

The topology of a quantale valued metric space

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Abstract

The ‘the’ in the title hides a subtlety. A metric space induces not one but four topologies - by means of open sets, closed sets, closure, and interior - they just so happen to coincide. The agreement between these four structures arising from a metric function $d: X \times X \rightarrow [0, \infty]$ is due to a combination of the metric axioms and the lattice structure of $[0, \infty]$. Further motivation materializes from Lawvere’s observation from 1973 to the effect that a (slightly generalized) metric space is a category enriched in $[0, \infty]$. Metric spaces taking values in structures other than $[0, \infty]$ are relevant for generalizations of metric spaces and find a natural home in Lawvere’s categorical setting. In particular, in recent years quantales emerged as structures occupying an important niche in between $[0, \infty]$ and arbitrary monoidal categories. Since a category enriched in a quantale Q is the same thing as a metric space taking values in Q one may ask whether such a thing belongs to algebra or geometry. Further, does the quadruplet of topologies associated to a Q -valued space/category still consist of identical siblings? We propose a litmus test for the geometricity of Q -valued spaces as we investigate these issues.

Keywords: metric space, quantale, induced topology, probabilistic metric space, generalized metric space, category enriched in a quantale

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1. Introduction

The interplay between metric spaces and topology underpins the birth of topology and is about a century old. Despite its maturity aspects of the relationship between the two concepts continue to occupy current research, as

detailed further below. In this introduction we merely note that the theme of volume 350 of Fuzzy Sets and Systems is Topology and Metric Spaces. Two of the articles published in that volume, namely Zhang (2018); Han et al. (2018), are related to quantales. Quantales are a natural axiomatization of the codomain of a metric function, yielding new classes of metric spaces. The relevance of quantales to fuzzy set theory, in a rather broad sense, is well established (e.g., Höhle and Kubiak (2011); Tao et al. (2014)). The more particular relevance of quantales from the perspective of this work for the fuzzy sets community has been recognized and studied in Gutiérrez García et al. (2017).

The article is comprised of three sections, the first of which addresses the topology of metrizable spaces in the context of quantale theory. It is self-contained and includes a survey of recent results as well as some new ones as it further serves to motivate and frame the contents of the second section. The second section is a detailed study of the relationship between a Q -valued metric space and its associated topology. In particular, conditions are given that assure the existence of a well defined unique such induced topology. The article closes with a section devoted to a re-examination of historical aspects in light of the new results.

2. Metrization - restrictive and extensive tales

The open ball topology refers to what is now for many years a perfectly standard construction that associates a topology to a metric space. The metric space axioms were laid down in 1906 by Fréchet in his Ph.D. dissertation Fréchet (1906) and have remained roughly unchanged: a metric space (X, d) is a set X together with a distance function $d: X \times X \rightarrow [0, \infty)$ satisfying

$$\begin{aligned} d(x, y) = 0 &\iff x = y \\ d(x, y) &= d(y, x) \\ d(x, z) &\leq d(x, y) + d(y, z) \end{aligned}$$

for all $x, y, z \in X$. Given a metric space (X, d) , a point $x \in X$, and $r > 0$ the *open ball* with *centre* x and *radius* r is the set $B_r(x) = \{y \in X \mid d(x, y) < r\}$. A subset $U \subseteq X$ is said to be *open* if for all $x \in X$ there exists $r > 0$ with $B_r(x) \subseteq U$. The collection $\mathcal{O}(X, d)$ of all open sets in X is the open ball topology, and so every metric space gives rise to a topology but not every topology comes from a metric. So much can be found in any standard textbook on topology.

Metrizability theory played an important role in the formation of point-set topology. Much effort went into identifying topological conditions that capture the metrizability of a space. From a modern perspective the metrizability problem can be phrased as follows. Consider the categories \mathbf{Met}_c of metric spaces and continuous functions and \mathbf{Top} of topological spaces. The open ball topology construction yields a functor $\mathcal{O}: \mathbf{Met}_c \rightarrow \mathbf{Top}$ and the problem of metrizability aims to characterize the essential image of \mathcal{O} . We shall refer to this classical meaning as the *restrictive metrization* problem. A different interpretation of the problem of metrizability received less attention. Instead of identifying a smaller portion of \mathbf{Top} that under \mathcal{O} is equivalent to \mathbf{Met}_c one may ask instead to enlarge \mathbf{Met}_c in such a way that an extension of \mathcal{O} yields an equivalence with \mathbf{Top} . This option shall be referred to as *extensive metrization*.

The first work in the extensive direction seems to be Kopperman (1988) where Kopperman, in 1988, introduced continuity spaces by replacing the non-negative reals as the codomain of the metric function with an axiomatically defined structure coined *value semigroup*. The main result is that by so doing every topological space (X, τ) is metrizable in the sense that there exists a value semigroup K_τ and a suitable metric function $d: X \times X \rightarrow K_\tau$ whose induced topology is precisely the original τ . It is natural to attempt to utilize Kopperman's continuity spaces to answer the extensive metrization problem, as follows.

We note that part of the structure of a value semigroup K is a specified subset $P \subseteq K$ of *positives* and we write $\varepsilon \succ 0$ to indicate that $\varepsilon \in P$. Let \mathbf{KMet}_c be the category whose objects are (X, K, d) with X a set, K a value semigroup, and $d: X \times X \rightarrow K$ a continuity space, i.e., $d(x, x) = 0$ and $d(x, z) \leq d(x, y) + d(y, z)$. The morphisms $f: (X_1, K_1, d_1) \rightarrow (X_2, K_2, d_2)$ are functions between the underlying sets satisfying the usual Cauchy continuity condition, namely for all $x \in X_1$ and $\varepsilon \succ 0$ in K_2 there exists $\delta \succ 0$ in K_1 such that $d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \varepsilon$. Kopperman notices that in the open ball topology associated with a continuity space, an open ball need not be an open set. To see how that occurs we mention that the non-negative extended real numbers $[0, \infty]$ with the usual structure constitute a value semigroup with the usual positive numbers as set of positives. With the element-wise structure the cartesian product $K = [0, \infty] \times [0, \infty]$ is a value semigroup. Let $X = \mathbb{R}^2$ as a set and define $d: X \times X \rightarrow K$ by $d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|, |y_1 - y_2|)$ to obtain a Kopperman continuity space. Its induced topology is the Euclidean one on \mathbb{R}^2 but a straightfor-

ward verification shows that the ‘open’ ball $B_{(1,1)}(0,0)$, even though it has a positive radius $(1,1)$, is not open; it is a closed square with the four corners removed. This deficiency sabotages the extensive metrization efforts since the existence of non-open open balls implies that $\mathcal{O}: \mathbf{KMet}_c \rightarrow \mathbf{Top}$ is not full and thus not an equivalence.

The metrizability of the objects without proper attention paid to the morphisms cannot be entirely satisfactory. This point was safely ignored in the context of restrictive metrization as it was in fact automatic from the very nature of the open ball topology. This claim, and more, form part of the tight compatibility between a metric and its open ball topology as witnessed by the following textbook exercises:

1. The collection $\mathcal{O}(X, d)$ of all open sets in X forms a topology.
2. The collection of all open balls is a base for $\mathcal{O}(X, d)$.
3. If a function $f: X \rightarrow Y$ between metric spaces is continuous, then f is continuous with respect to the induced topologies. In other words, $\mathcal{O}: \mathbf{Met}_c \rightarrow \mathbf{Top}$ is a functor.
4. If X and Y are metric spaces and $f: X \rightarrow Y$ is continuous with respect to the induced topologies, then f is metrically continuous. Consequently, the functor $\mathcal{O}: \mathbf{Met}_c \rightarrow \mathbf{Top}_m$ to the category of metrizable topological spaces is full and thus an equivalence of categories.
5. The point-to-set distance $d(x, S) = \bigwedge_{s \in S} d(x, s)$ agrees with the topological closure operator, namely $d(x, S) = 0$ if, and only if, $x \in \overline{S}$.
6. Every open ball $\{y \in X \mid d(x, y) < r\}$ is an open set.
7. Every closed ball $\{y \in X \mid d(y, x) \leq r\}$ is a closed set.

For Kopperman’s continuity spaces only the first three claims hold true. For instance, for the same continuity space X exhibiting an open ball that is not an open set let $S = \{(1,0), (0,1)\}$ and $x = (0,0)$. Then $d(x, S) = (0,0)$, the 0 element of $K = [0, \infty] \times [0, \infty]$, but clearly $x \notin \overline{S}$ - a disastrous point-to-set distance phenomenon.

The discussion so far points to the existence of special features of $[0, \infty]$ in the classical definition of metric space that make the open ball topology particularly well suited for a topological investigation of the given metric space. In any extension of the notion of metric space by allowing more general codomains for the metric, as in Kopperman’s continuity spaces, the question of the topological suitability of the codomain must be addressed. The mere metrizability of the space is not enough, e.g., the usual Euclidean

metrization of \mathbb{R}^2 is a continuity space where the close relationship between metric and topology holds while the same topology metrized by means of $[0, \infty] \times [0, \infty]$ suffers severe shortcomings.

A further step toward an extensive metrization of topology was taken in Flagg (1997) where Flagg, in 1997, introduced another meaning to continuity space. Instead of value semigroups Flagg introduced value quantales and proved a metrization result, namely that for every topological space (X, τ) there exists a value quantale F and a metric function $d: X \times X \rightarrow F$ whose induced topology is τ . A significant difference between value semigroups and value quantales is that in the former a set of positives must be provided externally while in the latter the positives are uniquely determined from the rest of the lattice structure. This is certainly a pleasant feature of value quantales but there is a much more foundational advantage; all of the properties listed above regarding the intimate metric-topology relationship hold for Flagg's continuity spaces. Most of these claims are proved by Flagg. Consequently, as noted in Weiss (2015), Flagg's continuity spaces provide a solution for the extensive metrization problem, as we now describe.

