# Linguistic Composition Based Modelling

# by Fuzzy Networks with Modular Rule Bases

Alexander Gegov*, Farzad Arabikhan*, Nedyalko Petrov*

*University of Portsmouth, School of Computing

Buckingham Building, Portsmouth PO1 3HE, United Kingdom

**Abstract:** This paper proposes a linguistic composition based modelling approach by networked fuzzy systems that are known as fuzzy networks. The nodes in these networks are modules of fuzzy rule bases and the connections between these modules are the outputs from some rule bases that are fed as inputs to other rule bases. The proposed approach represents a fuzzy network as an equivalent fuzzy system by linguistic composition of the network nodes. In comparison to the known multiple rule base approaches, this networked rule base approach reflects adequately the structure of the modelled process in terms of interacting sub-processes and leads to more accurate solutions. The approach improves significantly the transparency of the associated model while ensuring a high level of accuracy that is comparable to the one achieved by established approaches. Another advantage of this fuzzy network approach is that it fits well within the existing approaches with single rule base and multiple rule bases.

**Keywords:** fuzzy models, decision analysis, large-scale systems, transport management, retail management, linguistic models.

## 1. Introduction

Complexity is a versatile feature of existing systems that cannot be described by a single definition. In this context, complexity is usually associated with a number of attributes such as uncertainty, dimensionality and structure, which make the modelling of systems with these attributes more difficult. Therefore, the complexity of a given system can be accounted for by identifying the complexity related attributes that are to be found in this system.

Fuzzy logic has proved itself as a powerful tool for dealing with uncertainty as an attribute of systemic complexity. In this context, fuzziness is quite suitable for reflecting non-probabilistic uncertainty such as imprecision, incompleteness and ambiguity [1-3].

More recently, fuzzy logic has also become more effective in dealing with dimensionality as a systemic complexity attribute by means of rule base reduction and compression. Dimensionality in rule base reduction is associated with the number of rules, which is an exponential function of the number of system inputs and the number of linguistic terms per input [4-7]. In rule base compression, dimensionally is associated with the amount of on-line operations required during fuzzification, inference and defuzzification [8].

However, as far as structure is concerned, fuzzy logic is still unable to reflect adequately any interacting modules within a modelled process. This is due to the black-box nature of most fuzzy models that cannot take into account explicitly any interactions among sub-processes [9-12]. In this respect, the following paragraphs discuss some of the main approaches in fuzzy modelling and their ability to deal with structure as a systemic complexity attribute.

The most common type of fuzzy system is the one with a single rule base [13-15]. This type of system is referred to here as Standard Fuzzy System (SFS). The latter is characterised by a black-box nature whereby the inputs are mapped directly to the outputs without the consideration of any internal connections. The operation of SFS is based on a single Fuzzification-Inference-Defuzzification (FID) sequence and it is usually quite accurate for output modelling as it reflects the simultaneous influence of all inputs on the output. However, the efficiency and transparency of SFS deteriorate with the increase of the number of rules. Therefore, as the number of rules increases, it not only takes longer to simulate the model output but it is also less clear how this output is affected by the model inputs.

Another type of fuzzy system is the one with multiple rule bases [16-19]. This type of system is often described by cascaded rule bases and it is usually referred to as Chained Fuzzy System (CFS) or Hierarchical Fuzzy System (HFS). Both CFS and HFS are characterised by a white-box nature whereby the inputs are mapped to the outputs by means of some internal variables in the form of connections. The operation of CFS and HFS is based on multiple FID sequences whereby each connection links the FID sequences for two adjacent rule bases.

CFS has an arbitrary structure in terms of subsystems and the connections among them [20-22]. In this case, each subsystem represents an individual rule base whereas each interaction is represented by a connection linking a pair of adjacent rule bases. This connection is identical with an output from the first rule base and an input to the second rule base in the pair. CFS is usually used as a detailed presentation of SFS for the purpose of improving transparency by explicitly taking into account all subsystems and the interactions among them. Also, efficiency is improved because of the smaller number of inputs to the individual rule bases. However, accuracy may be lost due to the accumulation of errors as a result of the multiple FID sequences.

HFS is a special type of CFS that has a specific structure [23-27]. Each subsystem in HFS has two inputs and one output. Some connections represent identical mappings, which may propagate across parts of the system. HFS is often used as an alternative presentation of SFS for the purpose of improving transparency by explicitly taking into account all subsystems and the interactions among them. Efficiency is also improved by the reduction of the overall number of rules, which is a linear function of the number of inputs to the subsystems and the number of linguistic terms per input. However, these improvements are often at the expense of accuracy due to the accumulation of errors as a result of the multiple FID sequences.

A third type of fuzzy system is the one with networked rule bases. This type of fuzzy system has been recently introduced as a theoretical concept in [28]. This concept is referred there as Networked Fuzzy System (NFS) and it has been further extended in this work by more generic descriptions in the form of generalised Boolean matrices. NFS is characterised by a white-box nature whereby the inputs are mapped to the outputs by means of connections. Subsystems in NFS are represented by nodes and the interactions among subsystems are the connections among these nodes. NFS is a hybrid between SFS and CFS/HFS. On one hand, the structure of NFS is similar to the structure of CFS/HFS due to the explicit presentation of subsystems and the interactions among them. On the other hand, the operation of NFS resembles the operation of SFS as the multiple rule bases are simplified to a linguistically equivalent single rule base. This simplification is based on the linguistic composition approach that is described further in this work. As a hybrid concept, NFS has the potential of providing a trade-off between SFS and CFS/HFS.

Properties of fuzzy systems such as accuracy, efficiency and transparency are directly related to attributes of systemic complexity such as uncertainty, dimensionality and structure. In this respect, uncertainty is an obstacle to accuracy as it is harder to build an accurate model from uncertain data [29-32]. Furthermore, dimensionality represents an obstacle to efficiency because it is more difficult to reduce the amount of computations in a FID sequence for a large number of rules [33-36]. Finally, structure is an obstacle to transparency as it is harder to understand the behaviour of a black-box model that does not reflect the interactions among subsystems [37-40].

This paper introduces an advanced theoretical framework for NFS as a novel type of fuzzy system. The framework facilitates the validation of NFS as a modelling tool with respect to SFS and CFS/HFS. For clarity and simplicity, NFS is referred to as Fuzzy Network (FN) further in this paper whereby NFS and FN are equivalent in terms of performance. Besides this, the paper addresses several attributes of systemic complexity including uncertainty, dimensionality and structure and the associated properties of the above fuzzy systems such as accuracy, efficiency and transparency. This research methodology is more balanced than the one used in many current studies as they usually focus on only one attribute of systemic complexity and the associated property of the fuzzy system used.

The remaining part of this paper is structured as follows. Section 2 provides some theoretical preliminaries for fuzzy networks. Section 3 introduces the linguistic composition approach. Section 4 illustrates the application of this approach for a transport demand management case study. Section 5 evaluates the performance of the approach in a quantitative and comparative context. Section 6 summarises the main advantages of the approach and highlights future research directions.

## 2. Theoretical Preliminaries

A fuzzy system with $r$ rules, $m$ inputs $x_1 \ldots x_m$ taking linguistic terms from the input sets $\{A_{11},\ldots,A_{1r}\},\ldots,\{A_{m1},\ldots,A_{mr}\}$ and $n$ outputs $y_1 \ldots y_n$ taking linguistic terms from the output sets $\{B_{11},\ldots,B_{1r}\},\ldots,\{B_{n1},\ldots,B_{nr}\}$ can be represented by the following rule base

$$\begin{array}{c} \textit{Rule 1: If } x_1 \textit{ is } A_{11} \textit{ and } \ldots \textit{ and } x_m \textit{ is } A_{m1}\textit{, then } y_1 \textit{ is } B_{11} \textit{ and } \ldots \textit{ and } y_n \textit{ is } B_{n1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \textit{Rule r: If } x_1 \textit{ is } A_{1r} \textit{ and } \ldots \textit{ and } x_m \textit{ is } A_{mr}\textit{, then } y_1 \textit{ is } B_{1r} \textit{ and } \ldots \textit{ and } y_n \textit{ is } B_{nr} \end{array} \tag{1}$$

A fuzzy network with $p.q$ nodes $\{N_{11}\ldots N_{p1}\},\ldots,\{N_{1q}\ldots N_{pq}\}$, $p\times q$ node inputs $\{x_{11}\ldots x_{p1}\},\ldots,\{x_{1q}\ldots x_{pq}\}$ taking linguistic terms from any admissible input sets, $p\times q$ node outputs $\{y_{11}\ldots y_{p1}\},\ldots,\{y_{1q}\ldots y_{pq}\}$ taking linguistic terms from any admissible output sets, $p$ horizontal levels and $q$ vertical layers in the general grid structure for this network can be described by Equation (2)

$$
\begin{array}{lc}
 & \textit{Layer 1} \ldots\ldots\ldots\ldots\ldots \textit{Layer q} \\
\textit{Level 1} & N_{11}(x_{11}, y_{11}) \ldots\ldots\ldots N_{1q}(x_{1q}, y_{1q}) \\
\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\textit{Level p} & N_{p1}(x_{p1}, y_{p1}) \ldots\ldots\ldots N_{pq}(x_{pq}, y_{pq})
\end{array}
\qquad (2)
$$

where the subscripts for the nodes specify their location in the grid structure and the subscripts for the associated inputs and outputs are identical with the ones for their nodes.

Each node in the grid structure from Equation (2) is a separate fuzzy system as the one described by Equation (1). The levels in this grid structure represent a spatial hierarchy of the nodes in terms of subordination in space and the layers represent a temporal hierarchy in terms of consecutiveness in time. For completeness, the fuzzy network described by Equation (2) has a node in each cell of the grid structure but in general a grid structure may have empty cells.

The grid structure in Equation (2) does not give any information about the connections among the nodes in the fuzzy network. However, such information is contained by the sample connection structure in Equation (3) whereby the $p \times (q-1)$ node connections $\{z_{11,12} \ldots z_{p1,p2}\}, \ldots, \{z_{1q-1,1q} \ldots z_{pq-1,pq}\}$ take linguistic terms from the admissible sets for the associated node outputs and inputs

$$
\begin{array}{lc}
 & \textit{Layer 1} \ldots\ldots\ldots\ldots\ldots \textit{Layer q-1} \\
\textit{Level 1} & z_{11,12} = y_{11} = x_{12} \ldots\ldots\ldots z_{1q-1,1q} = y_{1q-1} = x_{1q} \\
\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\textit{Level p} & z_{p1,p2} = y_{p1} = x_{p2} \ldots\ldots\ldots z_{pq-1,pq} = y_{pq-1} = x_{pq}
\end{array}
\qquad (3)
$$

where for each connection $z$ the first subscript is identical with the subscript for its origin node and the second subscript is identical with the subscript for its destination node. Also, the first subscript for a particular connection $z$ is identical with the subscript for the associated output $y$ and the second subscript is identical with the subscript for the associated input $x$.

Like each node input and output from the general grid structure in Equation (2), each node connection from the sample connection structure in Equation (3) can be either of scalar or vector type. For simplicity, this interconnection structure describes only connections that are of feedforward type and among adjacent nodes in the same level but it can be easily extended for connections that are of feedback type or among non-adjacent nodes in different levels.

As a fuzzy network represents an extension of a fuzzy system, i.e. it can be viewed as a system of fuzzy systems or a network whose nodes are fuzzy systems, some of the general presentation techniques for fuzzy systems can be used also for fuzzy networks. However, other presentation techniques that are specific to fuzzy networks are required for the simplification of a fuzzy network to a linguistically equivalent fuzzy system. These techniques use compressed information about nodes in fuzzy networks and they are discussed further in this work.

