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Beyond Markowitz with multiple criteria decision aiding

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Abstract The paper is about portfolio selection in a non-Markowitz way, involving uncertainty modeling in terms of a series of meaningful quantiles of probabilistic distributions. Considering the quantiles as evaluation criteria of the portfolios leads to a multiobjective optimization problem which needs to be solved using a Multiple Criteria Decision Aiding (MCDA) method. The primary method we propose for solving this problem is an Interactive Multiobjective Optimization (IMO) method based on so-called Dominance-based Rough Set Approach (DRSA). IMO-DRSA is composed of two phases: computation phase, and dialogue phase. In the computation phase, a sample of feasible portfolio solutions is calculated and presented to the Decision Maker (DM). In the dialogue phase, the DM indicates portfolio solutions which are relatively attractive in a given sample; this binary classification of sample portfolios into 'good' and 'others' is an input preference information to be analyzed using DRSA; DRSA is producing decision rules relating conditions on particular quantiles with the qualification of supporting portfolios as 'good'; a rule that best fits the current DM's preferences is chosen to constrain the previous multiobjective optimization in order to compute a new sample in the next

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computation phase; in this way, the computation phase yields a new sample including better portfolios, and the procedure loops a necessary number of times to end with the most preferred portfolio. We compare IMO-DRSA with two representative MCDA methods based on traditional preference models: value function (UTA method) and outranking relation (ELECTRE IS method). The comparison, which is of methodological nature, is illustrated by a didactic example.

Keywords Portfolio selection · Uncertainty modeling · Multiple criteria decision aiding · Interactive multiobjective optimization · Dominance-based rough set approach

JEL Classification C61 · C63 · G11

1 Introduction

Portfolio selection is one of the most intensively studied optimization problems in Operational Research. Its classical formulation proposed by Harry Markowitz within Mean Variance Portfolio Theory (Markowitz 1952) involves two probabilistic evaluation criteria of the portfolios: expected return (ER) and standard deviation of the return (SD). From a mathematical perspective, this problem can be stated in more general terms as follows.

A set of alternatives (portfolios) $A = \{a, b, \ldots\}$ is evaluated on n criteria constituting a consistent family $F = \{g_1, \ldots, g_i, \ldots, g_n\}$, where $g_i : A \rightarrow \mathbb{R}$. One or more of these criteria make a probabilistic evaluation, i.e. for such criteria we have $g_i : A \rightarrow \mathbb{P}$, where $\mathbb{P}$ is a set of probability distributions on $\mathbb{R}$. These probability distributions can also be non-additive and even qualitative.

As a direct interpretation of probabilistic evaluation criteria, like ER and SD, may be too difficult for the Decision Maker (DM), following (Greco et al. 2010; Matos 2006), we propose to deal with the above problem by considering a set of quantiles on the distribution of criterion $g_i \in F$, corresponding to a set of meaningful probability levels $P_i = \{p_{i1}, \ldots, p_{ih_i}\}$. To each one of these quantiles correspond values g_i^p with $p \in P_i$, such that, for each $a \in A$, $g_i^p(a)$ means that alternative a gets at least value $g_i^p(a)$ with probability p. Therefore, this approach to uncertainty handling in a decision problem consists in replacing a probabilistic criterion g_i by a corresponding set of quantiles $g_i^p, p \in P_i$, playing the role of deterministic criteria in a multiple criteria decision problem.

As shown in (Greco et al. 2010), this approach can be extended to deal also with time preferences. In this case, one is considering, moreover, a set of meaningful time epochs $\mathbb{T} = \{t_1, \ldots, t_r\}$, such that, for each criterion $g_i \in F, p \in P_i$ and $t \in \mathbb{T}$, there is a quantile $g_i^{p,t}$. For each $a \in A$, to each one of these quantiles correspond values $g_i^{p,t}(a)$ which mean that alternative a gets at least value $g_i^{p,t}(a)$ with a probability p within time epoch t. Moreover, instead of quantiles, one can consider the expected value within the quantiles, denoted by $\mathbb{E}_p[g_i(\cdot)]$, such that $\mathbb{E}_p[g_i(a)], a \in A$, means that in p % of best cases, alternative a gets an average value of $\mathbb{E}_p[g_i(a)]$.

The resulting formulation of the multiple criteria decision problem can be dealt with any methodology already existing in Multiple Criteria Decision Aiding (MCDA) (for a collection of state-of-the-art surveys on MCDA, see Figueira et al. 2005), such as value function methods (Dyer 2005; Keeney and Raiffa 1976), outranking relation methods (Figueira et al. 2005; Roy 1991), decision rule methods based on Dominance-based Rough Set Approach (Greco et al. 2001a, 2005), interactive multiobjective optimization and evolutionary multiobjective optimization (Branke et al. 2008). The proposed way of handling uncertainty in decision problems is very useful not only for portfolio optimization considered in this paper, but also for other classical problems of Operational Research, such as inventory control, supply chain management, and scheduling. It translates uncertainty to directly interpretable and understandable deterministic evaluation criteria, which makes the original decision problem solvable using an MCDA methodology.

The primary method we propose for solving the portfolio selection problem with quantile representation of uncertainty is an Interactive Multiobjective Optimization (IMO) method based on so-called Dominance-based Rough Set Approach (DRSA) (Greco et al. 2001a). IMO-DRSA (Greco et al. 2008a) is composed of two alternating phases: computation phase, and dialogue phase. In the computation phase, a sample of feasible portfolio solutions is calculated and presented to the Decision Maker (DM). In the dialogue phase, the DM indicates portfolio solutions which are relatively attractive in a given sample; this binary classification of sample portfolios into ‘good’ and ‘others’ is an input preference information to be analyzed using DRSA; DRSA is producing decision rules relating conditions on particular quantiles with the qualification of supporting portfolios as ‘good’; a rule that best fits the current DM’s preferences is chosen to constrain the previous multiobjective optimization in the next computation phase; in this way, the computation phase yields a new sample including better portfolios, and the procedure loops until the most preferred portfolio is found, or until the DM concludes that there is no satisfactory solution for the current problem setting. We compare IMO-DRSA with two representative MCDA methods based on other preference models than the set of decision rules, i.e. a value function [UTA method (Jacquet-Lagrèze and Siskos 1982)] and an outranking relation [ELECTRE IS method (Roy 1991)]. The comparison, which is of methodological nature, is illustrated by a didactic example.

The paper is organized in the following way. In Sect. 2, we shortly review the research on portfolio selection problem. In Sect. 3, we provide arguments in favor of uncertainty handling using a set of meaningful quantiles, and we present this approach in the setting of portfolio selection. In Sect. 4, we give a brief description of the IMO-DRSA method, preceded by a reminder on the Dominance-based Rough Set methodology. Section 5 provides a paradigmatic example which shows the solution process of an illustrative portfolio selection problem with quantile representation of uncertainty using the IMO-DRSA method, the UTA method, and the ELECTRE IS method. The last section groups conclusions.

2 Portfolio selection problem

The so-called modern portfolio theory (see, e.g., Elton and Gruber 1995) usually concerns two types of problems: portfolio analysis, and models of prices and returns equilibrium in the capital markets. With respect to the last problem, some well known models have been built up; among them the most applied are the Capital Asset Pricing Model (Sharpe 1963, 1964; Lintner 1965; Mossin 1966) and the Arbitrage Pricing Theory (Ross 1976), presenting different forms of equilibrium relationships subject to various assumptions. In this framework, investors should be able to determine the (theoretical) prices at which assets/securities will be sold, taking into account their risk and/or a set of other factors. One of the most important approaches, at least in theory, to portfolio analysis is the Mean Variance Portfolio Theory, usually known as Markowitz Model (1952). The main aim of this theory is to determine the composition of an "optimum portfolio": given some assumptions and properties of each individual security, find the characteristics and composition of feasible portfolios that make them preferable to others for a particular DM. In order to select the best portfolio, the principle of utility maximization is usually employed. In this kind of model, assuming the existence of a set $J = \{1,\ldots,j,\ldots,z\}$ of risk securities (assets) in the capital market, each portfolio $\mathbf{x}$ is described by the corresponding allocation vector $\mathbf{x} = [x_1,\ldots,x_z]$, with $\sum_{j=1}^{z} x_i = 1$. If one knows all possible outcomes and corresponding probabilities of each security $j \in J$, one can easily compute the expected value $E(R_j)$ of the return R_j of security j. Let us denote by $R(\mathbf{x})$ the return of portfolio $\mathbf{x}$, i.e.

$$R(\mathbf{x}) = \sum_{j=1}^{z} x_j R_j.$$

The expected value of portfolio return $R(\mathbf{x})$ is therefore $E(R(\mathbf{x})) = \sum_{j=1}^{z} x_j\ E(R_j)$. The risk is usually measured in terms of variance $\sigma^2(R(\mathbf{x}))$ that is the expected squared deviation of the portfolio return $R(\mathbf{x})$ from its expected return $E(R(\mathbf{x}))$, i.e. $\sigma^2(R(\mathbf{x})) = E[(R(\mathbf{x}) - E(R(\mathbf{x})))^2]$. Supposing that returns of risk securities are normally distributed, one can write the variance $\sigma^2(R(\mathbf{x}))$ as follows:

$$\sigma^2(R(\mathbf{x})) = \sum_{j=1}^{z} x_j^2 \sigma_j^2 + 2\sum_{j=1}^{z-1}\sum_{h>j}^{z} \sigma_{jh} x_j x_h$$

where σ_j^2 is the variance of returns of risk security $j \in J$, and σ_{jh} is the covariance of returns of risk securities $j, h \in J$. The choice of variance as risk measure is due to its easy computation and because the mean and the variance of returns contain all relevant information about the distribution if returns are normally distributed, as usually assumed. As a substitute of variance, one is often using its square root $\sigma(R(\mathbf{x}))$, called standard deviation. The main idea of the mean variance model is that the risk of a combination of assets (portfolio) is very different from the weighted average of the risk of individual assets. This could happen only in the (absolutely didactic) case that the returns of all asset of portfolio move in the same direction and proportion, that is when they are perfectly positively correlated one another, i.e. when $\sigma_{jh} = \sigma_j \sigma_h$ for all $\{j, h\} \subseteq J$. In a lot of real cases, the variance of the portfolio is smaller than the variance of either of the individual assets considered. It depends on how the returns co-move together, possibly in opposite ways; in other words, on the reciprocal independence of the returns of the assets. Therefore, in this kind of model a crucial role is played by the covariance σ_{jh} between the returns R_j and R_h of any couple of securities $j, h \in J$, that is the expected value of the product of the deviations of the return R_j on security j from its mean value $E(R_j)$ and the deviation of the return R_h of security h from its mean value $E(R_h)$, i.e.

$$\sigma_{jh} = E[(R_j - E(R_j))(R_h - E(R_h))].$$

To apply the Markowitz model we need to know z expected returns and standard deviations (one for each security), and $\frac{z(z-1)}{2}$ covariances. In the Markowitz model, the problem of portfolio selection is a bi-objective optimization problem, where the two objectives are: maximization of the expected return $E(R(\mathbf{x}))$, and minimization of the variance $\sigma^2(R(\mathbf{x}))$. In such a context, the efficient frontier is composed of portfolios $\mathbf{x}$ for which it is not possible to increase the expected return $E(R(\mathbf{x}))$ without increasing also the variance $\sigma(R(\mathbf{x}))$, or, equivalently, it is not possible to decrease the variance $\sigma^2(R(\mathbf{x}))$ without decreasing also the expected return $E(R(\mathbf{x}))$. For a given value of variance α, a portfolio $\mathbf{x}$ is in the efficient frontier if it maximizes the expected return $E(R(\mathbf{x}))$, i.e. if $\mathbf{x}$ is a solution of the following problem (P1):

$$\text{maximize}\, E(R(\mathbf{x}))$$

under the constraints

$$\sigma(R(\mathbf{x})) = \alpha,$$

$$\sum_{j=1}^{z} x_j = 1.$$

Theoretically, the efficient frontier could be computed by solving problem (P1) for all feasible values of α. Analogously, for a given value of expected return $\beta \geq R^\star$, with $R^\star$ being the expected return of the minimum variance portfolio, portfolio $\mathbf{x}$ is in the efficient frontier if it minimizes the variance $\sigma^2(R(\mathbf{x}))$, i.e. if $\mathbf{x}$ is a solution of the following problem (P2):

$$\text{minimize}\, \sigma^2(R(\mathbf{x}))$$

under the constraints

$$E(R(\mathbf{x})) = \beta,$$

$$\sum_{j=1}^{z} x_j = 1.$$

Also in this case, theoretically, the efficient frontier could be computed by solving problem (P2) for all feasible values of $\beta \geq R^\star$.