A *value quantale* is a non-trivial complete lattice F together with a commutative and associative binary operation $+$ that distributes over meets, i.e., $a + \bigwedge S = \bigwedge(a + S)$ where $a + S = \{a + s \mid s \in S\}$, and has 0 as neutral element (0 is the least element of F , i.e., the empty join). Two further axioms use the *totally above* relation $b \succ a$. The meaning of $b \succ a$, for elements a, b in any complete lattice L , is that for all subsets $S \subseteq L$ with $a \geq \bigwedge S$ there exists $s \in S$ with $b \geq s$. We denote $\uparrow(a) = \{x \in L \mid x \succ a\}$. For F to be a value quantale it is required that $a = \bigwedge \uparrow(a)$ for all $a \in F$ and that $a \wedge b \succ 0$ whenever $a, b \succ 0$. The first of these two conditions is well known to characterize completely distributive complete lattices (see the classical Raney (1952, 1953, 1960) and the more modern Vickers (1993)). A Flagg continuity space (X, F, d) is a set X , a value quantale F , and a metric function $d: X \times X \rightarrow F$ satisfying $d(x, x) = 0$ and $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$. The open ball topology $\mathcal{O}(X, F, d)$ is the one where a set $U \subseteq X$ is open if for all $x \in U$ there exists $\varepsilon \succ 0$ with $B_\varepsilon(x) \subseteq U$, where $B_\varepsilon(x) = \{y \in X \mid d(x, y) \prec \varepsilon\}$.

We demonstrate the power of the value quantale axioms by proving that open balls are open sets and that the point-to-set distance disaster of Kopperman's formalism is avoided. Let us fix a Flagg continuity space (X, F, d) and $y \in B_\varepsilon(x)$ with $\varepsilon \succ 0$, namely $d(x, y) \prec \varepsilon$. It can be shown that $a \succ \bigwedge S$ implies the existence of $s \in S$ with $a \succ s$. Since $\varepsilon \succ d(x, y) = d(x, y) + 0 =$

$\bigwedge_{\delta \succ 0} d(x, y) + \delta$ it follows that $\varepsilon \succ d(x, y) + \delta$ for some $\delta \succ 0$. The inclusion $B_\delta(y) \subseteq B_\varepsilon(x)$ follows routinely from the triangle inequality, and so $B_\varepsilon(x)$ is open. Suppose now that $S \subseteq X$ and we show that $d(x, S) = 0$ if, and only if, $x \in \overline{S}$, where $d(x, S) = \bigwedge d(x, s)$ with the meet computed in F . If $d(x, S) = 0$ and $\varepsilon \succ 0$ is arbitrary, then from $\varepsilon \succ d(x, S)$ we obtain some $s \in S$ with $\varepsilon \succ d(x, s)$. Thus, every open set containing x intersects S , so $x \in \overline{S}$. For the converse, if $d(x, S) \neq 0$, then there exists $\varepsilon \succ 0$ with $d(x, S) \not\leq \varepsilon$. But then $B_\varepsilon(x) \cap S = \emptyset$ since any $y \in S$ with $d(x, y) \prec \varepsilon$ would imply that $d(x, S) \leq \varepsilon$, a contradiction, so we may conclude that $x \notin \overline{S}$.

All that is left to obtain a solution for the extensive metrization problem using Flagg's continuity spaces is a demonstration that every topological space is metrizable. The details rely on a family of value quantales, one for every set. For a set T let $\downarrow(T)$ be the collection of all finite subsets of T . Flagg defines the value quantale $\Omega(T) = \{a \subseteq \downarrow(T) \mid A \in a \implies \downarrow(A) \subseteq a\}$, ordered by reverse inclusion and with $+ = \cap$. Given a topological space (X, τ) define $d: X \times X \rightarrow \Omega(\tau)$ by $d(x, y) = \downarrow(\tau_{x \rightarrow y})$, where $\tau_{x \rightarrow y} = \{U \in \tau \mid x \in U \implies y \in U\}$. It is an exercise to verify that $\mathcal{O}(X, \Omega(\tau), d) = \tau$. As a corollary, let \mathbf{FMet}_c be the category of all Flagg continuity spaces with the usual Cauchy continuous functions, i.e., $f: (X_1, F_1, d_1) \rightarrow (X_2, F_2, d_2)$ is a function $f: X_1 \rightarrow X_2$ such that for all $x \in X$ and $\varepsilon \succ 0$ in F_2 there exists $\delta \succ 0$ in F_1 with $d(f(x), f(y)) \prec \varepsilon$ whenever $d(x, y) \prec \delta$. Then the functor $\mathcal{O}: \mathbf{FMet}_c \rightarrow \mathbf{Top}$ is an equivalence of categories.

Remark 1. *Flagg's metrization is clearly almost never symmetric. For both notions of continuity spaces it is the case that a topological space X admits a symmetric metrization, i.e., one satisfying $d(x, y) = d(y, x)$, if, and only if, X is completely regular. For Kopperman's continuity spaces this is shown in Kopperman (1988) and for Flagg's continuity spaces the details can be found in Bruno (2020).*

Success versus failure in the two attempts for an extensive metrization discussed above hinges on properties of the structures allowed as the codomains for metric functions that go beyond their versatility in metrizing all topological spaces. Since there are multiple options for axioms that carve out suitable codomains for metric functions it becomes pertinent to more systematically and precisely pinpoint the features of Flagg's value quantales that make the link between metric and topology tick so well. Before tending to that we complement the discussion with several observations regarding Flagg's formalism.

With non-metrizability no longer an option and with new kinds of metrizations it is natural to ask just how comfortably can a metrization fit a given topology. The next result shows that Flagg’s metrization is as snug as possible.

Theorem 1. *Let (X, τ) be a topological space and $(X, \Omega(\tau), d)$ its Flagg metrization. Then every open set $U \in \tau$ other than the empty set is an open ball.*

Proof. The collection $\varepsilon = \{\emptyset, \{U\}\}$ is an element in $\Omega(\tau)$ satisfying $\varepsilon \succ 0$. A straightforward calculation shows that $B_\varepsilon(x) = U$ for any $x \in U$. \square

For the next result note that the construction $T \mapsto \Omega(T)$ extends to a functor $\Omega: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{F}$ where \mathbf{F} is the category of value quantales with the evident notion of morphism. For a function $f: T_1 \rightarrow T_2$ the morphism part is given by the function $\Omega(f): \Omega(T_2) \rightarrow \Omega(T_1)$ with $\Omega(f)(b) = \{A \subseteq \downarrow(T_1) \mid f_{\rightarrow}(A) \in b\}$, using the direct image function f_{\rightarrow} . Let \mathbf{FMet}_s be the category of all Flagg continuity spaces (X, F, d) with morphisms $(f, \varphi): (X_1, F_1, d_1) \rightarrow (X_2, F_2, d_2)$ consisting of a function $f: X_1 \rightarrow X_2$ and a morphism $\varphi: F_1 \rightarrow F_2$ of value quantales such that $d_2(f(x), f(y)) \leq \varphi(d_1(x, y))$ for all $x, y \in X_1$. In this case we say that f is *short relative* to φ . Note that shortness relative to the identity takes the usual form of short mappings in metric space theory. The next result states that every topologically continuous function is short relative to itself.

Theorem 2. *Flagg’s metrization extends to a functor $\mathbf{Top} \rightarrow \mathbf{FMet}_s$.*

Proof. The precise claim being made is that if $f: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$ is a continuous function and $(X_i, \Omega(\tau_i), d_i)$, $i = 1, 2$, are the Flagg metrizations, then f is short relative to $\Omega(f^{\leftarrow})$ where $f^{\leftarrow}: \tau_2 \rightarrow \tau_1$ is the inverse image function. In other words, $(f, \Omega(f^{\leftarrow})): (X_1, \Omega(\tau_1), d_1) \rightarrow (X_2, \Omega(\tau_2), d_2)$ is a morphism in \mathbf{FMet}_s , yielding the desired functor. The verification is another exercise. \square

A classical result from 1928 is the Niemytzki–Tychonoff theorem characterizing compactness in metric terms: a classically metrizable topological space X is compact if, and only if, it is complete in every classical metric inducing its topology. The next result is established in Weiss (2018).

Theorem 3. *An arbitrary topological space X is compact if, and only if, it is complete in every Flagg continuity space inducing its topology.*

The next stop takes us to 1962 when Pervin (Pervin (1962)) showed that every topological space is quasi-uniformizable. Recall that for a classical metric space X the family $\{(x, y) \in X \times X \mid d(x, y) \leq \varepsilon\}$, indexed by $\varepsilon > 0$, is a base for the uniformity induced by the metric. If the symmetry axiom of d is dropped, then the same family is a basis for the induced quasi-uniformity. For a metric space X valued in a value quantale F , the family $\{(x, y) \in X \times X \mid d(x, y) \leq \varepsilon\}$, with $\varepsilon \succ 0$ in F , is similarly a basis for a quasi-uniformity associated to X . Every quasi-uniformity induces a topology and a topological space X is said to be (quasi) uniformizable if its topology is induced by a (quasi) uniformity. It is well known that the uniformizable spaces are the completely regular ones. Pervin showed that all topological spaces (X, τ) are quasi-uniformizable, as follows. For each $U \in \tau$ let $S_U = \{(x, y) \in X \times X \mid x \in U \implies y \in U\}$; the collection $\{S_U\}_{U \in \tau}$ generates a quasi-uniformity whose induced topology is τ .

