

Scale-independent R^2 inflation

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Weyl (scale)-invariant theories of scalars and gravity can generate all mass scales spontaneously. In this paper, we study a particularly simple version—scale-invariant R^2 gravity—and show that, during an inflationary period, it leads to fluctuations which, for a particular parameter choice, are almost indistinguishable from normal R^2 inflation. Current observations place tight constraints on the primary coupling constant of this theory and predict a tensor to scalar ratio, $0.0033 > r > 0.0026$, which is testable with the next generation of cosmic microwave background experiments.

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I. INTRODUCTION

Recently, there has been a resurgence of interest in scale (Weyl)-invariant theories as a possible solution to the hierarchy problem—the need to keep the Brout, Englert, Higgs (BEH) boson [1–3] light in the presence of gravity or large mass scales associated with physics beyond the Standard Model, such as grand unification. Such scale-invariant theories have to generate all dimension-full scales through spontaneous breaking of the symmetry, including the Planck scale and the electroweak breaking scale associated with the BEH boson. To generate a period of inflation, it is also necessary for the spontaneous scale symmetry breaking to give rise to the inflation scale and it has been shown that this is possible in the context of a model with two scalar fields nonminimally coupled to the Ricci scalar, R [4–22]. Such models can naturally lead to an acceptable period of inflation and, for the case in which one scalar is identified with the BEH boson, are similar to “Higgs” inflation [23].

In this paper, we are interested in a simpler possibility, generalizing $R + R^2$ inflation to a scale-independent form [24–32]. The Planck constraints on the inflationary observables are in remarkable agreement with the $R + R^2$ model of Starobinsky [33], proposed in 1980 and with a spectrum of CMB perturbations as analyzed by Mukhanov and Chibisov [34] shortly afterward. In particular, the model

predicts a spectrum with scalar perturbation index $n_s \sim 0.96$ and tensor to scalar ratio $r \sim 0.004$.

As an R^2 term is already scale invariant, to build a fully scale-invariant form, it is only necessary to add a single “Jordan Brans Dicke” [35,36] scalar field that, after inflation, is the dominant source of spontaneous breaking of the scale symmetry and is responsible for generating the Planck mass. Since the R^2 term involves fourth order derivatives, it implies the existence of an additional propagating scalar degree of freedom similar to the two scalar model mentioned above. In the locally symmetric case, one scalar is eliminated, providing the longitudinal degree of freedom of a massive “Weyl photon” [37–39], and this has recently been extended to include an R^2 term in [22].

In this paper, we analyze the inflationary properties of both the global and local scale-invariant $R + R^2$ models. Our analysis simplifies previous studies of scale-invariant R^2 inflation [24–32], through the observation that “inertial” spontaneous Weyl symmetry breaking leads to the decoupling of the dilaton/Weyl photon and results in single field inflation with the second scalar being either a massless dilaton or providing the longitudinal component of a massive Weyl photon. In the global case, the decoupling of the dilaton eliminates the usual “fifth-force” constraints on Jordan Brans Dicke models [17]. Moreover, the recently developed frame-independent approach [19,40] allows us to determine the inflationary parameters directly in the Jordan frame where the scale invariance is manifest. As a result, we are able to give analytic forms for the inflationary parameters. We find that the inflationary predictions of both the global and local Weyl invariant models are the same but

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differ from the original $R + R^2$ Starobinsky model, except in a particular limit.

In Sec. II, we construct the model that is globally invariant under scale (Weyl) symmetry. The origin of spontaneous “inertial” Weyl symmetry breaking is discussed in Sec. II B where we show that this occurs independently of the scalar potential. In Sec. II C, we identify the dilaton, the Goldstone boson associated with the spontaneous breaking of the Weyl symmetry, and show that it decouples from the inflaton sector. Section III analyzes the locally Weyl symmetric model and shows that it also undergoes inertial spontaneous symmetry breaking that generates a mass for the “Weyl photon.” The Weyl photon decouples from the inflaton sector which has the same form as found in the globally symmetric case. In Sec. IV, it is shown that in a significant region of parameter space, slow-roll inflation occurs in both the globally and locally Weyl symmetric cases and gives an analytic form for the inflationary observables. The results are compared to the original Starobinsky model. Finally, in Sec. V, we present a summary and our conclusions.