## 3. Linguistic Composition Approach

The proposed linguistic composition approach uses generalised Boolean matrices for the presentation of individual rule bases in fuzzy networks and operations on these matrices for manipulating the rule bases. A generalised Boolean matrix compresses the information from a rule base that is represented by a node. In this case, the row and column labels of the Boolean matrix are all possible permutations of linguistic terms of the inputs and the outputs for this rule base. The elements of the Boolean matrix are either '0's or '1's whereby each '1' reflects a present rule. The Boolean matrix presentation of the rule base from Equation (1) is given by Equation (4).

$$\begin{array}{cccc} & B_{11}\ldots B_{n1} & \ldots & B_{1r}\ldots B_{nr} \\ A_{11}\ldots A_{m1} & 1 & \ldots & 0 \\ \ldots & & \ldots & \\ A_{1r}\ldots A_{mr} & 0 & \ldots & 1 \end{array} \tag{4}$$

The proposed approach uses also topological expressions for the overall presentation of fuzzy networks and the connections among the individual rule bases. Like grid and interconnection structures, topological expressions describe the location of nodes and the connections among them. In this case, the subscripts of each node specify its location in the network whereby the first subscript gives the level number and the second subscript gives the layer number. Besides this, topological expressions specify all inputs, outputs and connections for the nodes. The topological expression presentation of the fuzzy network from Equations (2)-(3) is given by Equation (5).

$$\begin{aligned} &\{[N_{11}]\ (x_{11} \mid z_{11,12}=y_{11}=x_{12}) * \ldots * [N_{1q}]\ (z_{1q-1,1q}=y_{1q-1}= x_{1q} \mid y_{1q})\} + \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \{[N_{p1}]\ (x_{p1} \mid z_{p1,p2}=y_{p1}=x_{p2}) * \ldots * [N_{pq}]\ (z_{pq-1,pq}= y_{pq-1}= x_{pq} \mid y_{pq})\} \end{aligned} \tag{5}$$

As shown in Equation (5), each node in a topological expression is placed within a pair of square brackets '*[ ]*'. The inputs and the outputs for each node are placed within a pair of simple brackets '*( )*' right after the node. In this case, the inputs are separated from the outputs by a vertical slash '|'. Nodes in sequence are designated by the symbol '*' for horizontal relative location whereas nodes in parallel are designated by the symbol '+' for vertical relative location. Curly brackets '*{ }*' are used to specify the priority of linguistic composition operations in the fuzzy network, i.e. whether nodes with horizontal or vertical relative location have to be manipulated first.

Boolean matrices and topological expressions are very suitable for formal representation of fuzzy networks. While Boolean matrices describe fuzzy networks at a lower level of abstraction with respect to individual nodes, topological expressions describe these networks at a higher level of abstraction with respect to the whole network. In this context, Boolean matrices and topological expressions lend themselves easily to manipulation for the purpose of simplifying fuzzy networks to linguistically equivalent fuzzy systems using the linguistic composition approach. More details on this approach are presented below.

The linguistic composition approach is based mainly on the most common operations for horizontal and vertical merging of nodes in fuzzy networks. These operations are binary in that can be applied to a pair of sequential or parallel nodes. Other less common operations such as output merging of nodes with common inputs are not considered in this work as they are not applicable to the case study. For simplicity, the operations of horizontal and vertical merging are illustrated for nodes with scalar inputs, outputs and connections but their extension to the vector case is straightforward. The operations make use of Boolean matrices at the node level and topological expressions at the network level.

### 3.1 Horizontal merging of rule bases

Horizontal merging can be applied to a pair of sequential nodes, i.e. nodes located in the same level of the fuzzy network. This operation merges the operand nodes from the pair into a single product node in the context of the linguistic composition approach. The operation can be applied when the output from the first operand node is fed forward as an input to the second operand node in the form of a connection. In this case, the product node has the same input as the one to the first operand node and the same output as the one from the second operand node whereas the connection does not appear in the product node.

The horizontal merging operation is identical with Boolean matrix multiplication. The latter is similar to conventional matrix multiplication whereby each arithmetic multiplication is replaced by a '*min*' operation and each arithmetic addition is replaced by a '*max*' operation. In this case, the row labels of the product matrix are the same as the row labels of the first operand matrix whereas the column labels of the product matrix are the same as the column labels of the second operand matrix.

Therefore, if the first operand node is the rule base from Equation (1) that is presented by the Boolean matrix from Equation (4) and the second operand node is the rule base in Equation (6) that is presented by the generalised Boolean matrix in Equation (7)

$$\textit{Rule 1: If } y_1 \textit{ is } B_{11} \textit{ and} \ldots \textit{and } y_n \textit{ is } B_{n1}, \textit{ then } v_1 \textit{ is } C_{11} \textit{ and} \ldots \textit{and } v_g \textit{ is } C_{g1} \tag{6}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\textit{Rule r: If } y_1 \textit{ is } B_{1r} \textit{ and} \ldots \textit{and } y_n \textit{ is } B_{nr}, \textit{ then } v_1 \textit{ is } C_{1r} \textit{ and} \ldots \textit{and } v_g \textit{ is } C_{gr}$$

$$\begin{array}{cccc} & C_{11}\ldots C_{g1} & \ldots & C_{1r}\ldots C_{gr} \\ B_{11}\ldots B_{n1} & 1 & \ldots & 0 \\ \ldots & & \ldots & \\ B_{1r}\ldots B_{nr} & 0 & \ldots & 1 \end{array} \tag{7}$$

the product node is the rule base in Equation (8) that is presented by the generalised Boolean matrix in Equation (9)

$$\textit{Rule 1: If } x_1 \textit{ is } A_{11} \textit{ and} \ldots \textit{and } x_m \textit{ is } A_{m1}, \textit{ then } v_1 \textit{ is } C_{11} \textit{ and} \ldots \textit{and } v_g \textit{ is } C_{g1} \tag{8}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\textit{Rule r: If } x_1 \textit{ is } A_{1r} \textit{ and} \ldots \textit{and } x_m \textit{ is } A_{mr}, \textit{ then } v_1 \textit{ is } C_{1r} \textit{ and} \ldots \textit{and } v_g \textit{ is } C_{gr}$$

$$\begin{array}{cccc} & C_{11}\ldots C_{g1} & \ldots & C_{1r}\ldots C_{gr} \\ A_{11}\ldots A_{m1} & 1 & \ldots & 0 \\ \ldots & & \ldots & \\ A_{1r}\ldots A_{mr} & 0 & \ldots & 1 \end{array} \tag{9}$$

In this case, the fuzzy system described by the rule base in Equation (6) is with $r$ rules, $n$ inputs $y_1\ldots y_n$ taking linguistic terms from the input sets $\{B_{11},\ldots,B_{1r}\},\ldots,\{B_{n1},\ldots,B_{nr}\}$ and $g$ outputs $v_1\ldots v_g$ taking linguistic terms from the output sets $\{C_{11},\ldots,C_{1r}\},\ldots,\{C_{g1},\ldots,C_{gr}\}$. Similarly, the fuzzy system described by the rule base in Equation (8) is with $r$ rules, $m$ inputs $x_1\ldots x_m$ taking linguistic terms from the input sets $\{A_{11},\ldots,A_{1r}\},\ldots,\{A_{m1},\ldots,A_{mr}\}$ and $g$ outputs $v_1\ldots v_g$ taking linguistic terms from the output sets $\{C_{11},\ldots,C_{1r}\},\ldots,\{C_{g1},\ldots,C_{gr}\}$. In general, the operand rule bases may have a different number of rules but the number of rules in the product rule base is always equal to the number of rules in the first operand rule base.

The horizontal merging operation above can be described by the block-scheme in Figure 1 and the topological expression in Equation (10)

$$[N_{11}]\ (x_1,\ldots,x_m \mid y_1,\ldots,y_n) * [N_{12}]\ (y_1,\ldots,y_n \mid v_1,\ldots,v_g) = [N_{11*12}]\ (x_1,\ldots,x_m \mid v_1,\ldots,v_g) \quad (10)$$

where $N_{11}$ and $N_{12}$ are the two operand nodes from the fuzzy network and $N_{11*12}$ is the product node for the fuzzy system. For simplicity, the notations used in Figure 1 are in a vector form where the vectors $x$, $y$ and $v$ are of dimension $n$, $m$ and $g$, respectively.

Vertical merging can be applied to a pair of parallel nodes, i.e. nodes located in the same layer of the fuzzy network. This operation merges the operand nodes from the pair into a single product node. The operation can be applied when the outputs from the operand nodes are not fed as inputs to these nodes.

**3.2 Vertical merging of rule bases**

Vertical merging can be applied to a pair of parallel nodes, i.e. nodes located in the same layer of the fuzzy network. This operation merges the operand nodes from the pair into a single product node in the context of the linguistic composition approach. The operation can be applied when the inputs and the outputs of the two operand nodes are independent, i.e. there are no outputs that are connected with any inputs and vice versa. In this case, the inputs to the product node represent the union of the inputs to the operand nodes whereas the outputs from the product node represent the union of the outputs from the operand nodes.

The vertical merging operation is identical with Boolean matrix Kroneker product that represents an expansion of the first operand matrix along its rows and columns. In particular, the product matrix is obtained by expanding each non-zero element from the first operand matrix to a block that is the same as the second operand matrix and by expanding each zero element from the first operand matrix to a zero block of the same dimension as the second operand matrix. In this case, the row labels of the product matrix are all possible permutations of row labels of the operand matrices whereas the column labels of the product matrix are all permutations of column labels of the operand matrices.

Therefore, if the first operand node is the rule base from Equation (1) that is presented by the Boolean matrix from Equation (4) and the second operand node is the rule base in Equation (11) that is presented by the generalised Boolean matrix in Equation (12)

$$\begin{array}{c} \textit{Rule 1: If } v_1 \textit{ is } C_{11} \textit{ and} \dots \textit{and } v_g \textit{ is } C_{g1}\textit{, then } w_1 \textit{ is } D_{11} \textit{ and} \dots \textit{and } w_h \textit{ is } D_{h1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \textit{Rule s: If } v_1 \textit{ is } C_{1s} \textit{ and} \dots \textit{and } v_g \textit{ is } C_{gs}\textit{, then } w_1 \textit{ is } D_{1s} \textit{ and} \dots \textit{and } w_h \textit{ is } D_{hs} \end{array} \quad (11)$$

$$\begin{array}{cccc} & D_{11}\dots D_{h1} & \dots & D_{1s}\dots D_{hs} \\ C_{11}\dots C_{g1} & 1 & \dots & 0 \\ \dots & & \dots & \\ C_{1s}\dots C_{gs} & 0 & \dots & 1 \end{array} \quad (12)$$

the product node is the rule base in Equation (13) that is presented by the generalised Boolean matrix in Equation (14)

$$\begin{array}{c} \textit{Rule 1: If } x_1 \textit{ is } A_{11} \textit{ and} \dots \textit{and } x_m \textit{ is } A_{m1} \textit{ and } v_1 \textit{ is } C_{11} \textit{ and} \dots \textit{and } v_g \textit{ is } C_{g1}\textit{,} \\ \textit{then } y_1 \textit{ is } B_{11} \textit{ and} \dots \textit{and } y_n \textit{ is } B_{n1} \textit{ and } w_1 \textit{ is } D_{11} \textit{ and} \dots \textit{and } w_h \textit{ is } D_{h1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \textit{Rule r. s: If } x_1 \textit{ is } A_{1r} \textit{ and} \dots \textit{and } x_m \textit{ is } A_{mr} \textit{ and } v_1 \textit{ is } C_{1s} \textit{ and} \dots \textit{and } v_g \textit{ is } C_{gs}\textit{,} \\ \textit{then } y_1 \textit{ is } B_{1r} \textit{ and} \dots \textit{and } y_n \textit{ is } B_{nr} \textit{ and } w_1 \textit{ is } D_{1s} \textit{ and} \dots \textit{and } w_h \textit{ is } D_{hs} \end{array} \quad (13)$$

$$\begin{array}{cccc} & B_{11}\dots B_{n1}D_{11}\dots D_{h1} & \dots & B_{1r}..B_{nr}D_{1s}..D_{hs} \\ A_{11}\dots A_{m1}C_{11}\dots C_{g1} & 1 & \dots & 0 \\ \dots & & \dots & \\ A_{1r}\dots A_{mr}C_{1s}\dots C_{gs} & 0 & \dots & 1 \end{array} \quad (14)$$