Theorem 4. *For all topological spaces (X, τ) the quasi-uniformity induced by Flagg’s metrization $(X, \Omega(\tau), d)$ is Pervin’s quasi-uniformization.*

Proof. One more exercise. □

Remark 2. *Failure to address the behavior of morphisms in the presence of a quasi-uniformization of the objects of **Top** has led to some confusion. In Fleischer (2006), titled “continuity space”=quasi-uniform space, the author notes that Kopperman’s continuity spaces also naturally give rise to a quasi-uniformization of topological spaces and concludes that continuity spaces and quasi-uniform spaces are one and the same thing (presumably, therefore, implying that one of the two concepts should be dispensed with). The same argument, on the same grounds, can be made of Flagg’s continuity spaces. But, as was made clear in the extensive metrization discussion, the two continuity space concepts are very different when one brings morphisms into the picture. The same vindication of the concepts holds in an analogous quasi-uniformization debate.*

A metric structure is much richer than a topology and so it may be simpler to introduce certain constructions in the presence of a metric. A construction may depend on auxiliary information, for instance a *scale*. If (X, F, d) is a continuity space, then a scale on it is a function $R: X \rightarrow F$ with the sole condition that $R(x) \succ 0$. Suppose that a construction $A(X, F, d, R)$ is given that depends on the scale. Varying the scale, but keeping (X, F, d) fixed,

produces a diagram of objects, each a scale invariant. In Weiss (2017) a precise meaning is given to the intuitive idea that the limit of the diagram as the scale vanishes gives rise to a topological invariant. Invariants that arise in this manner include variants of homotopy groups and homology groups as well as ordinary topological connectedness (explored extensively in Weiss (2016)).

The final result of this part is an observation regarding the uniqueness of the open ball topology as the solution to a problem of equivalency between two formalisms of continuity. Consider now the categories \mathbf{FMet}_c and \mathbf{Top} as concrete categories over \mathbf{Set} via the evident forgetful functors to the underlying sets of points. A concrete functor $\mathbf{FMet}_c \rightarrow \mathbf{Top}$ is one that commutes with the forgetful functors, simply amounting to the requirement that the functor maps a Flagg continuity space (X, \dots) to a topological space (X, \dots) and formally maps a morphism f to itself. Recall that any two-point set $\mathbb{S} = \{a, b\}$ admits four topologies, two of which, namely $\{\emptyset, \{a\}, \mathbb{S}\}$ and $\{\emptyset, \{b\}, \mathbb{S}\}$, are known as Sierpiński space. Classically, of course, Sierpiński space is not metrizable but it is in Flagg’s formalism. It is interesting to note that a metric Sierpiński space arises naturally in the theory of Flagg’s continuity spaces, so we present a short detour of far greater generality than required for the mere definition of the space.

The distributivity of $+$ over arbitrary meets in a value quantale F implies the existence of a left adjoint to the function $x \mapsto a + x$, for any fixed $a \in F$. The left adjoint is denoted by $x \mapsto x - a$ and is characterized by the condition $x \leq a + y \iff x - a \leq y$. In particular, it follows that $x - x = 0$ and that $x - z \leq (x - y) + (y - z)$, that in turn imply that F is canonically a continuity space valued in F via $d(x, y) = y - x$. When $F = \mathcal{B}$, the value quantale of boolean truth values $\mathcal{B} = \{0 < 1\}$ (see Example 2 below for details), the resulting continuity space is a metric analogue of the Sierpiński space. The distances in \mathcal{B} are $\mathcal{B}(0, 0) = \mathcal{B}(1, 1) = \mathcal{B}(1, 0) = 0$ and $\mathcal{B}(0, 1) = 1$. The open balls in \mathcal{B} are thus $B_1(0) = B_1(1) = B_0(1) = \mathcal{B}$ and $B_0(0) = \{0\}$, so the open ball topology is the Sierpiński space $(\mathcal{B}, \{\emptyset, \{0\}, \mathcal{B}\})$.

We are now in position to characterize the open ball topology as the unique solution to a functorial construction.

Theorem 5. *The open ball topology functor $\mathcal{O}: \mathbf{FMet}_c \rightarrow \mathbf{Top}$ is the unique full concrete functor between these categories (in the stated direction).*

Proof. Suppose that $\mathcal{T}: \mathbf{FMet}_c \rightarrow \mathbf{Top}$ is full and concrete and let $S = \mathcal{T}(\mathcal{B})$ where \mathcal{B} is the metric Sierpiński space. Since \mathcal{T} is concrete $\mathcal{T}(\mathcal{B})$ is a

topology on $\{0, 1\}$ and since \mathcal{T} is fully faithful $\mathbf{Top}(S, S) \cong \mathbf{FMet}_c(\mathcal{B}, \mathcal{B})$. Direct inspection reveals that of the two non-constant functions $\mathcal{B} \rightarrow \mathcal{B}$ only the identity function is continuous. Of the four topologies on $\{0, 1\}$ the discrete and indiscrete ones allow for four continuous mappings $S \rightarrow S$ and so these two options are excluded, namely S is one of the two Sierpiński space structures on \mathcal{B} with either $\{0\}$ open, denoted by \mathbb{S}_0 , or with $\{1\}$ open, denoted by \mathbb{S}_1 .

Suppose that $\mathcal{T}(\mathcal{B}) = \mathbb{S}_0$ and let (X, F, d) be an arbitrary Flagg continuity space. For $x \in X$ and $\varepsilon \succ 0$ in F let $f_{\varepsilon, x}: X \rightarrow S$ be given by $f_{\varepsilon, x}(y) = 0$ if $d(x, y) \prec \varepsilon$ and 1 otherwise. We claim that $f_{\varepsilon, x}$ is a continuous function between X and \mathcal{B} as continuity spaces. In \mathcal{B} , $0 \succ 0$ so in order to confirm continuity at a point y it suffices to find $\delta \succ 0$ in F with $d(y, z) \prec \delta \implies \mathcal{B}(f_{\varepsilon, x}(y), f_{\varepsilon, x}(z)) = 0$. With that said, continuity at a point $y \notin B_\varepsilon(x)$ is immediate. If $y \in B_\varepsilon(x)$, then, since open balls are open sets, there is $\delta \succ 0$ in F with $B_\delta(y) \subseteq B_\varepsilon(x)$ and a straightforward verification shows this δ fits the bill. Thus $f_{\varepsilon, x}: X \rightarrow \mathcal{B}$ is a morphism in \mathbf{FMet}_c and functoriality of \mathcal{T} implies that $f_{\varepsilon, x}: \mathcal{T}(X) \rightarrow \mathbb{S}_0$ is continuous. Since $\{0\}$ is open $B_\varepsilon(x) = f_{\varepsilon, x}^{\leftarrow}(\{0\})$ is an open set in $\mathcal{T}(X)$. Therefore, the topology of $\mathcal{T}(X)$ contains the open ball topology $\mathcal{O}(X)$. Conversely, suppose U is an open set in $\mathcal{T}(X, F, d)$ and consider the function $f_U: X \rightarrow S$ that satisfies $f_U^{\leftarrow}(\{0\}) = U$ and note that $f_u: \mathcal{T}(X, F, d) \rightarrow \mathbb{S}_0$ is continuous. Since \mathcal{T} is full $f_u: (X, F, d) \rightarrow \mathcal{B}$ is continuous. If $x \in U$, then continuity at x implies the existence of $\delta \succ 0$ in F with $d(x, y) \prec \delta \implies \mathcal{B}(0, f_U(y)) = 0$, showing that $f_U(y) = 0$ and so $y \in U$. In other words, $B_\delta(x) \subseteq U$, making U open in the open ball topology. To summarize, if $\mathcal{T}(\mathcal{B}) = \mathbb{S}_0$, then $\mathcal{T} = \mathcal{O}$.

Suppose now that $\mathcal{T}(\mathcal{B}) = \mathbb{S}_1$ and let (X, F, d) be an arbitrary continuity space. As above, the function $f_{\varepsilon, x}: X \rightarrow \mathbb{S}_1$ is continuous and this time the conclusion is that $f_{\varepsilon, x}^{\leftarrow}(\{1\}) = X \setminus B_\varepsilon(x)$ is open. In other words, each open ball $B_\varepsilon(x)$ is closed in the topology of $\mathcal{T}(X)$. Given a closed set C in $\mathcal{T}(X)$ and f_C with $f_C^{\leftarrow}(\{0\}) = C$, which is continuous as a function $f_C: X \rightarrow \mathbb{S}_1$ and thus also as $f_C: X \rightarrow \mathcal{B}$, the $\varepsilon - \delta$ condition implies that C is open in $\mathcal{O}(X)$. In other words, every closed set in $\mathcal{T}(X)$ is open in $\mathcal{O}(X)$. It follows that the intersection of any collection of open balls is open. But this is certainly not the case for all continuity spaces (X, F, d) , and so $\mathcal{T}(\mathcal{B}) = \mathbb{S}_1$ is impossible. \square

Corollary 1. *The category \mathbf{FMet}_c is uniquely concretely equivalent to \mathbf{Top} .*

Remark 3. *This medley of topologically flavored results relies on certain an-*

alytic properties of value quantales absent in general quantales. By contrast, a fixed point result for symmetric continuity spaces valued in a quantale is given in Ackerman (2016). There the requirements on the quantale are more algebraic than analytic. The dependency of all of these results on various aspects of the quantale in question is a facet that only surfaces when the classical quantale $[0, \infty]$ is abandoned; this is partly the motivation for this work.