In fact, to determine the efficient frontier it is not necessary to solve infinitely many optimization problems of type (P1) or (P2). Using some properties of the efficient frontier in the Markowitz model, one can determine the efficient frontier more easily. For example, if **x** and **y** are portfolios in the efficient frontier, also the portfolio $\gamma\, \mathbf{x} + (1 - \gamma)\, \mathbf{y}$ is a portfolio in this frontier for any $\gamma \in [0, 1]$.

Very often, however, the portfolio selection problem is more complex than those formulated as (P1) and (P2). This is for several reasons, the main ones being the following:

- Short sales, i.e. negative values of $x_j, j = 1, \ldots, z$, are allowed only within given limits or completely disallowed. In these cases, to problems (P1) and (P2), one has to add the constraint $x_j \geq c_j$, where c_j is the minimum value that x_j can attain, $j = 1, \ldots, z$. If $c_j = 0$ for all $j \in J$, then short sales are not allowed.
- Presence of riskless lending or borrowing rates: in this case, it is possible that the investor could lend or could borrow from the market at a riskless rate that can be the same or different for lending and borrowing, i.e. the lending rate is equal or smaller than the borrowing rate.

Finally, to select the best portfolio, that is to find the optimal allocation vector **x**, a utility function $U(E(R(\mathbf{x})), \sigma^2(R(\mathbf{x})))$ representing the preferences of the investor with respect to expected return $E(R(\mathbf{x}))$ and variance $\sigma^2(R(\mathbf{x}))$ has to be introduced. The utility function $U(E(R(\mathbf{x})), \sigma^2(R(\mathbf{x})))$ has to be nondecreasing with respect to $E(R(\mathbf{x}))$ and nonincreasing with respect to $\sigma^2(R(\mathbf{x}))$. It is generally supposed that indifference curves of $U(E(R(\mathbf{x})), \sigma^2(R(\mathbf{x})))$, in the 'standard deviation–expected return' plane are convex functions. Of course, the elicitation of such a function is a very difficult task. Once setting the utility function, the selection of optimum portfolio is obtained by maximizing the utility function on the set of the efficient portfolios.

3 Uncertainty handling using a set of meaningful quantiles

The Markowitz model is of fundamental importance for finance theory. It suffers, however, from several problems that have been largely discussed in the literature and that have to be dealt with in real life portfolio selection. In the following, we are remembering some of the weak points of the Markowitz model:

- The mean-variance model of Markowitz fails to satisfy monotonicity properties, such that there can be a situation where the DM adopting the mean-variance approach can prefer investment *a* having a prospect dominated by a prospect of another investment *b*, i.e. the DM selects *a* even if it gives not better returns than *b* in each state of the nature. This can happen when some additional unit of return increases the average payoff while making the distribution of payoffs more dispersed around the mean, thus increasing the variance. If the positive influence of the increase in the mean on the utility function $U(E(R(\mathbf{x})),\sigma^2(R(\mathbf{x})))$ is compensated by the negative influence of the increase in the variance, the mean-variance approach can select a dominated investment. This weakness has been described, e.g., in (Bigelow 1993; Dybvig and Ingersoll 1982; Jarrow and Madan 1997). The violation of monotonicity of the mean-variance model cannot happen if the returns are normally distributed. However, this is not often the case and, therefore, the problem of monotonicity of the mean-variance model becomes relevant for portfolio selection on real financial market. Recently, Maccheroni et al. (2009) proposed a portfolio selection model based on a class of monotone preferences. On the domain of monotonicity, this model coincides with mean-variance preference model of the type

$$U(E(R(\mathbf{x})), \sigma^2(R(\mathbf{x}))) = E(R(\mathbf{x})) - \frac{\theta}{2}\sigma^2(R(\mathbf{x}))$$

with θ being an index of aversion to variance. The model differs, however, when mean-variance preferences fail to be monotone, and then becomes not economically meaningful. Observe, however, that this model leaves very little flexibility to represent DM's preferences, because it can be tuned only through the parameter θ.

- The hypothesis of normal distributions of returns, on which the Markowitz approach is largely based, has been put in doubt by the work of Mandelbrot (1963, 1967) and Fama (1963, 1965a, b) who proposed the stable Pareto distribution as a statistical model for asset return. Indeed, differently from the normal distribution that depends only on two parameters being the mean and the variance, the Pareto stable distribution is much more complex and depends on four parameters related to asymmetry, concentration, shift and skewness. With the Pareto stable distribution, the variance looses its meaning of a value of risk measure in terms of dispersion of its return around the mean typical for the normal distribution, and thus the Markowitz approach cannot be applied (Rachev et al. 2004).
- The variance as a risk measure presents several problems, the first of which is that it considers negative deviations from the average that are not desirable, but it considers also positive deviations from the average that, instead, are desirable; for this reason the same Markowitz (1959) proposed to consider the semivariance based on negative deviations only. In fact, recently, many alternative risk measures have been considered. The most well known is the so-called Value at Risk (VaR) (Jorion 2006), that is the worst α % quantile of returns, the coherent measures of risk (Artzner et al. 1999), among which the most well known risk measures are the Conditional Value at Risk (CVaR) [also known as Average Value at Risk (AVaR) and expected tail loss (ETL)] (Acerbi and Tasche 2002; Rockafellar and Uryasev 2000), and the spectral measures of risk (Acerbi 2002). Observe that apart from VaR, that is a quantile by definition, also CVaR and the spectral measures of risk are related to the concept of quantile. Indeed, CVaR is the expected value of the worst α % of cases and can be seen as an average of the quantiles of the worst α % of cases, and the spectral measure of risk is a weighted average of the quantiles in the worst α % of cases. Observe, finally, that the Cumulative Prospect Theory (CPT) model (Tversky and Kahnemann 1992), which is currently so much appreciated with respect to decision under uncertainty, can be seen as a difference of a weighted average of quantiles related to gains and losses.

Taking into account the above critics of the Markowitz model, we are proposing in this paper to handle the uncertainty of returns through consideration of a limited number of quantiles on the distribution of returns. The preferences of the decision maker will be built on those quantiles which are well understandable by her/him also in case (s)he has no statistical and financial background. This is not possible if one considers the technical parameters of the Markowitz model, i.e. expected return $E(R(\mathbf{x}))$ and variance of portfolio returns $\sigma^2(R(x))$.

It is worth stressing that our approach to portfolio selection, which takes into account DM's preferences with respect to a selected set of meaningful quantiles, overcomes all three weak points of the Markowitz model discussed above. Precisely, let us observe that:

- With respect to the monotonicity, our approach is safe because if an investment *a* is dominating another investment *b*, in the sense that it gives not smaller returns in all states of the world, then *a* dominates also *b* with respect to any subset of considered quantiles.
- With respect to non-normal distribution of returns, our model can work with all kinds of distribution because it requires only the estimation of a small number of meaningful quantiles of the portfolio return distribution.
- With respect to risk measures and Cumulative Prospect Theory, our model can be seen as a simplified model of risk that takes into account only a limited number of quantiles. Observe, however, that in our model, much more attention is paid to representation of the preferences of investors. For example, in Cumulative Prospect Theory model, only four parameters are estimated, while in our approach we can estimate as many parameters as required to represent DM's preferences with a satisfactory level of detail. Moreover, in Cumulative Prospect Theory, the parameters are estimated on the base of statistic tests trying to determine values of parameters that should be valid in average for a majority of investors, while we propose to elicit preference information by each single investor, which permits to build his/her specific preference model.

4 Dominance-based rough set approach and interactive multiobjective optimization (IMO-DRSA)

4.1 Dominance-based rough set approach (DRSA)

The Rough Set Theory proposed by Pawlak (1991) has proved to be an excellent mathematical tool for reasoning about data affected by a kind of inconsistency caused by ambiguity. This theory cannot deal, however, with data describing decision problems, because it does not take into account the preference order in data, which is so characteristic for evaluations of alternatives on considered criteria. Greco et al. (2001a) proposed a generalization of rough set theory, called Dominance-based Rough Set Approach (DRSA) which overcomes this limitation.

DRSA thus became a methodology of Multiple Criteria Decision Aiding aiming at obtaining a representation of the DM's preferences in terms of easily understandable "*if* ..., *then* ..." decision rules, on the basis of some exemplary decisions (past decisions or simulated decisions) given by the DM. In case of classification problems, which will be considered in the dialogue phase of the interactive procedure for multiobjective portfolio optimization, exemplary decisions are *classification examples*, i.e. alternatives described by a set of criteria and assigned to preference ordered classes (qualifiers), for example, classification of alternatives into 'good' and 'others'.

Criteria and the class assignment considered within DRSA correspond to the *condition attributes* and the *decision attribute*, respectively, in the classical Rough Set Approach (Pawlak 1991). Differently from the classical Rough Set Approach, in DRSA we are considering criteria that are condition attributes with value sets ordered according to decreasing or increasing preference.

Let us consider the set of n criteria $F = \{g_1, \ldots, g_n\}$, the set of their indices $I = \{1, \ldots, n\}$, and a finite universe of alternatives (objects, solutions, actions) U, such that, without loss of generality, $g_i : U \rightarrow \mathbb{R}$ for each $i = 1, \ldots, n$, and, for all objects $x, y \in U, g_i(x) \geq g_i(y)$ means that "x is at least as good as y with respect to criterion i", which is denoted as $x \succeq_i y$. Therefore, we suppose that $\succeq_i$ is a complete preorder, i.e. a strongly complete and transitive binary relation, defined on U on the basis of evaluations $g_i(\cdot)$. Note that in the context of multiobjective optimization, g_i corresponds to an objective function. Furthermore, we assume that there is a decision attribute d which makes a partition of U into a finite number of decision classes, called *classification*, **Cl** = $\{Cl_1, \ldots, Cl_m\}$, such that each alternative $x \in U$ belongs to one and only one class $Cl_t, t = 1, \ldots, m$. We suppose that the classes are preference ordered, i.e. for all $r, s = 1, \ldots, m$, such that $r > s$, the alternatives from Cl_r are preferred to the alternatives from Cl_s. More formally, if $\succeq$ is a *comprehensive weak reference relation* on U, i.e. if for all $x, y \in U, x \succeq y$ reads "x is at least as good as y", then we suppose

$$[x \in Cl_r,\ y \in Cl_s,\ r > s] \Rightarrow x \succ y,$$

where $x \succ y$ means $x \succeq y$ and *not* $y \succeq x$. The above assumptions are typical for consideration of a multiple criteria classification problem (also called ordinal classification with monotonicity constraints).

In DRSA, the explanation of the assignment of alternatives to preference ordered decision classes is made on the base of their evaluation with respect to a subset of criteria $P \subseteq I$. This explanation is called *approximation* of decision classes with respect to P. Of course, the most commonly considered case is that when $P = I$. In order to take into account the order of decision classes, in DRSA the classes are not considered one by one but, instead, unions of classes are approximated: *upward union* from class Cl_t to class Cl_m, denoted by $Cl_t^{\geq}$, and *downward union* from class Cl_t to class Cl_1, denoted by $Cl_t^{\leq}$, i.e.:

$$Cl_t^{\geq} = \bigcup_{s \geq t} Cl_s, \quad Cl_t^{\leq} = \bigcup_{s \leq t} Cl_s, \quad t = 1, \ldots, m.$$

The statement $x \in Cl_t^{\geq}$ reads "x belongs to at least class Cl_t", while $x \in Cl_t^{\leq}$ reads "x belongs to at most class Cl_t". Let us remark that $Cl_1^{\geq} = Cl_m^{\leq} = U, Cl_m^{\geq} = Cl_m$ and $Cl_1^{\leq} = Cl_1$. Furthermore, for $t = 2, \ldots, m$, we have:

$$Cl_{t-1}^{\leq} = U - Cl_t^{\geq} \text{ and } Cl_t^{\geq} = U - Cl_{t-1}^{\leq}.$$

For example, in multiple criteria classification of cars, the considered cars are the alternatives evaluated on such criteria as maximum speed, acceleration, price and fuel consumption, and they are to be assigned to one of three classes of overall evaluation: 'bad', 'medium', 'good'. Then, the upward unions are: $Cl^{\geq}_{medium}$, that is the set of all the cars classified at least 'medium' (i.e. the set of cars classified 'medium' or 'good'), and $Cl^{\geq}_{good}$, that is the set of all the cars classified at least 'good' (i.e. the set of cars classified 'good'), while the downward unions are: $Cl^{\leq}_{medium}$, that is the set of all the cars classified at most 'medium' (i.e. the set of cars classified 'medium' or 'bad'), and $Cl^{\leq}_{bad}$, that is the set of all the cars classified at most 'bad' (i.e. the set of cars classified 'bad'). Notice that, formally, also $Cl^{\geq}_{bad}$ is an upward union, as well as $Cl^{\leq}_{good}$ is a downward union, however, as 'bad' and 'good' are extreme classes, these two unions boil down to the whole universe U.

The key idea of the rough set approach is explanation (approximation) of knowledge generated by the decision attributes, by *granules of knowledge* generated by condition attributes.