In this case, the fuzzy system described by the rule base in Equation (11) is with $s$ rules, $g$ inputs $v_1\dots v_g$ taking linguistic terms from the input sets $\{C_{11},\dots,C_{1s}\},\dots,\{C_{g1},\dots,C_{gs}\}$ and $h$ outputs $w_1\dots w_h$ taking linguistic terms from the output sets $\{D_{11},\dots,D_{1s}\},\dots,\{D_{h1},\dots,D_{hs}\}$. However, the fuzzy system described by the rule base in Equation (13) is with $r.s$ rules, $m+g$ inputs $x_1\dots x_m$, $v_1\dots v_g$ taking linguistic terms from the input sets $\{A_{11},\dots,A_{1r}\},\dots,\{A_{m1},\dots,A_{mr}\}$,

$\{C_{11},\ldots,C_{1s}\},\ldots,\{C_{g1},\ldots,C_{gs}\}$ and $n+h$ outputs $y_1\ldots y_g$, $w_1\ldots w_h$ taking linguistic terms from the output sets $\{B_{11},\ldots,B_{1r}\},\ldots,\{B_{n1},\ldots,B_{nr}\}$, $\{D_{11},\ldots,D_{1s}\},\ldots,\{D_{h1},\ldots,D_{hs}\}$. The number of rules in the product rule base is equal to the product of the number of rules in the operand rule bases.

The vertical merging operation above can be described by the block-scheme in Figure 2 and the topological expression in Equation (15)

$$[N_{11}]\ (x_1,\ldots,x_m \mid y_1,\ldots,y_n) + [N_{21}]\ (v_1,\ldots,v_g \mid w_1,\ldots,w_h) = \\ [N_{11+21}]\ (x_1,\ldots,x_m, v_1,\ldots,v_g \mid y_1,\ldots,y_n, w_1,\ldots,w_h) \tag{15}$$

where $N_{11}$ and $N_{21}$ are the two operand nodes from the fuzzy network and $N_{11+21}$ is the product node for the fuzzy system. For simplicity, the notations used in Figure 2 are in a vector form where the vectors $x$, $y$, $v$ and $w$ are of dimension $n$, $m$, $g$ and $h$, respectively.

### 3.3 Associativity of rule base merging

The horizontal and vertical merging operations on nodes introduced above are quite basic in that they can be applied only to fairly simple fuzzy networks with a pair of nodes. However, a more complex fuzzy network may be with a large number of sequential and parallel nodes that have to be merged horizontally and vertically using the linguistic composition approach. This is possible due to the associativity property of the horizontal and vertical merging operations. These properties are proved below by theorems for scalar inputs, outputs and connections but the extension of the proofs to the vector case is straightforward.

The proofs presented below are based on binary relational presentation of Boolean matrices. A binary relation compresses further the information from a Boolean matrix representation of a rule base. In this case, the pairs in the binary relation are the permutations of linguistic terms of the inputs and the outputs from the row and column labels for the Boolean matrix. Therefore, each pair in the binary relation reflects a rule from the rule base. In this case, the Boolean matrices from Equations (4), (7), (9), (12) and (14) can be presented by the binary relations in Equations (16)-(20).

$$\{(A_{11}\ldots A_{m1}, B_{11}\ldots B_{n1}),\ \ldots\ ,(A_{1r}\ldots A_{mr}, B_{1r}\ldots B_{nr})\} \tag{16}$$

$$\{(B_{11}\ldots B_{n1}, C_{11}\ldots C_{g1}),\ \ldots\ ,(B_{1r}\ldots B_{nr}, C_{1r}\ldots C_{gr})\} \tag{17}$$

$$\{(A_{11}\ldots A_{m1}, C_{11}\ldots C_{g1}),\ \ldots\ ,(A_{1r}\ldots A_{mr}, C_{1r}\ldots C_{gr})\} \tag{18}$$

$$\{(C_{11}\ldots C_{g1}, D_{11}\ldots D_{h1}),\ \ldots\ ,(C_{1s}\ldots C_{gs}, D_{1s}\ldots D_{hs})\} \tag{19}$$

$$\{(A_{11}\ldots A_{m1}\, C_{11}\ldots C_{g1}, B_{11}\ldots B_{n1}\, D_{11}\ldots D_{h1}),\ \ldots\ ,(A_{1r}\ldots A_{mr}\, C_{1s}\ldots C_{gs}, B_{1r}\ldots B_{nr}\, D_{1s}\ldots D_{hs})\} \tag{20}$$

As binary relations are an alternative to Boolean matrices for representing nodes in fuzzy networks, they can also be used for horizontal and vertical merging operations on these nodes. In this case, horizontal merging is identical with standard relational composition whereas vertical merging is identical with a modified type of Cartesian product that is applied separately to the first and second elements from the pairs of the operand relations. These details of binary relations are used in Theorems 1-2 further below whose proofs are presented in the Appendix.

When the property of associativity is related to the operation of horizontal merging, the latter is applied to three sequential nodes for the purpose of merging them into a single node. In particular, this property allows the merging of three operand nodes *A*, *B* and *C* into a product node *A**B*C* to take place as a sequence of two binary merging operations that can be applied either from left to right or from right to left, as shown in Figure 3. The property can be applied when the output from the first node *A* is fed forward as an input to the second node *B* in the form of a connection and the output from the second node *B* is fed forward as an input to the third node *C* in the form of another connection. In this case, the product node *A**B*C* has the same input as the input to the first operand node *A* and the same output as the output from the third operand node *C* whereas the two connections do not appear in the product node.

**Theorem 1:** The operation of horizontal merging denoted by the symbol '*' is associative in accordance with Equation (21)

$$(A*B)*C = A*(B*C) \tag{21}$$

whereby the horizontal merging of any three operand nodes *A*, *B* and *C* from left to right is equivalent to their horizontal merging from right to left.

When the property of associativity is related to the operation of vertical merging, the latter is applied to three parallel nodes for the purpose of merging them into a single node. In particular, this property allows the merging of three operand nodes *A*, *B* and *C* into a product node *A+B+C* to take place as a sequence of two binary merging operations that can be applied either from top to bottom or from bottom to top, as shown in Figure 4. The property can be applied when none of the outputs from any of the three nodes *A*, *B* and *C* are fed as any of the three inputs to these nodes. In this case, the input set to the product node *A+B+C* is the union of the inputs to the operand nodes *A*, *B* and *C* whereas the output set from the product node is the union of the outputs from the operand nodes.

**Theorem 2:** The operation of vertical merging denoted by the symbol '+' is associative in accordance with Equation (22)

$$(A+B)+C = A+(B+C) \tag{22}$$

whereby the vertical merging of any three operand nodes *A*, *B* and *C* from top to bottom is equivalent to their vertical merging from bottom to top.

Although Theorems 1-2 prove the associativity property only for fuzzy networks with three sequential and parallel nodes, respectively, this property can be trivially extended for fuzzy networks with an arbitrary number of nodes. Therefore, this property can be viewed in the context of the linguistic composition approach as the glue that makes the building blocks for simplification of a fuzzy network to a fuzzy system, i.e. the horizontal and merging operations on nodes, stick together. In this case, the generalisation of the associativity property for horizontal and vertical merging can be presented by Equations (23)-(24)

$$(((\ldots((A*B)*C*)\ldots*X)*Y)*Z) = (A*(B*(C*\ldots*(X*(Y*Z))\ldots))) \tag{23}$$

$$(((\ldots((A+B)+C+)\ldots+X)+Y)+Z) = (A+(B+(C+\ldots+(X+(Y+Z))\ldots))) \tag{24}$$

where *A*, *B*, *C*, … , *X*, *Y*, *Z* are operand nodes from a fuzzy network with a single level and layer, respectively.

The associativity property of horizontal and merging operations from Theorems 1-2 provides the basis for the application of the linguistic composition approach to complex fuzzy networks with an arbitrary number of nodes. In particular, the nodes can be merged quite flexibly, i.e. from left to right or right to left within the same level and from top to bottom or from bottom to top within the same layer. In this case, the resulting single equivalent system is the same irrespective of the order of application of the binary merging operations.

**3.4 Application of rule base merging**

The linguistic composition approach can be applied in the context of the three types of fuzzy systems discussed earlier – with single rule base, multiple rule bases and networked rule bases. This process consists of two stages whereby a multiple rule base system such as HFS is first converted into a networked fuzzy system such as FN and then the latter is composed into a single rule base system such as SFS. The theoretical validity of the above two-stage process is proved by means of topological expressions in Theorem 3 below whose proof is presented in the Appendix.

**Theorem 3:** A HFS with set of $m$ inputs $\{x_1, x_2,\ldots, x_m\}$, a set of $m-1$ network nodes $\{N_{11}, N_{12},\ldots, N_{1,m-1}\}$, a set of $m-2$ connections $\{z_1, z_2,\ldots, z_{m-2}\}$ and a single output $y$, as described by the block-scheme in Figure 5 and the topological expression in Equation (25)

$$[N_{11}]\ (x_1, x_2 \mid z_1) * [N_{12}]\ (z_1, x_3 \mid z_2) * \ldots * [N_{1,m-1}]\ (z_{m-2}, x_m \mid y) \tag{25}$$

can be represented as a SFS with the same set of $m$ inputs, a single network node N, no connections and the same single output, as described by the block-schemes in Figures 6-7 and the topological expression in Equation (26)

$$[\prod_{p=1}^{m-1} (N_{1p} + \sum_{q=p+1}^{m-1} I_{qp})]\ (x_1, x_2, \ldots, x_m \mid y) \tag{26}$$

where $N = \prod_{p=1}^{m-1} (N_{1p} + \sum_{q=p+1}^{m-1} I_{qp})$.

Theorem 3 is applicable only to single-output systems but it can be extended trivially for multiple-output systems. In this case, the HFS would have a set of $n$ outputs $\{y_1, y_2, \ldots, y_n\}$ and it could be presented as a set of $n$ independent systems. Therefore, the two-step process from the theorem above would be repeated for each independent system and its output.

### 3.5 Model performance indicators

As opposed to most existing approaches where the focus is to improve efficiency by representing a SFS as a HFS with rule bases of smaller size, the focus of the linguistic composition approach is to maintain accuracy by representing a HFS as a SFS with a single FID sequence while improving transparency by means of the modular rule bases that reflect the subsystems of the modelled system. This is not the case in most existing approaches where the HFS is a mathematical approximation of the SFS that does not reflect the subsystems of the modelled system.

When SFS, HFS and FN are used for modelling, the quality of the associated models can be quantified using performance indicators. In particular, three model performance indicators are introduced further below. They are called Accuracy Index (AI), Efficiency Index (EI) and Transparency Index (TI). These performance indicators represent modifications of performance indicators used for fuzzy systems that can also be used for fuzzy networks.