3. Matrices and their induced topologies

The absolute necessary requirements from a codomain of a metric function needed to interpret the metric space axioms are very minimal; a partial ordering, a designated 0, and a binary operation $+$. However, a useful general theory at such a level of generality is hopeless. Moreover, the preceding section demonstrates that some care is required before one announces that a ‘metric space’ entity X whose metric takes values in some such generic structure is rightfully to be considered a geometric entity. The ability to induce an open ball style topology is far from sufficient and so we turn to investigate just how geometric such metric spaces are.

On the spectrum of possible axioms for the codomain of a metric we focus on quantales. A *complete lattice* L is a poset admitting all joins \bigvee and all meets \bigwedge . A *quantale* $(Q, \cdot, 1)$ is a complete lattice Q together with an associative binary operation \cdot with unit 1 and such that the distributive laws $x \cdot \bigvee S = \bigvee x \cdot S$ and $(\bigvee S) \cdot x = \bigvee (S \cdot x)$ hold for all $x \in Q$ and $S \subseteq Q$, where $x \cdot S = \{x \cdot s \mid s \in S\}$ and a similar meaning for $S \cdot x$.

Remark 4. *In the same manner that a ring R is a monoid object in the category \mathbf{Ab} of abelian groups so is a quantale Q a monoid object in the category \mathbf{CSLat} of complete join semilattices. The analogy between ring theory and quantale theory can be traced to Joyal and Tierney (1984) and is central to the approach taken in Eklund et al. (2018).*

The definition of quantale conflicts with Flagg’s value quantales encountered above. In a value quantale the operation, named $+$, is required to distribute over meets rather than joins. The reason for the conflict is tradition. In metric space theory one speaks of a distance function and Flagg designed value quantales to mesh well with the established geometric language. In quantale theory the common language exhibits a preference to colimits over

limits and so one develops the theory using joins. In particular, a small category enriched in a quantale Q amounts to a set X and for all $x, y \in X$ an element $X(x, y) \in Q$ such that $1 \leq X(x, x)$ and $X(x, y) \cdot X(y, z) \leq X(x, z)$. The dual F^{op} of a value quantale F obtained by reversing the ordering in F , is a quantale in the usual sense. The two approaches are simply dual to each other, rendering the conflict one of syntax. The next example illustrates the clash of traditions.

Example 1. *The set of real numbers $[0, 1]$ with the usual ordering and multiplication is a quantale. The extended non-negative real numbers $[0, \infty]$ with reverse ordering and addition is also a quantale. It is clear that the function $x \mapsto e^{-x}$ is a structure preserving bijection $[0, \infty] \rightarrow [0, 1]$. The opposite of $[0, \infty]$, namely with the usual ordering, fails to be a quantale since distributivity of addition over the empty join entails that $x + 0 = 0$ must hold for all $x \in [0, \infty]$. We call $\mathcal{F}_+ = ([0, \infty], \geq, +, 0)$ the additive Fréchet quantale and $\mathcal{F} = ([0, 1], \leq, \cdot, 1)$ the multiplicative Fréchet quantale. A classical (non-symmetric) metric space is (Lawvere (1973)) precisely a small category enriched in \mathcal{F}_+ . On the other hand, enrichment in \mathcal{F} amounts to the axioms $s(x, x) = 1$ and $s(x, y) \cdot s(y, z) \leq s(x, z)$ that can be interpreted as expressing similarity rather than distance. The isomorphism $\mathcal{F}_+ \cong \mathcal{F}$ reflects the obvious fact that the theories of metric spaces and similarity spaces are isomorphic. A preference for one formalism over the other is a matter of taste, however there are situations where the multiplicative variant is more useful. For instance, in the recent work Leinster (2013) on the magnitude of a metric space the first step is to apply the transformation $x \mapsto e^{-x}$.*

We choose not to choose sides and instead notationally embrace both cultures. We speak of a quantale Q ; reversing the order yields the *op-quantale* Q^{op} . A small category enriched in a quantale Q is the same as a metric space valued in Q^{op} and we use $+$ and 0 in the latter case. The *totally above* relation $b \succ a$ in a quantale Q , meaning $a \geq \bigwedge S$ implies $b \geq s$ for some $s \in S$, is equivalent in Q^{op} to the *totally below* relation $b \ll a$, that reads as $a \leq \bigvee S$ implies $b \leq s$ for some $s \in S$. Viewing $a \in Q$ we write $\downarrow(a) = \{x \in Q \mid x \ll a\}$ and viewing $a \in Q^{\text{op}}$ we write $\uparrow(a) = \{x \in Q^{\text{op}} \mid x \succ a\}$. The two sets are identical.

For enrichment/metrization the following are quantales of interest.

Example 2. *The quantale of distance distribution functions Δ consists of all monotone functions $f: [0, \infty) \rightarrow [0, 1]$ that are continuous on the left, ordered*

pointwisely. The quantale product is given by $(f \cdot g)(t) = \bigvee_{u+v=t} f(u) \cdot g(v)$. Metric spaces taking values in Δ are known as probabilistic metric spaces (see Hofmann and Reis (2013) for a quantalic approach). The boolean truth values quantale is $\mathcal{B} = \{0 < 1\}$ with quantale product given by disjunction. Flagg's value quantales $\Omega(S)$ are, to be precise, op-quantales so we should rightfully write $\Omega(S)^{\text{op}}$ for Flagg's construction. Finally, any frame, i.e., a complete lattice L satisfying the infinite distributivity law $x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$ is a quantale with $\cdot = \wedge$. In fact, quantales were introduced in Mulvey (1986) precisely to generalize frames.

3.1. A tropical view of the triangle inequality

A function $d: X \times X \rightarrow [0, \infty]$ can be thought of as an X by X matrix. Tropical geometry studies such matrices by defining addition in $[0, \infty]$ by \bigwedge and multiplication by addition. Matrix multiplication is then re-stated using these operations, so in particular $d^2(x, z) = \bigwedge_{y \in X} d(x, y) + d(y, z)$. The triangle inequality for the distance function d is thus equivalent to the matrix inequality $d \leq d^2$ interpreted entry-wise. More accurately, in the terminology of Willerton (2013), $[0, \infty]$ with \bigwedge and $+$ is a *semi-tropical semi-ring*. The fully tropical case is $(-\infty, \infty]$ and in that context the characterization of the triangle inequality in terms of idempotency is well known (Develin and Sturmfels (2004); Izhakian et al. (2016); Kambites and Johnson (2014)). Replacing $[0, \infty]$ by the Fréchet quantale \mathcal{F} interchanges meets and joins as well as $+$ and \cdot so matrix multiplication takes the more familiar form $s^2(x, z) = \bigvee_{y \in X} s(x, y) \cdot s(y, z)$ and consequently the triangle inequality becomes $s^2 \leq s$. Denoting the identity matrix by I , the condition $d(x, x) = 0$ becomes $s(x, x) = 1$ and so can be written as $s \geq I$. The two inequalities together imply $s^2 = s$.

To conclude, a function $A: X \times X \rightarrow [0, \infty]$, can be considered as a matrix valued in $[0, \infty]$, in $\mathcal{F}_+ = [0, \infty]^{\text{op}}$, or in $\mathcal{F} = [0, 1]$. In the latter two cases A satisfies $A(x, x) = 0$ and, respectively, $A(x, x) = 1$, if, and only if, $A \geq I$. Such an A further satisfies the triangle inequality if, and only if, $A^2 = A$. The metric symmetry axiom is, of course, the symmetry of the matrix. The study of metric spaces is thus the study of symmetric idempotent matrices A with $A \geq I$. In the context of this work we drop the symmetry requirement.

3.2. Q -valued matrices

Matrices with entries in a commutative (i.e., $a \cdot b = b \cdot a$) quantale Q were introduced in Krüml (2002). In this subsection we further assume that Q is

affine, namely that the unit 1 is also the top element of Q . If X, Y are sets, then an $X \times Y$ matrix A over Q is a function $A: X \times Y \rightarrow Q$. With entry-wise ordering the set $M_Q(X, Y)$ of all such matrices is a complete lattice in which meets and joins are computed entry-wise. Given matrices $A \in M_{X \times Y}(Q)$ and $B \in M_{Y \times Z}(Q)$ their *product* is the matrix $AB \in M_{X \times Z}(Q)$ given by $(AB)(x, z) = \bigvee_{y \in Y} A(x, y) \cdot B(y, z)$. Endowed with this product, $M_X(Q) = M_{X \times X}(Q)$, for non-empty X , is a quantale (in fact $M_X(Q)$ is a *quantale algebra* over Q , a concept developed in Solovyov (2008), in much the same way that ordinary matrices form an algebra). The usual decomposition of the product of matrices as a sum of rank 1 matrices extends to the quantale setting to give $AB = \bigvee_{y \in Y} A(-, y) \cdot B(y, -)$ where $A(-, y)$ is viewed as an $X \times \{y\}$ matrix and $B(y, -)$ as a $\{y\} \times Z$ matrix. In particular, it is clear that $A(x, y)B(y, z) \leq (AB)(x, z)$ for all $x, y, z \in X$.

The usual point-to-set distance in a metric space is given by $d(x, W) = \bigwedge \{d(x, w) \mid w \in W\}$. From the dual perspective of similarity one would define $s(x, W) = \bigvee \{s(x, w) \mid w \in W\}$. Naturally, in the context of a matrix $A \in M_X(Q)$, the corresponding definition is $A(x, W) = \bigvee \{A(x, w) \mid w \in W\}$. Note that this arises as a special case of matrix multiplication, as follows. Let \bullet stand for a singleton set whose sole element, by abuse of notation, is \bullet . Let $\chi_W: X \times \bullet \rightarrow Q$ be the indicator function with $\chi_W(x, \bullet) = 1$ if $x \in W$ and 0 otherwise. Then $A(x, W) = (A \cdot \chi_W)(x, \bullet)$, where χ_W is treated as an $X \times \bullet$ matrix.