In DRSA, where condition attributes are criteria and decision classes are preference ordered, the knowledge to be explained is the assignment of alternatives to *upward* and *downward unions of classes* and the granules of knowledge are sets of alternatives contained in *dominance cones* defined in the space of evaluation criteria.

We say that alternative x *dominates* alternative y with respect to $P \subseteq I$ (shortly, x *P-dominates* y), denoted by xD_Py, if for every criterion $i \in P$, $g_i(x) \geq g_i(y)$. The relation of P-dominance is reflexive and transitive, that is, it is a partial preorder.

Given a set of criteria $P \subseteq I$ and alternative $x \in U$, the granules of knowledge used for approximation in DRSA are:

- a set of alternatives dominating x, called *P-dominating set*, $D^+_P(x) = \{y \in U : yD_Px\}$,
- a set of alternatives dominated by x, called *P-dominated set*, $D^-_P(x) = \{y \in U : xD_Py\}$.

In terms of multiple criteria evaluations, the above definitions correspond to:

$$D^+_P(x) = \{y \in U : g_i(y) \geq g_i(x), \quad \text{for all} \quad i \in P\},$$

$$D^-_P(x) = \{y \in U : g_i(y) \leq g_i(x), \quad \text{for all} \quad i \in P\}.$$

Let us recall that the dominance principle requires that an alternative x dominating alternative y with respect to considered criteria (i.e. x having evaluations at least as good as y on all considered criteria) should also dominate y on the decision (i.e. x should be assigned to at least as good decision class as y). This principle is the only objective principle that is widely agreed upon in the multiple criteria comparisons of objects.

Given $P \subseteq I$, the inclusion of an alternative $x \in U$ to the upward union of classes $Cl^{\geq}_t$, $t = 2, \ldots, m$, is *inconsistent with the dominance principle* if one of the following conditions holds:

- x belongs to class Cl_t or better but it is P-dominated by an alternative y belonging to a class worse than Cl_t, i.e. $x \in Cl_t^{\geq}$ but $D_P^+(x) \cap Cl_{t-1}^{\leq} \neq \emptyset$,
- x belongs to a worse class than Cl_t but it P-dominates an alternative y belonging to class Cl_t or better, i.e. $x \notin Cl_t^{\geq}$ but $D_P^-(x) \cap Cl_t^{\geq} \neq \emptyset$.

If, given a set of criteria $P \subseteq I$, the inclusion of $x \in U$ to $Cl_t^{\geq}$, where $t = 2, \ldots, m$, is inconsistent with the dominance principle, we say that x belongs to $Cl_t^{\geq}$ *with some ambiguity*. Thus, x belongs to $Cl_t^{\geq}$ *without any ambiguity* with respect to $P \subseteq I$, if $x \in Cl_t^{\geq}$ and there is no inconsistency with the dominance principle. This means that all alternatives P-dominating x belong to $Cl_t^{\geq}$, i.e. $D_P^+(x) \subseteq Cl_t^{\geq}$.

Furthermore, alternative *x possibly belongs to* $Cl_t^{\geq}$ with respect to $P \subseteq I$ if one of the following conditions holds:

- according to decision attribute d, x belongs to $Cl_t^{\geq}$,
- according to decision attribute d, x does not belong to $Cl_t^{\geq}$, but it is inconsistent in the sense of the dominance principle with an alternative y belonging to $Cl_t^{\geq}$.

In terms of ambiguity, x possibly belongs to $Cl_t^{\geq}$ with respect to $P \subseteq I$, if x belongs to $Cl_t^{\geq}$ with or without any ambiguity. Due to the reflexivity of the dominance relation D_P, the above conditions can be summarized as follows: *x possibly belongs* to class Cl_t or better, with respect to $P \subseteq I$, if among the alternatives P-dominated by x there is an alternative y belonging to class Cl_t or better, i.e. $D_P^-(x) \cap Cl_t^{\geq} \neq \emptyset$.

The *P-lower approximation* of $Cl_t^{\geq}$, denoted by $\underline{P}(Cl_t^{\geq})$, and the *P-upper approximation* of $Cl_t^{\geq}$, denoted by $\overline{P}Cl_t^{\geq}$, are defined as follows ($t = 2, \ldots, m$):

$$\underline{P}(Cl_t^{\geq}) = \{x \in U : D_P^+(x) \subseteq Cl_t^{\geq}\},$$
$$\overline{P}(Cl_t^{\geq}) = \{x \in U : D_P^-(x) \cap Cl_t^{\geq} \neq \emptyset\}.$$

Analogously, one can define the *P-lower approximation* and the *P-upper approximation* of $Cl_t^{\leq}$ as follows ($t = 1, \ldots, m - 1$):

$$\underline{P}Cl_t^{\leq} = \left\{x \in U : D_P^-(x) \subseteq Cl_t^{\leq}\right\},$$
$$\overline{P}(Cl_t^{\leq}) = \left\{x \in U : D_P^+(x) \cap Cl_t^{\leq} \neq \emptyset\right\}.$$

The *P-boundaries* of $Cl_t^{\geq}$ and $Cl_t^{\leq}$, denoted by $Bn_P(Cl_t^{\geq})(t = 2, \ldots, m)$ and $Bn_P(Cl_t^{\leq})(t = 1, \ldots, m - 1)$, respectively, are defined as follows:

$$Bn_P\left(Cl_t^{\geq}\right) = \overline{P}\left(Cl_t^{\geq}\right) - \underline{P}\left(Cl_t^{\geq}\right), \quad Bn_P\left(Cl_t^{\leq}\right) = \overline{P}\left(Cl_t^{\leq}\right) - \underline{P}\left(Cl_t^{\leq}\right).$$

For every $P \subseteq I$, the alternatives that are consistent in the sense of dominance principle with all upward and downward unions of classes are *P-correctly classified*. For every $P \subseteq I$, the *quality of classification* **Cl** by a set of criteria $P \in I$ is defined as the ratio of the number of alternatives P-consistent with the dominance principle and the number of all the alternatives in U. Since the P-consistent alternatives are those which do not belong to any P-boundary $Bn_P(Cl_t^{\geq})$ or $Bn_P(Cl_t^{\leq})$, and $Bn_P(Cl_t^{\geq}) = Bn_P(Cl_{t-1}^{\leq})$ for $t = 2, \ldots, m$, the quality of classification Cl by a set of criteria $P \in I$ can be written as

$$\gamma_P(\mathbf{Cl}) = \frac{\left|U - \left(\bigcup_{t\in\{2,\dots,m\}} Bn_P(Cl_t^{\geq})\right)\right|}{|U|} = \frac{\left|U - \left(\bigcup_{t\in\{1,\dots,m-1\}} Bn_P(Cl_t^{\leq})\right)\right|}{|U|}.$$

γ_P (**Cl**) can be seen as a degree of consistency of the classification examples, where P is the set of criteria and **Cl** is the considered classification.

Each minimal (in the sense of inclusion) subset $P \subseteq I$, such that $\gamma_P(\mathbf{Cl}) = \gamma_I(\mathbf{Cl})$, is called a *reduct* of classification Cl, and is denoted by $RED_{\mathbf{Cl}}$. Let us remark that for a given set of classification examples one can have more than one reduct. The intersection of all reducts is called the *core*, and is denoted by $CORE_{\mathbf{Cl}}$. Criteria in $CORE_{\mathbf{Cl}}$ cannot be removed from consideration without deteriorating the quality of classification **Cl**. This means that, in set I, there are three categories of criteria:

- *indispensable* criteria included in the core,
- *exchangeable* criteria included in some reducts, but not in the core,
- *redundant* criteria, neither indispensable nor exchangeable, and thus not included in any reduct.

The dominance-based rough approximations of upward and downward unions of decision classes can serve to induce a generalized description of classification decisions in terms of "*if* ..., *then* ..." decision rules. For a given upward or downward union of classes, $Cl_t^{\geq}$ or $Cl_s^{\leq}$, the decision rules induced under a hypothesis that alternatives belonging to $\underline{P}(Cl_t^{\geq})$ or $\underline{P}(Cl_s^{\leq})$ are positive examples (that is alternatives that have to be matched by the induced decision rules), and all the others are negative (that is alternatives that have to be not matched by the induced decision rules), suggest a *certain* assignment to "class Cl_t or better", or to "class Cl_s or worse", respectively. On the other hand, the decision rules induced under a hypothesis that alternatives belonging to $\overline{P}(Cl_t^{\geq})$ or $\overline{P}(Cl_s^{\leq})$ are positive examples, and all the others are negative, suggest a *possible* assignment to "class Cl_t or better", or to "class Cl_s or worse", respectively. Finally, the decision rules induced under a hypothesis that alternatives belonging to the intersection $\overline{P}(Cl_s^{\leq}) \cap \overline{P}(Cl_t^{\geq})$ are positive examples, and all the others are negative, suggest an assignment to some classes between Cl_s and Cl_t ($s < t$). These rules are matching inconsistent alternatives $x \in U$, which cannot be assigned without doubts to classes Cl_r, $s < r < t$, because $x \notin \underline{P}(Cl_r^{\geq})$ and $x \notin \underline{P}(Cl_r^{\leq})$ for all r such that $s < r < t$.

Given the preference information in terms of classification examples, it is meaningful to consider the following five types of decision rules:

1. *certain* $D_{\geq}$ *-decision rules*, providing lower profiles (i.e. sets of minimal values for considered criteria) of alternatives belonging to $\underline{P}(Cl_t^{\geq}), P = \{i_1, \dots, i_p\} \subseteq I$: *if* $g_{i_1}(x) \geq r_{i_1}$ *and...and* $g_{i_p}(x) \geq r_{i_p}$, *then* $x \in Cl_t^{\geq}, t = 2, \dots, m, r_{i_1}, \dots, r_{i_p} \in \mathbb{R}$;
2. *possible* $D_{\geq}$ *-decision rules*, providing lower profiles of alternatives belonging to $\overline{P}(Cl_t^{\geq}), P = \{i_1, \dots, i_p\} \subseteq I$: *if* $g_{i_1}(x) \geq r_{i_1}$ *and...and* $g_{i_p}(x) \geq r_{i_p}$, *then* x *possibly belongs to* $Cl_t^{\geq}, t = 2, \dots, m, r_{i_1}, \dots, r_{i_p} \in \mathbb{R}$;

3. *certain* $D_{\leq}$ *-decision rules*, providing upper profiles (i.e. sets of maximal values for considered criteria) of alternatives belonging to $\underline{P}(Cl_t^{\leq}), P=\{i_1,\ldots,i_p\}\subseteq I$: *if* $g_{i_1}(x)\leq r_{i_1}$ *and…and* $g_{i_p}(x)\leq r_{i_p}$, *then* $x\in Cl_t^{\leq}$, $t=1,\ldots,m-1, r_{i_1},\ldots, r_{i_p}\in\mathbb{R}$;
4. *possible* $D_{\leq}$ *-decision rules*, providing upper profiles of alternatives belonging to $\overline{P}(Cl_t^{\leq}), P=\{i_1,\ldots,i_p\}\subseteq I$: *if* $g_{i_1}(x)\leq r_{i_1}$ *and…and* $g_{i_p}(x)\leq r_{i_p}$, then *x possibly belongs to* $Cl_t^{\leq}$, $t=1,\ldots,m-1, r_{i_1},\ldots,r_{i_p}\in\mathbb{R}$;
5. *approximate* $D_{\geq\leq}$ *-decision rules*, providing simultaneously lower and upper profiles of alternatives belonging to $Cl_s\cup Cl_{s+1}\cup\ldots\cup Cl_t$, without possibility of discerning to which class: *if* $g_{i_1}(x)\geq r_{i_1}$ *and…and* $g_{i_k}(x)\geq r_{i_k}$ *and* $g_{i_{k+1}}(x)\leq r_{i_{k+1}}$ *and… and* $g_{i_p}(x)\leq r_{i_p}$, *then* $x\in Cl_s\cup Cl_{s+1}\cup\ldots\cup Cl_t$, $\{i_1,\ldots,i_p\}\subseteq I$, $s,t\in\{1,\ldots,m\}, s<t, r_{i_1},\ldots,r_{i_p}\in\mathbb{R}$.

The rules of type (1) and (3) represent certain knowledge extracted from the decision table, while the rules of type (2) and (4) represent possible knowledge. Rules of type (5) represent doubtful knowledge. Algorithms for induction of decision rules from rough approximations of upward and downward unions of decision classes have been described in (Błaszczyński et al. 2011; Greco et al. 2001b).

More detailed surveys and tutorials on DRSA can be found in (Greco et al. 2005; Słowiński et al. 2005, 2009, 2012).