The first performance indicator AI reflects the accuracy of the model by means of the absolute difference between the model and the data, as shown by Equation (27)

$$AI = \sum_{i=1}^{nl} \sum_{j=1}^{qil} \sum_{k=1}^{vji} (|y_{ji}^{k} - d_{ji}^{k}| / vji) \quad (27)$$

The notations in Equation (27) are as follows: *nl* is the number of nodes in the last layer, *qil* is the number of outputs from the *i*-th node in the last layer, *vji* is the number of discrete values for the *j*-th output from the *i*-th node in the last layer, $y_{ji}^{k}$ is the simulated *k*-th discrete value for the *j*-th output from the *i*-th node in the last layer and $d_{ji}^{k}$ is the measured *k*-th discrete value for the *j*-th output from the *i*-th node in the last layer. Identity nodes are included in this indicator alongside any other nodes in the last layer because their outputs also have to be compared with the data. As a model is more accurate when the absolute difference between the model and the data given by Equation (27) is smaller, a lower AI implies better accuracy.

The second performance indicator EI reflects the efficiency of the model by means of the overall number of rules, as shown by Equation (28)

$$EI = \sum_{i=1}^{n} (q_i^{FID} \times r_i^{FID}) \quad (28)$$

The notations in Equation (28) are as follows: *n* is the number of non-identity network nodes, $q_i^{FID}$ is the number of outputs from the *i*-th non-identity node with an associated FID sequence and $r_i$ is the number of rules for the *i*-th non-identity node with an associated FID sequence. Identity nodes are excluded from this indicator because they are virtual nodes for converting a HFS into a FN that do not affect the efficiency. As a model is more efficient when the overall number of rules given by Equation (28) is smaller, a lower EI implies better efficiency.

The third performance indicator TI reflects the transparency of the model by means of the extent of its opaqueness from the inside, as shown by Equation (29)

$$TI = (p + q) / (n + m) \quad (29)$$

The notations in Equation (29) are as follows: $p$ is the overall number of inputs, $q$ is the overall number of outputs, $n$ is the number of non-identity nodes and $m$ is the number of non-identity connections. Identity nodes are excluded from this indicator as they are virtual nodes for converting a HFS into a FN that do not affect the transparency. As a model is more transparent when the extent of its opaqueness from the inside given by Equation (29) is smaller, i.e. the overall number of inputs and outputs is bigger while at the same time the number of sub-models and connections is smaller, a lower TI implies better transparency.

## 4. Simulation Results

The linguistic composition approach is applied to two case studies from different industries. The first case study is on transport demand management and the second one is on retail product management.

### 4.1 Transport demand management

The main goal in this case study is to model preferences of employees to telecommuting. The data is based on a survey that has been obtained from several government organisations located in the central district of the capital city of Tehran, Iran.

The inputs taken into account for determining preferences of employees are computer time usage, phone/fax time usage, travel time from home to work, travel time from work to home, travel cost from home to work, travel cost from work to home and age. The output is the number of days on which each employee prefers to telecommute from satellite offices.

The preferences of employees to telecommuting can be modelled by a SFS, as shown by the topological expression in Equation (30) and the block-scheme in Figure 8. The notations used are as follows: $N$ is the rule base for the SFS, the inputs $x_1$ and $x_2$ are computer and phone/fax time usage, the inputs $x_3$ and $x_4$ are travel times from home to work and work to home, the inputs $x_5$ and $x_6$ are travel costs from home to work and work to home, the input $x_7$ is age and the output $y$ is the preferred number of telecommuting days.

$$[N]\ (x_1, x_2, x_3, x_4, x_5, x_6, x_7 \mid y) \tag{30}$$

The preferences of employees to telecommuting can also be modelled by a HFS, as shown by the topological expression in Equation (31) and the block-scheme in Figure 9. The notations used are as follows: $N_{12}$, $N_{31}$, $N_{41}$, $N_{32}$ and $N_{13}$ are rule bases for the HFS, the inputs $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$, $x_7$ and the output $y$ are the same as the ones for the SFS, whereas the connections have the following meanings: $z_{12,13}$ shows employees involvement with computer and phone/fax equipment, $z_{31,32}$ reflects employees travel time, $z_{41,32}$ represents employees travel cost whereas $z_{32,13}$ stands for both employees travel time and cost.

$$\begin{gathered}\{[N_{12}]\ (x_1, x_2 \mid z_{12,13}) + \{\ \{[N_{31}]\ (x_3, x_4 \mid z_{31,32}) + \\ [N_{41}]\ (x_5, x_6 \mid z_{41,32})\} * [N_{32}]\ (z_{31,32},\ z_{41,32} \mid z_{32,13})\}\ \} * \\ [N_{13}]\ (z_{12,13}, z_{32,13}, x_7 \mid y)\end{gathered} \tag{31}$$

The preferences of employees to telecommuting can be modelled by a FN as well, as shown by the topological expression in Equation (32) and the block-scheme in Figure 10. Most notations used are the same as the ones for the HFS. The new notations are the identity rule bases $I_{11}$, $I_{21}$, $I_{51}$ and $I_{52}$ representing the propagation of the identity mapping $x_1$, $x_2$, $x_7$ and $x_7$ through the first and second layers of the grid structure. In this context, $N_{12}$, $N_{31}$, $N_{41}$, $N_{32}$ and $N_{13}$ are the network rules bases and they are usually of non-identity type.

$$\begin{gathered}\{\{\{[I_{11}](x_1 \mid x_1) + [I_{21}](x_2 \mid x_2)\} * [N_{12}]\ (x_1, x_2 \mid z_{12,13})\}\ + \\ \{\ \{[N_{31}]\ (x_3, x_4 \mid z_{31,32}) + [N_{41}]\ (x_5, x_6 \mid z_{41,32})\}\ * \\ [N_{32}]\ (z_{31,32}, z_{41,32} \mid z_{32,13})\} + [I_{51}](x_7 \mid x_7) * [I_{52}](x_7 \mid x_7)\ \} * \\ [N_{13}]\ (z_{12,13}, z_{32,13},\ x_7 \mid y)\end{gathered} \tag{32}$$

Using the proposed linguistic composition approach, the HFS with multiple rule bases can be converted first to a FN with networked rule bases. The latter can then be simplified to a SFS with a single rule base, as shown by the topological expression in Equation (33). In this equation, the composite rule base $[(I_{11} + I_{21})* N_{12} + (N_{31}+N_{41})* N_{32} + I_{51}*I_{52}]* N_{13}$ for the

SFS is derived along the lines of the topological expression in Equation (26) by means of the associated merging operations for rule bases that are presented by Boolean matrices.

$$[((I_{11}+ I_{21} )* N_{12} + (N_{31}+N_{41})* N_{32} + I_{51}*I_{52} )* N_{13}]\ (x_1, x_2, x_3, x_4, x_5, x_6, x_7 \mid y) \qquad (33)$$

For simplicity, the inputs are presented by three linguistic terms each, as shown in Figures 11-17. These terms belong to the set *{low, medium, high}* and they are represented by triangular fuzzy membership functions that cover uniformly the whole variation range for the inputs. For consistency, the variation ranges for $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$, $x_7$ are normalised between 0 and 100.

For consistency with the inputs, the output and the connections are presented by the same three linguistic terms *{low, medium, high}*, as shown in Figures 18-19. As opposed to connections whose variation range is normalised between 0 and 100, the variation range for $y$ is normalised between 0 and 5, i.e. the number of days on which employees prefer to telecommute.

For further simplicity, the linguistic terms for the inputs, the connections and the output in all rule bases presented further are encoded as positive integers, i.e. *low=1, medium=2, high=3*. These rule bases are derived from survey based data and can be used as adequate models for determining the telecommuting preferences.

Due to the large number of rules, the rule base for the SFS is shown partially in Table 1 where only the first and the last nine rules are presented. This rule base is derived from survey based global data about preferences of employees to telecommuting and in accordance with Equation (30).

The five rule bases for the HFS are shown in Tables 2-6. These rule bases are derived from survey based local data about preferences of employees to telecommuting and in accordance with Equation (31).

Due to the large number of rules, the rule base for the FN is shown partially in Table 7 where only the first and the last nine rules are presented. This rule base is derived from survey based local data about preferences of employees to telecommuting and in accordance with Equation (33).

The simulation results for the SFS, the HFS and the FN are shown in Figures 20-22 where the survey based observation and the model output are presented together. In this case, each of the three models is simulated in terms of its output, i.e. the preferred number of telecommuting days, for the relevant permutations of the crisp values of the inputs for each of the 245 interviewed employees, i.e. computer and phone/fax time usage, travel times from home to work and work to home, travel costs from home to work and work to home as well as age. The comparative evaluation of the SFS, the HFS and the FN for this case study is given in Table 8 and it is discussed further in the text.

**4.2 Retail product management**

The main goal in this case study is to model prices of retail products. The data is based on a survey that has been obtained from several retail companies in the central district of the capital city of London, UK.

The inputs taken into account for the determination of the price are the expected selling price of the product, the margin, i.e. the relative difference between the price and the cost of the product, and the expected sell through, i.e. the relative quantity of the product expected to be sold. The output from this process is the maximum cost of the product.

The product price determination process can be modelled by a SFS, as shown by the topological expression in Equation (34) and the block-scheme in Figure 23. The notations used are as follows: $N$ is the rule base for the SFS, the first input $x_1$ is the expected selling price, the second input $x_2$ is the margin, the third input $x_3$ is the expected sell through and the output $y$ is the maximum cost.

$$[N]\ (x_1, x_2, x_3 \mid y) \tag{34}$$

The product price determination process can also be modelled by a HFS, as shown by the topological expression in Equation (35) and the block-scheme in Figure 24. The notations used are as follows: $N_{11}$ is the first rule base for the HFS, $N_{12}$ is the second rule base for the HFS, the inputs $x_1$, $x_2$, $x_3$ and the output $y$ are the same as the ones for the SFS, whereas the connection $z$ has the same meaning as the output $y$ for the SFS but it represents the provisional maximum cost of the product.

$$[N_{11}]\ (x_1, x_2 \mid z) * [N_{12}]\ (z, x_3 \mid y) \tag{35}$$

The product price determination process can be modelled by a FN as well, as shown by the topological expression in Equation (36) and the block-scheme in Figure 25. Most notations used are the same as the ones for the HFS. The only new notation is the identity rule base $I_{21}$ representing the propagation of the identity mapping $x_3$ through the first layer of the grid structure. In this context, $N_{11}$ and $N_{12}$ are the network rules bases and they are usually of non-identity type.

$$\{[N_{11}]\ (x_1, x_2 \mid z) + I_{21}\, (x_3 \mid x_3)\} * [N_{12}]\ (z, x_3 \mid y) \tag{36}$$

Using the proposed linguistic composition approach, the HFS with multiple rule bases can be converted first to a FN with networked rule bases. The latter can then be simplified to a SFS with a single rule base, as shown by the topological expression in Equation (37). In this equation, the composite rule base $(N_{11} + I_{21}) * N_{12}$ for the SFS is derived in accordance with the topological expression in Equation (26) and the associated merging operations for rule bases by means of Boolean matrices.

$$[(N_{11} + I_{21}) * N_{12}]\ (x_1, x_2, x_3 \mid y) \tag{37}$$

The inputs $x_1$, $x_2$, $x_3$ are presented by five linguistic terms each, as shown in Figures 26-28. These terms belong to the set *{very low, low, average, high, very high}* and they are represented by triangular fuzzy membership functions that cover uniformly the whole variation range for the inputs. For consistency, the variation ranges for $x_1$, $x_2$, $x_3$ are normalised between 0 and 100.

The output $y$ and the connection $z$ are presented by eleven linguistic terms each, as shown in Figures 29-30. These terms belong to the set *{low5, low4, low3, low2, low1, average, high1, high2, high3, high4, high5}* and they are also represented by triangular fuzzy membership functions that cover uniformly the whole variation range for the output and the connection. The variation ranges for $y$ and $z$ are also normalised between 0 and 100.

