon_1} \tag{74}
\]
where \(\varepsilon_1 = \frac{\lambda_{min}(Q_1)}{\lambda_{max}(P_1)}, \varepsilon_2 = \frac{\lambda_{min}(Q_2)}{\lambda_{max}(P_2)}\), \(\Lambda = 1 - \bar{a}\), \(\sigma = \bar{b} - \ddot{b}\), and \(\varepsilon_1 \leq \frac{\lambda_{min}(P_1)}{Q_1}, Q_2, P_1, P_2\) are nonsingular, symmetric, and positive-definite matrices and satisfied by \(P_1A_1 + A_1^T = Q_1, P_2A_2 + A_2^T = Q_2\). \(\Box\)

#### 3.2.2 Control design for the translational dynamics

The tracking error vector for the position is defined by:
\[
\begin{align*}
e_p &= P - P_d, \\
e_p &= \dot{P} - \dot{P}_d.
\end{align*} \tag{75}
\]
where \(P_d = [x_d, y_d, z_d]^T \in \mathbb{R}_+^3\) and \(P = [x, y, z]^T\) are respectively the desired cartesian trajectory and the actual positions, and \(e_p := e_x, e_y, e_z\) are the desired errors. By differentiating (75), the error dynamics is obtained as:
\[
\begin{align*}
e_p &= e_p, \\
\dot{e}_p &= \ddot{P} - \ddot{P}_d.
\end{align*} \tag{76}
\]
The control objective is that the position’s states \(P(t)\) track robustly the desired signals \(P_d(t)\), i.e., to render \(e_p(t) \equiv 0\)
\[ P_d(t) - P(t) \equiv 0, \forall t \geq T_p \text{ in fixed-time}. \] This by designing continuous-SMC control law \( u_c(t) \).

Taking Assumption 1 into consideration (\( f_p = 0 \)), the desired virtual controls \( F^d_p \in \mathbb{R}^3, P = \{x, y, z\} \) for the position subsystem can be derived in the same manner as for the attitude subsystem. Then, the desired virtual controls are designed as:

\[
F^d_p := -c_p [\dot{e}_p]^{a_1} - c_p [\dot{e}_p]^{a_2} - c_p [\dot{e}_p]^{a_3} - k_p [e_p]^{b_1} - k_p [e_p]^{b_2} - k_p [e_p]^{b_3} - d_p + P_d + u_{np},
\]

where

\[
\dot{u}_{np} := -\lambda_p [s_p] i - \mu_p [s_p] e.
\]

and

\[
s_p := \dot{e}_p + c_p [\dot{e}_p]^{a_1} + c_p [\dot{e}_p]^{a_2} + c_p [\dot{e}_p]^{a_3} + k_p [e_p]^{b_1} + k_p [e_p]^{b_2} + k_p [e_p]^{b_3},
\]

Here \( \lambda_p, \mu_p \in \mathbb{R}^+ \), and \( c_p = [c_{p_1}, c_{p_2}, c_{p_3}]^T, k_p = [k_{p_1}, k_{p_2}, k_{p_3}]^T \in \mathbb{R}^3_j \). The stability analysis for the position subsystem can be derived in the same manner as for the attitude subsystem. Where it can be shown that the position tracking errors can be stabilized to the origin \( e_p = [e_x, e_y, e_z]^T \rightarrow 0 \) along \( s_p = 0 \) within fixed-time \( T_p \).

3.2.3 Thrust force control and desired attitude computation

Now, the thrust force \( u_t \) and the desired attitude angles \( \Phi_d, \theta_d \) are defined in function of the desired force control inputs \( F^d_p = [F^d_x, F^d_y, F^d_z]^T \) of the translational subsystem. The physical meaning of the control vector \( F^d_p \) is that it corresponds to the desired forces that make the quadrotor moves along the \( x, y, \) and \( z \) directions. The thrust force \( u_t \) constitutes the magnitude of these forces, whereas their corresponding orientations are determined by the desired attitude \( (\Phi_d, \theta_d, \psi_d) \), where \( \Phi_d, \theta_d \) and \( \psi_d \) are the desired roll, pitch, and yaw angles.

Recalling the expressions of \( F_p = [F_x, F_y, F_z]^T \) from (30) and combining them with \( F^d_p = [F^d_x, F^d_y, F^d_z]^T \) in (77), it yields:

\[
\begin{align*}
F_x &= -u_x m^{-1}(c \Phi \psi \Theta \Theta \psi s \phi s \psi) + s \Phi \psi \Theta \Theta \psi s \phi s \psi \psi := F^d_{\Theta x}, \\
F_y &= -u_y m^{-1}(c \Phi \psi \Theta \Theta \psi s \phi - s \Phi \psi \Theta \Theta \psi c \phi \psi) := F^d_{\Theta y}, \\
F_z &= -u_x m^{-1}(c \Phi \psi \Theta \Theta \psi c \phi) + g := F^d_{\Theta z}.
\end{align*}
\]

Then, after some algebraic manipulations of the above expressions, the thrust force control \( u_t \) and the desired roll and pitch angles \( (\Phi_d, \theta_d) \) can be defined as follows:

\[
u_t := m \sqrt{(F^d_x)^2 + (F^d_y)^2 + (F^d_z - g)^2},
\]

\[
\begin{align*}
\Phi_d := \arcsin \left( \frac{-m}{u_t} (F^d_x \Theta \Theta \psi_d - F^d_y \phi \psi_d) \right), \\
\theta_d := \arctan \left( \frac{1}{F^d_z - g} (F^d_z \phi \psi_d + F^d_y \Theta \Theta \psi_d) \right).
\end{align*}
\]

The desired heading angle (yaw) \( \psi_d \) is given within the desired reference trajectory. It has been set to 0 (\( \psi_d = 0 \)).

4 Experimental results and discussions

To validate the obtained theoretical results and demonstrate the feasibility of our study, the proposed control approach has been applied to a real quadrotor system. The obtained results provide important insights into two major aspects. First, the chattering has been remarkably alleviated. Second, convergence speed, accuracy, and robustness have been improved.