Similarly, one may define matrices with entries in an op-quantale. Clearly then $M_X(Q)^{\text{op}} = M_X(Q^{\text{op}})$ and the identity function $M_X(Q) \rightarrow M_X(Q^{\text{op}})$ is an anti-isomorphism that facilitates the change from similarity to distance syntax. For notational clarity we denote this anti-isomorphism by $A \mapsto A^{\text{op}}$. So, by our convention, $A(x, W) = 1$ is equivalent to $A^{\text{op}}(x, W) = 0$.

3.3. Set operators

We shall be interested in the relationship between a matrix $A \in M_X(Q)$ and its associated closure and interior operators. Let X be a set and consider the set Ope_X of all monotone functions $O: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, i.e., if $S \subseteq T$, then $O(S) \subseteq O(T)$. The point-wise ordering $O_1 \leq O_2$ when $O_1(S) \subseteq O_2(S)$ for all $S \subseteq X$ endows Ope_X with the structure of a complete lattice. With composition as binary operation Ope_X is nearly a quantale with unit the identity function Id ; of the two distributivity conditions only $(\bigvee O_i) \circ O = \bigvee (O_i \circ O)$ holds. Thus we obtain the *left quantale* Ope_X of set operators on

X . Two left sub-quantales of interest are Ope_X^\cup of operators that preserve finite unions and Ope_X^\cap of operators that preserve finite meets.

Recall that a closure operator \mathcal{F} on X is an operator that is extensive, namely $\mathcal{F} \geq \text{Id}$, and satisfies $\mathcal{F} \circ \mathcal{F} = \mathcal{F}$. A Kuratowski closure operator is a closure operator \mathcal{F} that preserves finite unions.

3.4. Idempotents

The tropical view of the triangle inequality points at the importance of idempotents in the quantale $M_X(Q)$ of matrices; the objects of interest satisfy $x \geq 1$ and $x^2 = x$. Similarly, an operator $O \in \text{Ope}_X$ is a closure operator precisely when $O \geq \text{I}$ and $O^2 = O$. Further, the Kuratowski closure operators are precisely the elements in Ope_X^\cup satisfying $x \geq \text{I}$ and $x^2 \geq x$. The common abstraction, a two-step passage from a left quantale first to the elements satisfying $x \geq 1$ and then to the idempotent ones, merits its own discussion.

Let Q be a left quantale and write $1^+(Q) = \{x \in Q \mid x \geq 1\}$ and $1^-(Q) = \{x \in Q \mid x \leq 1\}$. The assignments $x \mapsto x \vee 1$ and $x \mapsto x \wedge 1$ yield the functions $1^+(Q) \xleftarrow{-\vee 1} Q \xrightarrow{-\wedge 1} 1^-(Q)$, commuting with the dualities $1^+(Q^{\text{op}}) = 1^-(Q)$ and $1^-(Q^{\text{op}}) = 1^+(Q)$. For $x \in Q$ let $x^0 = 1$ and $x^{n+1} = x \cdot x^n$ for all natural numbers $n \geq 0$. When $x \geq 1$ we extend to transfinite powers by defining $x^{\beta+1} = x \cdot x^\beta$ for all ordinals β and $x^\lambda = \bigvee_{\beta < \lambda} x^\beta$ for limit ordinals λ . If $x \leq 1$, then its powers are defined similarly except that $x^\lambda = \bigwedge x^\beta$ in the limit case. Situations where x and 1 are incomparable do not concern us. Note that if Q is a left quantale, then Q^{op} is a left op-quantale and x^α is independent of whether $x \in Q$ or $x \in Q^{\text{op}}$. Consequently, any fact about $x \geq 1$ yields a dual fact about $x \leq 1$.

Proposition 1. *Let Q be a left quantale. For all $x \geq 1$ the transfinite sequence $\{x^\alpha\}$ is monotonically increasing and, dually, for $x \leq 1$ the sequence is monotonically decreasing. In either case, the sequence stabilizes.*

Proof. It is easily seen that the quantale product is monotone in each variable. A routine transfinite induction completes the argument. \square

We typically refrain from making the dual statement for $x \leq 1$ explicit.

Definition 1. *Let Q be a left quantale. For $x \geq 1$ the ordinal of stabilization for x , namely the least ordinal α with $x^\alpha = x^{\alpha+1}$, is the power dimension of x , denoted by $\pi(x)$, and we write $K(x) = x^{\pi(x)}$. Clearly, $x \cdot K(x) = K(x)$.*

For a left quantale Q we write $\triangleright(1^+(Q)) = \{x \in 1^+(Q) \mid x^2 = x\}$ and $\triangleright(1^-(Q)) = \{x \in 1^-(Q) \mid x^2 = x\}$.

Proposition 2. *The function $K: 1^+(Q) \rightarrow \triangleright(1^+(Q))$ is a reflection.*

Proof. First we verify that $K(x)$ is idempotent. If we knew that $x^\alpha x^\beta = x^{\beta+\alpha}$, then $K(x) \cdot K(x) = x^{\pi(x)} \cdot x^{\pi(x)} = x^{\pi(x)+\pi(x)} = x^{\pi(x)} = K(x)$, as needed. We show the auxiliary equality by transfinite induction on α . The case $\alpha = 0$ is clear and since $x^{\alpha+1}x^\beta = x \cdot x^\alpha \cdot x^\beta = x \cdot x^{\beta+\alpha} = x^{(\beta+\alpha)+1} = x^{\beta+(\alpha+1)}$ the successor ordinal case is dealt with. Assume now that λ is a limit ordinal and that the equality holds for all $\alpha < \lambda$. By the distributivity available in a left quantale $x^\lambda x^\beta = \bigvee_{\alpha < \lambda} x^\alpha x^\beta = \bigvee_{\alpha < \lambda} x^{\beta+\alpha}$. The ordinal $\beta + \lambda$ is also a limit ordinal and, since ordinal addition is left cancelable, $\beta + \lambda = \bigcup_{\alpha < \beta+\lambda} \alpha = \bigcup_{\alpha < \lambda} \beta + \alpha$. Since the sequence of transfinite powers is monotone this leads to $\bigvee_{\alpha < \lambda} x^{\beta+\alpha} = \bigvee_{\alpha < \beta+\lambda} x^\alpha = x^{\beta+\lambda}$.

The claim that K is a reflection, namely that $K(x) \leq y \iff x \leq y$ for all $x \in 1^+(Q)$ and $y \in \triangleright(1^+(Q))$, follows at once since $x \leq K(x)$ is clear and so $K(x) \leq y$ implies $x \leq y$. That K is a monotone operation is also clear and so if $x \leq y$, then $K(x) \leq K(y) = y$. \square

Remark 5. *If Q is a quantale, then distributivity easily implies, for all $x \geq 1$, that $\pi(x) = \omega$, the first countable ordinal, and $K(x)$ takes the simpler form $K(x) = \bigvee_{n \geq \mathbb{N}} x^n$. In the context of a comparison between $M_X(Q)$ and Ope_X the computational complexity of $K(x)$ is significant. In particular, for $A \in M_X(\mathcal{F}_+)$ the matrix $K(A)$ is the familiar metric function generated by the entries of A viewed as weights - a computable metric used often in applications (e.g., while not stated in terms of quantales, in Carlsson and Mémoli (2013) $K(A)$ is used with the quantale $[0, \infty]^{\text{op}}$ with meet as addition). The transfinite steps required in Ope_X render it less computable.*

Theorem 6. *If Q is a left quantale, then $\triangleright(1^+(Q))$ is a complete lattice with joins given by $\bigvee_{\triangleright(1^+(Q))} S = K(\bigvee_Q S)$ and K is a complete join homomorphism.*

Proof. The left adjoint K preserves all joins in $1^+(Q)$ and thus all joins in $\triangleright(1^+(Q))$ are computed via the inclusion and K . \square

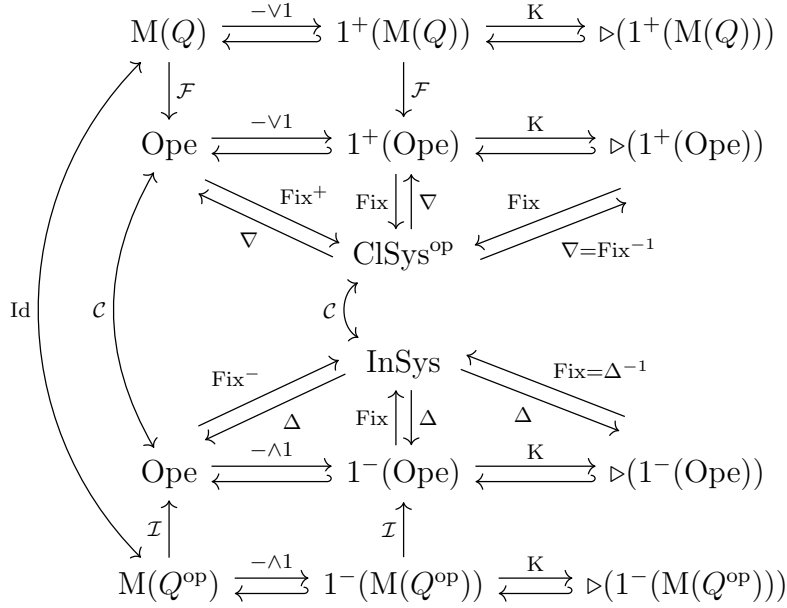
Remark 6. *Generally, $\triangleright(1^+(Q))$ does not inherit a multiplication from Q , and so usually fails to be a (left) quantale. A well known exception is when Q is a commutative ($a \cdot b = b \cdot a$) and affine (1 is the top element) quantale.*

3.5. Closure, interior, open sets, and closed sets associated with a matrix

A classical metric space (X, d) induces four entities: the collection of closed sets, the collection of open sets, a closure operator, and an interior operator. The final aim of this work is to define the corresponding notions for matrices with entries in Q and study how certain properties of Q affect their relationships. We employ similarity syntax for closedness and distance syntax for openness.