II. GLOBALLY WEYL INVARIANT $R + R^2$ MODEL

A. A minimal model

The model consists of a real Jordan-Brans-Dicke scalar, ϕ nonminimally coupled to the Ricci scalar [35,36] plus the R^2 term introduced by Starobinsky [33]. The action is given by

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} \phi^4 - \frac{1}{12} \alpha_1 \phi^2 R + \frac{1}{6f_0^2} R^2 \right). \quad (1)$$

This theory is the most general one relevant to the inflationary era that can be constructed from these fields that is invariant under the global Weyl (scale) transformation

$$\begin{aligned} \phi &\rightarrow e^{-\epsilon} \phi \\ g_{\mu\nu} &\rightarrow e^{2\epsilon} g_{\mu\nu}. \end{aligned} \quad (2)$$

Note that other, quadratic, terms could be included (involving the Ricci and Riemann tensors) but which are absent (or do not contribute to the overall dynamics) during the inflationary regime.

Since the R^2 term involves fourth order derivatives, it contains an additional (scalar) degree of freedom. To make this explicit, it is convenient to reduce the fourth order derivatives to second order by introducing the auxiliary field η^1 [41,42] with the action now given by

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} \phi^4 - \frac{1}{12} \alpha_1 \phi^2 R - \frac{1}{12} \alpha_2 \eta^2 R - \frac{\xi}{4} \eta^4 \right). \quad (3)$$

Here, the equation of motion for η gives

$$\eta^2 = -\frac{\alpha_2}{6\xi} R, \quad (4)$$

which, substituted in Eq. (3), gives Eq. (1) for

$$\xi = 6f_0^2 \left(\frac{\alpha_2}{12} \right)^2. \quad (5)$$

The action is very close to that found in the two-scalar model analyzed by several groups [4–7,15–20]; in the Jordan frame, the difference is the absence of a kinetic term for η .

B. Inertial symmetry breaking

Under the Weyl transformation, $\eta \rightarrow e^{-\epsilon} \eta$, the Weyl current is given by

$$K_\mu \equiv \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \partial^\mu \epsilon} = (1 - \alpha_1) \phi \partial_\mu \phi - \alpha_2 \eta \partial_\mu \eta. \quad (6)$$

It can be written as $K_\mu = \partial_\mu K$, where the kernel K is given by

$$K = \frac{1}{2} ((1 - \alpha_1) \phi^2 - \alpha_2 \eta^2) \quad (7)$$

and is covariantly conserved

$$D^\mu K_\mu = 0. \quad (8)$$

Consider now a patch of the Universe which can be described as approximately spatially constant, but time dependent. These regions can be described by the Friedman-Robertson-Walker metric corresponding to $g_{\mu\nu} = [1, -a^2(t), -a^2(t), -a^2(t)]$. In terms of the kernel, the conservation law of Eq. (8) becomes

$$\ddot{K} + 3H\dot{K} = 0, \quad (9)$$

where $H = \dot{a}/a$, giving

$$K(t) = c_1 + c_2 \int_{t_0}^t \frac{dt'}{a^3(t')}, \quad (10)$$

where c_1 and c_2 are constants. Thus, in an expanding universe, $K(t)$ will evolve to a constant value, $\bar{K} = K(t \rightarrow \infty)$ that, from Eq. (7) implies the scalar fields acquire constant vacuum expectation values, spontaneously breaking Weyl symmetry [18].

Note that this inertial Weyl symmetry breaking does not rely on a scalar potential, the value of the kernel being determined by the chaotic initial conditions. Note also that a

¹By redefining the fields on moving to the Einstein frame, one obtains an explicit kinetic energy term for the new field η with the canonical form for a scalar field coupled to gravity.

constant kernel implies a relation between the scalar fields which is the reason the model results in single field inflation.

C. Dilaton decoupling

To identify the Goldstone mode associated with the spontaneous breaking of the global Weyl symmetry, we change variables to

$$\begin{aligned}\phi &= e^{-\sigma(x)/f} \hat{\phi} \\ \eta &= e^{-\sigma(x)/f} \hat{\eta} \\ g_{\mu\nu} &= e^{2\sigma(x)/f} \hat{g}_{\mu\nu},\end{aligned}\quad (11)$$

where f has dimensions of mass. Note that $\hat{\phi}$, $\hat{\eta}$, and $\hat{g}_{\mu\nu}$ are invariant under the global Weyl symmetry and only σ transforms, $\sigma \rightarrow \sigma + c$ where $c = \epsilon f$. In terms of the new metric,

$$R(g) = e^{2\sigma/f} (R(\hat{g}) + 6\hat{D}^2\sigma/f + 6(\hat{D}\sigma/f)^2), \quad (12)$$