To highlight the improvement attained with the proposed control strategy, comparative experiments are performed for the attitude control system and cartesian trajectory tracking control. To better illustrate the effectiveness of our control approach, the experiments contain a variety of detailed scenarios. In the first step, a set of experiments including six different controllers is presented for the attitude subsystem. In the second step, PIL experiments are conducted to validate the proposed flight control system for the cartesian trajectory tracking in 3D space. A comparative analysis is performed by applying four different flight control systems. To conduct the above experiments, we have followed a convenient MBD framework (the details are in the following subsection). The MBD approach is well-known and widely used by engineers and scientists in control system design and real-time validation owing to its advantages such as time-saving, reliability, and inherent robustness against coding errors.

In the following the software and hardware setup for the experiments will be explained, then the experimental results and discussions will be presented.

4.1 Experimental setup

To carry out the experiments, an inexpensive quadrotor research experimental platform has been established. A 3-DoF testbed is developed based on the quadrotor to test the attitude control system. The aircraft is mounted on a 3-DoF spherical cardan joint as shown in Fig. 3. This configuration gives the aircraft unrestricted yaw movement and around ±35 [deg] of roll and pitch.

![Fig. 3. The experiment setup.](image)

The experimental platform consists of the following parts: DJI 450 mechanical frame, Sunnysky A2212-13 980KV brushless motors, Hobbywing Xrotor 20A electronic speed
controllers (ESCs), 5300 mAh-30 11.1V351P 58.5 Wh LiPo battery, 9450 DJI propellers, propellers’ protectors, landing gear, M8N GPS module with a compass, power module of the flight controller board, power distribution board, 915 MHz radio telemetry module, Radiolink AT-9S 2.4G remote controller with R9DS receiver, and a commercial-off-the-shelf (COTS) onboard flight controller.

The flight controller is open-source Pixhawk® autopilot [59] includes a main system-on-chip: 32-bit STM32F427, CPU: 180 MHz, ARM® Cortex® M4 with single-precision floating-point unit (FP), RAM: 256 KB SRAM (L1), 2 MB Flash; failsafe system-on-chip: STM32F100, CPU: 24 MHz ARM® Cortex® M3, RAM: 8 KB SRAM. The processor is running NuttX real-time operating system (RTOS). Besides, the autopilot board includes the following sensors: inertial measurement unit (IMU): ST Micro L3GD20H 16-bit gyroscope, ST Micro LSM303D 14-bit accelerometer/magnetometer, InvenSense MPU 6000 3-axis accelerometer/gyroscope; barometer: MEAS MS5611.

To implement the control algorithms, we have created a convenient MBD framework (see Fig. 4). This framework is manifested in the following four steps: quadrotor modeling, controller design, plant and controller simulation, and finally controller deployment to Pixhawk® onboard computer.

The designed and compared controllers are initially tested in MATLAB®/Simulink® (R2019b version) interfaced with FlightGear® flight simulator (2018.3.1 version) (Fig. 5). The latter is an open-source flight simulator that acquires the quadrotor’s position and orientation from Simulink® to visualize a dynamic simulation of the vehicle in 6-DoF within a realistic environment. This simulation is an efficient way to test the flight algorithms safely during the early phase of control design. We have used the trial free version of the AC3D® software to design the 3D graphic for the flying quadrotor in the FlightGear® flight simulator. We have also used the FlightGear® model provided in [60].

Once the control algorithm is verified within the simulation environment, the embedded coder support package for PX4® autopilots is used to generate and deploy the C++ code of the control algorithm to the Pixhawk® onboard computer for the experimental validation on the physical quadrotor platform or for the PIL experiment. The vehicle’s orientation $\eta = [\phi, \theta, \psi]^T$ and angular velocities $\zeta = [\rho, \sigma, \varphi]^T$ are measured by the onboard IMU. We use the external mode of Simulink® to connect to the Pixhawk® autopilot target. This feature allows tuning the control parameters in real-time as the experiment is running. All the signals, including orientation, angular velocities, and control inputs are logged to a 32GB SanDisk Ultra micro-SD card connected to the Pixhawk® flight controller. The logged data will be used in the control performance analysis using a MATLAB® script that we have established for this purpose.

All the numerical simulations and experiments have been implemented using Euler’s method with a fixed integration step equal to 0.004 s. The physical parameters of the quadrotor used in this implementation are summarized as follows: $m = 1.636$ kg, $I_{xx} = 0.0232$ kg m$^2$, $I_{yy} = 0.0249$ kg m$^2$, $I_{zz} = 0.0342$ kg m$^2$, $l = 0.225$ m. We have estimated the moments of inertia $(I_{xx}, I_{yy}, I_{zz})$ by the bifilar pendulum experiment [61] as shown in Fig. 6. The moment of inertia is calculated by the following expression:

$$J_{PP} := mgd_w^2T_{oscil}^2/16\pi^2l_w^2 \quad [\text{kg m}^2].$$

(83)

where $J_{PP} = \{I_{xx}, I_{yy}, I_{zz}\}$, $l_w$ is the length of the two suspending wires (filars) in [m], $d_w$ is the distance between the two wires in [m], and $T_{oscil}$ is the period of one oscillation in [s]. The period for 10 oscillations are measured using a stopwatch for 5 trials with an initial swing angle equal to 10 [deg]. The average value was then used to calculate $T_{oscil}$.

The control parameters of the attitude controller are set to be:

- $k_{\phi_1} = k_{\phi_2} = k_{\phi_3} = k_{\theta_1} = k_{\theta_2} = 10$ ,
- $k_{\theta_3} = k_{\varphi_3} = 0.6$,
- $c_{\theta_1} = c_{\theta_2} = c_{\theta_3} = 10$, $c_{\phi_1} = c_{\phi_2} = c_{\phi_3} = c_{\psi_1} = c_{\psi_2} = 15, v = 0.4$ ,
- $\lambda_4 = 3.5, \mu_4 = 12, \xi = 1.5, \varepsilon = 0.001$. The parameters of the FTO observer are set to be: $\rho_1^n = 20, \rho_2^n = 15, \rho_3^n = 2, l_{d\eta} = 3$. 

![Fig. 4. Block diagram of the implementation process from the simulation environment to the real autopilot hardware. An MBD framework is adopted to implement the control algorithms. Embedded coder support package for PX4® autopilots is used to generate and deploy the appropriate C++ code of the control algorithm to the Pixhawk® onboard computer.](image-url)
4.2 Comparative experiments for attitude control

Four groups of experiments are conducted in a laboratory environment as shown in Fig. 3. The following six different controllers are compared in these experiments: FTO-CNTSMC, DLISMC controller [11], IBSMC controller [12], proportional-integral-derivative (PID) controller, backstepping controller (BSC), and robust adaptive nonsingular fast terminal sliding mode controller (RANFTSMC) [46]. This group of experiments includes origin stabilization, transient response, attitude regulation, and attitude tracking.