Definition 2. Let A be a matrix in $M_X(Q)$. Associated with A are the closure operator $\mathcal{F}(A)$ and the interior operator $\mathcal{I}(A)$: for $S \subseteq X$ we have $\mathcal{F}(A)(S) = \{x \in X \mid A(x, S) = 1\}$ and $\mathcal{I}(A)(S) = \{x \in X \mid \exists \varepsilon \succ 0: B_\varepsilon(x) \subseteq S\}$.

Here $B_\varepsilon(x) = \{y \in X \mid A(x, y) \prec \varepsilon\}$, using distance syntax. The classical notions are recovered when interpreting in the additive Fréchet quantale \mathcal{F}_+ . It is clear that \mathcal{F} and \mathcal{I} are operators and, conforming to our convention, $\mathcal{F}: M_X(Q) \rightarrow \text{Ope}_X$ while $\mathcal{I}: M_X(Q^{\text{op}}) \rightarrow \text{Ope}_X$. In fact, quite trivially, and without further conditions on Q , $\mathcal{F}(A)$ preserves arbitrary intersections and $\mathcal{I}(A)$ preserves arbitrary unions. However, these are minor details that should not clutter the discussion or notation; we make note of it and proceed. These two functions fit in the diagram



which we now describe in some detail. Each vertex is a complete lattice and all but three arrows are monotone mappings. The exceptions are the three curved arrows, each of which is an anti-isomorphism and its own inverse. In fact, these anti-isomorphisms preserve the (left) quantale structures; they preserve the multiplication and they interchange meets and joins. Each pair of arrows in the diagram is an adjunction with left adjoint on the left or on top. A missing adjoint indicates the given arrow has no adjoint. The identity anti-isomorphism, which in fact is three arrows, one from each entry in the top row to the corresponding one in the bottom row, represents the change of perspective that comes with considering a matrix as representing similarities (in the top row) versus distances (in the bottom row). We omit the subscript X throughout the diagram. The second row consists, from left to right, of operators, extensive operators, and closure operators. Its corresponding row in the bottom half consists of operators, shrinking operators, and interior operators. The depicted anti-isomorphism between these two rows is again a triplet of anti-isomorphisms each defined by complementation, namely $\mathcal{C}(O)(S) = O(S^c)^c$. All of the horizontal arrows from right to left are inclusions. The set ClSys is the lattice of closure systems on X , i.e., collections of subsets of X that are closed under arbitrary intersections. Similarly, InSys is the lattice of interior systems, namely collections of subsets of X closed under arbitrary unions. The anti-isomorphism is again by complementation, i.e., $\{C_i\} \mapsto \{C_i^c\}$. For an operator O the closure system $\text{Fix}^+(O)$ consists of all $S \subseteq X$ with $O(S) \subseteq S$ while $\text{Fix}^-(O)$ consists of those $S \subseteq X$ with $S \subseteq O(S)$. If O is extensive (resp. shrinking), then $S \in \text{Fix}^+(O)$ (resp. $S \in \text{Fix}^-(O)$) if, and only if, $O(S) = S$, namely when S is fixed by O . This describes the six arrows that lead to the centre of the diagram, which is very well known. In particular, each of these six arrows has a right adjoint. For a closure system \mathcal{C} the operator $\Delta(\mathcal{C})$ maps S to $\bigcap\{C \in \mathcal{C} \mid C \supseteq S\}$ and for an interior system \mathcal{I} the operator $\nabla(\mathcal{I})$ sends S to $\bigcup\{U \in \mathcal{I} \mid U \subseteq S\}$. The composition $\text{Fix}^+ \circ \mathcal{F}$ associates with a matrix A its collection of *closed* sets, i.e., those $C \subseteq X$ for which $A(x, C) = 1$ implies $x \in C$. The composition $\text{Fix}^- \circ \mathcal{I}$ associates with A its *open* sets, i.e., those $U \subseteq X$ for which if $x \in U$, then there exists $\varepsilon \succ 0$ with $B_\varepsilon(x) \subseteq U$.

An exhaustive list of commutativity relations in the diagram would be daunting. We therefore limit the discussion to the effects on the diagram bourn by properties of Q .

Definition 3. *A commutative affine quantale Q is*

1. Kuratowski if $a \vee b = 1$ implies $a = 1$ or $b = 1$;
2. Sierpiński if the set $\Downarrow(1)$ is an ideal, i.e., $x, y \ll 1 \implies x \vee y \ll 1$; and
3. triangular if $1 = \bigvee \Downarrow(1)$.

Stated in Q^{op} these conditions in distance syntax are

1. $a \wedge b = 0$ implies $a = 0$ or $b = 0$;
2. $x, y \succ 0$ implies $x \wedge y \succ 0$; and
3. $0 = \bigwedge \Uparrow(0)$.

Remark 7. *The terminology is inspired by the effects on the diagram. Historically, Kuratowski used the closedness notion as primitive while Sierpiński used openness. Without further conditions \mathcal{F} need not be a Kuratowski closure operator, and thus determine a topology, and dually nor does \mathcal{I} determine a topology. The sufficient condition to obtain an induced topology in each case is christened accordingly.*

Proposition 3. *Let Q be a commutative affine quantale. If Q is triangular, then the Kuratowski and Sierpiński conditions are equivalent.*

Proof. Suppose Q is Sierpiński and assume $a, b \in Q$ exist with $a \vee b = 1$ while $a, b < 1$. Since $a < 1$ there must exist $t_a \in \Downarrow(1)$ with $a \not\leq t_a$, since otherwise $a \geq \bigvee \Downarrow(1) = 1$. Similarly, there exists $t_b \in \Downarrow(1)$ with $b \not\leq t_b$. The ideal assumption implies $t_a \vee t_b \ll 1 = a \vee b$ and so either $t_a \vee t_b \leq a$ or $t_a \vee t_b \leq b$. But the first option leads to $t_a \leq a$ while the second one to $t_b \leq b$, neither of which is possible.

For the converse we switch to distance notation, so the op-quantale Q^{op} satisfies $\bigwedge \Uparrow(0) = 0$ and if $a \wedge b = 0$, then $a = 0$ or $b = 0$. Let $\varepsilon_1, \varepsilon_2 \succ 0$ be given and we show that $\varepsilon_1 \wedge \varepsilon_2 \succ 0$. Consider the sets $S_i = \{x \in Q^{\text{op}} \mid \varepsilon_i \not\leq x\}$, $i = 1, 2$, and note that $\alpha_i = \bigwedge S_i$ satisfies $\alpha_i > 0$ since otherwise the condition $\varepsilon_i \succ \alpha_i$ would imply that $\varepsilon_i \geq x$ for some $x \in S_i$. The Kuratowski condition implies that $\alpha_1 \wedge \alpha_2 > 0$ and so there exists $\delta \succ 0$ with $\alpha_1 \wedge \alpha_2 \not\leq \delta$. It thus follows that $\delta \notin S_i$ and so $\varepsilon_i \geq \delta$, $i = 1, 2$. In particular, $\varepsilon_1 \wedge \varepsilon_2 \geq \delta \succ 0$, completing the argument. \square

Proposition 4. *Let Q be a commutative affine quantale. If Q is triangular, then $\mathcal{F}(A) \circ \mathcal{F}(B) \leq \mathcal{F}(AB)$ and $\mathcal{I}(AB) \leq \mathcal{I}(A) \circ \mathcal{I}(B)$ for all $A, B \in \mathbf{M}_X(Q)$.*

Proof. Suppose $x \in \mathcal{F}(A)(\mathcal{F}(B)(S))$ but $x \notin \mathcal{F}(AB)(S)$, i.e., $A(x, \mathcal{F}(B)(S)) = 1$ while $(AB)(x, S) < 1$, and so there exists $t \ll 1$ with $t \not\leq (AB)(x, S)$.

From $t \ll 1 \leq A(x, \mathcal{F}(B)(S)) = \bigvee_{y \in \mathcal{F}(B)(S)} A(x, y)$ we obtain $y \in \mathcal{F}(B)(S)$ with $t \leq A(x, y)$. We shall thus arrive at a contradiction by showing that $A(x, y) \leq (AB)(x, S)$. From $B(y, S) = 1$ follows $A(x, y) = A(x, y) \cdot B(y, S) = \bigvee_{s \in S} A(x, y)B(y, s) \leq \bigvee_{s \in S} (AB)(x, s) = (AB)(x, S)$, as required.