4.2 Interactive multiobjective optimization guided by rules generated by DRSA

Representation of preferences in terms of decision rules generated by DRSA can be fruitfully exploited in course of an Interactive Multiobjective Optimization (IMO) procedure, as proposed in (Greco et al. 2008a). An interactive procedure is composed of two alternating phases: computation phase and dialogue phase. In the computation phase, a sample of feasible solutions is calculated and presented to the DM. Then, in the dialogue phase, the DM is criticizing the proposed solutions unless one of them is completely satisfactory. In the latter case the procedure stops. Otherwise, the critic evaluation of proposed solutions is used as preference information to build a preference model of the DM. This model is used to calculate a new sample of feasible solutions in the next computation phase, with the intention to better fit the DM's preferences. In some IMO procedures the preference model appearing between the dialogue stage and the computation phase is implicit. However, it is useful when it can be explicitly shown to the DM for her approval. For this, the preference model should be easily understandable and the treatment of preference information leading to the model should be intelligible for the DM. The decision rules stemming from DRSA fulfill both these requirements.

The IMO-DRSA procedure can be summarized in the following steps, where $\mathbf{x}$ represents a solution, X is a set of solutions, and $g_i : X\rightarrow\mathbb{R}, i=1,\ldots,n$, are the considered objective functions (criteria) to be maximized:

Step 1. Generate a representative sample of feasible solutions.

Step 2. Present the sample to the DM.

Step 3. If the DM is satisfied with one solution from the sample, then this is the compromise solution and the procedure stops. Otherwise continue.

Step 4. Ask the DM to indicate a subset of 'good' solutions in the sample.

Step 5. Apply DRSA to the current sample of solutions classified into 'good' and 'others', in order to induce a set of decision rules with the following syntax: "$if \;\; g_{i_1}(\mathbf{x}) \geq \alpha_{i_1} and \ldots and \, g_{i_p}(\mathbf{x}) \geq \alpha_{i_p}, then \, solution \, \mathbf{x} \;\; is \;\; good$", $\{i_1, \ldots, i_p\} \subseteq I$.

Step 6. Present the obtained set of rules to the DM.

Step 7. Ask the DM to select the most important decision rule for her in the set.

Step 8. Adjoin the constraints $g_{i_1}(\mathbf{x}) \geq \alpha_{i_1}, \ldots, g_{i_p}(\mathbf{x}) \geq \alpha_{i_p}$ coming from the rule selected in *Step* 7 to the set of constraints of the optimization problem at hand, in order to focus on the region of feasible solutions more interesting from the point of view of DM's preferences.

Step 9. Go back to *Step* 1.

When applying the above procedure to portfolio selection problem, it is important to note that if the sample of portfolios generated in *Step* 1, and presented to the DM in *Step* 2, would be located in the efficient frontier, then it could happen that a decision rule chosen in *Step* 7 is supported by only one efficient portfolio, and therefore, in the following iteration, the constraints imposed by this decision rule would reduce the set of feasible portfolios to only one solution, being the efficient portfolio supporting this rule. This could prematurely conclude the decision process, without a sufficient exploration of the set of feasible portfolios. To avoid this risk, we present to the DM a sample of portfolios that are not in the efficient frontier.

We advise that the sample of representative feasible portfolios is built in co-operation with the DM (possibly assisted by an analyst) who can check different feasible allocations to the considered assets until the quantiles of the return distribution of a portfolio get some interesting values. Such an interesting feasible portfolio is then stored with its description and the process can restart to build a new interesting feasible portfolio. When the number of portfolios built in this way is sufficiently large (usually a dozen or two), then the DM can pass to *Step* 3, and the procedure either stops if one of these portfolios is completely satisfactory, or continues in *Step* 4 to indicate a subset of portfolios relatively 'good' among the current sample. Starting from the second iteration, the procedure for building the sample of representative feasible portfolios has to take into account the constraints imposed in terms of minimum values of some quantiles by decision rules selected in *Step* 7 of previous iterations. Indeed, each time a selected decision rule adds new constraints to the set of constraints defining feasible portfolios, the procedure reduces the set of feasible solutions to a smaller subset of feasible portfolios containing a smaller part of the efficient frontier. This gradual reduction of the feasible set and its corresponding efficient frontier ensures convergence of the whole procedure.

When the DM stops the procedure because (s)he is satisfied, the selected portfolio $\mathbf{x}^\circ$. is, in general, not efficient, and thus it has to be projected on the efficient frontier, giving an efficient portfolio $\mathbf{x}^\star$. The projection can be done in several ways, e.g.:

- select $\mathbf{x}^{\star} = \mathbf{x}^{\circ}_{E(R(\mathbf{x}))}$ that satisfies all the constraints imposed by selected decision rules and, while keeping the variance of $\mathbf{x}^{\circ}$, maximizes the expected return,
- select $\mathbf{x}^{\star} = \mathbf{x}^{\circ}_{\sigma(R(\mathbf{x}))}$ that satisfies all the constraints imposed by selected decision rules and, while keeping the expected return of $\mathbf{x}^{\circ}$, minimizes the variance,
- select $\mathbf{x}^{\star} = \mathbf{x}^{\circ}_{R_{p\%}(\mathbf{x})}$ that satisfies all the constraints imposed by selected decision rules and while keeping the expected return and the variance of $\mathbf{x}^{\circ}$, maximizes quantile $R_{p\ \%}$ considered the most important.

As acknowledged by practical experiments, the final portfolio $\mathbf{x}^{\star}$ obtained in this way is very close to $\mathbf{x}^{\circ}$ and, practically, equivalent for the DM. If this would not be the case, one should better come back to the interaction procedure and continue the exploration of the set of feasible portfolios in order to get closer with the sample of feasible portfolios to the efficient frontier.

The syntax of rules induced in *Step* 5 corresponds to maximization of objective functions. In case of minimization of an objective function g_i, the condition concerning this objective in the decision rule should have the form $g_i(\mathbf{x}) \leq \alpha_i$.

Restriction of the set of feasible solutions cannot be considered as irreversible. Indeed, the DM can retract to the set of feasible solutions considered in one of previous iterations and continue from this point, exploring a different region of the feasible set. This is in the spirit of a learning oriented conception of multiobjective interactive optimization, i.e. it agrees with the idea that the interactive procedure permits the DM to learn about his/her preferences and about the "shape" of the set of feasible solutions (Belton et al. 2008).

5 The portfolio selection problem as a paradigmatic example

We consider three assets, s_1, s_2 and s_3, having the following expected returns, respectively: $R_1 = 12\ \%$, $R_2 = 14\ \%$ and $R_3 = 16\ \%$. Let us suppose that we have the variance-covariance matrix Σ shown in Table 1. Each portfolio is characterized by the vector $\mathbf{x} = [x_1, x_2, x_3], x_1, x_2, x_3 \in \mathbb{R}$, $x_1 + x_2 + x_3 = 1$, where x_1, x_2 and x_3 represents the percentage invested in assets s_1, s_2, s_3, respectively. Since we suppose that the decision maker wants a portfolio without short sales, we add the constraints $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. For the sake of simple calculation of quantiles of the portfolio return distribution, we assume that returns are normally distributed. From portfolio theory (Markowitz 1952) we know that, under the hypothesis that the expected returns are normally distributed, the portfolio $\mathbf{x}$ has a return normally distributed with the expected return

$$E(R(\mathbf{x})) = x_1R_1 + x_2R_2 + x_3R_3$$

and standard deviation

Table 1 Return variance-covariance matrix

	s_1	s_2	s_3
s_1	100	50	−20
s_2	50	200	10
s_3	−20	10	300

$$\sigma(R(\mathbf{x})) = \sqrt{\sum_{i=1}^{3} x_i^2\sigma_i^2 + 2\sum_{i,j;i<j} \sigma_{ij}x_ix_j}.$$

$\sigma(R(\mathbf{x}))$ is interpreted as a risk measure to be minimized. Therefore, usually, the financial portfolio is selected by maximizing the expected return $E(R(\mathbf{x}))$ under the constraint $\sigma(R(\mathbf{x})) \leq v$, where v is the standard deviation of the portfolios considered acceptable. Alternatively, one can minimize $\sigma(R(\mathbf{x}))$, under the constraint $E(R(\mathbf{x})) \geq t$, where t is a minimal level of expected return.

In fact, the concept of variance is not quite clear for an investor who, usually, is not an expert in finance. We believe that it is much more understandable for an investor to reason in terms of minimal or maximal gain or loss attached to a set of meaningful probability levels, such as "the minimum gain with 75 % probability" or the "maximum loss with 99 % probability". In this perspective, we consider the following set of meaningful probability levels $\Pi = \{1\,\%, 25\,\%, 50\,\%, 75\,\%, 99\,\%\}$. Since to each portfolio $\mathbf{x}$ corresponds a normal distribution of returns with expected return $E(R(\mathbf{x}))$ and standard deviation $\sigma(R(\mathbf{x}))$, quantiles of the portfolio return distribution corresponding to probability $p \in \Pi$, denoted by $R_p(\mathbf{x})$, can be computed as follows:

$$\begin{aligned} R_{1\,\%}(\mathbf{x}) &\approx E(R(\mathbf{x})) + 2.33\sigma(R(\mathbf{x})), \\ R_{25\,\%}(\mathbf{x}) &\approx E(R(\mathbf{x})) + 0.67\sigma(R(\mathbf{x})), \\ R_{50\,\%}(\mathbf{x}) &= E(R(\mathbf{x})), \\ R_{75\,\%}(\mathbf{x}) &\approx E(R(\mathbf{x})) - 0.67\sigma(R(\mathbf{x})), \\ R_{99\,\%}(\mathbf{x}) &\approx E(R(\mathbf{x})) - 2.33\sigma(R(\mathbf{x})). \end{aligned}$$

Let us remark that quantiles $R_p(\mathbf{x})$ can be estimated also for other probability distributions, such as stable Pareto distribution. Thus, differently from the original Markowitz model, our approach can be applied to any probability distribution of portfolio returns.

5.1 IMO-DRSA

The first sample of feasible portfolios shown in Table 2 was generated and presented to the DM. The DM evaluates the portfolios and indicates those which are relatively 'good', as shown in the last column of Table 2.

Using DRSA to preference information contained in Table 2, one obtains 19 decision rules describing portfolios with Overall Evaluation 'good'. The most interesting decision rules, which are supported by a relatively large number of

Table 2 Set of representative feasible portfolios in the first iteration

Portfolio	x_1	x_2	x_3	$E(R(\mathbf{x}))$	$\sigma(R(\mathbf{x}))$	$R_{1\,\%}(\mathbf{x})$	$R_{25\,\%}(\mathbf{x})$	$R_{50\,\%}(\mathbf{x})$	$R_{75\,\%}(\mathbf{x})$	$R_{99\,\%}(\mathbf{x})$	Class
P1	0.39	0.29	0.32	13.86	8.43	33.5	19.51	13.86	8.21	−5.78	*
P2	0.21	0.22	0.57	14.71	10.64	39.49	21.84	14.71	7.58	−10.07	*
P3	0.01	0.48	0.51	15.01	11.39	41.55	22.64	15.01	7.37	−11.54	*
P4	0.61	0.04	0.35	13.5	8.3	32.82	19.05	13.5	7.94	−5.83	*
P5	0.43	0.39	0.18	13.52	8.58	33.51	19.27	13.52	7.77	−6.48	Good
P6	0.51	0.46	0.03	13.04	9.58	35.37	19.46	13.04	6.62	−9.29	*
P7	0.52	0.2	0.29	13.54	8.03	32.24	18.92	13.54	8.16	−5.16	Good
P8	0.54	0.04	0.42	13.75	8.7	34.03	19.58	13.75	7.92	−6.53	Good
P9	0.34	0.21	0.45	14.22	9.16	35.57	20.36	14.22	8.08	−7.13	*
P10	0.54	0.22	0.23	13.38	7.99	32.01	18.74	13.38	8.03	−5.24	Good
P11	0.6	0.15	0.25	13.28	7.94	31.78	18.6	13.28	7.97	−5.21	*
P12	0.53	0.19	0.28	13.5	8	32.14	18.86	13.5	8.14	−5.14	Good
P13	0.37	0.26	0.37	14	8.62	34.09	19.78	14	8.224	−6.09	Good
P14	0.21	0.34	0.46	14.5	9.79	37.3	21.06	14.5	7.94	−8.3	Good
P15	0.04	0.41	0.54	15	11.33	41.39	22.59	15	7.41	−11.39	*
P16	0	0.25	0.75	15.5	13.6	47.19	24.61	15.5	6.39	−16.19	*
P17	0	0	1	16	17.32	56.36	27.6	16	4.4	−24.36	Good

'good' portfolios, are the following (in the parentheses there are identifiers of portfolios matching the corresponding rule):

Rule 1.1: If $R_{1\,\%}(\mathbf{x}) \geq 32.01\,\%$ and $R_{99\,\%}(\mathbf{x}) \geq -5.24\,\%$, then portfolio $\mathbf{x}$ is 'good', ($P7$, $P10$, $P12$)