To test the robustness and disturbance rejection capabilities of the controllers, the regulation and tracking tasks have been conducted under the influence of sustained external wind disturbances. Besides, considering the simplification in Assumption 1, model uncertainties are also present as disturbances. Also, the uncertainty in the moments of inertia parameters ($J_{xx}, J_{yy}, J_{zz}$) is inevitable in practice due to the estimation errors.

The origin stabilization experiment is considered as an introductive elementary test which is necessarily established as a first step. It aims to assess its capability of the designed control law is stabilizing the attitude to the origin from a random given initial configuration and to show the fixed-time convergence property. Then, transient response is addressed as a second experiment. The objective of this experiment is to evaluate the settling-time and overshoot of the controllers. These two indexes are principal determining factors for the controller speed. Once the controller shows its superiority and satisfactory results are obtained in this test, regulation and tracking tests could then be performed as the last step. This aims to better illustrate the workability and effectiveness of the proposed control system in challenging conditions.

4.2.1 Stabilization experiment

This experiment demonstrates the ability of the proposed control law in stabilizing the attitude variables to the origin $\eta_\theta = [0,0,0]^T [\text{deg}]$ within a fixed settling-time from different initial conditions $\eta_0 = [\phi_0, \theta_0, \psi_0]^T [\text{deg}]$. In practice, such a situation could be a hovering flight, where the quadrotor holds its position to accomplish some missions such as photography, video streaming, communication relay, or coverage.

To validate the theoretical results of the fixed-time convergence of the proposed control law, our designed controller is compared with an asymptotic convergent method. Thus, the controller in [12] has been taken as an example. Two scenarios for different initial values of the angles are considered:

(i) $[\phi_0, \theta_0, \psi_0]^T = [7, -7, -5]^T [\text{deg}]$;
(ii) $[\phi_0, \theta_0, \psi_0]^T = [15, -15, -10]^T [\text{deg}]$.

The experimental attitude responses for the proposed FTO-CNTSMC controller and IBSMC controller are depicted in Fig. 7. The control inputs of the FTO-CNTSMC controller for the first scenario are shown in Fig. 8. It can be observed from Fig. 7 how starting from initial angles far from the origin, both controllers can force all the attitude states ($\Phi, \theta, \psi$) to zero where they maintain steadily in the neighborhood of the origin. However, from Fig. 7(a), it is observed that the origin is reached almost in the same settling-time by the fixed-time convergent FTO-CNTSMC controller even different initial conditions are set for the attitude states. There is a difference of only 0.23 [s] which is reasonable within the adopted simplification of small-angle approximation. By contrast to that, from Fig. 7(b), the settling-time of the asymptotically convergent IBSMC controller grows as the initial conditions become larger. The difference in the convergence time for the IBSMC controller is 2.38 [s].

The real voltages of the three phases of brushless motor 1 are shown in Fig. 9. This graph is recorded in real-time by a Tektronix MDO 3024 oscilloscope.

![Fig. 6. Bifilar pendulum setup.](image1)

![Fig. 7. Origin stabilization of the ($\Phi, \theta, \psi$) states. (a) Convergence of the ($\Phi, \theta$) angles for the FTO-CNTSMC. (b) Convergence of the ($\Phi, \theta$) angles for the IBSMC. (c) Convergence of the ($\psi$) angle for both FTO-CNTSMC and IBSMC controllers.](image2)
of ±5% ((14.25, 15.75) [deg] in this case) of its final value. While the overshoot refers to an output exceeding its final steady-state value. Fig. 10 depicts the transient response of the angles using a large step reference signal of 15 [deg] for each controller. Table 1 compares the experimental results obtained from this step response.

<table>
<thead>
<tr>
<th>Control strategy</th>
<th>Performance index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Setting-time [s]</td>
</tr>
<tr>
<td></td>
<td>Φ</td>
</tr>
<tr>
<td>PID</td>
<td>0.92</td>
</tr>
<tr>
<td>BSC</td>
<td>1.21</td>
</tr>
<tr>
<td>LESO-DLISMC</td>
<td>0.54</td>
</tr>
<tr>
<td>IBSMC [12]</td>
<td>0.77</td>
</tr>
<tr>
<td>RANFTSMC [46]</td>
<td>3.22</td>
</tr>
<tr>
<td>FTO-CNTSMC</td>
<td>4.96</td>
</tr>
</tbody>
</table>

As Table 1 shows, there is a significant improvement achieved by the proposed controller in the transient response regarding the settling-time and overshoot. Surprisingly, RANFTSMC presents a smaller overshoot than the proposed controller for the yaw orientation. However, overall the FTO-CNTSMC controller exhibits the best performance notably for the response speed, fast convergence to the setpoint, and the smallest overshoot for the Φ and θ orientations. Both PID and BSC controllers have a settling-time that exceeds 10 [s].

In the following regulation and tracking experiments, a set of analyses is made to examine the stability of the controllers while encountering environmental disturbances, namely wind gusts. A 140 W electric fan with a diameter of 0.5 m is used to generate the wind from t = 0 s. A wind field is created where the cyan arrow represents wind direction as shown in Fig. 3. The wind is applied over the diagonal of O_BX_B, O_BY_B axes.

The initial conditions for the quadrotor’s orientations and angular velocities are η_0 = [Φ_0, θ_0, ψ_0]^T = [0, 0, 0]^T [deg], ζ_0 = [p_0, q_0, r_0]^T = [0, 0, 0]^T [rad/s].

### 4.2.3 Regulation experiment

To better illustrate the workability of the proposed control system, we have provided a piecewise continuous reference altitude trajectory with abrupt changes of references along with a wide range in the values of the angles. The experimental results are presented from Fig. 11 to Fig. 15. It can be seen from Fig. 11 that all the controllers display a robust convergence of the orientation variables to the desired set-points when abrupt changes of reference and sustained wind gusts are applied to the quadrotor. However, it seems that the PID control of the ψ orientation is very sensitive to the disturbances and exhibits a bad behavior. Fig. 12 displays the orientations’ errors. To provide a more precise quantitative comparison of the attained results, some useful performance indexes have been used. On the one hand, the RMSE and the ISE performance indexes are used in the analysis of the error signals. They are defined as:

\[ RMSE = \sqrt{\frac{1}{T_f-T_i} \int_{T_i}^{T_f} e_\eta(t)^2 dt} \quad ISE = \int_{T_i}^{T_f} e_\eta(t)^2 dt. \] (84)
where \( t_i \), \( t_f \) denote the initial and the final instants respectively. These performance indexes are computed and summarized in Table 2 where the best performances are shown in bold text.