The linguistic terms in the rule bases for the SFS, the HFS and the FN are represented by positive integers. In this case, the substitutions are in accordance with Equations (38)-(39)

$$very\ low = 1,\ low = 2,\ average = 3,\ high = 4,\ very\ high = 5 \tag{38}$$

$$\begin{gathered} low5 = 1,\ low4 = 2,\ low3 = 3,\ low2 = 4,\ low1 = 5,\ average = 6, \\ high1 = 7,\ high2 = 8,\ high3 = 9,\ high4 = 10,\ high5 = 11 \end{gathered} \tag{39}$$

The rule base for the SFS is shown in two parts in Tables 9-10. This rule base is derived from data about the product pricing process and in accordance with Equation (34). The derivation is done using a clustering approach whereby the rules represent an approximation of the input-output data points from the data set for the process.

The two rule bases for the HFS are shown in Tables 11-12. These rule bases are derived from data about the two sub-processes within the product pricing process and in accordance with Equation (35). The derivation is done using a clustering approach whereby the rules represent an approximation of the input-output data points from the data sets for the sub-processes.

The rule base for the FN is shown in two parts in Tables 13-14. This rule base is derived in accordance with Equation (37).

The simulation results for the SFS, the HFS and the FN are shown in Figures 31-33 where the data and the model output are presented together. In this case, each of the three models is simulated in terms of its output, i.e. the maximum cost of a retail product, for all 125 possible permutations of the crisp values 0, 25, 50, 75, 100 of the inputs, i.e. the expected selling price of the product, the margin and the expected sell through. The comparative evaluation of the SFS, the HFS and the FN for this case study is given in Table 15 and it is discussed further in the text.

**5. Performance Evaluation**

The proposed linguistic composition approach is evaluated comparatively in terms of accuracy, efficiency and transparency. In particular, a FN that uses the linguistic composition approach and a single FID sequence is compared to a SFS that uses a single FID sequence and a HFS that uses a multiple FID sequence. The evaluation uses the performance indicators from Equations (27)-(29).

The comparative evaluation of the SFS, the HFS and the FN for the first case study on transport demand management is presented in Table 8. The latter shows that in terms of accuracy, the FN is slightly inferior to the SFS and the HFS. As far as efficiency is concerned, the FN is equivalent to the SFS but inferior to the HFS. And finally, in terms of transparency, the FN is superior to the SFS and equivalent to the HFS.

The comparative evaluation of the SFS, the HFS and the FN for the second case study on retail product management is presented in Table 15. The latter shows that in terms of accuracy, the FN is slightly inferior to the SFS and slightly superior the HFS. As far as efficiency is concerned, the FN is equivalent to the SFS but inferior to the HFS. And finally, in terms of transparency, the FN is superior to the SFS and equivalent to the HFS.

For both case studies, the accuracy of the FN can be improved by increasing the number of linguistic terms for the inputs, the connections and the output or adapting the fuzzy membership functions for these variables. In this case, the accuracy of the FN can get better than the one of the SFS and the HFS.

For both case studies, the efficiency of the FN is the same as the one of the SFS due to the same size of the rule base but it is worse than the one of the HFS due to the larger size of the rule base. However, the efficiency of the FN can be improved by rule base reduction or compression in which case it can get better than the one of the HFS.

For both case studies, the transparency of the FN is the same as the one of the HFS due to the use of the same modular rule bases but it is better than the one of the SFS which uses a single rule base. However, the transparency of the FN can be further improved by increasing the number of modular rule bases in which case it can get much better than the one of the SFS.

## 6. Conclusion

The proposed linguistic composition approach provides a novel theoretical framework for fuzzy systems with networked rule bases called fuzzy networks. These networks compare well in terms of accuracy, efficiency and transparency with established fuzzy systems such as standard fuzzy systems with a single rule base and hierarchical fuzzy systems with multiple rule bases. The approach is suitable for modelling processes characterised by uncertainty, dimensionality and structure and can be easily extended to improve performance indicators such as accuracy, efficiency and transparency.

The framework shows a novel application of discrete mathematics and systems theory. It uses generalised Boolean matrices and binary relations for representing network nodes as well as topological expressions and connectionism concepts for representing whole networks. In this framework, a fuzzy network represents an extension of a standard fuzzy system and a hierarchical fuzzy system. In particular, a fuzzy network is a compact way of representing a hierarchical fuzzy system by means of a standard fuzzy system whereby structure is dealt with during the linguistic composition process.

Apart from being an extension, a fuzzy network acts like a bridge between a standard fuzzy system and a hierarchical fuzzy system by means of the linguistic composition process. The latter allows a hierarchical fuzzy system first to be converted into a fuzzy network which can then be composed into a standard fuzzy system. During this process some performance indicators can be improved without deteriorating other indicators. Therefore, this bridging capability of fuzzy networks improves the flexibility of fuzzy systems in terms of modelling depending on the specific requirements to these models.

The linguistic composition approach can be used in a wide range of application areas where the knowledge or data about the modelled process can be provided in a modular fashion, i.e. for each interacting sub-process by means of individual rule bases. Such modular processes are quite common in many areas such as decision making, manufacturing, communications and transport. In this case, the interacting modules can be decision units, manufacturing cells, communication nodes or traffic junctions. To achieve better results, the proposed approach can be further extended for learning and optimisation of the structure and parameters of fuzzy networks in the context of real-world applications.

Also, the approach can be easily extended to other types of rule based systems such as the ones using deterministic and probabilistic logic. These non-fuzzy rule based systems can be represented by deterministic and probabilistic graphical models, respectively.

## Appendix

**Proof of Theorem 1:** The proof is based on the use of binary relations for representing the operand nodes $A$, $B$ and $C$. In this case, the elements of the relational pairs are denoted by the letter $a$ in $A$, the letters $a$ and $c$ in $B$, and the letter $c$ in $C$, as shown in Equations (40)-(42). For clarity, all pairs in the middle relation $B$ are assumed to be composable with pairs from the left relation $A$ and the right relation $C$. This is why the first and the second element of each pair in $B$ are denoted by $a$ and $c$, respectively, and not by $b$.

$$A = \{(a_1^1, a_2^1),\ldots,(a_1^p, a_2^p)\} \quad (40)$$

$$B = \{(a_2^1, c_1^1),\ldots,(a_2^1, c_1^q),\ldots,(a_2^p, c_1^1),\ldots,(a_2^p, c_1^q)\} \quad (41)$$

$$C = \{(c_1^1, c_2^1),\ldots,(c_1^q, c_2^q)\} \quad (42)$$

The first and the second element of any relational pair in $A$ and $C$ are denoted by the subscripts '*1*' and '*2*', respectively. However, the superscripts for the first and the second element of any relational pair in $A$ and $C$ are identical as they indicate the corresponding number for each pair. In particular, the relation $A$ has $p$ pairs and the relation $C$ has $q$ pairs. The subscripts for the first and the second element of any relational pair in $B$ are '*2*' and '*1*', respectively. This is due to the requirement for left and right composability of $B$, i.e. the first element of each pair in $B$ must be identical with a second element of a pair in $A$ whereas the second element of each pair in $B$ must be identical with a first element of a pair in $C$. In this case, the superscripts for the elements of the relational pairs in $B$ do not have to be identical and therefore the relation $B$ is assumed to have $p\times q$ pairs.

The horizontal composition of the operand relations $A$ and $B$ gives the temporary relation $A*B$, as shown in Equation (43)

$$A*B = \{(a_1^1, c_1^1),\ldots,(a_1^1, c_1^q),\ldots,(a_1^p, c_1^1),\ldots,(a_1^p, c_1^q)\} \quad (43)$$

Further on, the horizontal composition of the temporary relation $A*B$ and the operand relation $C$ gives the product relation $(A*B)*C$, as shown in Equation (44)

$$(A*B)*C = \{(a_1^1, c_2^1),\ldots,(a_1^1, c_2^q),\ldots,(a_1^p, c_2^1),\ldots,(a_1^p, c_2^q)\} \quad (44)$$

On the other hand, the horizontal composition of the operand relations $B$ and $C$ gives the temporary relation $B*C$, as shown in Equation (45)

$$B*C = \{(a_2^1, c_2^1),\ldots,(a_2^1, c_2^q),\ldots,(a_2^p, c_2^1),\ldots,(a_2^p, c_2^q)\} \quad (45)$$

In this case, the horizontal composition of the operand relation $A$ and the temporary relation $B*C$ gives the product relation $A*(B*C)$. As the latter is identical with the product relation $(A*B)*C$ from Equation (44), this implies Equation (21) and concludes the proof.

**Proof of Theorem 2:** The proof is based on the use of binary relations for representing the operand nodes $A$, $B$ and $C$. In this case, the elements of the relational pairs are denoted by the letter $a$ in $A$, the letter $b$ in $B$ and the letter $c$ in $C$, as shown in Equations (46)-(48)

$$A = \{(a_1^1, a_2^1),\ldots,(a_1^p, a_2^p)\} \quad (46)$$

$$B = \{(b_1^1, b_2^1),\ldots,(b_1^q, b_2^q)\} \quad (47)$$

$$C = \{(c_1^1, c_2^1),\ldots,(c_1^r, c_2^r)\} \quad (48)$$

The first and the second element of any relational pair in $A$, $B$ and $C$ are denoted by the subscripts '*1*' and '*2*', respectively. However, the superscripts for the first and the second element of any relational pair in $A$, $B$ and $C$ are identical as they indicate the corresponding number for each pair. In particular, the relation $A$ has $p$ pairs, the relation $B$ has $q$ pairs and the relation $C$ has $r$ pairs.

The vertical composition of the operand relations $A$ and $B$ gives the temporary relation $A+B$, as shown in Equation (49)

$$A+B = \{(a_1^1\, b_1^1, a_2^1\, b_2^1),\ldots,(a_1^1\, b_1^q, a_2^1\, b_2^q),\ldots,(a_1^p\, b_1^1, a_2^p\, b_2^1),\ldots,(a_1^p\, b_1^q, a_2^p\, b_2^q\,)\} \quad (49)$$

Further on, the vertical composition of the temporary relation $A+B$ and the operand relation $C$ gives the product relation $(A+B)+C$, as shown in Equation (50)

$$\begin{aligned}(A+B)+C = \{&(a_1^1\, b_1^q\, c_1^1, a_2^1\, b_2^q\, c_2^1),\ldots,(a_1^1\, b_1^q\, c_1^r, a_2^1\, b_2^q\, c_2^r),\ldots,\\ &(a_1^p\, b_1^1\, c_1^1, a_2^p\, b_2^1\, c_2^1),\ldots,(a_1^p\, b_1^1\, c_1^r, a_2^p\, b_2^1\, c_2^r),\ldots,\end{aligned} \quad (50)$$

$$(a_1^p\, b_1^q\, c_1^1,\, a_2^p\, b_2^q\, c_2^1),\ldots,(a_1^p\, b_1^q\, c_1^r,\, a_2^p\, b_2^q\, c_2^r))\}$$

On the other hand, the vertical composition of the operand relations *B* and *C* gives the temporary relation *B+C*, as shown in Equation (51)

$$B+C = \{(b_1^1\, c_1^1,\, b_2^1\, c_2^1),\ldots,(b_1^1\, c_1^r,\, b_2^1\, c_2^r),\ldots,(b_1^q\, c_1^1,\, b_2^q\, c_2^1),\ldots,(b_1^q\, c_1^r,\, b_2^q\, c_2^r)\} \quad (51)$$

In this case, the vertical composition of the operand relation *A* and the temporary relation *B+C* gives the product relation *A+(B+C)*. As the latter is identical with the product relation *(A+B)+C* from Equation (50), this implies Equation (22) and concludes the proof.