Rule 1.2: If $R_{25\,\%}(\mathbf{x}) \geq 18.74\,\%$ and $R_{99\,\%}(\mathbf{x}) \geq -5.24\,\%$, then portfolio $\mathbf{x}$ is 'good', ($P7$, $P10$, $P12$)

Rule 1.3: If $R_{50\,\%}(\mathbf{x}) \geq 13.38\,\%$ and $R_{99\,\%}(\mathbf{x}) \geq -5.24\,\%$, then portfolio $\mathbf{x}$ is 'good', ($P7$, $P10$, $P12$)

Rule 1.4: If $R_{75\,\%}(\mathbf{x}) \geq 8.03\,\%$ and $R_{99\,\%}(\mathbf{x}) \geq -5.24\,\%$, then portfolio $\mathbf{x}$ is 'good', ($P7$, $P10$, $P12$)

Rule 1.5: If $R_{1\,\%}(\mathbf{x}) \geq 33.51\,\%$ and $R_{99\,\%}(\mathbf{x}) \geq -6.48\,\%$, then portfolio $\mathbf{x}$ is 'good', ($P5$, $P13$)

Rule 1.6: If $R_{1\,\%}(\mathbf{x}) \geq 34.03\,\%$ and $R_{99\,\%}(\mathbf{x}) \geq -6.53\,\%$, then portfolio $\mathbf{x}$ is 'good', ($P8$, $P13$)

Rule 1.7: If $R_{75\,\%}(\mathbf{x}) \geq 16\,\%$, then portfolio $\mathbf{x}$ is good, ($P17$)

Rule 1.8: If $R_{50\,\%}(\mathbf{x}) \geq 14.5\,\%$ and $R_{99\,\%}(\mathbf{x}) \geq -8.3\,\%$, then portfolio $\mathbf{x}$ is 'good'. ($P14$)

The DM found rule 1.4 to be the most representative of his/her preferences and, consequently, the following constraints were added to the original optimization problem:

$$R_{75\,\%}(\mathbf{x}) = E(R(\mathbf{x})) - 0.67\sigma(R(\mathbf{x})) \geq 8.03\,\%,$$

$$R_{99\,\%}(\mathbf{x}) = E(R(\mathbf{x})) - 2.33\sigma(R(\mathbf{x})) \geq -5.24\,\%.$$

Table 3 Set of representative feasible portfolios in the second iteration

Portfolio	x_1	x_2	x_3	$E(R(\mathbf{x}))$	$\sigma(R(\mathbf{x}))$	$R_{1\,\%}(\mathbf{x})$	$R_{25\,\%}(\mathbf{x})$	$R_{50\,\%}(\mathbf{x})$	$R_{75\,\%}(\mathbf{x})$	$R_{99\,\%}(\mathbf{x})$	Class
P1′	0.52	0.2	0.29	13.86	8.03	32.24	18.92	13.54	8.16	−5.16	*
P2′	0.54	0.19	0.27	14.71	7.98	32.04	18.8	13.45	8.11	−5.13	Good
P3′	0.54	0.2	0.26	15.01	7.98	32.05	18.8	13.45	8.1	−5.15	*
P4′	0.5	0.23	0.27	13.5	8.05	32.29	18.93	13.53	8.14	−5.22	Good
P5′	0.53	0.18	0.29	13.52	8.02	32.2	18.89	13.52	8.15	−5.16	Good
P6′	0.57	0.16	0.27	13.04	7.96	31.93	18.72	13.39	8.06	−5.14	Good
P7′	0.54	0.16	0.3	13.54	8.02	32.2	18.89	13.51	8.14	−5.18	*
P8′	0.52	0.21	0.27	13.75	8.01	32.14	18.85	13.49	8.12	−5.17	*
P9′	0.59	0.12	0.29	14.22	7.99	32	18.74	13.39	8.04	−5.22	*
P10′	0.59	0.12	0.3	13.38	8	32.06	18.78	13.42	8.05	−5.23	*
P11′	0.58	0.16	0.26	13.35	7.94	31.86	18.67	13.35	8.03	−5.16	*
P12′	0.49	0.2	0.3	13.62	8.1	32.49	19.05	13.62	8.2	−5.24	Good
P13′	0.57	0.17	0.27	13.4	7.96	31.94	18.73	13.4	8.07	−5.14	*
P14′	0.55	0.18	0.27	13.45	7.97	32.03	18.79	13.45	8.11	−5.13	Good
P15′	0.53	0.18	0.28	13.5	8	32.14	18.86	13.5	8.14	−5.14	*
P16′	0.5	0.2	0.3	13.6	8.07	32.41	19.01	13.6	8.19	−5.21	Good

Then, the second sample of representative feasible portfolios shown in Table 3 was generated and presented to the DM. Upon reflection, the DM indicated those portfolios in the second sample which are relatively ‘good’. This information is displayed in the last column of Table 3.

Using DRSA to preference information contained in Table 3, one obtains the following 5 decision rules describing portfolios with Overall Evaluation ‘good’:

Rule 2.1: If $R_{1\,\%}(\mathbf{x}) \geq 32.29\,\%$, then portfolio $\mathbf{x}$ is ‘good’, ($P4'$, $P12'$, $P16'$)

Rule 2.2: If $R_{25\,\%}(\mathbf{x}) \geq 18.93\,\%$, then portfolio $\mathbf{x}$ is ‘good’, ($P4'$, $P12'$, $P16'$)

Rule 2.3: If $R_{50\,\%}(\mathbf{x}) \geq 13.6\,\%$, then portfolio $\mathbf{x}$ is ‘good’, ($P12'$, $P16'$)

Rule 2.4: If $R_{50\,\%}(\mathbf{x}) \geq 8.19\,\%$, then portfolio $\mathbf{x}$ is ‘good’, ($P12'$, $P16'$)

Rule 2.5: If $R_{99\,\%}(\mathbf{x}) \geq -5.13\,\%$, then portfolio $\mathbf{x}$ is ‘good’. ($P2'$, $P14'$)

Now, the DM considered rule 2.2 to be the most representative of his/her preferences and, consequently, the following constraint was added to the original optimization problem:

$$R_{25\,\%}(\mathbf{x}) = E(R(\mathbf{x})) - 0.67\sigma(R(\mathbf{x})) \geq 18.93\,\%.$$

Next, the third sample of representative feasible portfolios shown in Table 4 was generated and presented to the DM. Upon a new reflection, the DM indicated those portfolios in the third sample which are relatively ‘good’. This information is displayed in the last column of Table 4.

The DM considered portfolio P12″ as the most satisfactory one, and the procedure stopped here. Since portfolio P12″ is not in the efficient frontier, the following efficient portfolios that improve P12″ have been computed:

Table 4 Set of representative feasible portfolios in the third iteration

Portfolio	x_1	x_2	x_3	$E(R(\mathbf{x}))$	$\sigma(R(\mathbf{x}))$	$R_{1\,\%}(\mathbf{x})$	$R_{25\,\%}(\mathbf{x})$	$R_{50\,\%}(\mathbf{x})$	$R_{75\,\%}(\mathbf{x})$	$R_{99\,\%}(\mathbf{x})$	Class
P1″	0.5	0.2	0.3	13.59	8.07	32.38	18.99	13.59	8.18	–5.2	*
P2″	0.49	0.2	0.3	13.62	8.09	32.48	19.04	13.62	8.2	–5.24	*
P3″	0.5	0.19	0.31	13.62	8.09	32.47	19.04	13.62	8.2	–5.23	*
P4″	0.51	0.2	0.29	13.55	8.03	32.27	18.93	13.55	8.17	–5.17	*
P5″	0.5	0.22	0.28	13.55	8.05	32.31	18.95	13.55	8.16	–5.2	*
P6″	0.5	0.21	0.28	13.55	8.04	32.29	18.94	13.55	8.16	–5.19	*
P7″	0.52	0.17	0.3	13.56	8.04	32.3	18.95	13.56	8.17	–5.19	*
P8″	0.5	0.21	0.29	13.59	8.07	32.38	18.99	13.59	8.18	–5.21	*
P9″	0.49	0.23	0.28	13.58	8.07	32.39	18.99	13.58	8.17	–5.23	*
P10″	0.5	0.2	0.3	13.56	8.05	32.33	18.96	13.56	8.16	–5.21	*
P11″	0.52	0.19	0.29	13.55	8.03	32.26	18.93	13.55	8.17	–5.17	*
P12″	0.49	0.2	0.3	13.62	8.1	32.49	19.05	13.62	8.2	–5.24	Best
P13″	0.51	0.2	0.29	13.57	8.05	32.33	18.96	13.57	8.18	–5.19	*
P14″	0.5	0.2	0.3	13.6	8.07	32.41	19.01	13.6	8.19	–5.21	*

- portfolio $P12''_{E(R(\mathbf{x}))}$ satisfying all the constraints imposed by selected decision rules ($R_{75\,\%}(\mathbf{x}) \geq 8.03\,\%$, $R_{99\,\%}(\mathbf{x}) \geq -5.24\,\%$ and $R_{25\,\%}(\mathbf{x}) \geq 18.93\,\%$) and, while keeping the variance of $P12''$, maximizes the expected return,
- portfolio $P12''_{\sigma(R(\mathbf{x}))}$ satisfying all the constraints imposed by selected decision rules ($R_{75\,\%}(\mathbf{x}) \geq 8.03\,\%$, $R_{99\,\%}(\mathbf{x}) \geq 5.24\,\%$ and $R_{25\,\%}(\mathbf{x}) \geq 18.93\,\%$) and, while keeping the expected return of $P12''$, minimizes the variance,
- portfolios $P12''_{R_{p\%}(\mathbf{x}))}$ satisfying all the constraints imposed by selected decision rules and, while keeping the expected return and the variance of $P12''$, maximizes quantiles $R_{p\,\%} = 1\,\%, 25\,\%, 75\,\%, 99\,\%$.

All the above portfolios were practically equal to $P12''$ which is, therefore, the best portfolio for the DM.

Observe, moreover, that decision rules 1.4 and 2.2 give explanations and justifications of the portfolio selection, i.e. the arguments for the final selection of portfolio $P12''$ are the following:

- in 75 % of cases it gives a return not smaller than 8.03 % (first condition in rule 1.4),
- in 99 % of cases it gives a return not smaller than −5.24 % (second condition in rule 1.4),
- in 25 % of cases it gives a return not smaller than 18.93 % (the condition in rule 2.2).

5.2 UTA

The same portfolio selection problem has been dealt with by the ordinal regression method *UTA* (Jacquet-Lagrèze and Siskos 1982).

5.2.1 A short reminder on the UTA method

The UTA (UTilités Additives) method, proposed by Jacquet-Lagrèze and Siskos (1982), aims at inferring one or more additive value functions from a given ranking on a reference set of alternatives A^R. The method uses special linear programming techniques to assess these functions so that the ranking(s) obtained through these functions on A^R is (are) as consistent as possible with the given one.

The criteria aggregation model in UTA is assumed to be an additive value function of the following form: for any $a \in A$,

$$U[\mathbf{g}(a)] = \sum_{i=1}^{n} p_i u_i[g_i(a)] \tag{1}$$

subject to normalization constraints:

$$\begin{cases} \sum_{i=1}^{n} p_i = 1 \\ u_i(g_{i^*}) = 0, \quad u_i(g_i^*) = 1, \quad \forall i = 1, 2, \cdots, n. \end{cases} \tag{2}$$

where $u_i, i = 1, 2, \cdots, n$, are non-decreasing real valued functions, named marginal value or utility functions, which are normalized between 0 and 1, p_i is the weight of u_i, and g_{i^*} and g_i^* are the worst and the best considered values of criterion g_i, respectively.

Both the marginal and the comprehensive value functions have the monotonicity property which, in the case of the comprehensive value function, has the following form: for any $a, b \in A$,

$$\begin{cases} U[\mathbf{g}(a)] > U[\mathbf{g}(b)] \Leftrightarrow a \succ b \, (\textit{preference}) \\ U[\mathbf{g}(a)] = U[\mathbf{g}(b)] \Leftrightarrow a \sim b \, (\textit{indifference}) \end{cases} \tag{3}$$

The UTA method infers an unweighted form of the additive value function, equivalent to the form defined by (1) and (2), as follows: for any $a \in A$,

$$U[\mathbf{g}(a)] = \sum_{i=1}^{n} u_i[g_i(a)] \tag{4}$$

subject to normalization constraints:

$$\begin{cases} \sum_{i=1}^{n} u_i(g_i^*) = 1 \\ u_i(g_{i^*}) = 0, \qquad \forall i = 1, 2, \cdots, n. \end{cases} \tag{5}$$

Of course, the existence of such a preference model assumes the preferential independence of the criteria for the DM (Keeney and Raiffa 1976).