On the other hand, the IAU and the IADU criteria are used for the control signal (depicted in Fig. 13) analysis. They are given as:

\[
\text{IAU} = \int_{t_i}^{t_f} |u_\eta(\tau)| \, d\tau, \quad \text{IADU} = \int_{t_i}^{t_f} \frac{d|u_\eta(\tau)|}{d\tau} \, d\tau. \tag{85}\]

The IADU performance index is very appropriate to check the smoothness of the control signal and thereby indicates chattering alleviation capability for the control input [62]. Taken together, the control signal results are compared in Table 3.

It is apparent from this table that the smoothness is improved by the proposed FTO-CNTSMC control strategy for all control signals compared to the SMC-based controllers, i.e., LESO-DLISMC, RANFTSMC, and IBSMC. However, RANFTSMC also allows for mitigating the chattering effect presents in the IBSMC which is consistent with the comparative study in [46].

As shown in Fig. 14, the FTO observer can precisely identify the sustained external time-varying wind disturbances and the internal uncertainties. Since the wind gusts are applied over the diagonal of \( O_{pX_R}, O_{pY_R} \) axes, the \( \Phi \) and \( \theta \) orientations are most affected by the wind gusts than the \( \psi \) orientation. This difference could be also explained in part by the fact that the yaw dynamics are not influenced by the gyroscopic effect moment.

### Table 2

RMSE and ISE performance indexes analysis for the regulation task.

<table>
<thead>
<tr>
<th>Control strategy</th>
<th>Performance index</th>
<th>RMSE</th>
<th>ISE</th>
</tr>
</thead>
<tbody>
<tr>
<td>PID</td>
<td>( \phi )</td>
<td>1.324</td>
<td>0.978</td>
</tr>
<tr>
<td></td>
<td>( \theta )</td>
<td>0.289</td>
<td>0.153</td>
</tr>
<tr>
<td>BSC</td>
<td>( \phi )</td>
<td>0.692</td>
<td>0.531</td>
</tr>
<tr>
<td></td>
<td>( \theta )</td>
<td>0.097</td>
<td>0.068</td>
</tr>
<tr>
<td>LESO-DLISMC [12]</td>
<td>( \phi )</td>
<td>0.748</td>
<td>0.655</td>
</tr>
<tr>
<td></td>
<td>( \theta )</td>
<td>0.415</td>
<td>0.068</td>
</tr>
<tr>
<td>IBSMC [11]</td>
<td>( \phi )</td>
<td>0.759</td>
<td>0.532</td>
</tr>
<tr>
<td></td>
<td>( \theta )</td>
<td>0.199</td>
<td>0.092</td>
</tr>
<tr>
<td>RANFTSMC [46]</td>
<td>( \phi )</td>
<td>0.780</td>
<td>0.655</td>
</tr>
<tr>
<td></td>
<td>( \theta )</td>
<td>0.415</td>
<td>0.097</td>
</tr>
<tr>
<td>FTO-CNTSMC</td>
<td>( \phi )</td>
<td>0.642</td>
<td>0.365</td>
</tr>
<tr>
<td></td>
<td>( \theta )</td>
<td>0.490</td>
<td>0.066</td>
</tr>
</tbody>
</table>

What stands out in this table is that the FTO-CNTSMC control strategy has the smallest RMSE and ISE performance indexes for the \( \Phi \) and \( \theta \) states. Unfortunately, it does not offer the best performance for the \( \psi \) orientation. However, the difference is still small compared to RANFTSMC which has the best performance for the yaw angle. The PID and BSC controllers are observed to have the worst behavior. It also can be noted that the IBSMC controller offers satisfactory results. The LESO-DLISMC controller provides better performance than the IBSMC controller. This is due to the LESO observer which attenuates the disturbances.
Since the main drawback of the SMC techniques is the chattering effect, in contrast to other comparative works, we introduce the relative percentage difference (RPD) for the IADU index. It is used to indicate the ratio of alleviated chattering by our controller compared to the other three SMC-based control strategies used in this comparative study. An RPD between two control methods for the IADU performance index is defined as follows:

\[
\text{RPD}_{\text{method}} \% \downarrow = \frac{|\text{IADU}_{\text{method}} - \text{IADU}_{\text{proposed}}|}{\text{IADU}_{\text{method}}} \times 100. \tag{86}
\]

As a result, the chattering of conventional SMC in the IBSMC controller is reduced considerably \((u_\Phi \downarrow 82\%, u_\Theta \downarrow 49\%, u_\Psi \downarrow 55\%)\). Compared to the RANFTSMC control strategy, the chattering has been alleviated by \((u_\Phi \downarrow 52\%, u_\Theta \downarrow 40\%, u_\Psi \downarrow 54\%)\) and \((u_\Phi \downarrow 56\%, u_\Theta \downarrow 52\%, u_\Psi \downarrow 82\%)\) compared to LESO-DLISMC. What is interesting about this data is that there is a clear trend of decreasing the chattering effect by the proposed control strategy.

Furthermore, to make the comparative results of the errors and control signals more comprehensive, a representation in bar graph of Table 2 and can be found in Fig. 15.

4.2.4 Tracking experiment

For the experiments depicted in this subsection, the attitude variables are commanded to track a time-varying reference trajectory given by:

\[
\Phi_d = -5\sin(\omega t), \Theta_d = 5\sin(\omega t), \Psi_d = 7.5\cos(\omega t) \tag{87}
\]

where \(\omega = 2\pi f\), where \(f = 1/25\) denotes the frequency in [Hz].

The experimental results are given in Fig. 16-Fig. 19. The actual and desired tracking states are displayed in Fig. 16 where Fig. 17 shows the corresponding tracking errors. From Fig. 16 and Fig. 17, it can be observed that all the SMC-based control strategies, i.e., LESO-DLISMC, IBSMC, RANFTSMC, and FTO-CNTSMC, guarantee a robust tracking of the reference trajectory. In contrast to that, BSC and PID controllers are very influenced by the disturbances, notably in the yaw tracking for the PID controller. Fig. 18 presents the control signals for all controllers. The most interesting aspect of these graphs is that the control inputs of the proposed FTO-CNTSMC control system have no noticeable control chatters. Moreover, the magnitudes of the inputs are within the admissible ranges for all the controllers.
The results obtained from the analysis of tracking errors using the previously mentioned performance indexes are compared in Table 4, where those of the control inputs are summarized in Table 5. The best performances are shown in bold text.