**Proof of Theorem 3:** The HFS from Equation (25) can first be converted into a FN by representing all identity mappings propagating through any layers in the grid structure with the set of identity nodes $\{I_{21}\}$, … ,$\{I_{m-1,1},\ I_{m-1,2},\ \ldots\}$. This FN can be described by the topological expression in Equation (52)

$$\begin{gathered}\{[N_{11}]\ (x_1, x_2 \mid z_1) + [I_{21}]\ (x_3 \mid x_3) + \ldots + [I_{m-1,1}]\ (x_m \mid x_m)\} \ * \\ \{[N_{12}]\ (z_1, x_3 \mid z_2) + \ldots + [I_{m-1,2}]\ (x_m \mid x_m)\} \ * \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ * \\ [N_{1,m-1}]\ (z_{m-2}, x_m \mid y)\end{gathered} \quad (52)$$

where each network node has two inputs and one output as opposed to each identity node that has one input and one output. In this case, the input to each identical node is identical with the output from the same node, as shown by the block-scheme in Figure 7.

The FN can then be composed into a SFS by merging first vertically and then horizontally all network and identity nodes into a single network node $N = \prod_{p=1}^{m-1} (N_{1p} + \sum_{q=p+1}^{m-1} I_{qp})$. In this case, the SFS is like a single node FN with the same set of *m* inputs $\{x_1, x_2,\ldots, x_m\}$ and the same single output *y* as the HFS. This SFS can de described by the topological expression from Equation (26) that uses prefix notation for the horizontal merging operation and a mixture of infix/prefix notation for the vertical merging operation. This concludes the proof.

## Acknowledgement

The first author would like to thank the Faculty of Technology at the University of Portsmouth for the granted research sabbatical that made possible the writing of this paper.

## References

[1] Fernandez-Caballero A (2009) Contribution of fuzziness and uncertainty to modern artificial intelligence. Fuzzy Sets and Systems 160/2:129-129

[2] Garibaldi J M, Ozen T (2007) Uncertain fuzzy reasoning: a case study in modelling expert decision making. IEEE Transactions on Fuzzy Systems 15/1:16-30

[3] Gil-Aluja (2004) Fuzzy sets in the management of uncertainty. Springer, Berlin

[4] Chen B, Liu X (2005) Fuzzy approximate disturbance decoupling of MIMO nonlinear systems by backstepping and application to chemical processes. IEEE Transactions on Fuzzy Systems 13/6:832-847

[5] Chen B, Tong S, Liu X (2007) Fuzzy approximate disturbance decoupling of MIMO nonlinear systems by back stepping approach. Fuzzy sets and systems 158/10:1097-1125

[6] Wang W J, Lin W W (2005) Decentralized PDC for large-scale fuzzy systems. IEEE Transactions on Fuzzy Systems 13/6:779-786

[7] Wang W J, Luoh L (2004) Stability and stabilization of fuzzy large-scale systems. IEEE Transactions on Fuzzy Systems 12/3:309-315

[8] Gegov A (2007) Complexity management in fuzzy systems: a rule base compression approach. Springer, Berlin

[9] Kim K J, Cho S B (2007) Personalized mining of web documents link structures and fuzzy concept networks. Applied Soft Computing 7/1:398-410

[10] Xu C, Shin Y C (2007) Interaction analysis of MIMO nonlinear systems based on fuzzy basis function network model. Fuzzy sets and systems 158/18:2013-2015

[11] Zhang H, Dang C, Li C (2009) Decentralized H-infinity filter design for discrete-time interconnected fuzzy systems. IEEE Transactions on Fuzzy Systems 17/6:1428-1440

[12] Zhang T P (2001) Stable adaptive fuzzy sliding mode control of interconnected systems. Fuzzy Sets and Systems 122/1:5-19

[13] Buckley, J J (2005) Simulating fuzzy systems. Springer, Berlin

[14] Piegat A (2001) Fuzzy modelling and control. Springer, Berlin

[15] Ross T (2004) Fuzzy logic with engineering applications. Wiley, Chichester

[16] Ji-Cheng Duan, Fu-Lai Chung (2001) Cascaded fuzzy network model based on syllogistic fuzzy reasoning. IEEE Transactions on Fuzzy Systems 9/2:293-306

[17] Lendek Z, Babuska R, De Schutter B (2009) Stability of cascaded fuzzy systems and observers. IEEE Transactions on Fuzzy Systems 17/3:641-653

[18] Mar J, Lin H T (2005) A car-following collision prevention control device based on the cascaded fuzzy inference system. Fuzzy Sets and Systems 150/3:457-473

[19] Yeh Z M, Li K H (2004) A systematic approach for designing multistage fuzzy control systems. Fuzzy Sets and Systems 143/2:251-273

[20] Bucolo M, Fortuna L, La Rosa M (2004) Complex dynamics through fuzzy chains. IEEE Transactions on Fuzzy Systems 12/3:289-295

[21] Hall L O (2001) Rule chaining in fuzzy expert systems. IEEE Transactions on Fuzzy Systems 9/6:822-828

[22] Igel C, Temme K H (2004) The chaining syllogism in fuzzy logic. IEEE Transactions on Fuzzy Systems 12/6:849-853

[23] Aja-Fernandez S, Alberola-Lopez C (2008) Matrix modelling of hierarchical fuzzy systems. IEEE Transactions on Fuzzy Systems 16/3:585-599

[24] Chen Y, Yang B, Abraham A, Peng L (2007) Automatic design of hierarchical Takagi-Sugeno type fuzzy systems using evolutionary algorithms. IEEE Transactions on Fuzzy Systems 15/3:385-397

[25] Cheong F, Lai R (2007) Designing a hierarchical fuzzy logic controller using the differential evolution approach. Applied Soft Computing 7/2:481-491

[26] Joo M G, Lee J S (2005) A class of hierarchical fuzzy systems with constraints on the fuzzy rules. IEEE Transactions on Fuzzy Systems 13/2:194-203

[27] Lee M L, Chung H Y, Yu F M (2003) Modelling of hierarchical fuzzy systems. Fuzzy Sets and Systems 135/2:343-361

[28] Gegov A (2010) Fuzzy networks for complex systems: a modular rule base approach. Springer, Berlin

[29] Landago M, Rio M J, Perez A (2001) A note on smooth approximation capabilities of fuzzy systems. IEEE Transactions on Fuzzy Systems 9/2:229-237

[30] Li Y M, Shi Z K, Li Z H (2002) Approximation theory of fuzzy systems based upon genuine many-valued implications –MIMO cases. Fuzzy Sets and Systems 130/2:159-174

[31] Moreno-Velo F J, Baturone I, Barriga A, Sanchez-Solano S (2007) Automatic tuning of complex fuzzy systems with X-fuzzy. Fuzzy sets and systems 158/18:2026-2038

[32] Wan F, Shang H, Wang L X, Sun Y X (2005) How to determine the minimum number of fuzzy rules to achieve given accuracy: a computational geometric approach to SISO case. Fuzzy Sets and Systems 150/2:199-209

[33] Gil-Aluja (2004) Fuzzy sets in the management of uncertainty. Springer, Berlin

[34] Joo M G, Sudkamp T (2009) A method of converting a fuzzy system to a two-layered hierarchical fuzzy system and its run-time efficiency. IEEE Transactions on Fuzzy Systems 17/1:93-103

[35] Pal N R, Eluri V K, Mandal G K (2002) Fuzzy logic approaches to structure preserving dimensionality reduction. IEEE Transactions on Fuzzy Systems 10/3:277-286

[36] Xiong N, Litz L (2002) Reduction of fuzzy control rules by means of premise learning – method and case study. Fuzzy Sets and Systems 132/2:217-231

[37] Fiordaliso A (2001) A constrained Takagi-Sugeno fuzzy system that allows for better interpretation and analysis. Fuzzy Sets and Systems 118/2:307-318

[38] Guillaume S (2001) Designing fuzzy inference systems from data: an interpretability-oriented review. IEEE Transactions on Fuzzy Systems 9/3:426-443

[39] Kumar M, Stoll R, Stoll N (2006) A robust design criterion for interpretable fuzzy models with uncertain data. IEEE Transactions on Fuzzy Systems 14/2:314-328

[40] Zhou S M, Gan J Q (2008) Low-level interpretability and high-level interpretability: a unified view of data-driven interpretable fuzzy system modelling. Fuzzy sets and systems 159/23:3091-3131