Table 5 Set of reference portfolios presented to the DM

Investments	$E(R(\mathbf{x}))$ (%)	$\sigma(R(\mathbf{x}))$ (%)	$R_{1\,\%}(\mathbf{x})$ (%)	$R_{25\,\%}(\mathbf{x})$ (%)	$R_{50\,\%}(\mathbf{x})$ (%)	$R_{75\,\%}(\mathbf{x})$ (%)	$R_{99\,\%}(\mathbf{x})$ (%)	Preference order
P1	12	4	21.32	14.68	12.00	9.32	2.68	2°
P2	17	7	33.31	21.69	17.00	12.31	0.69	1°
P3	5	1	7.33	5.67	5.00	4.33	2.67	5°
P4	10	3	16.99	12.01	10.00	7.99	3.01	3°
P5	15	10	38.30	21.70	15.00	8.30	–8.30	7°
P6	20	12	47.96	28.04	20.00	11.96	–7.96	6°
P7	7	2	11.66	8.34	7.00	5.66	2.34	4°

Table 6 Values of the marginal value functions in the breakpoints of considered meaningful quantiles (values normalized to 100)

$R_{1\,\%}$ (%)	$u_{1\,\%}$ ($R_{1\,\%}$)	$R_{25\,\%}$ (%)	$u_{25\,\%}$ ($R_{25\,\%}$)	$R_{50\,\%}$ (%)	$u_{50\,\%}$ ($R_{50\,\%}$)	$R_{75\,\%}$ (%)	$u_{75\,\%}$ ($R_{75\,\%}$)	$R_{99\,\%}$ (%)	$u_{99\,\%}$ ($R_{99\,\%}$)
5	0	5	0	0	0	0	0	−10.00	0
15	0	10	0	5	0	4	0	−7.50	0
30	0	15	8.02	10	0	8	0	−5.00	0
45	0	20	8.02	15	1.77	12	5.4	−2.50	0
60	0	25	13.08	20	1.77	16	5.4	0.00	48.93
		30	43.9	25	1.77			2.50	48.93
								5.00	48.93

5.2.2 Application of the UTA method to multiple criteria portfolio selection

We are considering a value function assigning to each portfolio $\mathbf{x}$ the following comprehensive evaluation:

$$U(\mathbf{x}) = \sum_{p\in\Pi} u_p(R_p(\mathbf{x})).$$

Considering the same set of meaningful probabilities as before, $\Pi = \{1\,\%, 25\,\%, 50\,\%, 75\,\%, 99\,\%\}$, the value function (4) becomes:

$$U(\mathbf{x}) = u_{1\,\%}(R_{1\,\%}(\mathbf{x})) + u_{25\,\%}(R_{25\,\%}(\mathbf{x})) + u_{50\,\%}(R_{50\,\%}(\mathbf{x})) + u_{75\,\%}(R_{75\,\%}(\mathbf{x})) + u_{99\,\%}(R_{99\,\%}(\mathbf{x})).$$

To build the value function via ordinal regression, the DM is asked to preference order seven uncertain investments (reference portfolios) with normal distributions of returns, presented in Table 5.

Table 6 contains the inferred values of the marginal utility functions in the breakpoints of the meaningful quantiles corresponding to criteria of the problem at hand.

Using the information contained in Table 6, one can calculate for each portfolio the overall value assigned to it by the comprehensive value function, as well as the marginal values of all considered quantiles. For example, considering portfolio **x** defined by investing $x_1 = 37$ % on s_1, $x_2 = 26$ % on s_2, and $x_3 = 37$ % on s_3, we get an expected return $E(R(\mathbf{x})) = 14$ % and standard deviation $\sigma(\mathbf{x}) = 8.7$ %. Thus, the meaningful quantiles take for this portfolio the following values:

- $R_{1\,\%}(\mathbf{x}) = 34$ %, i.e. there is 1 % probability of gaining not less than 34 %,
- $R_{25}\,(\mathbf{x}) = 20$ %, i.e. there is 25 % probability of gaining not less than 20 %,
- $R_{50\,\%}(\mathbf{x}) = 14$ %, i.e. there is 50 % probability of gaining not less than 14 %,
- $R_{75\,\%}(\mathbf{x}) = 8$ %, i.e. there is 75 % probability of gaining not less than 8 %,
- $R_{99\,\%}(\mathbf{x}) = -6$ %, i.e. there is 99 % probability of gaining not less than -6 %.

The value function gives to each one of these quantiles the following values on a scale ranging from 0 to 100:

- $u_{1\,\%}(R_{1\,\%}(\mathbf{x})) = u_{1\,\%}(34\,\%) = 0$, i.e. the gain not smaller than 34 % with 1 % probability gets marginal value 0,
- $u_{25\,\%}(R_{25\,\%}(\mathbf{x})) = u_{25\,\%}(20\,\%) = 8.02$, i.e. the gain not smaller than 20 % with 25 % probability gets marginal value 8.02,
- $u_{50\,\%}(R_{50\,\%}(\mathbf{x})) = u_{50\,\%}(14\,\%) = 1.41$, i.e. the gain not smaller than 14 % with 50 % probability gets marginal value 1.41,
- $u_{75\,\%}(R_{75\,\%}(\mathbf{x})) = u_{75\,\%}(8\,\%) = 0.3$, i.e. the gain not smaller than 8 % with 75 % probability gets marginal value 0.3,
- $u_{99\,\%}(R_{99\,\%}(\mathbf{x})) = u_{99\,\%}(-6\,\%) = 0$, i.e. the gain not smaller than -6 % with 99 % probability gets marginal value 0.

Therefore, the overall value given by the value function U to portfolio **x** is:

$$\begin{aligned} U(\mathbf{x}) &= u_{1\,\%}(R_{1\,\%}(\mathbf{x})) + u_{25\,\%}(R_{25\,\%}(\mathbf{x})) + u_{50\,\%}(R_{50\,\%}(\mathbf{x})) + u_{75\,\%}(R_{75\,\%}(\mathbf{x})) + u_{99\,\%}(R_{99\,\%}(\mathbf{x})) \\ &= u_{1\,\%}(34\,\%) + u_{25\,\%}(20\,\%) + u_{50\,\%}(14\,\%) + u_{75\,\%}(8\,\%) + u_{99\,\%}(-6\,\%) = 9.43. \end{aligned}$$

In Table 7, there are some representative portfolios of the mean-variance efficient frontier. For each portfolio **x**, there is given the composition, the expected return $E(R(\mathbf{x}))$, the standard deviation $\sigma(\mathbf{x})$ and the meaningful quantiles. The same portfolios are presented in Table 8, where there are marginal values $u_p(R_p(\mathbf{x}))$ of all quantiles $R_p(\mathbf{x})$, as well as the overall value $U(\mathbf{x})$ of each portfolio.

5.3 ELECTRE IS

5.3.1 A short reminder on ELECTRE I methods

The methods of ELECTRE I family (Figueira et al. 2005) have been developed in view of aiding DMs in selecting a subset of alternatives, as small as possible, in such a way that a single best compromise alternative may finally be chosen. Below, following (Figueira et al. 2005), we remind the development of ELECTRE I methods, from I to Iv to Is.

Table 7 Representative portfolios in the mean-variance efficient frontier

Portfolio	x_1	x_2	x_3	$E(R(\mathbf{x}))$	$\sigma(\mathbf{x})$	$R_{1\,\%}(\mathbf{x})$	$R_{25\,\%}(\mathbf{x})$	$R_{50\,\%}(\mathbf{x})$	$R_{75\,\%}(\mathbf{x})$	$R_{99\,\%}(\mathbf{x})$
$\mathbf{x}^1$	0.60	0.15	0.25	13.30	7.94	31.79	18.62	13.30	7.98	–5.19
$\mathbf{x}^2$	0.57	0.17	0.27	13.40	7.96	31.94	18.73	13.40	8.07	–5.14
$\mathbf{x}^3$	0.53	0.18	0.28	13.50	8.00	32.14	18.86	13.50	8.14	–5.14
$\mathbf{x}^4$	0.50	0.20	0.30	13.60	8.07	32.41	19.01	13.60	8.19	–5.21
$\mathbf{x}^5$	0.47	0.22	0.32	13.70	8.17	32.75	19.18	13.70	8.22	–5.35
$\mathbf{x}^6$	0.43	0.23	0.33	13.80	8.30	33.14	19.36	13.80	8.24	–5.54
$\mathbf{x}^7$	0.40	0.25	0.35	13.90	8.45	33.59	19.56	13.90	8.24	–5.79
$\mathbf{x}^8$	0.37	0.26	0.37	14.00	8.62	34.09	19.78	14.00	8.22	–6.09
$\mathbf{x}^9$	0.34	0.28	0.39	14.10	8.82	34.64	20.01	14.10	8.19	–6.44
$\mathbf{x}^{10}$	0.30	0.29	0.40	14.20	9.03	35.25	20.25	14.20	8.15	–6.85
$\mathbf{x}^{11}$	0.27	0.31	0.42	14.30	9.27	35.89	20.51	14.30	8.09	–7.29
$\mathbf{x}^{12}$	0.24	0.32	0.44	14.40	9.52	36.58	20.78	14.40	8.02	–7.78
$\mathbf{x}^{13}$	0.21	0.34	0.46	14.50	9.79	37.30	21.06	14.50	7.94	–8.30
$\mathbf{x}^{14}$	0.17	0.35	0.47	14.60	10.07	38.06	21.35	14.60	7.85	–8.86
$\mathbf{x}^{15}$	0.14	0.37	0.49	14.70	10.36	38.85	21.64	14.70	7.76	–9.45
$\mathbf{x}^{16}$	0.11	0.38	0.51	14.80	10.67	39.67	21.95	14.80	7.65	–10.07
$\mathbf{x}^{17}$	0.08	0.40	0.53	14.90	10.99	40.52	22.27	14.90	7.53	–10.72
$\mathbf{x}^{18}$	0.04	0.41	0.54	15.00	11.33	41.39	22.59	15.00	7.41	–11.39
$\mathbf{x}^{19}$	0.01	0.43	0.56	15.10	11.67	42.28	22.92	15.10	7.28	–12.08
$\mathbf{x}^{20}$	0.00	0.40	0.60	15.20	12.03	43.24	23.26	15.20	7.14	–12.84
$\mathbf{x}^{21}$	0.00	0.35	0.65	15.30	12.48	44.38	23.66	15.30	6.94	–13.78
$\mathbf{x}^{22}$	0.00	0.30	0.70	15.40	13.01	45.71	24.12	15.40	6.68	–14.91
$\mathbf{x}^{23}$	0.00	0.25	0.75	15.50	13.60	47.19	24.61	15.50	6.39	–16.19
$\mathbf{x}^{24}$	0.00	0.20	0.80	15.60	14.25	48.81	25.15	15.60	6.05	–17.61
$\mathbf{x}^{25}$	0.00	0.15	0.85	15.70	14.96	50.56	25.72	15.70	5.68	–19.16
$\mathbf{x}^{26}$	0.00	0.10	0.90	15.80	15.71	52.40	26.33	15.80	5.27	–20.80
$\mathbf{x}^{27}$	0.00	0.05	0.95	15.90	16.50	54.34	26.95	15.90	4.85	–22.54
$\mathbf{x}^{28}$	0.00	0.00	1.00	16.00	17.32	56.36	27.60	16.00	4.40	–24.36

ELECTRE I The ELECTRE I method (Roy 1968) is very simple and should be applied only when all the criteria have been coded in numerical scales with identical ranges. In such a situation, one can assert that an alternative *a outranks b* (that is, *a is at least as good as b*), denoted by aSb, only when two following conditions hold.