The single most striking observation to emerge from the data comparison of Table 4 is that the proposed controller has the best RMSE and ISE performance indexes, while the PID and BSC have the worst performance. As Table 5 shows, there is a significant difference between the SMC-based controllers regarding the IADU index, where strong evidence of chattering attenuation has been found using the FTO-CNTSMC control approach. This significant improvement is given by the RPD index as follows: \( u_\theta \downarrow 67\%, u_\psi \downarrow 47\%, u_\phi \downarrow 61\% \) compared to IBSMC, \( u_\theta \downarrow 3\%, u_\psi \downarrow 34\%, u_\phi \downarrow 55\% \) compared to RANFTSMC, and \( u_\theta \downarrow 20\%, u_\psi \downarrow 52\%, u_\phi \downarrow 81\% \) compared to LESO-DLISMC.

A representation in a bar graph of Table 4 and Table 5 is depicted in Fig. 19. These results provide important insights at least in two major aspects. First, the chartering has been remarkably alleviated. Second, accuracy and robustness against disturbances have been improved. As a result, the FTO-CNTSMC controller shows the best behavior in all the employed performance indexes related to the attitude tracking errors and the control inputs.

### Table 4

<table>
<thead>
<tr>
<th>Control strategy</th>
<th>Performance index</th>
<th>RMSE</th>
<th>ISE</th>
</tr>
</thead>
<tbody>
<tr>
<td>PID</td>
<td>( \phi )</td>
<td>1.792</td>
<td>0.545</td>
</tr>
<tr>
<td>BSC</td>
<td>( \theta )</td>
<td>2.337</td>
<td>0.385</td>
</tr>
<tr>
<td>LESO-DLISMC</td>
<td>( \psi )</td>
<td>0.341</td>
<td>0.229</td>
</tr>
<tr>
<td>IBSMC [12]</td>
<td>( \phi )</td>
<td>0.669</td>
<td>0.263</td>
</tr>
<tr>
<td>RANFTSMC [46]</td>
<td>( \theta )</td>
<td>0.631</td>
<td>0.196</td>
</tr>
<tr>
<td>FTO-CNTSMC</td>
<td>( \psi )</td>
<td>0.309</td>
<td>0.086</td>
</tr>
</tbody>
</table>

### Table 5

<table>
<thead>
<tr>
<th>Control strategy</th>
<th>Performance index</th>
<th>IAU</th>
<th>IADU</th>
</tr>
</thead>
<tbody>
<tr>
<td>PID</td>
<td>( \phi )</td>
<td>0.817</td>
<td>0.513</td>
</tr>
<tr>
<td>BSC</td>
<td>( \theta )</td>
<td>0.918</td>
<td>0.576</td>
</tr>
<tr>
<td>LESO-DLISMC</td>
<td>( \psi )</td>
<td>1.316</td>
<td>0.740</td>
</tr>
<tr>
<td>IBSMC [12]</td>
<td>( \phi )</td>
<td>2.277</td>
<td>0.845</td>
</tr>
<tr>
<td>RANFTSMC [46]</td>
<td>( \theta )</td>
<td>1.007</td>
<td>0.740</td>
</tr>
<tr>
<td>FTO-CNTSMC</td>
<td>( \psi )</td>
<td>0.911</td>
<td>0.618</td>
</tr>
</tbody>
</table>

### 4.3 Comparative experiments for cartesian trajectory tracking

The validation of the flight control algorithm for the cartesian trajectory tracking in 3D state-space is conducted by a PIL experiment within a co-simulation scheme. In this implementation scheme, the compiled C++ flight control algorithm is running in the Pixhawk® autopilot hardware, whereas the quadrotor’s 6-DoF model is simulated in MATLAB®/Simulink® (see Fig. 4). The quadrotor model used in the PIL experiment is reliable and very accurate. As an example, simulation and experimental data used in the identification process for the quadrotor dynamics are plotted in Fig. 20. We can observe from this comparison the big matching between the simulation model and the real physical system.

Moreover, a comparative analysis is performed by applying four different flight control systems (FCS) (see Table 6).
Controller 3 is the standard FCS used in the Pixhawk® autopilot.

### Table 6

<table>
<thead>
<tr>
<th>Flight control system</th>
<th>Position controller</th>
<th>Attitude controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>FCS 1 (proposed)</td>
<td>FXESO-CNTSMC</td>
<td>FTO-CNTSMC</td>
</tr>
<tr>
<td>FCS 3 (Pixhawk®)</td>
<td>PID</td>
<td>PID</td>
</tr>
<tr>
<td>FCS 4 [12]</td>
<td>IBSMC</td>
<td>IBSMC</td>
</tr>
</tbody>
</table>

To better evaluate the FCSs, two different flight scenarios for the cartesian trajectory tracking are performed. Moreover, to test the robustness and disturbance rejection capabilities of the proposed FCSs, the cartesian trajectory tracking control has been conducted under the influence of parameter variation, internal model uncertainties as well as external time-varying wind disturbances. This approach is more realistic in practice since all these disturbances and uncertainties are present in the dynamic model of quadrotor at the same time. This aims to test the control system in the most complex and challenging flying situation that could be encountered in a real scenario. The total disturbances acting on the quadrotor are given as follows:

#### (i) External wind disturbances:

**(i.a) Flight scenario 1:** Wind disturbances on the aerodynamic forces and moments are applied at $t = 16$ s. These disturbances are given in the form of linear and angular accelerations as follows [46]:

$$d_{\text{ext}}^x(t) = -1.6 \sin(0.1t) + 0.8 \sin(0.44t) + 0.16 \sin(1.57t) + 0.112 \sin(0.28t), \text{m/s}^2$$

$$d_{\text{ext}}^y(t) = 1.5 \sin(0.4t) + 1.5 \cos(0.7t), \text{m/s}^2$$

$$d_{\text{ext}}^z(t) = \cos(0.7t), \text{m/s}^2$$

$$d_{\text{ext}}^W(t) = 0.5 \cos(0.4t), \text{rad/s}^2$$

$$d_{\text{ext}}^W(t) = 0.5 \sin(0.5t), \text{rad/s}^2$$

(88)

**(i.b) Flight scenario 2:** To provide more realistic wind disturbances, the Dryden wind model is considered in this scenario. Its component forces $d_{\text{ext}}^x(t), P \in \{x, y, z\}$ along each axis is expressed as follows [63]:

$$d_{\text{ext}}^x(t) = -k_{d,p} (v_p - v_{wp})^2 \text{sign}(v_p - v_{wp})$$

in which $k_{d,p} \in \mathbb{R}_+$. Also, the terms $v_p$ and $v_{wp}$ are respectively the velocities of the quadrotor and the wind along $x, y$, and $z$ axes. The pics of the wind gusts can reach a speed of around 4 m/s at some given instants, which is more challenging in practice.