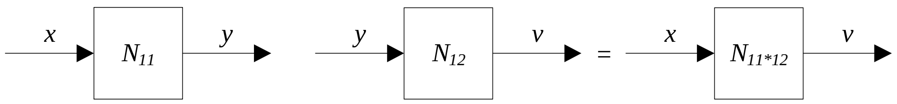


**Figure 1:** Horizontal merging of rule bases

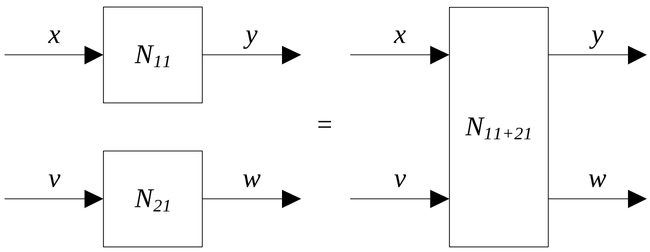


**Figure 2:** Vertical merging of rule bases

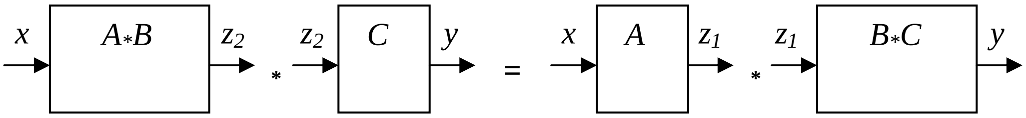


**Figure 3:** Associativity property of horizontal merging

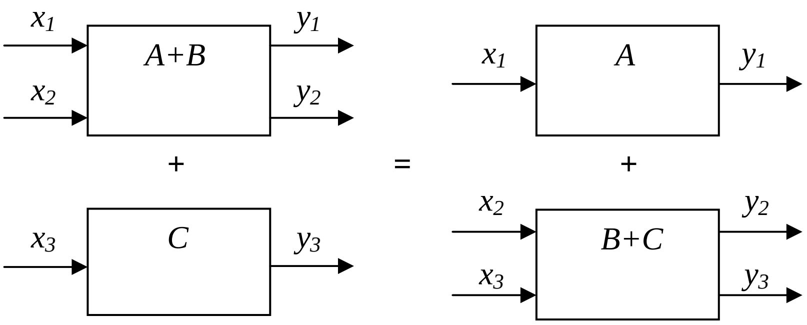


**Figure 4:** Associativity property of vertical merging

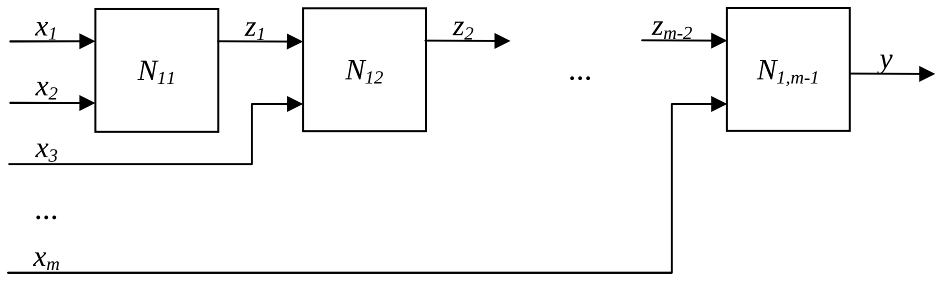


**Figure 5:** Hierarchical fuzzy system

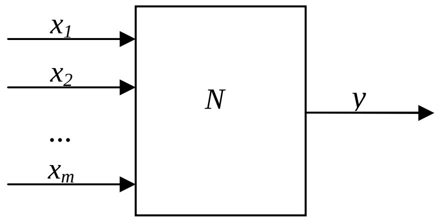


**Figure 6:** Standard fuzzy system

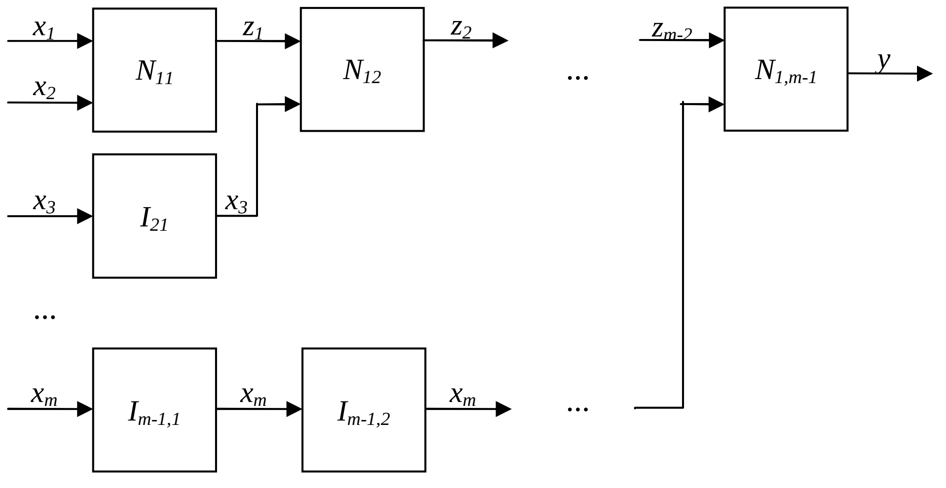


**Figure 7:** Fuzzy network

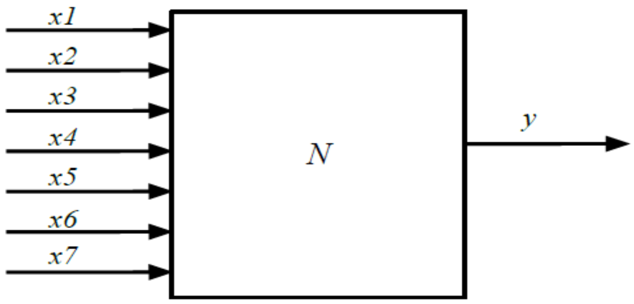


**Figure 8:** Standard fuzzy system for case study 1

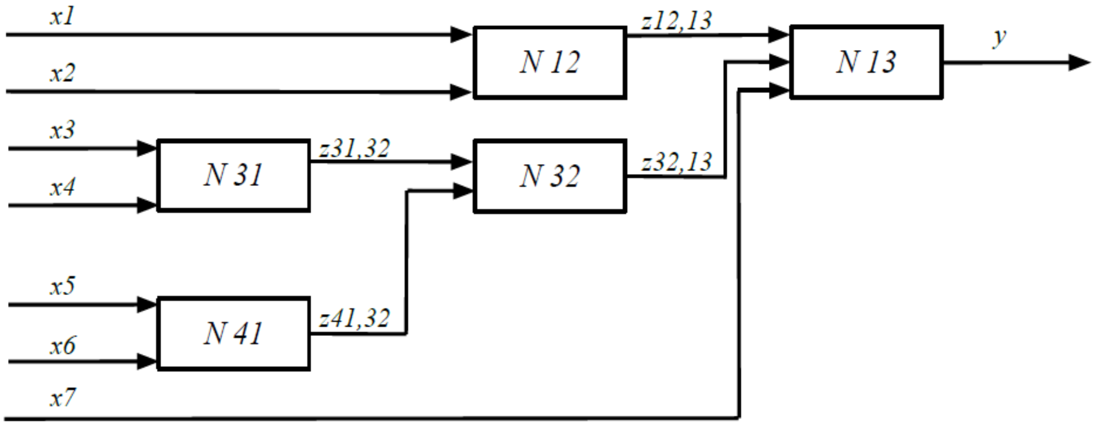


**Figure 9:** Hierarchical fuzzy system for case study 1

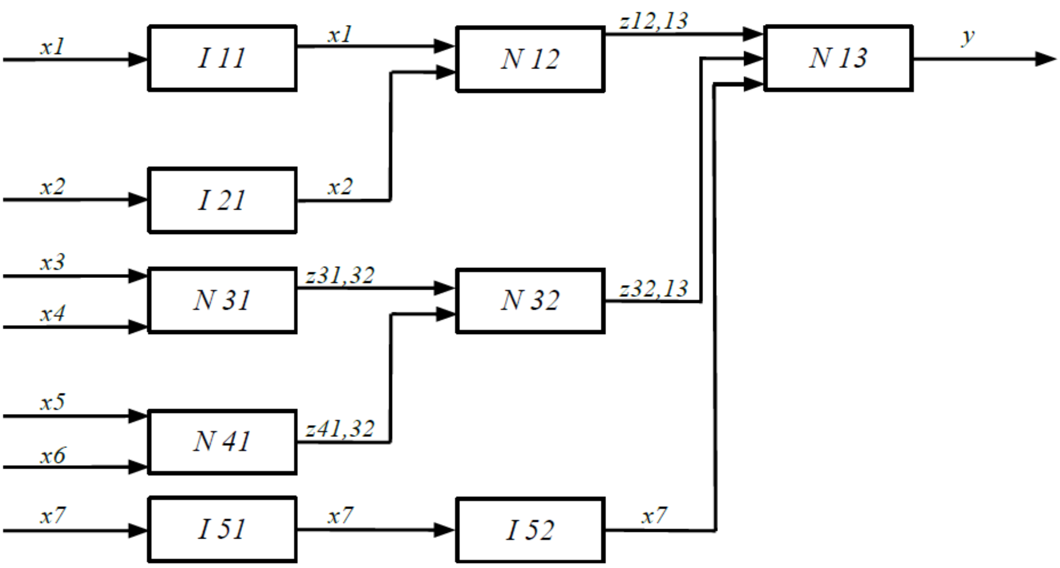


**Figure 10:** Fuzzy network for case study 1

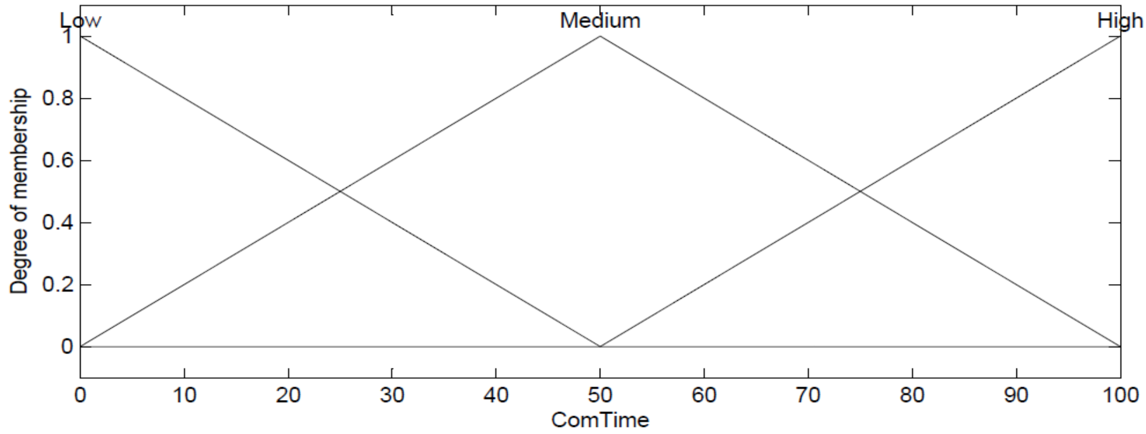