giving

$$\begin{aligned}S &= \int d^4x \sqrt{-g} \left(-\frac{1}{12} (\alpha_1 \hat{\phi}^2 + \alpha_2 \hat{\eta}^2) \hat{R} + \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} \right. \\ &\quad \left. + \frac{1}{f^2} \hat{K} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{f} \partial_\mu \sigma \partial^\mu \hat{K} - \frac{\lambda}{4} \hat{\phi}^4 - \frac{\xi}{4} \hat{\eta}^4 + \lambda_L C(\hat{\phi}, \hat{\eta}) \right),\end{aligned}\quad (13)$$

where we have added a Lagrange multiplier, λ_L , and the constraint

$$C(\hat{\phi}, \hat{\eta}) = \bar{K} - \frac{1}{2} ((1 - \alpha_1) \hat{\phi}^2 - \alpha_2 \hat{\eta}^2). \quad (14)$$

For the constant \bar{K} , this gives the ‘‘Einstein frame’’ form of the theory

$$\begin{aligned}S &= \int d^4x \sqrt{-\hat{g}} \left(-\frac{1}{12} (\alpha_1 \hat{\phi}^2 + \alpha_2 \hat{\eta}^2) \hat{R} + \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} \right. \\ &\quad \left. - \frac{\lambda}{4} \hat{\phi}^4 - \frac{\xi}{4} \hat{\eta}^4 + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \lambda_L C(\hat{\phi}, \hat{\eta}) \right),\end{aligned}\quad (15)$$

where the dilaton is canonically normalized with the choice $f^2 = 2\bar{K}$.

The field σ is the massless dilaton. We see that it decouples from the hatted scalar fields. The same is true if one adds vector and fermion fields to the theory [18]. As a result, one avoids the severe astrophysical constraints on the fifth force normally associated with the dilaton.

Moreover, note that the dilaton now yields the Weyl current in the Einstein frame. We see that under the Weyl transformation $\delta\sigma/f = \delta\epsilon$, where all hatted variables are invariant, the Weyl current is now

$$K_\mu \equiv \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \partial^\mu \epsilon} = f \partial_\mu \sigma, \quad (16)$$

which is the familiar form of the current of a Nambu-Goldstone boson. Inertial symmetry breaking can be understood in the ‘‘Einstein frame’’ as a chaotic initial $K(x)$ parametrized by an arbitrary constant \bar{K} and an initially chaotic dilaton field $\sigma(x)$, as $K = e^{2\sigma(x)/f} \bar{K}$, with $f^2 = 2\bar{K}$. This dilaton form of the current, $f \partial_\mu \sigma$, is covariantly conserved, equivalent to the massless dilaton equation of motion, $D^2\sigma = 0$. This implies that the dilaton redshifts to a constant, yielding the spontaneously broken scale symmetry.

III. LOCALLY WEYL INVARIANT $R + R^2$ MODEL

For the case of local Weyl transformations, $\epsilon = \epsilon(x)$ in Eq. (2), the action, Eq. (1), is no longer invariant. To correct this, it is necessary to introduce a vector field, A_μ , the ‘‘Weyl photon’’ that transforms as

$$A_\mu \rightarrow A_\mu - \partial_\mu \epsilon(x). \quad (17)$$

With this, one can define a Weyl derivative acting on scalar fields, ϕ , by $\tilde{D}_\mu \phi \equiv (\partial_\mu - A_\mu) \phi$ that transforms covariantly

$$\tilde{D}_\mu \phi \rightarrow e^{-\epsilon(x)} \tilde{D}_\mu \phi \quad (18)$$

and replaces the Riemannian derivative when constructing a locally invariant version of Eq. (1). Similarly, it is necessary to replace the Riemannian Ricci scalar, R , with its Weyl geometric version, \tilde{R} defined by

$$\tilde{R} = R - 6D_\mu A^\mu - 6A_\mu A^\mu, \quad (19)$$

where D_μ is the Riemannian covariant derivative. Now \tilde{R} transforms covariantly under local Weyl transformations

$$\tilde{R} \rightarrow e^{-2\epsilon(x)} \tilde{R}. \quad (20)$$

Using these covariant forms, the locally invariant version of Eq. (1) is [21,22]

$$\begin{aligned}S_L &= \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \tilde{D}_\mu \phi \tilde{D}_\nu \phi - \frac{\lambda}{4} \phi^4 \right. \\ &\quad \left. - \frac{1}{12} \alpha_1 \phi^2 \tilde{R} + \frac{1}{6f_0^2} \tilde{R}^2 - \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} \right),\end{aligned}\quad (21)$$

where we have added to the Lagrangian the Weyl invariant Weyl photon kinetic term, $-\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Introducing an auxiliary field, as above, this can be rewritten as

$$\begin{aligned}S_L &\equiv \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \tilde{D}_\mu \phi \tilde{D}_\nu \phi - \frac{\lambda}{4} \phi^4 \right. \\ &\quad \left. - \frac{1}{12} (\alpha_1 \phi^2 + \alpha_2 \eta^2) \tilde{R} - \frac{\xi}{4} \eta^4 - \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} \right).\end{aligned}\quad (22)$$