**(ii) Considering the simplifications in Assumption 1, model uncertainties are also present as disturbances.**

**(iii) Parameters’ uncertainties of +30% are introduced in the nominal values of the moments of inertia $I_{xx}, I_{yy}, I_{zz}$.

Furthermore, to make the simulation results close to reality and better evaluate the performance of the control algorithm, the measurement noise effects are added to the gyroscope sensor. Also, the measurement errors are added to the GPS and barometer sensors.

The output of the gyroscope sensor is modeled as follows:

$$\zeta(t) = \zeta(t) + \beta_\zeta + N(t) \text{ [rad/s]}.$$  
(90)

where $\zeta = [p, q, r]^T$ [rad/s] is the final signal used by the control law, $\zeta$ is the true measurement in [rad/s], $\beta_\zeta$ is the measurement bias in [rad/s], $N(t)$ is a random noise with the standard deviation $\sigma_p$ in [rad/s] and the mean 0, that is $N = \sigma_p \cdot \text{rand}()$, where rand() is a MATLAB® function that generates a random number in the interval (0,1).

The GPS and barometer signals are obtained as follows:

$$P(t) = \bar{P}(t) + \text{rand}()P_e \text{ [m]}.$$  
(91)

$\bar{P}$ is the true measurement and $P_e$ is the position error in [m].

The parameters of the position controller are chosen as: $k_{p_1} = c_{p_1} = 1.35, k_{p_2} = c_{p_2} = 1.5, k_{p_3} = c_{p_3} = 1.2, \lambda_p = 0.7, \mu_p = 3.2$. Whereas, the FXESO observer gains are selected to be: $\epsilon_1 = 0.26, \bar{a} = 0.7, \epsilon_2 = 0.2, \bar{b} = 1.2, \rho_1^x = \rho_2^x = 5, \rho_1^y = \rho_2^y = 10, \rho_1^z = \rho_2^z = 12, L_\text{d, p} = 5$. The initial conditions of the states of the quadrotor are: $P_0 = [x_0, y_0, z_0]^T = [0, 0, 0]^T$ [m], $v_0 = [v_{x,0}, v_{y,0}, v_{z,0}]^T$ [m/s], $\eta_0 = [\Phi_0, \beta_0, \psi_0]^T = [0, 0, 0]^T$ [deg], $\zeta_0 = [p_0, q_0, r_0]^T = [0, 0, 0]^T$ [rad/s].

#### 4.3.1 Flight scenario 1

In this flight scenario, a practical and realistic example for trajectory tracking is provided. The quadrotor is required to accomplish a flight mission by following a set of waypoints given in the form of $(x_{wp}, y_{wp}, z_{wp})$ triples. The waypoints are listed as follows: (0,0,0), (0,0,1), (10,0,5), (20,20,5), (25,35,5), (26,60,5), (20,80,10), (5.90,15), (-10,90,15), (-17,70,15), (-20,40,15), (-15,10,1), (0,0,1), (0,1,0). In practice, these waypoints can be set manually or geneated by an appropriate path planning algorithm such as rapidly exploring random tree (RRT) [64]. The B-spline polynomial algorithm is used to generate a dynamically feasible and smooth cartesian trajectory $(x_d, y_d, z_d)$ that passes through the given waypoints. This trajectory generator is provided in MATLAB®/Simulink® library as a block named “Polynomial Trajectory”.

The results of the PIL experiments for trajectory tracking are presented from Fig. 21 to Fig. 27. The way in which the aircraft follows the reference flight trajectory for the different FCSs is presented in 3D space in Fig. 21. The first impression from this figure is that the default flight controller of the Pixhawk® autopilot seems to be unable to maintain the quadrotor on the predetermined flight trajectory in the presence of various disturbances. The aircraft leaves the trajectory at a given moment and it never reaches the reference trajectory again.

The actual and desired tracking positions are depicted in Fig. 22 where Fig. 23 shows the position tracking errors. It can be seen from Fig. 22 that all the controllers display a robust convergence of the translational variables to the desired setpoints when abrupt changes of reference and sustained wind gusts are applied to the quadrotor except the FCS 3 which is a PID-based controller. A closer inspection of the tracking errors in Fig. 23 shows that the proposed control method is more accurate regarding the reference trajectory tracking on the $x, y$. 
and z axes comparably to other controllers. It can be observed that the tracking errors converge to zero and maintain steadily in the neighborhood of the origin. We can see that all of the FCSs are affected by the disturbances except our proposed FCS, which exhibits an effective disturbance rejection capability. This due to the adopted control approach including the robust CNTSMC, DOBC, and ADRC control strategies. It seems that the PID controller is sensitive to the disturbances and suffers from a lack of robustness with large tracking errors.

To provide a more precise quantitative comparison of the attained results, the RMSE index is used for the error signal analysis. The RMSE criterion is computed and summarized in Table 7 where the best performances are shown in bold text. A representation in a bar graph of Table 7 can be found in Fig. 24.

We can see from Table 7 and Fig. 24 that the proposed flight controller has an overall superiority than the compared FCSs. Also, both FCS 2 and FCS 4 provide satisfactory results.

The outputs provided by the FXESO observer can be seen in Fig. 25 and Fig. 26, respectively. Fig. 25 displays the estimated total disturbances along the three axes. It can be observed in this figure that the disturbances can be exactly estimated after they have been applied to the quadrotor system at $t = 16$ s. Fig. 26 presents the estimated velocities of the quadrotor. From which we can see that the FXESO observer provides a precise velocity estimate even under disturbances.