On the one hand, the *strength of the concordant coalition* of criteria must be powerful enough to support the above assertion. By strength of the concordant coalition, we mean the sum of the weights associated to the criteria forming that coalition. It can be defined by the following *concordance index* (assuming, for the sake of formulae simplicity, that $\sum_{i=1}^{n} w_i = 1$):

Table 8 Marginal and overall values of representative portfolios

Portfolio	$u_{1\,\%}(R_{1\,\%}(\mathbf{x}))$	$u_{25\,\%}(R_{25\,\%}(\mathbf{x}))$	$u_{50\,\%}(R_{50\,\%}(\mathbf{x}))$	$u_{75\,\%}(R_{75\,\%}(\mathbf{x}))$	$u_{99\,\%}(R_{99\,\%}(\mathbf{x}))$	$U(\mathbf{x})$
$\mathbf{x}^1$	0	8.02	1.17	0	0	9.19
$\mathbf{x}^2$	0	8.02	1.20	0.09	0	9.32
$\mathbf{x}^3$	0	8.02	1.24	0.19	0	9.45
$\mathbf{x}^4$	0	8.02	1.27	0.26	0	9.55
$\mathbf{x}^5$	0	8.02	1.31	0.30	0	9.63
$\mathbf{x}^6$	0	8.02	1.35	0.32	0	9.69
$\mathbf{x}^7$	0	8.02	1.38	0.32	0	9.72
$\mathbf{x}^8$	0	8.02	1.42	0.30	0	9.74
$\mathbf{x}^9$	0	8.03	1.45	0.26	0	9.74
$\mathbf{x}^{10}$	0	8.27	1.49	0.20	0	9.96
$\mathbf{x}^{11}$	0	8.53	1.52	0.12	0	10.18
$\mathbf{x}^{12}$	0	8.81	1.56	0.03	0	10.39
$\mathbf{x}^{13}$	0	9.09	1.59	0	0	10.68
$\mathbf{x}^{14}$	0	9.38	1.63	0	0	11.01
$\mathbf{x}^{15}$	0	9.68	1.66	0	0	11.35
$\mathbf{x}^{16}$	0	9.99	1.70	0	0	11.69
$\mathbf{x}^{17}$	0	10.31	1.73	0	0	12.05
$\mathbf{x}^{18}$	0	10.64	1.77	0	0	12.41
$\mathbf{x}^{19}$	0	10.97	1.77	0	0	12.74
$\mathbf{x}^{20}$	0	11.32	1.77	0	0	13.09
$\mathbf{x}^{21}$	0	11.73	1.77	0	0	13.50
$\mathbf{x}^{22}$	0	12.18	1.77	0	0	13.95
$\mathbf{x}^{23}$	0	12.69	1.77	0	0	14.46
$\mathbf{x}^{24}$	0	14.01	1.77	0	0	15.78
$\mathbf{x}^{25}$	0	17.54	1.77	0	0	19.31
$\mathbf{x}^{26}$	0	21.25	1.77	0	0	23.02
$\mathbf{x}^{27}$	0	25.12	1.77	0	0	26.89
$\mathbf{x}^{28}$	0	29.14	1.77	0	0	30.91

$$c(aSb) = \sum_{\{i \in I : g_i(a) \geq g_i(b)\}} w_i,$$

where I is the set of indices of the criteria, $\{i \in I : g_i(a) \geq g_i(b)\}$ is the set of indices for all the criteria belonging to the concordant coalition with the outranking relation aSb.

In other words, the value of the concordance index must be greater than or equal to a given *concordance level* s, whose value generally falls within the range $[0.5, 1 - \min_{i \in I}\{w_i\}]$, i.e., $c(aSb) \geq s$.

On the other hand, no *discordance* against the assertion *a is at least as good as b* may occur. The discordance is measured by a *discordance level* defined as follows:

$$d(aSb) = \max_{\{i \in I : g_i(a) < g_i(b)\}} \{g_i(b) - g_i(a)\}.$$

This level measures in some way the power of the discordant coalition, meaning that if its value surpasses a given level v, the assertion is no longer valid. Discordant coalition exerts no power whenever $d(aSb) \leq v$.

Both concordance and discordance indices have to be computed for every pair of alternatives (a, b) in set A, where $a \neq b$.

It is easy to see that such a computing procedure leads to a binary relation in comprehensive terms (taking into account the whole set of criteria) on set A. Hence, for each pair of alternatives (a, b), only one of the following situations may occur:

- $a\ S\ b$ and not $b\ S\ a$, i.e., $a\ P\ b$ (a is strictly preferred to b).
- $b\ S\ a$ and not $a\ S\ b$, i.e., $b\ P\ a$ (b is strictly preferred to a).
- $a\ S\ b$ and $b\ S\ a$, i.e., $a\ I\ b$ (a is indifferent to b).
- Not $a\ S\ b$ and not $b\ S\ a$, i.e., $a\ R\ b$ (a is incomparable to b).

This preference-indifference framework with the possibility to resort to incomparability, says nothing about how to select the best compromise alternative, or a subset of alternatives the DM will focus his attention on. The outranking relation can be represented by a graph on A where nodes correspond to alternatives and arcs to the outranking relations.

The procedure for exploiting the above outranking graph in view of identifying a small subset $\hat{A}$ of interesting alternatives is based on the *graph kernel* concept, K_G, such that $\hat{A}$ has to satisfy the two following properties

- internal stability, i.e. there is no $a, b \in \hat{A}$ such that aSb,
- external stability, i.e. for all $a \notin \hat{A}$, there exists at least one $b \in \hat{A}$ such that bSa.

When the graph contains no direct cycles, there exists always a unique kernel; otherwise, the graph contains no kernels or several. If graph G contains direct cycles, a preprocessing step must take place where maximal direct cycles are reduced to singleton elements, forming thus a partition on A. Let $\bar{A}$ denote that partition. Each class on $\bar{A} = \{\bar{A}_1, \bar{A}_2, \ldots\}$ is now composed of a set of (considered) equivalent alternatives. It should be noticed that a new preference relation $\succ$ is defined on $\bar{A}$ as follows:

$$\bar{A}_p \succ \bar{A}_q \Leftrightarrow \exists a \in \bar{A}_p \text{ and } \exists b \in \bar{A}_q \text{ such that } aSb \text{ for } \bar{A}_p \neq \bar{A}_q$$

In ELECTRE I all the alternatives which form a cycle are considered indifferent.

ELECTRE Iv The name ELECTRE Iv means ELECTRE I with veto threshold. This method is equipped with a different but extremely useful tool. The new tool made possible for analysts and DMs to overcome the difficulties related to the heterogeneity of scales. Whichever the scales type, this method is always able to select the best compromise alternative or a subset of alternatives to be analyzed by DMs.

This tool is the *veto threshold* v_i, that can be attributed to certain criteria g_i belonging to the family of criteria F. The concept of veto threshold is related in some way to the definition of an upper bound beyond which the discordance about the assertion *a outranks b* cannot surpass and allow an outranking. In practice, the idea of threshold is, however, quite different from the idea of the discordance level like in ELECTRE I. Indeed, while discordance level is related to the scale of criterion g_i in absolute terms for an alternative $a \in A$, threshold veto is related to the preference differences between $g_i(a)$ and $g_i(b)$.

In terms of structure and formulae, little changes occur when moving from ELECTRE I to ELECTRE Iv. The only difference being the discordance condition, now called *no veto condition*, which may be stated as follows:

$$g_i(a) + v_i(g_i(a)) \geq g_i(b), \quad \forall i \in I.$$

To validate the assertion *a outranks b* it is necessary that, among the minority of criteria that are opposed to this assertion, none of them puts its veto.

ELECTRE Iv uses the same exploitation procedure as ELECTRE I.

ELECTRE IS The main novelty of ELECTRE IS (Roy and Skalka 1984) is the use of pseudo-criteria instead of true-criteria. A pseudo-criterion is a real-valued function g_i associated with two threshold functions, $q_i(\cdot)$ and $p_i(\cdot)$, satisfying the following condition: for all ordered pairs of alternatives $(a, a') \in A \times A$, such that $g_i(a) \geq g_i(a'), g_i(a) + p_i(g_i(a'))$ and $g_i(a) + q_i(g_i(a'))$ are non-decreasing monotone functions of $g_i(a')$, such that $p_i(g_i(a')) \geq q_i(g_i(a')) \geq 0$ for all $a \in A$. For more details about the concept of pseudo-criterion see (Roy 1991; Roy and Vincke 1984).

This method is an extension of the previous one aiming at taking into account a double objective: primarily the use of possible no nil indifference and preference thresholds for certain criteria belonging to F and, correlatively, a backing up (reinforcement) of the veto effect when the importance of the concordant coalition decreases. Both concordance and no veto conditions change. Let us present separately the formulae for each one of theses conditions.

- *Concordance condition* Let us start by building the following two sets of indices:

  (i) concerning the coalition of criteria in which aSb

  $$I^S = \Big\{ i \in I : g_i(a) + q_i\big(g_i(a)\big) \geq g_i(b) \Big\},$$

  (ii) concerning the coalition of criteria in which bQa

  $$I^Q = \Big\{ i \in I : g_i(a) + q_i\big(g_i(a)\big) < g_i(a) \leq g_i(b) + p_i\big(g_i(b)\big) \Big\}.$$

  The concordance condition will be:

  $$c(aSb) = \sum_{i \in I^S} w_j + \sum_{i \in I^Q} \varphi_i w_i \geq s$$

  where,

  $$\varphi_i = \frac{g_i(a) + p_i\big(g_i(a)\big) - g_i(b)}{p_i\big(g_i(a)\big) - q_i\big(g_i(a)\big)}$$

  the coefficient φ_i decreases linearly from 1 to 0, when g_i describes the range $[g_i(a) + q_i\big(g_i(a)\big), g_i(a) + p_i\big(g_i(a)\big)]$.

- *No veto condition* The no veto condition can be stated as follows:

$$g_i(a) + v_i\big(g_i(a)\big) \geq g_i(b) + q_i\big(g_i(b)\big)\eta_i,$$

where

$$\eta_i = \frac{1 - c(aSb) - w_i}{1 - s - w_i}.$$

One can apply the same exploitation procedure as in case of ELECTRE I, however, there is also another version of the kernel search for ELECTRE IS, where alternatives belonging to a cycle are no longer considered as indifferent. This version is based on the concept of degree of robustness of *a outranks b*. It is a reinforcement of veto effect and allows one to build true classes of *ex æquo* and thus define an acyclic graph over these classes; in such a graph there is always a single kernel.

Table 9 Preferential parameters

Quantile	q_j	p_j	v_j	k_j
$R_{1\ \%}$	5	10	15	0.10
$R_{25\ \%}$	3	5	10	0.15
$R_{50\ \%}$	1	2	4	0.25
$R_{75\ \%}$	1	2	4	0.25
$R_{99\ \%}$	1	3	5	0.25

Table 10 Concordance index

	$\mathbf{x}^3$	$\mathbf{x}^8$	$\mathbf{x}^{13}$	$\mathbf{x}^{18}$	$\mathbf{x}^{23}$	$\mathbf{x}^{28}$
$\mathbf{x}^3$	1	1	1	0.74	0.5	0.5
$\mathbf{x}^8$	1	1	1	0.95	0.63	0.5
$\mathbf{x}^{13}$	0.75	0.85	1	1	0.86	0.63
$\mathbf{x}^{18}$	0.75	0.75	0.75	1	0.98	0.75
$\mathbf{x}^{23}$	0.56	0.54	0.61	0.74	1	0.91
$\mathbf{x}^{28}$	0.5	0.5	0.5	0.5	0.5	1

Table 11 Veto relation

	$\mathbf{x}^3$	$\mathbf{x}^8$	$\mathbf{x}^{13}$	$\mathbf{x}^{18}$	$\mathbf{x}^{23}$	$\mathbf{x}^{28}$
$\mathbf{x}^3$	0	0	0	0	1	1
$\mathbf{x}^8$	0	0	0	0	0	1
$\mathbf{x}^{13}$	0	0	0	0	0	1
$\mathbf{x}^{18}$	1	1	0	0	0	0
$\mathbf{x}^{23}$	1	1	1	0	0	0
$\mathbf{x}^{28}$	1	1	1	1	1	0

Table 12 Outranking relation

	$\mathbf{x}^3$	$\mathbf{x}^8$	$\mathbf{x}^{13}$	$\mathbf{x}^{18}$	$\mathbf{x}^{23}$	$\mathbf{x}^{28}$
$\mathbf{x}^3$	1	1	1	0	0	0
$\mathbf{x}^8$	1	1	1	1	0	0
$\mathbf{x}^{13}$	0	0	1	1	0	0
$\mathbf{x}^{18}$	0	0	0	1	1	0
$\mathbf{x}^{23}$	0	0	0	0	1	1
$\mathbf{x}^{28}$	0	0	0	0	0	1

5.3.2 Application of the ELECTRE IS method to multiple criteria portfolio selection

We applied the ELECTRE IS method to the same portfolio selection problem as before, but, for the sake of the simplicity, we took into account a subset of six portfolios among the sample of portfolios representing the mean-variance efficient frontier, presented in Table 7. More precisely, we considered set A composed of portfolios $\mathbf{x}^3$, $\mathbf{x}^8$, $\mathbf{x}^{13}$, $\mathbf{x}^{18}$, $\mathbf{x}^{23}$ and $\mathbf{x}^{28}$. Also for the sake of simplicity, we computed the *no veto condition* as in ELECTRE Iv and we applied the exploitation procedure of the basic ELECTRE I method.

We considered the preferential parameters (indifference threshold q_i, preference threshold p_i, veto threshold v_i, and weight k_i of each criterion, $i = 1, \ldots, n$) shown in Table 9. Moreover, the concordance threshold was fixed at $s = 0.9$.

In result of calculations, we got the concordance indices and the veto relation shown in Tables 10 and 11, respectively. The comprehensive outranking relation is shown in Table 12.

The kernel of the outranking graph, i.e. the subset of portfolios recommended for selection, is composed of $\mathbf{x}^3$, $\mathbf{x}^8$ and $\mathbf{x}^{23}$.