Substituting \tilde{R} , using Eq. (19), integrating by parts, and rescaling $A_\mu \rightarrow eA_\mu$ gives

$$S_L = \int d^4x \sqrt{-g} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} e A_\mu K^\mu + \frac{1}{2} e^2 K A_\mu A^\mu - \frac{1}{12} (\alpha_1 \phi^2 + \alpha_2 \eta^2) R - \frac{\lambda}{4} \phi^4 - \frac{\xi}{4} \eta^4 \right), \quad (23)$$

where

$$K_\mu = \partial_\mu K, \quad K = \frac{1}{2} ((1 + \alpha_1) \phi^2 + \alpha_2 \eta^2). \quad (24)$$

Note that the variation of the action or Eq. (23) with respect to $\partial_\mu \epsilon(x)$, with $\delta e A_\mu = \partial_\mu \epsilon(x)$, now yields the vanishing identity

$$\frac{1}{\sqrt{-g}} \frac{\delta S_L}{\delta \partial^\mu \epsilon} = K_\mu - \partial_\mu K = 0. \quad (25)$$

In any local gauge theory, the variation of the full action with respect to the local gauge angle always produces zero, since this is the very definition of the symmetry.

The conserved Weyl current in the global case is now replaced by the Weyl photon equation of motion

$$\frac{1}{\sqrt{-g}} \frac{\delta S_L}{\delta A^\mu} = D_\nu F^{\mu\nu} - 2A_\mu e^2 K - e K_\mu = 0. \quad (26)$$

Changing variables as in Eq. (11), for constant \bar{K} , the action becomes

$$S_L = \int \sqrt{g} \left(-\frac{1}{12} (\alpha_1 \hat{\phi}^2 + \alpha_2 \hat{\eta}^2) \hat{R} + \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{\lambda}{4} \hat{\phi}^4 - \frac{\xi}{4} \hat{\eta}^4 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} e^2 \bar{K} B_\mu B^\mu + \lambda_L C \right), \quad (27)$$

where $B_\mu = A_\mu - \frac{e}{f} \partial_\mu \sigma$.

The spontaneous breaking of the Weyl symmetry has generated a mass for the Weyl photon, $M^2 = e^2 \bar{K}$, with the would-be Goldstone mode, σ , providing its longitudinal component. As for the dilaton in the global case, the Weyl photon decouples from the hatted fields. In the present local case, the dilaton, σ , and the gauge field, A_μ , are not separately physical. They are replaced by the combined (Stueckelberg) field B_μ , which is physical and invariant under local Weyl transformations.

Note that the conserved Weyl current is now given by the divergence of the field B_μ itself, which satisfies the Weyl Maxwell equation

$$D_\nu F^{\mu\nu} - M^2 B_\mu = 0 \quad M^2 = e^2 \bar{K}. \quad (28)$$

A massive spin-one field necessarily obeys the Lorentz gauge condition $D_\mu B^\mu = 0$ owing to the antisymmetry of $F_{\mu\nu}$ (note that a massive spin-one field has mass term $-M^2 B_\mu$, contrary to a scalar $+M^2 \phi$). Hence, an arbitrary initial physical B_μ will redshift to zero, yielding the spontaneously broken scale symmetry vacuum, with $B_\mu = 0$, and which is the analog of the redshifting of $K_\mu \rightarrow 0$ and $K_\mu \rightarrow \bar{K}$ in the global case. The vacuum is analogous to a superconductor, where the massive B_μ has the solution $B_\mu = 0$, which is the analog of the London equation for a superconductor, $e\vec{A} - \vec{j} = 0$.

IV. INFLATION

It is now straightforward to analyze the possibility that there is a period of inflation in both the global and local cases. The decoupled fields, the dilaton in the global case or the Weyl photon in the local case, play no role in the inflationary era. In their absence, the action for the hatted fields is the same for both cases and so their inflationary structure is the same.