A direct proof for the claim about \mathcal{I} can be given but we postpone the proof to obtain it as a corollary of Theorem 8. \square

Theorem 7. *Let Q be a commutative affine quantale.*

1. *If Q is Kuratowski, then $\mathcal{F}: M_X(Q) \rightarrow \text{Ope}_X$ lands in Ope_X^\cup and $\mathcal{F}: 1^+(M_X(Q)) \rightarrow 1^+(\text{Ope}_X)$ lands in $1^+(\text{Ope}_X^\cup)$.*
2. *If Q is Sierpiński, then $\mathcal{I}: M_X(Q^{\text{op}}) \rightarrow \text{Ope}_X$ lands in Ope_X^\cap and $\mathcal{I}: 1^-(M_X(Q)) \rightarrow 1^-(\text{Ope}_X)$ lands in $1^-(\text{Ope}_X^\cap)$.*

Proof. The first claim follows since $A(x, S \cup T) = A(x, S) \vee A(x, T)$ for all $x \in X$ and $S, T \subseteq X$. The second claim is the textbook proof that the intersection of two open sets U_1 and U_2 in a metric space is open; given open balls $B_{\varepsilon_i}(x) \subseteq U_i$, $i = 1, 2$, the Sierpiński condition guarantees that $\varepsilon_1 \wedge \varepsilon_2$ is admissible as a radius for a ball contained in the intersection. \square

Consequently, the first item in the theorem states that when Q is Kuratowski any matrix $A \in M_X(Q)$ has an associated Kuratowski topology $\tau_K = \mathcal{C}(\text{Fix}^+(\mathcal{F}(A)))$. The second item furnishes A with an associated Sierpiński topology $\tau_S = \text{Fix}^-(\mathcal{I}(A^{\text{op}}))$. When $Q = \mathcal{F}_+$ these constructions recover the usual topology induced by a classical metric space, so in particular they coincide. The next result underpins the relationship between the induced topologies in general. Here the similarity and distance notations interact so recall the convention regarding viewing A as a similarity matrix while writing A^{op} for distance notation.

Theorem 8. *Let Q be a commutative affine quantale. Then $\mathcal{I} \circ \text{Id} \leq \mathcal{C} \circ \mathcal{F}$, namely $\mathcal{I}(A^{\text{op}})(S) \subseteq (\mathcal{F}(A)(S^c))^c$ for all $A \in M_X(Q)$. If Q is triangular, then $\mathcal{I} \circ \text{Id} = \mathcal{C} \circ \mathcal{F}$.*

Proof. If $x \in \mathcal{I}(A^{\text{op}})(S)$ yet $x \in \mathcal{F}(A)(S^c)$, then $B_\varepsilon(x) \subseteq S$ for some $\varepsilon \succ 0$ as well as $A(x, S^c) = 1$. In terms of similarity there exists $t \ll 1$ with $A(x, y) \gg t$ implies $y \in S$. But then from $t \ll A(x, S^c)$ it follows that $t \ll A(x, y)$ for some $y \notin S$; a contradiction that implies the claimed inclusion. For the converse, in the presence of triangularity, suppose $x \notin \mathcal{F}(A)(S^c)$, namely $A(x, S^c) < 1 = \bigvee \downarrow(1)$, and so $A(x, S^c) \not\geq t$ for some $t \ll 1$. In

terms of distances there exists $\varepsilon \succ 0$ with $A^{\text{op}}(x, S^c) \not\leq \varepsilon$. It follows that if $A^{\text{op}}(x, y) \prec \varepsilon$ then $y \in S$ since otherwise $A^{\text{op}}(x, S^c) \leq A(x, y) \leq \varepsilon$. In other words, the reverse inclusion $(\mathcal{F}(A)(S^c))^c \subseteq \mathcal{I}(A)(S)$ holds. \square

Corollary 2. *Let Q be a commutative affine quantale that is both Kuratowski and Sierpiński. For a matrix $A \in M_X(Q)$ the induced Kuratowski topology τ_K and the induced Sierpiński topology τ_S satisfy $\tau_S \subseteq \tau_K$, with equality if Q is triangular.*

Remark 8. *The equality $\mathcal{F}(A) = \mathcal{C}(\mathcal{I}(A))$ implies that the point-to-set distance disaster does not occur for metric spaces valued in a triangular Q .*

Remark 9. *To complete the proof of Proposition 4 apply the equality of the theorem to the proved equality from the proposition.*

Generally, if $A \in \triangleright(1^+(M_X(Q)))$, i.e., $A \geq I$ and $A^2 = A$, it need not follow that any of the associated operators $\mathcal{F}(A), \mathcal{I}(A)$ is idempotent (as the missing vertical arrows on the right side of the diagram indicate). Triangularity of Q suffices to ensure that this will be the case.

Theorem 9. *Let Q be a commutative affine quantale.*

1. *If Q is triangular, then the restriction of \mathcal{F} to $\triangleright(1^+(M_X(Q)))$ lands in $\triangleright(1^+(\text{Ope}_X))$; if Q is also Kuratowski, then this \mathcal{F} lands in $\triangleright(1^+(\text{Ope}_X^{\cup}))$.*
2. *If Q is triangular, then the restriction of \mathcal{I} to $\triangleright(1^-(M_X(Q)))$ lands in $\triangleright(1^-(\text{Ope}_X))$; if Q is also Sierpiński, then this \mathcal{I} lands in $\triangleright(1^-(\text{Ope}_X^{\cap}))$.*

Proof. For the first claim we have to show for $A \in 1^+(M_X(Q))$ that $\mathcal{F}(A)$ is idempotent when A is. By Proposition 4, $\mathcal{F}(A) \circ \mathcal{F}(A) \leq \mathcal{F}(A^2) = \mathcal{F}(A)$ and since $\mathcal{F}(A) \geq \text{Id}$ it follows that $\mathcal{F}(A)^2 = \mathcal{F}(A)$. The argument for \mathcal{I} follows similarly, as well as by duality. \square

Consequently, when Q is Kuratowski, Sierpiński, and triangular the dia-

gram becomes

$$\begin{array}{ccccc}
M(Q) & \xleftrightarrow{-\vee 1} & 1^+(M(Q)) & \xleftrightarrow{K} & \triangleright(1^+(M(Q))) \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
\text{Ope}^\cup & \xleftrightarrow{-\vee 1} & 1^+(\text{Ope}^\cup) & \xleftrightarrow{K} & \triangleright(1^+(\text{Ope}^\cup)) \\
& \swarrow \text{Fix}^+ & \text{Fix} \downarrow \nabla & \searrow \text{Fix} & \\
& & \text{CSTop}^{\text{op}} & & \\
& \swarrow \nabla & \uparrow \text{Fix} & \swarrow \nabla = \text{Fix}^{-1} & \\
& & \text{OSTop} & & \\
& \swarrow \text{Fix}^- & \text{Fix} \downarrow \Delta & \swarrow \text{Fix} = \Delta^{-1} & \\
& & \text{Ope}^\cap & \xleftrightarrow{-\wedge 1} & 1^-(\text{Ope}^\cap) \\
& \swarrow \Delta & \uparrow \text{Fix} & \swarrow \Delta & \\
& & \text{Ope}^\cap & \xleftrightarrow{-\wedge 1} & 1^-(\text{Ope}^\cap) \\
& \swarrow \text{Fix}^- & \uparrow \mathcal{I} & \swarrow \text{Fix}^- & \\
M(Q^{\text{op}}) & \xleftrightarrow{-\wedge 1} & 1^-(M(Q^{\text{op}})) & \xleftrightarrow{K} & \triangleright(1^-(M(Q^{\text{op}}))) \\
\uparrow \mathcal{I} & & \uparrow \mathcal{I} & & \uparrow \mathcal{I}
\end{array}$$

Id (curved arrow from $M(Q)$ to $M(Q^{\text{op}})$)
 c (curved arrow from Ope^\cup to Ope^\cap)
 c (curved arrow from CSTop^{op} to OSTop)

where the column on the right depicts the lattices of metric structures on X , similarity structures on X , the familiar models of topology in terms of closure and interior operators, and, in the middle, the familiar models in terms of closed sets and open sets. Generally, the rectangles involving K do not commute, however Proposition 4 implies that $K \circ \mathcal{F} \leq \mathcal{F} \circ K$ and, dually, $K \circ \mathcal{I} \geq \mathcal{I} \circ K$ (cf. Remark 5 above).

Remark 10. For a matrix $A \geq I$ one may consider three routes to an associated Kuratowski topology, i.e., $\text{Fix}(\mathcal{F}(A))$, $\text{Fix}(K(\mathcal{F}(A)))$, and $\text{Fix}(\mathcal{F}(K(A)))$. Without the condition $A^2 = A$ these topologies, while related, may be distinct. Similarly, there is a three-fold path to an associated Sierpiński topology.

Theorem 10. Let Q be a commutative affine quantale, $A \in M_X(Q)$ with $A \geq I$ and $A^2 = A$, $x \in X$, and $r \in Q$. If Q is triangular and Sierpiński, then the open ball $B_r(x) = \{y \in X \mid A^{\text{op}}(x, y) \prec r\}$ is open in the induced Sierpiński topology. If Q is triangular and Kuratowski, then the closed ball $B^r(x) = \{y \in X \mid A(y, x) \geq r\}$ is closed in the induced Kuratowski topology.

Proof. The argument for the closed ball is as follows. Suppose $A(z, B^r(x)) = 1$ and fix $t \ll 1$. Then $t \leq A(z, y)$ for some $y \in B^r(x)$ so that $A(z, x) \geq$

$A(z, y) \cdot A(y, x) \geq t \cdot r$. Since t was arbitrary and Q is triangular it follows that $A(z, x) \geq r$, so $z \in B^r(x)$. The argument for the open ball was essentially given in section 1. \square

Note that this last theorem is a crucial component in the proof of the theorem leading to Corollary 1 - the uniqueness of the open ball topology construction. We conclude this discussion with the observation that the Kuratowski, Sierpiński, and triangularity of Q imply the agreement of the four main ways of obtaining a topology from a metric and that this common induced topology is the only one that captures the notion of continuity.