![Fig. 21. 3D trajectory tracking: Flight scenario 1.](image1)

![Fig. 22. Profiles of the position of the quadrotor (translational variables $(x, y, z)$): Flight scenario 1.](image2)

![Fig. 23. Evolution of the trajectory tracking errors $(e_x, e_y, e_z)$: Flight scenario 1.](image3)

![Fig. 24. Properties of the RMSE performance index: Flight scenario 1.](image4)

![Fig. 25. Profiles of the estimated total lumped disturbances by the FXESO observer: Flight scenario 1.](image5)

![Fig. 26. Profiles of the estimated velocities $(\dot{v}_x, \dot{v}_y, \dot{v}_z)$ by the FXESO observer: Flight scenario 1.](image6)

Finally, the four control signals of the proposed flight control system are presented in Fig. 27. These control inputs are three torque inputs $(u_\phi, u_\theta, u_\psi)$ and one force input $(u_z)$. 

<table>
<thead>
<tr>
<th>Flight control system</th>
<th>Performance index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x$</td>
</tr>
<tr>
<td>FCS 1 (proposed)</td>
<td>0.0057</td>
</tr>
<tr>
<td>FCS 2</td>
<td>0.0663</td>
</tr>
<tr>
<td>FCS 3 (Pixhawk®)</td>
<td>5.6960</td>
</tr>
<tr>
<td>FCS 4 [12]</td>
<td>0.0823</td>
</tr>
</tbody>
</table>
By contrast to \cite{62} and \cite{63}, no replacement of the sign(·) function is required in the control laws to avoid the chattering problem. The control signal provided by our proposed approach is naturally continuous and smooth. Such replacement of the sign(·) function in practice could inevitably degrade the robustness of the control system leading to a chattering-robustness tradeoff. Moreover, the work in \cite{63} has the shortcoming of not providing simulation results under fast-time varying external wind disturbances. It would be more interesting if \cite{63} considered such disturbances to better show the effectiveness of the controller in dealing with complex flying conditions encountered in practical scenarios.

Compared to the work provided by the authors in \cite{66}, their control signals are noticeably corrupted by chattering and undesirable oscillations which could lead to the degradation of the motors. Furthermore, our proposed control method provides small tracking errors. Also, their study is limited to the regulation tests, and no tracking of time-varying reference trajectory under windy and challenging conditions is conducted.

It can be seen from these figures that the designed FCS succeeds in tracking the desired flight trajectory with high accuracy while external disturbances, parameter variation, and model uncertainty effects are well compensated. Moreover, the proposed position controller ensures faster convergence than the other controllers. This is due to the proposed sliding surface and reaching control.

The estimated total disturbances by the FXESO observer are presented in Fig. 33. Although the disturbances are fast-time-varying as in a real situation, the FXESO observer shows a high ability in identifying them precisely.

The results obtained from the analysis of tracking errors using the RMSE index are compared in Table 8. The best performances are shown in bold text. A representation in the bar graph of Table 8 is depicted in Fig. 34. As in the previous flight scenario, the obtained results confirm the effectiveness of the proposed FCS in dealing with trajectory tracking even under fast-time varying external disturbances and model uncertainties. Moreover, the CNTSMC-based FCS shows the best behavior in the employed performance index related to the position tracking errors.

4.3.2 Flight scenario 2

The quadrotor is required to track the following time-varying vertical helix reference cartesian trajectory:

\[
\begin{align*}
(x_d, y_d, z_d, \psi_d) &= (2 \sin(0.2t), 2 \cos(0.2t), t, 0)^T.
\end{align*}
\]

The results of this flight scenario are presented in Fig. 28.- Fig. 34. The velocities of the simulated Dryden wind are displayed in Fig. 28. It can be observed that the evolution of the wind speeds along the \(x, y, z\) axes seem to be more realistic. Also, the pics of the wind gusts can reach a speed of 8 m/s at some instants.

The 3D trajectory tracking for different flight control systems is presented in Fig. 29 and the evolution of the quadrotor’s trajectory in the \(x – y\) plane is given in Fig. 30. The profiles of the translational variables \((x, y, z)\) and their corresponding tracking errors are depicted in Fig. 31 and Fig. 32, respectively.
In this paper, a solution has been worked out for the robust fixed-time attitude and position control problems of a quadrotor system subject to multiple disturbances. Thus, a new flight control system has been proposed to ensure an accurate cartesian trajectory tracking control in 3D state-space. The 6-DoF equations of motion of the quadrotor are derived based on the Newton-Euler formula. Subsequently, DOBC and ADRC control approaches have been creatively proposed. Basically, a homogeneous full-order TSM surface has been designed to develop a CNTSMC control algorithm. Such choice of the sliding surface together with a pertinent reaching law allows stabilizing the tracking error at the origin in a fixed-time uniformly with respect to the initial condition which leads to improving the system’s robustness. The design of the flight control system follows a hierarchical structure, namely, attitude-loop control and position-loop control. In the inner-loop, a robust fixed-time FTO-CNTSMC control has been constructed to stabilize the attitude states. In addition, a TD has been introduced to supply smooth and bounded attitude reference signals. In the outer-loop, an output-feedback control has been synthesized by combining the CNTSMC scheme and an FXESO observer. Rigorous mathematical proofs are established in the stability analysis of the closed-loop control system by using the Lyapunov theorem, bi-limit homogeneity theory, and ISS notion.

By following a convenient MBD framework, experimental tests conducted on a real quadrotor system besides PIL implementations on Pixhawk® autopilot hardware have corroborated the theoretical findings. Furthermore, a comparative study has been made to evaluate the performance and show the improvements attained by the developed control system. The obtained results confirm the effectiveness and superiority of the proposed control method regarding fast transient response, accuracy, robustness, and chattering alleviation. In addition, the control law is smooth without undesirable singularity or chattering effect. Also, the most distinguishing feature of the proposed flight control system lies in that it achieves fixed-time trajectory tracking in the presence of modeling errors, parameters’ uncertainties, and fast time-varying wind gusts disturbances. On the other hand, it has been confirmed that the RANFTSMC controller allows for chattering mitigation, which is in line with previous studies and thus supports earlier research.

Further work could usefully investigate the control design by considering actuator faults, input saturation, and time-delay analysis. Also, outdoor flight experiments could be conducted.