In the Jordan frame, the Klein-Gordon equations resulting from Eq. (27) are given by

$$D^2 \hat{\phi} + \lambda \hat{\phi}^3 + \frac{1}{6} \alpha_1 \hat{\phi} R = 0$$

$$\xi \hat{\eta}^3 + \frac{1}{6} \alpha_2 \hat{\eta} R = 0. \quad (29)$$

Thus,

$$D^2 \frac{\hat{\phi}}{\hat{\phi}} = -\alpha_2 \lambda \hat{\phi}^2 + \alpha_1 \xi \hat{\eta}^2. \quad (30)$$

Note that there is an infrared stable fixed point that determines the ratio of the field vacuum expectation values (VEVs) given by

$$\frac{\hat{\eta}^2}{\hat{\phi}^2} = \frac{\alpha_2 \lambda}{\alpha_1 \xi} \quad (31)$$

that is approached after any period of inflation [15].

Taking the slow-roll limit with the ‘‘inflation derivative’’ (i.e., in terms of the number of e-foldings), $D^2 \hat{\phi} \rightarrow 3H \dot{\hat{\phi}} = 3H^2 \partial_N \hat{\phi}$ where $N = \ln a(t)$ gives

$$\frac{3}{2} H^2 \alpha_2 \frac{\partial_N \hat{\phi}^2}{\hat{\phi}^2} = -\alpha_2 \lambda \hat{\phi}^2 + \alpha_1 \xi \hat{\eta}^2. \quad (32)$$

H^2 may be eliminated using the (00) Einstein equation in the slow-roll limit,

$$G_{00} = 3H^2 \approx -\frac{3}{2} g_{00} \frac{\lambda \hat{\phi}^4 + \xi \hat{\eta}^4}{\alpha_1 \hat{\phi}^2 + \alpha_2 \hat{\eta}^2}. \quad (33)$$

From Eq. (14), one sees that the constraint that the Kernel should be constant corresponds to the fields being constrained to move on an ellipse. Without loss of generality, we can choose the ellipse $\bar{K} = 1$ and map the first quadrant of the ellipse into the variables

$$x = (1 - \alpha_1)\hat{\phi}^2, \quad y = -\alpha_2\hat{\eta}^2, \quad (34)$$

so that the ellipse becomes the line segment $x + y = 1$. Using Eqs. (30)–(34), we find

$$\partial_N x = \frac{S(x)}{W(x)} x(1-x)(x-x_0), \quad (35)$$

where

$$S(x) = \frac{1}{3} \times \frac{-\alpha_2^2 \lambda + \alpha_1(1-\alpha_1)\xi}{(1-\alpha_1)^2 \alpha_2^2} \times \frac{1-x-\alpha_1}{1-x} \quad (36)$$

$$x_0 = \frac{\xi(1-\alpha_1)}{\xi(1-\alpha_1) - \alpha_2^2 \lambda / \alpha_1} \quad (37)$$

and

$$W(x) = \frac{\lambda x^2}{4(1-\alpha_1)^2} + \frac{\xi(1-x)^2}{4\alpha_2^2}. \quad (38)$$

Here we consider the possibility of slow-roll inflation for the case where one coupling in the scalar potential dominates, the case relevant to the generation of a large hierarchy. For the case in which ξ dominates, $\partial_N x \propto \frac{\alpha_1}{1-\alpha_1}$ and slow-roll can occur for $|\alpha_1|$ small.² For the case in which λ dominates, because of the absence of the kinetic term in the second of Eq. (29), there is no dependence of $\partial_N x$ on α_i and no slow-roll. Returning to the first case,³ keeping only ξ nonzero, we have

$$\partial_N x = -\frac{4}{3} \alpha_1 x \quad (39)$$

$$H^2 \approx \frac{\xi y^2}{2\alpha_2^2(y - \alpha_1 x)}, \quad (40)$$

and $x(t)$ will roll from an initial $x(0)$ toward $x(t_E) \equiv x_E \approx 1$. Here t_E is the end of inflation given by the inflation parameter, ϵ , approaching unity $\epsilon = -\frac{1}{2} \frac{d \ln H^2}{dN} \approx 1$, corresponding to $y \approx -\frac{2}{3} \alpha_1$ or, equivalently, $\alpha_1 \hat{\phi}^2 \approx \alpha_2 \hat{\eta}^2$. Thus,

²For large α_1 , slow-roll generically fails to take place. Phrased in terms of the frame-invariant formalism of [40], which we will use extensively below, the η slow-roll parameter becomes large once $\alpha_1 \gg 1$.

³The case with $\lambda \ll \xi$ and α_1 small is the same limit studied in the two-scalar model studied in [19], making it easy to compare the two models.

$$x = x_E e^{-\nu N_J}, \quad y = 1 - x_E e^{-\nu N_J}, \quad \nu = -\frac{4}{3} \alpha_1, \quad (41)$$

where N_J is the number of e-foldings until the end of inflation in the Jordan frame. This is the same form as was found for the two-scalar model.