Remark 11. *It is interesting to note that complete distributivity in Flagg's value quantales does not play a role. What does play a role though is the use of the totally below relation. We make note here of several aspects of weakening the totally below relation.*

There is an important class of lattices in which the Sierpiński condition is automatic. In a lattice L the way below relation $a \triangleleft b$ means that if $b \leq \bigvee S$ for a directed set $S \subseteq L$, then $a \leq s$ for some $s \in S$ (S is directed if it is non-empty and if it contains some upper bound for any two of its elements). The lattice L is continuous if $a = \bigvee \{x \in L \mid x \triangleleft a\}$ for all $a \in L$. The formal difference in the definitions of complete distributivity and continuity is that the former uses the totally below relation while the latter makes use of the way below relation. Since totally below implies way below it follows easily that a completely distributive lattice is continuous. It is well known, and elementary to show, that in a continuous lattice $a, b \triangleleft c$ implies $a \vee b \triangleleft c$. In particular, if L is a quantale whose underlying lattice is continuous, then the Sierpiński condition holds.

This may indicate that, for the purposes of the metrization of topology, quantales with completely distributive underlying lattices are unnecessarily restrictive, opting instead for quantales with continuous underlying lattices. It is instructive to consider again the metrization of \mathbb{R}^2 by means of $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow [0, \infty] \times [0, \infty]$ with $d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|, |y_1 - y_2|)$. This time though we consider the codomain as a continuous lattice and use the way above relation \triangleright dual to the way below relation. When we previously looked at this example the open ball $B_{(1,1)}((0,0))$ was not an open set. However, the way above relation sorts this out and this time $B_{(1,1)}((0,0))$ is the interior of a square. In fact, the proof that open balls are open sets in Flagg's formalism relies on the property that if $a \triangleright \bigwedge S$, then $a \triangleright s$ for some $s \in S$. In a continuous lattice the same property holds true under the provision that S is

directed. It is easily seen that the set used when showing that an open ball is open is directed, and so the proof holds verbatim. By contrast, inspection of the proof that $d(x, S) = 0$ if, and only if, $x \in \overline{S}$ reveals that the same property is used again only without any guarantee that the set is directed. In fact, it is still the case for \mathbb{R}^2 with the current metrization in $[0, \infty] \times [0, \infty]$ viewed as a quantale with an underlying continuous lattice, that the induced topology is the Euclidean one yet, as before, $d((0, 0), \{(1, 0), (0, 1)\}) = 0$, exhibiting the same problematic phenomenon.

We thus conclude that the topological symbiosis presented for metric spaces valued in a quantale, and in particular the coincidence of the four induced topologies, does not survive weakening the totally below relation to the way below relation.

Remark 12. *The issues studied above are relevant in the more general context of quantaloid enrichment. For instance, in Hofmann and Stubbe (2018) the authors present the case of partial metric spaces in a quantaloid setting and obtain a closure operator associated with an enrichment as they note that under mild conditions on the quantaloid the construction lands in topological spaces. We expect quantaloid variants of the quantalic conditions we presented would guarantee a uniqueness of the induced topology for quantaloid enrichment.*

4. Historical epilogue

In this final section we allow the results above to interact with the historical developments of topology, particularly the birth of the topological space definitions.

4.1. Shallow and deep dualities

By history we do not mean here the fascinating tracing out of ideas that led to topology (as in, e.g., James (1999)) but rather the convoluted chains (and anti-chains) of events that culminated in the definition of a topological space. More details can be found in Moore (2008) from which we mention a quote of Carathéodori: “This duality between closed point-sets and those which consist purely of interior points is, as we will see, a very deep one”. This depth can now be measured; Theorem 8 guarantees the duality only when Q is triangular; a rather deep property of the quantale Q . The formalisms of interior operators and interior systems agree and so do the formalisms of

closure operators and closure systems. These agreements are more shallow as the Kuratowski and Sierpiński conditions are less profound than triangularity.

4.2. *Modern indifference to competing definitions and the triumph of open sets*

Bearing in mind that any attempt to summarize the birth of a central branch of modern mathematics in a couple of paragraphs is bound to grossly miss the mark, we dare say that the early definitions of what constitutes a topological space were the result of taking different approaches of synthesizing what the topology of a metric space was, and then abstracting from that. The closedness condition $d(x, C) = 0 \implies x \in C$ gives rise to definitions in terms of closure operator and closed sets. The familiar openness condition gives rise to definitions in terms of interior operator and open sets. The former was advocated by Kuratowski and the latter by Sierpiński. Theorem 7 shows that different conditions on the quantale Q are required to assure that these approaches yield topologies, and then Theorem 8 identifies the triangularity of Q as the reason for the indifference between the two approaches. As textbooks were written down these and other occupants of the primordial topological ocean were subject to evolutionary pressures resulting (with the influence of Kelly’s famous *General Topology*) with a winner; the open sets formalism. In that context we note that the analysis carried out above reveals some differences between the classical approaches that are invisible when $Q = \mathcal{F}_+$, the additive Fréchet quantale.

Despite the equivalency and the prevalence of a single formalism through survival of the fittest (or is it the ubiquity of the popular?) the relationship between a metric and its topology continued to sustain interest. In *Distance Functions and Topologies* (Galvin and Shore (1991)) a treatment is offered ending with “Thus, it seems that we have finally arrived at the intrinsic connection (even for metrics) between a distance function and its topology”. We addressed the same issue but argued differently and reached different conclusions.

4.3. *A plethora of generalizations*

A multiplicity of equivalent definitions of a given concept suggests different paths toward generalization and abstraction which often bifurcate to non-equivalent new concepts. For instance, the study of locales is motivated by the open sets definition of a topology (Johnstone (1983)). The closure

operator perspective led (Dikranjan and Giuli (1987)) to the notion of closure operator on a category (see Castellini (2012) for an extensive treatise) while the interior operator route gave birth to interior operators in a category (Vorster (2000)). Interestingly, in a general category interior and closure operators are not dual and so interior operators are investigated independently (Castellini (2011)). On a similar note, the importance of understanding the conglomerate of topologies on a given set, was recognized and used by several key figures in the development of topology (see, e.g., the Introduction of Dikranjan and Giuli (1987) for more detail). The lattice of topologies has been extensively studied from a topological perspective (Larson and A. (1975)) and closure systems received a fair amount of attention as well but from different angles (Caspard and Monjardet (2003, 2005)), witnessing again the richness provided by the duplicated definitions. On this note we mention other approaches to topology have been considered as well, for instance Razafindrakoto and Holgate (2014) and Hofmann et al. (2014).

In Lawvere (1973) Lawvere famously made a fundamental connection between metric space theory and enriched categories. The slogan *a metric space is a category enriched in $[0, \infty]$* naturally raises the question as to the precise role of $[0, \infty]$ in this connection as opposed to any other monoidal category; a question that appears not to have been explicitly investigated earlier. Indirectly, this question is lurking under the surface in recent research (Bruno (2020); Chand and Weiss (2015); Flagg (1997); Hofmann and Reis (2013); Hofmann and Waszkiewicz (2011, 2012); Tholen (2018); Weiss (2015, 2016, 2017, 2018, 2019); Zhang (2007)) on quantale enriched categories.

In the context of a lattice-theoretic approach to topology it is natural to study the lattice of classical metric structures on a set (Bruno and Weiss (2016)). In fact, (Arkhangelskii and Pontryagin, 1990, page 22) strongly resonates with the work above: “The impetus to describe a topology using a distance function is justified not only by the intuitive nature of distance considerations, but also because it allows one to use the apparatus of the real numbers and, in particular, their continuity and order properties in investigations. In order to encompass the widest range of topological spaces, one can either omit one of the metric axioms or consider metrics with values in structures different from the field \mathbb{R} of real numbers”. A step in this direction was accomplished in Bruno and Szeptycki (2017).

4.4. *An axiomatic anomaly*

Fréchet’s axiomatization of metric spaces was forged at a time when the axiomatic approach, in the modern sense, was still new. At the time, function spaces became prominent and the need to be able to speak of functions operating on other functions, rather than points, required a suitable language. However, from a modernly very naïve perspective, it is natural to seek an axiomatization of the metric space concept that avoids any explicit mention of structure such as the field of real numbers, even if the most important examples would still employ \mathbb{R} . After all, even if one is interested only in real (or complex) vector spaces it is still desirable to think of (and teach) vector spaces abstractly. Today the commonplace definition of a metric space cannot be seen as fully axiomatized. The fact that up to isomorphism the field \mathbb{R} is the unique complete ordered field probably contributes to this state of affairs. Two aspects should invalidate a hasty acceptance of \mathbb{R} as the mandatory codomain of a metric though. Firstly, most of the field axioms are irrelevant for metric purposes; the negatives are ignored and multiplication plays no role. Second, and with more dramatic consequences, the categoricity of \mathbb{R} strongly depends on the ordering being a total order; but there is no a-priori reason to assume all distance measurements must be comparable. We believe the work above supports the elimination of \mathbb{R} from the metric space definition.

Quoting from Baulieu (1989), during the early topological discussions, Claude Chevalley argued that pedagogically one needs “to put general topological spaces before metric spaces, since the notion of a metric appears more and more ridiculous to me.” and André Weil in Weil (1938) writes: “The notion of metric is used in many works of topology, and it is hard to explain why it has come to play such a role in a branch of mathematics where it is, properly speaking, merely an intrusion... This hypothesis [second countable] appears here as a maleficent parasite that infests many books and memoirs, of which it weakens the import while hindering a clear understanding of the phenomena”. Standing on the shoulders of many giants we find ourselves in stark disagreement with these two giants. The fault lies not in the metric space concept but rather in not having phrased it at the right level of generality.

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