**5 Conclusion**

**Table 8**

<table>
<thead>
<tr>
<th>Flight control system</th>
<th>Performance index</th>
<th>( \dot{e}_{x_1} )</th>
<th>( \dot{e}_{x_2} )</th>
<th>( \dot{e}_{x_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>FCS 1 (proposed)</td>
<td></td>
<td>0.0076</td>
<td>0.1171</td>
<td>0.0419</td>
</tr>
<tr>
<td>FCS 2 [11]</td>
<td></td>
<td>0.0180</td>
<td>0.1353</td>
<td>0.0596</td>
</tr>
<tr>
<td>FCS 3 (Pixhawk®)</td>
<td></td>
<td>0.1980</td>
<td>0.3969</td>
<td>0.0771</td>
</tr>
<tr>
<td>FCS 4 [12]</td>
<td></td>
<td>0.0650</td>
<td>0.1962</td>
<td>0.0615</td>
</tr>
</tbody>
</table>

**Appendix**

**A.1. Control allocation for a quadrotor with X-configuration**
The following expressions are used to calculate the pulse width modulation (PWM) values generated by the controller implemented in Pixhawk flight controller:\(^1\):

\[
PWM_{m_1} = 10^3 (u_x - u_y + u_z + u_y) + 10^3,
\]

\[
PWM_{m_2} = 10^3 (u_x + u_y - u_z + u_y) + 10^3,
\]

\[
PWM_{m_3} = 10^3 (u_x + u_y + u_z - u_y) + 10^3,
\]

\[
PWM_{m_4} = 10^3 (u_x - u_y - u_z - u_y) + 10^3.
\]

where \(PWM_{m_i}\) corresponds to \(I_i\) (i = \(1,4\)), and \(10^3 \leq PWM_{m_i} \leq 2 \times 10^3\). These expressions are related to a quadrotor with X-configuration as the one shown in Fig. 1.

### A.2. Flight control algorithm

The following table presents the flight control algorithm:

<table>
<thead>
<tr>
<th>Table 9</th>
<th>Algorithm of the proposed flight control system for the quadrotor UAV.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algorithm:</strong> Flight control algorithm</td>
<td><strong>Given</strong></td>
</tr>
<tr>
<td>Define quadrotor’s parameters: (m, l, I_x, I_y, I_z);</td>
<td>Define quadrotor’s dynamics: (F, g, \theta, \phi, \psi);</td>
</tr>
<tr>
<td>Define the parameters of the position and attitude controllers: (k_{p}, c_{I}, \alpha, \beta, \gamma);</td>
<td>Define the parameters of the FTO observer: (p_1, p_2, p_3, \phi_d;)</td>
</tr>
<tr>
<td>Define the parameters of the FXESO observer: (a_0, b_0, c_0, \phi_d;)</td>
<td>Define the upper and lower saturation limits for the discrete integrators;</td>
</tr>
<tr>
<td>Define the upper and lower saturation limits for the saturation blocks;</td>
<td><strong>While (1): Repeat</strong></td>
</tr>
</tbody>
</table>

#### Position control loop

**CNTSMC controller**

- Acquire current position and reference trajectory: \(x, y, z\);
- Errors computation: \(e_x = x - x_d, e_y = y - y_d, e_z = z - z_d\);
- Errors’ derivatives: \(\dot{e}_x = \frac{dx}{dt}, \dot{e}_y = \frac{dy}{dt}, \dot{e}_z = \frac{dz}{dt};\)
- Define the sliding surfaces: \(s_i, s_j, s_k;\)
- Calculate the control inputs: \(\eta_{eq}, \eta_{eq}, \eta_{eq};\)
- Calculate the virtual controls: \(F_d = u_{eq} + u_{eq};\)
- Calculate the thrust force control: \(u_z = m \left( (F_d)^2 + (F_d)^2 + (F_d - g)^2 \right)\)
- Calculate the reference attitude signals: \(\phi_d = 0;\)
- \(\Phi = \arcsin \left( \frac{2}{w} (F_d^2 \psi_d - F_d^2 \psi_d) \right), \theta_d = \arctan \left( \frac{1}{2} F_d^2 \psi_d \right);\)

#### FXESO observer

**Initial conditions:** \(F_d = [0,0,0]^T\), \(i = 1,2,3\); \(\Phi_d, P_d;\)

**Observer’s output:** \(\hat{d}_\phi, \hat{d}_\beta\) where \(d_d = \left[ \dot{d}_x, \dot{d}_y, \dot{d}_z \right]^T\) and \(\hat{u}_d = \left[ \hat{u}_x, \hat{u}_y, \hat{u}_z \right]^T;\)

**End position control loop**

**Attitude control loop**

**CNTSMC controller**

- Acquire the attitude reference signals from the position-loop: \(\Phi_d, \theta_d;\)
- Acquire current angles: \(\Phi, \theta, \psi;\)
- Acquire current angular velocities: \(p, q, r;\)
- Calculate \(\hat{\Phi}_d, \hat{\theta}_d\) and \(\hat{\Phi}_d, \hat{\theta}_d\) using the tracking differentiator (TD); Calculate the second derivative of the current states: \(\ddot{\Phi}, \ddot{\theta}, \ddot{\psi};\)
- Calculate: \(s_p, s_q, s_r, \eta_{eq}, \eta_{eq}, \eta_{eq}, \eta_{eq}, \eta_{eq}, \eta_{eq};\)
- \(u_\Phi = g_\Phi \left( \eta_{eq} + u_{eq} \right);\)

**Saturation block:** \(u_\Phi \in \{-1, -1\}, u_\Phi \in \{1, -1\}, u_\Phi \in \{1, -1\};\)

---

\(^1\) These expressions can be found within the implementation file provided at: https://github.com/mechallomar/pixhawk_attitude_control_implementation/blob/main/pixhawk_attitude_control_implementation_github.rar

### Declaration of competing interest

The authors declare that there is no competing interest to disclose.

### Acknowledgment

The authors would like to express the sincerest gratitude to the Editor-in-Chief, the Associate Editor, and the anonymous reviewers whose insightful comments have helped to improve the quality of this paper considerably.

The authors would like to thank Pr. Jamshed Iqbal, Department of Electrical and Electronic Engineering, University of Jeddah, Saudi Arabia, for his assistance and suggestions.

This work was supported by the National Natural Science Foundation of China under Grant No. 61803075, the Fundamental Research Funds for the Central Universities under Grants No. ZYGX2018KYQD211 and No. ZYGX2019J084

### References