For clarity, the discussion to date has been in the Jordan frame. To determine the slow-roll parameters, we use the frame-independent formalism introduced in [40] and as applied in [19]. Denoting $\theta_1 = \hat{\phi}$, $\theta_2 = \hat{\eta}$, the frame-invariant field-space metric G_{AB} and potential U are given by

$$G_{AB} = \frac{\delta_{11}}{f} + \frac{3 f_{,A} f_{,B}}{f^2}, \quad U = \frac{\lambda \theta_1^4 + \xi \theta_2^4}{4f^2}, \quad (42)$$

where the key difference to the two-field model considered in [19] is the occurrence of δ_{11} (instead of δ_{AB}) and where

$$f \equiv M_P^2 \equiv -\frac{1}{6} \sum_{A=1}^2 \alpha_A \theta_A^2. \quad (43)$$

Defining the scalar (tensor) spectral index, n_S (n_T), the running of the scalar (tensor) spectral index, α_S (α_T), and the tensor to scalar ratio, r , we have that the frame-independent analysis proceeds as before, giving for the inflation observables

$$\begin{aligned} n_S &= 1 + \frac{4\alpha_1(e^{-\nu N_J} + 1)}{3(1 - e^{-\nu N_J})} + \mathcal{O}(\alpha_1^2), \\ r &= \frac{64\alpha_1^2 e^{-\nu N_J}}{3(e^{-\nu N_J} - 1)^2} + \mathcal{O}(\alpha_1^3), \\ \alpha_S &= -\frac{32\alpha_1^2 e^{-\nu N_J}}{9(e^{-\nu N_J} - 1)^2} + \mathcal{O}(\alpha_1^3), \\ n_T &= -\frac{8\alpha_1^2 e^{-\nu N_J}}{3(e^{-\nu N_J} - 1)^2} + \mathcal{O}(\alpha_1^3), \\ \alpha_T &= -\frac{32\alpha_1^3 e^{-\nu N_J} (1 + e^{-\nu N_J})}{9(e^{-\nu N_J} - 1)^3} + \mathcal{O}(\alpha_1^3), \end{aligned} \quad (44)$$

where the relation between the number of e-foldings in the Einstein and Jordan frames is given by

$$\begin{aligned} N_E &= N_J + \frac{1}{2} \ln \left(\frac{2\alpha_1}{(e^{-\nu N_J} - 1)} \right) + \mathcal{O}(\alpha_1) \\ &= N_J + \frac{1}{2} \ln \left(\frac{3}{2N_J} \right) + \mathcal{O}(\alpha_1), \end{aligned} \quad (45)$$

where we have expanded the exponential for small α_1 in the second line and assume $N_J \gg 1$. For $N_J = 60$, we therefore obtain $N_E \sim 58$. One feature that is immediately apparent is the absence of α_2 in (44). Indeed, we have explicitly checked up to tenth order in α_1 that no such

dependence is present. In Appendix A, we show why this is the case, but here we simply emphasize that despite the appearance of four parameters $(\alpha_1, \alpha_2, \lambda, \xi)$, in the $\xi \gg \lambda$ limit we are considering here only one single parameter controls all of the above observables. Working in this limit then also enforces a tight observational bound on α_1 : current constraints on n_S (Planck 2018 [43,44] finds $n_S = 0.9649 \pm 0.0042$) enforce $|\alpha_1| < 0.019$.⁴ The tensor-to-scalar ratio r , on the other hand, is always comfortably within current observational bounds ($r \lesssim 0.064$; see [43,44]). In fact, the precise value for r can be accurately predicted for our model $0.0033 > r > 0.0026$, where the lower bound is a consequence of the n_S induced bound on α_1 and the upper bound is saturated for $\alpha_1 \sim 0$ (and we again assume $N_J \simeq 60$)

The remaining observables in (44) satisfy $a_S \sim -2/N_J^2 \sim -5 \times 10^{-4}$, $a_T \sim -10^{-5}$, and $n_T \sim -4 \times 10^{-4}$, for all allowed choices of α_1 . Also, to reproduce the observed magnitude of the fluctuations, it is necessary to choose $\frac{\xi}{\alpha_2^2} = \mathcal{O}(10^{-10})$.