6 Conclusions

We presented a novel approach to portfolio selection problem that goes beyond the Markowitz model in several aspects:

- our approach is based on the idea that the preferences of the DM have to be expressed with respect to a set of meaningful quantiles which are much more easily understandable for the DM than the expected return and the variance of portfolio;
- our approach can be used with any probability distribution of portfolio returns because quantiles maintain the same meaning for any probability distribution of portfolio returns, while expected return and variance are no more fully meaningful in case of a non-normal distribution of returns, such as the Pareto stable distribution;
- our approach gives a fundamental importance to quantiles according to most of the risk measures proposed in the literature (VaR, CVaR and expected shortfall,

spectral measures of risk), as well as to the most appreciated model of decisions under risk (Cumulative Prospect Theory);

- our approach permits to build a preference model "tailor made" for each specific DM, while this is not the case of mean-variance models involving the concept of the Markowitz efficient frontier: those models are generally based on the estimation of very few parameters (very often only one) that measure the risk aversion of the DM;
- our approach permits to apply several multiple criteria decision aiding methods based on different premises, and this gives a further space of freedom to the DM and to the analyst, who can organize the decision aiding process in a way adequate to the technical and psychological profiles of these actors;
- our approach permits to integrate expected returns and variance to the set of criteria according to which different portfolios are evaluated: thus, for an expert DM, accustomed to work with the basic Markowitz model, our approach is not restrictive or alternative but, instead, can be seen as an opportunity to add other perspectives (the quantiles) to the classical points of view (expected return and variance of the portfolio);
- our approach permits also to integrate the many methods that have already been proposed to deal with the portfolio selection problem in terms of MCDA [some comprehensive state-of-the-art surveys on this subject are given in (Spronk et al. 2005; Steuer and Na 2003; Steuer et al. 2005)]: indeed our approach aims at modeling preferences strictly related to the portfolio return distribution described in terms of quantiles; this aspect can be integrated, however, with many other aspects represented by criteria such as maximum investment proportion weight, social responsibility, number of securities or economic sectors in the portfolio, short selling, and so on).

We envisage the following future developments of our approach:

- adoption of some recently proposed methods of MCDA based on so-called robust ordinal regression, which involve the whole set of instances of a preference model compatible with the preference information given by the DM (Greco et al. 2008b, 2011; Figueira et al. 2009);
- integration of UTA or ELECTRE methods with DRSA in portfolio selection, in view of explaining the results obtained by UTA or ELECTRE in terms of decision rules (Greco et al. 2013);
- consideration of interaction between criteria in the preference model assessed by UTA (Greco et al. 2012b) or ELECTRE methods (Figueira et al. 2009); it is also possible to use some other preference models, such as the Choquet integral (Choquet 1953–1954) or other nonadditive integrals (Grabisch and Labreuche 2005) permitting to model properly complex interactions among criteria;
- consideration of the hierarchy of criteria in order to help the DM in understanding preferences with respect to subsets of homogeneous criteria (Corrente et al. 2012, 2013): e.g., one could consider
  - one criterion related to possible losses, having as sub-criteria the quantiles related to the worst 1 %, 5 % and 10 %,

- one criterion related to average return, having as sub-criteria the quantiles at 25 %, 50 % and 75 %,
- one criterion related to possible gains, having as sub-criteria the quantiles related to the best 1 %, 5 % and 10 %;

- dealing with distribution of returns over time, by consideration of meaningful quantiles at several reference future dates as criteria (Greco et al. 2010).

Finally, let us observe that an analogous approach can be applied in any decision problems where consequences of a decision are uncertain and possibly distributed over time. This is the case of many classical problems of Operational Research, such as inventory control, supply chain management, and scheduling [for an application of the methodology proposed in this paper to inventory control see (Greco et al. 2012a)].

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References

Acerbi C (2002) Spectral measures of risk: a coherent representation of subjective risk aversion. J Bank Financ 26(7):1505–1518
Acerbi C, Tasche D (2002) Expected shortfall: a natural coherent alternative to value at risk. Econ Notes 31:379–388
Artzner Ph, Delbaen F, Eber JM, Heath D (1999) Coherent measures of risk. Math Financ 9(3):203–228
Belton V, Branke J, Eskelinen P, Greco S, Molina J, Ruiz F, Słowiński R (2008) Interactive multiobjective optimization from a learning perspective, chapter 15. In: Branke J, Deb K, Miettinen K, Słowiński R (eds) Multiobjective optimization: interactive and evolutionary approaches. LNCS vol 5252, state-of-the-art surveys. Springer, Berlin, pp 405–434
Bigelow JP (1993) Consistency of mean-variance analysis and expected utility analysis: a complete characterisation. Econ Lett 43:187–192
Blaszczyński J, Słowiński R, Szelag J (2011) Sequential covering rule induction algorithm for variable consistency rough set approaches. Inf Sci 181:987–1002
Branke J, Deb K, Miettinen K, Słowiński R (eds) (2008) Multiobjective optimization: interactive and evolutionary approaches. In: LNCS, vol 5252, state-of-the-art surveys. Springer, Berlin
Choquet G (1953–1954). Theory of capacities. Ann Inst Fourier 5:131–295
Corrente S, Greco S, Słowiński R (2012) Multiple criteria hierarchy process in robust ordinal regression. Decis Support Syst 53:660–674
Corrente S, Greco S, Słowiński R (2013) Multiple criteria hierarchy process with ELECTRE and PROMETHEE. Omega 41:820–846
Dybvig PH, Ingersoll JE (1982) Mean-variance theory in complete markets. J Business 55:233–251
Dyer JS (2005) MAUT-multiattribute utility theory, chapter 7. In: Figueira J, Greco S, Ehrgott M (eds) Multiple criteria decision analysis: state-of-the-art surveys. Springer, New York, pp 265–295
Elton EJ, Gruber MJ (1995) Modern portfolio theory and investment analysis. Wiley, New York
Fama E (1963) Mandelbrot and the stable Paretian hypothesis. J Business 36:420–429
Fama E (1965a) The behavior of stock market prices. J Business 38:34–105
Fama E (1965b) Portfolio analysis in a stable Paretian market. Manage Sci 11:404–419
Figueira J, Greco S, Ehrgott M (eds) (2005) Multiple criteria decision analysis, state-of-the-art surveys. Springer, New York
Figueira J, Greco S, Roy B (2009) ELECTRE methods with interaction between criteria: an extension of the concordance index. Eur J Oper Res 199:478–495
Figueira J, Greco S, Słowiński R (2009) Building a set of additive value functions representing a reference preorder and intensities of preference: GRIP method. Eur J Oper Res 195:460–486

Figueira J, Mousseau V, Roy B (2005) ELECTRE methods, chapter 14. In: Figueira J, Greco S, Ehrgott M (eds) Multiple criteria decision analysis: state-of-the-art surveys. Springer, New York, pp 133–162

Grabisch M, Labreuche Ch (2005) Fuzzy measures and integrals in MCDA, chapter 5. In: Figueira J, Greco S, Ehrgott M (eds) Multiple criteria decision analysis: state-of-the-art surveys, Springer, New York, pp 563–608

Greco S, Kadziński M, Mousseau V, Słowiński R (2011) ELECTRE^{GKMS}: robust ordinal regression for outranking methods. Eur J Oper Res 214:118–135

Greco S, Matarazzo B, Słowiński R (2001a) Rough sets theory for multicriteria decision analysis. Eur J Oper Res 129:1–47

Greco S, Matarazzo B, Słowiński R, Stefanowski J (2001b) An algorithm for induction of decision rules consistent with dominance principle. In: Ziarko W, Yao Y (eds) Rough sets and current trends in computing. LNAI, vol 2005. Springer, Berlin, pp 304–313

Greco S, Matarazzo B, Słowiński R (2005) Decision rule approach, chapter 13. In: Figueira J, Greco S, Ehrgott M (eds) Multiple criteria decision analysis: state-of-the-art surveys. Springer, New York, pp 507–562

Greco S, Matarazzo B, Słowiński R (2008a) Dominance-based rough set approach to interactive multiobjective optimization, chapter 5. In: Branke J, Deb K, Miettinen K, Słowiński R (eds) Multiobjective optimization: interactive and evolutionary approaches. LNCS, vol 5252, State-of-the-Art Surveys. Springer, Berlin, pp 121–155

Greco S, Mousseau V, Słowiński R (2008b) Ordinal regression revisited: multiple criteria ranking using a set of additive value functions. Eur J Oper Res 191:415–435

Greco S, Matarazzo B, Słowiński R (2010) Dominance-based rough set approach to decision under uncertainty and time preference. Ann Oper Res 176:41–75

Greco S, Matarazzo B, Słowiński R, Vaccarella G (2012a) Inventory control using interactive multiobjective optimization guided by dominance-based rough set approach (submitted)

Greco S, Mousseau V, Słowiński R (2012b) UTA^{GMS} – INT: robust ordinal regression for value functions handling interacting criteria (submitted)

Greco S, Słowiński R, Zielniewicz P (2013) Putting dominance-based rough set approach and robust ordinal regression together. Decis Support Syst 54:891–903

Jacquet-Lagrèze E, Siskos Y (1982) Assessing a set of additive utility functions for multicriteria decision making: the UTA method. Eur J Oper Res 10:151–164

Jarrow RA, Madan DB (1997) Is mean-variance analysis vacuous: or was beta still born? Eur Finance Rev 1:15–30

Jorion Ph (2006) Value at risk: the new benchmark for managing financial risk, 3rd edn. McGraw Hill, New York

Keeney RL, Raiffa H (1976) Decisions with multiple objectives: preferences and value tradeoffs. Wiley, New York

Lintner J (1965) The valuation of risky assets and the selection of risky investments in stock portfolios and capital budget. Rev Econ Stat 47:13–37

Maccheroni F, Marinacci M, Rustichini A, Taboga M (2009) Portfolio selection with monotone mean-variance preferences. Math Financ 19(3):487–521

Mandelbrot B (1963) The variation of certain speculative prices. J Bus 36(4):394–419

Mandelbrot B (1967) The variation of some other speculative prices. J Bus 40:393–413

Markowitz HM (1952) Portfolio selection. J Financ 7:77–91

Markowitz HM (1959) Portfolio selection: efficient diversification of investments. Wiley/Yale University Press, 1970, Basil/Blackwell, 1991

Matos MA (2006) Decision under risk as a multicriteria problem. Eur J Oper Res 181:1516–1529

Mossin J (1966) Equilibrium in a capital asset market. Econometrica 34:768–783

Pawlak Z (1991) Rough sets. Theoretical aspects of reasoning about data. Kluwer, Dordrecht

Rachev S, Ortobelli S, Schwartz E (2004) The problem of optimal asset allocation with stable distributed returns. In: Krinik A, Swift R (eds) Stochastic processes and functional analysis: a volume of recent advances in honor of M.M. Rao. Lecture notes in pure and applied mathematics. Marcel Dekker, New York, pp 295–347

Rockafellar RT, Uryasev SP (2000) Optimization of conditional value-at-risk. J Risk 2:21–42

Ross S (1976) The arbitrage theory of capital pricing. J Econ Theory 13:341–360

Roy B (1968) Classement et choix en présence de points de vue multiples (la méthode ELECTRE). RIRO 8:57–75

Roy B (1991) The outranking approach and the foundations of ELECTRE methods. Theor Decis 31:49–73

Roy B, Skalka J (1984) ELECTRE IS: Aspects méthodologiques et guide d'utilisation. Document du LAMSADE 30, Université Paris Dauphine, Paris

Roy B, Vincke Ph (1984) Relational systems of preference with one or more pseudo-criteria: some new concepts and results. Manage Sci 30:1323–1335

Sharpe WF (1963) A simplified model for portfolio analysis. Manage Sci 9:277–293

Sharpe WF (1964) Capital asset prices: a theory of market equilibrium under conditions of risk. J Financ 19:425–442

Słowiński R, Greco S, Matarazzo B (2005) Rough set based decision support, chapter 16. In: Burke EK, Kendall G (eds), Search methodologies: introductory tutorials in optimization and decision support techniques. Springer, New York, pp 475–527

Słowiński R, Greco S, Matarazzo B (2009) Rough sets in decision making. In: Meyers RA (ed) Encyclopedia of complexity and systems science. Springer, New York, pp 7753–7786

Słowiński R, Greco S, Matarazzo B (2012) Rough set and rule-based multicriteria decision aiding. Pesqui Oper 32:213–269

Spronk J, Steuer RE, Zopounidis C (2005) Multicriteria decision analysis/aid in finance, chapter 20. In: Figueira J, Greco S, Ehrgott M (eds) Multiple criteria decision analysis: state-of-the-art surveys. Springer, New York, pp 799–857

Steuer RE, Na P (2003) Multiple criteria decision making combined with finance: a categorized bibliography. Eur J Oper Res 150:496–515

Steuer RE, Qi Y, Hirschberger M (2005) Multiple objectives in portfolio selection. J Financ Decis Mak 1:5–20

Tversky A, Kahnemann D (1992) Advances in prospect theory: cumulative representation of uncertainty. J Risk Uncertain 5(4):297–323