The above results may be compared to the original Starobinsky model. Focusing on n_S and r , there one finds

$$n_s - 1 \approx -\frac{2}{N_E}, \quad r \approx \frac{12}{N_E^2}. \quad (46)$$

Expanding the observables in (44) in the limit $\alpha_1 \rightarrow 0$ for general N_J (assumed to satisfy $N_J \gg 1$), the scale-invariant model gives

$$n_s - 1 \approx -\frac{2}{N_J}, \quad r \approx \frac{12}{N_J^2} \quad (47)$$

at leading order in N_J^{-1} , so that the predictions are very close to the original Starobinsky model (recall that N_J and N_E only differ by $\sim 3\%$ in the scale-invariant model investigated here, assuming close to 60 e-folds of inflation).

Finally, a note on observables beyond the two-point function and on isocurvature perturbations. The essential features here are identical to the ones of the scale-independent two field models considered in [19]: the local non-Gaussian parameter f_{NL} satisfies [45,46]

$$f_{NL}^{\text{local}} \approx \frac{5}{6} \frac{N^{,A} N^{,B} (\nabla_A \nabla_B N)}{(N_{,C} N^{,C})^2}, \quad (48)$$

where N is the frame-covariant number of e-folds ($N = N_E$) [40]. Evaluating this expression and expanding for small α_1 , we find

⁴For the model considered here, there is also an upper bound on n_S , namely $n_S \lesssim 1 - 2/N_J \simeq 0.9663$ (where we are assuming $N_J \simeq 60$), which is saturated as $\alpha_1 \rightarrow 0$.

$$f_{NL}^{\text{local}} \approx \frac{5\alpha_1(e^{-\nu N_J} + 1)}{9(e^{-\nu N_J} - 1)} + \mathcal{O}(\alpha_1^2) \sim 1.5 \times 10^{-2}, \quad (49)$$

so no sizeable non-Gaussian signature. Indeed, this is to be expected from the existence of a physically equivalent single scalar theory with a canonical kinetic term (see Appendix B). The effective single field nature then mandates suppressed local non-Gaussianity [47,48], while the canonical nature of the kinetic term ensures the absence of sizeable equilateral non-Gaussianity (because, for general single-field models, $f_{NL}^{\text{equil}} \lesssim 1/c_s^2$). Significant isocurvature perturbations are also absent, since the acceleration vector between paths in the geodesic flow

$$\omega^A = X^B \nabla_B \left[\frac{X^A}{\sqrt{X_C X^C}} \right] = 0 \quad (50)$$

for the models considered here (up to second order in slow-roll), where we have defined $X_A \equiv (\ln U)_{,A}$. When $\omega^A = 0$, the transfer function that converts curvature perturbations into isocurvature perturbations vanishes [19,40], so no sizeable isocurvature perturbations are generated for these models.

V. SUMMARY AND CONCLUSIONS

We have studied the scale-free version of R^2 inflation, setting it within the framework of the scale-invariant theories extensively studied in [4–22]. It can be shown to correspond to the two field model but without a canonical kinetic term for one of the scalar fields in the Jordan frame (for a more careful analysis of this limit, see Appendix A). As a consequence, we have shown that scale-free R^2 inflation is endowed with inertial symmetry breaking of Weyl invariance and that the techniques that were developed in [15,16] can be applied seamlessly to this scenario and used to derive predictions for the inflationary observables. The model we have considered here is, to some extent, the simplest nonminimally coupled theory which is scale invariant; it falls into the class of models considered in [49] which are favored by the Planck constraints.

The effect of radiative corrections on the inflationary era has been briefly discussed in the context of the two scalar model [16,17], and the general structure applies to the model considered here. In particular, if the scale invariance is preserved by quantum corrections by a suitable choice of the regularization procedure that does not introduce an extrinsic mass scale, inertial symmetry breaking still occurs and the dilaton still decouples. There will still be a period of slow-roll inflation, but the form of the constraint between the hatted fields will change and the ellipse on which they move will be distorted [17], giving a small change in the inflationary predictions provided the couplings are in the perturbative range.

We have shown that the inflationary observables— n_S , r , α_S , n_T , and α_T —are uniquely determined in terms of the small parameter, α_1 , which controls the nonminimal coupling of the extra scalar field. As a result, and with current constraints on the scalar spectral index, n_S , we can place tight bounds, $|\alpha_1| < 0.019$, which in turn leads to a very clear prediction for the tensor to scalar ratio $0.0033 > r > 0.0026$. This small range of values is within reach of the constraining power of future experiments: while the Simons Observatory [50] has a forecast sensitivity of $\sigma(r) = 0.003$, the LITEBIRD Mission will push the sensitivity to $\sigma(r) = 0.001$, while the S4 CMB consortium will aim for $\sigma(r) = 0.0005$ [51]. This means that this model offers a characteristic and precise prediction that will be experimentally testable within the next decade.

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APPENDIX A: SCALE-INDEPENDENT R^2 INFLATION AS A LIMIT OF A BISCALAR THEORY

Here we formally relate the results obtained for the scale-independent R^2 models considered throughout this paper to the scale-independent two field models considered in [19]. Recall the auxiliary field representation (3)

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{12} (\alpha_1 \hat{\phi}^2 + \alpha_2 \hat{\eta}^2) R + \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{\lambda}{4} \hat{\phi}^4 - \frac{\xi}{4} \hat{\eta}^4 \right), \quad (\text{A1})$$

and add an explicit kinetic term for the η field in the Jordan frame. One then finds

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{12} (\alpha_1 \hat{\phi}^2 + \alpha_2 \hat{\eta}^2) R + \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} + \frac{1}{2} \kappa^2 \partial_\mu \hat{\eta} \partial^\mu \hat{\eta} - \frac{\lambda}{4} \hat{\phi}^4 - \frac{\xi}{4} \hat{\eta}^4 \right), \quad (\text{A2})$$

where we have introduced a constant parameter κ . We may now canonically normalize the $\hat{\eta}$ field by sending $\hat{\eta} \rightarrow \hat{\eta}/\kappa$ and then find

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{12} (\alpha_1 \hat{\phi}^2 + \hat{\alpha}_2 \hat{\eta}^2) R + \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} + \frac{1}{2} \partial_\mu \hat{\eta} \partial^\mu \hat{\eta} - \frac{\lambda}{4} \hat{\phi}^4 - \frac{\hat{\xi}}{4} \hat{\eta}^4 \right), \quad (\text{A3})$$

which is precisely of the type considered in [19] and where

$$\hat{\alpha}_2 \equiv \frac{\alpha_2}{\kappa^2}, \quad \hat{\xi} \equiv \frac{\xi}{4\kappa^4}. \quad (\text{A4})$$

The model (A1), i.e., the $\kappa \rightarrow 0$ limit of (A2), is therefore analogous to sending $\hat{\alpha}_2$ and $\hat{\xi}$ to $\pm\infty$ in (A3) (with the appropriate scalings determined by their κ dependence). Observables computed for (A1) and (A3) can therefore be related by taking appropriate scaling limits of the $\hat{\alpha}$ and $\hat{\xi}$ parameters and indeed, as expected from this argument, the expressions (44) are precisely the $|\alpha_2| \rightarrow \infty$ limit of the analogous expressions obtained in [19]. Note that there remains an α_2 dependence in determining the magnitude of the fluctuations as it is proportional to $\frac{\xi}{\alpha_2^2} \equiv \frac{\hat{\xi}}{\hat{\alpha}_2^2}$.

APPENDIX B: EFFECTIVE SINGLE SCALAR THEORY

Using Eq. (14) with \bar{K} constant, we can express the field $\hat{\eta}$ in terms of $\hat{\phi}$ and describe the hatted field action that drives inflation in the scale-independent R^2 theory as a (single-)scalar-tensor theory

$$S = \int d^4x \sqrt{-g} \left[\frac{\hat{f}(\hat{\phi})}{2} \hat{R} + \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \hat{W}(\hat{\phi}) \right]. \quad (\text{B1})$$

As discussed in Sec. IV, the form of (B1) immediately ensures the absence of a sizeable non-Gaussian signature for this model and hence for (15).

The functions $\hat{f}(\hat{\phi})$ and $\hat{W}(\hat{\phi})$ are given by

$$\hat{f} = \frac{1}{6} (2\bar{K} - \hat{\phi}^2), \quad \hat{W} = \frac{\xi}{\alpha_2^2} (2\bar{K} + (\alpha_1 - 1) \hat{\phi}^2)^2, \quad (\text{B2})$$

where we may now define the frame-invariant potential $\hat{U} \equiv \hat{W}/\hat{f}^2$ and can also evaluate the frame-invariant field space metric \hat{G} , c.f. (42), which here is a simple scalar function (since the scalar field space is one-dimensional now). We find

$$\hat{G} = \frac{12\bar{K}}{(\hat{\phi}^2 - 2\bar{K})^2}, \quad (\text{B3})$$

where the frame-invariant inverse “metric” satisfies $\hat{G}^{-1} = 1/\hat{G}$. Using the same frame-independent analysis as above (and as explained in detail in [19,40]), we can now compute the inflationary observables n_S , r , α_S , n_T , α_T and indeed find identical predictions for the effective single field model (B1) when compared with the results in (44).

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