**Figure 11:** Linguistic terms for first input in case study 1

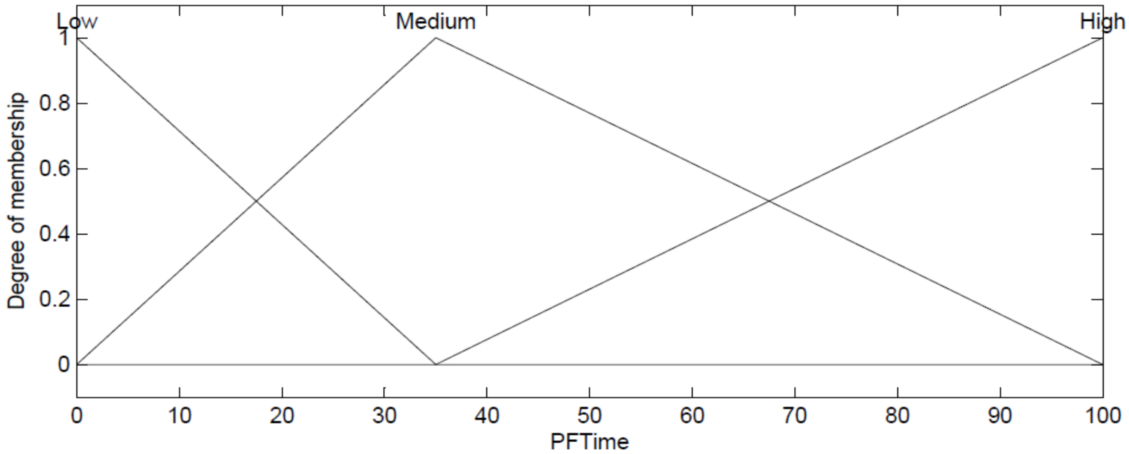


**Figure 12:** Linguistic terms for second input in case study 1

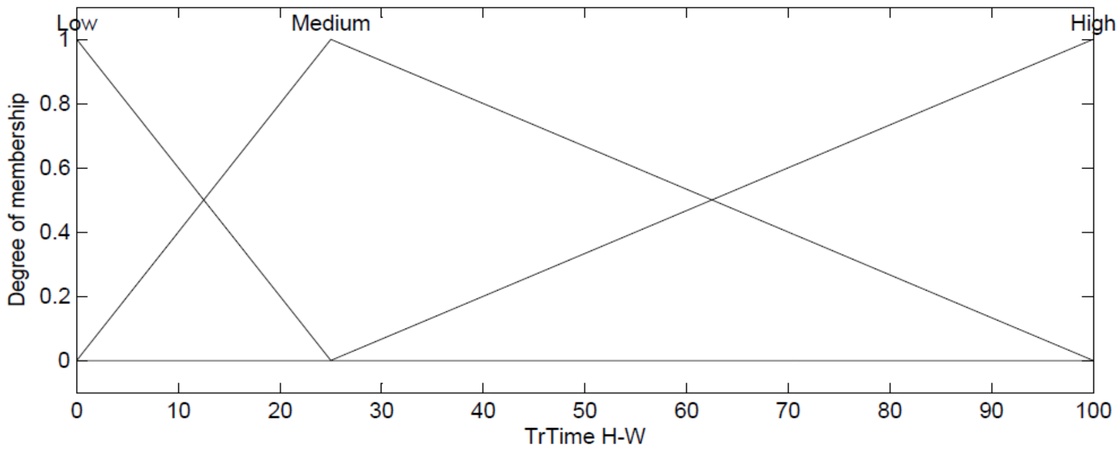


**Figure 13:** Linguistic terms for third input in case study 1

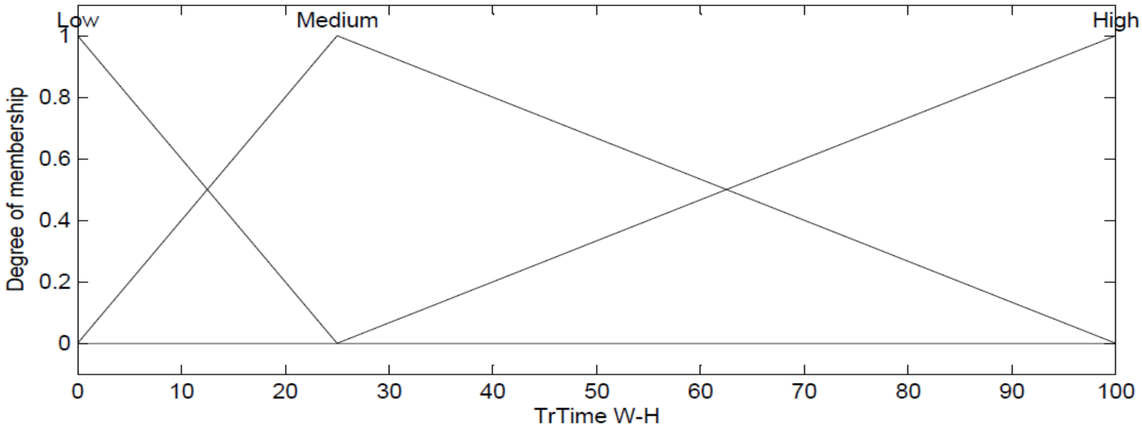


**Figure 14:** Linguistic terms for fourth input in case study 1

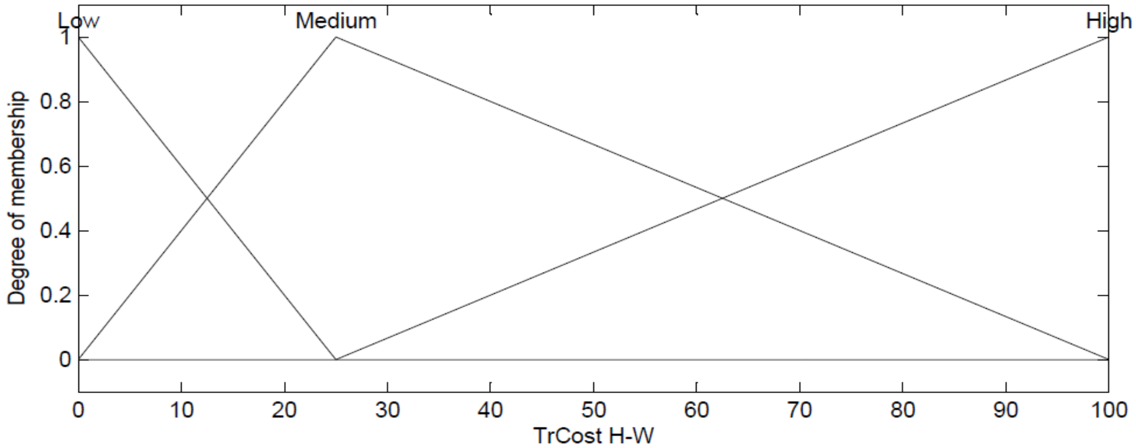


**Figure 15:** Linguistic terms for fifth input in case study 1

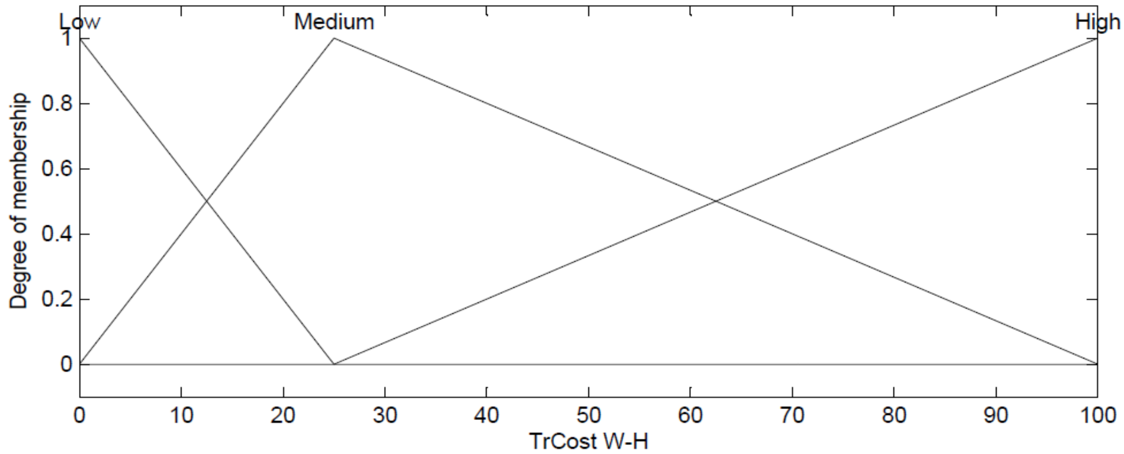


**Figure 16:** Linguistic terms for sixth input in case study 1

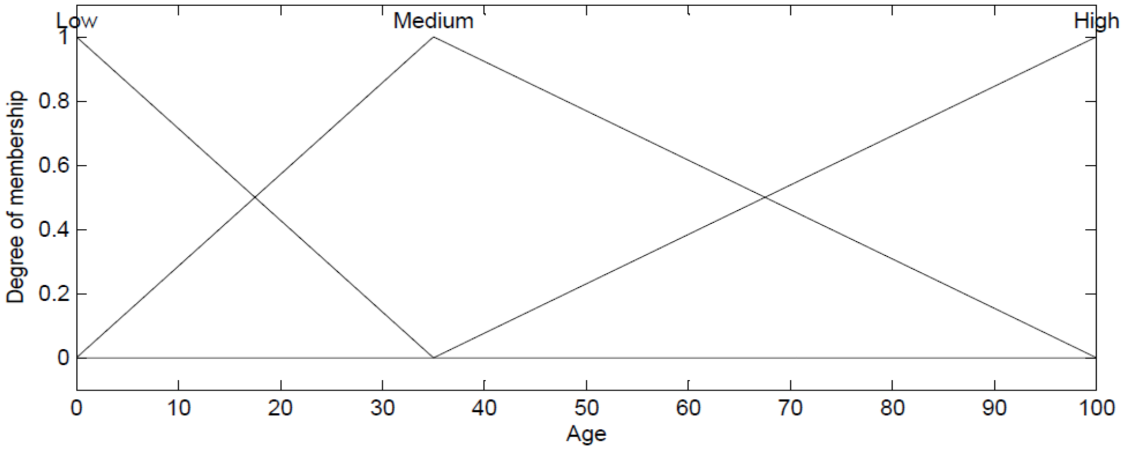


**Figure 17:** Linguistic terms for seventh input in case study 1

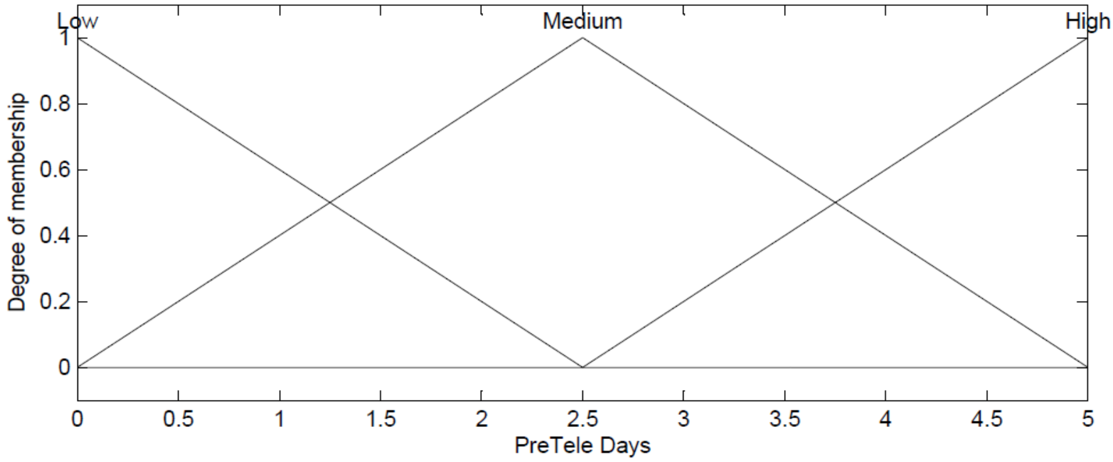


**Figure 18:** Linguistic terms for output in case study 1

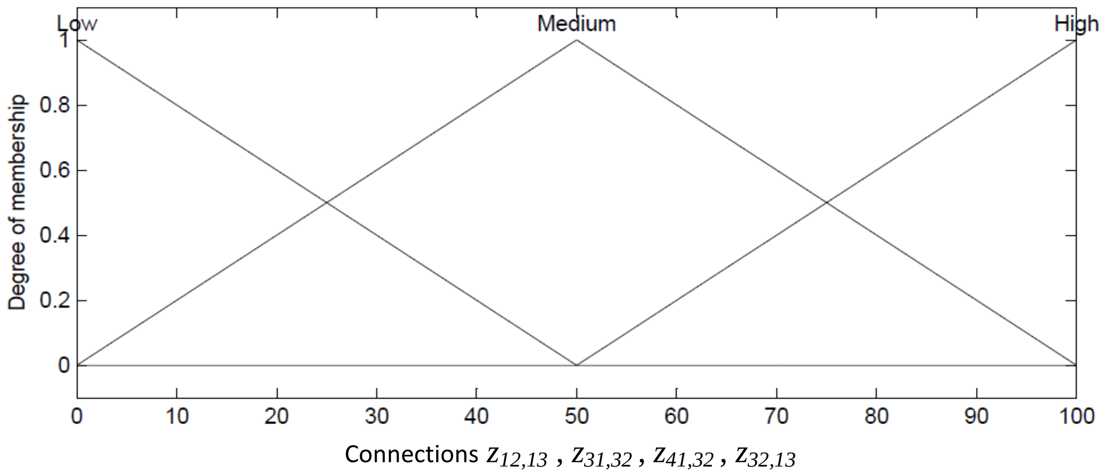


**Figure 19:** Linguistic terms for connections in case study 1

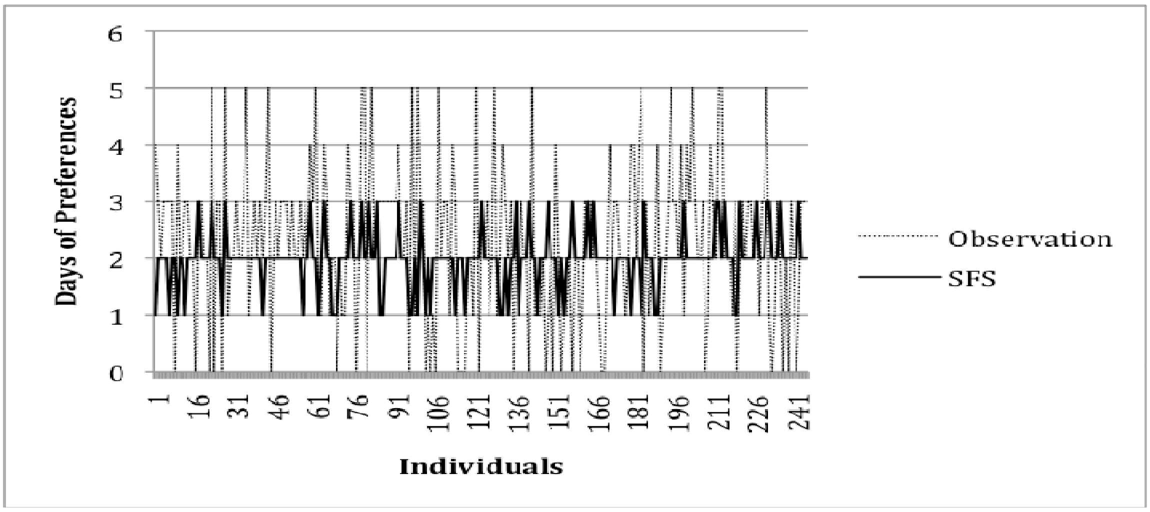


**Figure 20:** Simulation results for standard fuzzy system in case study 1

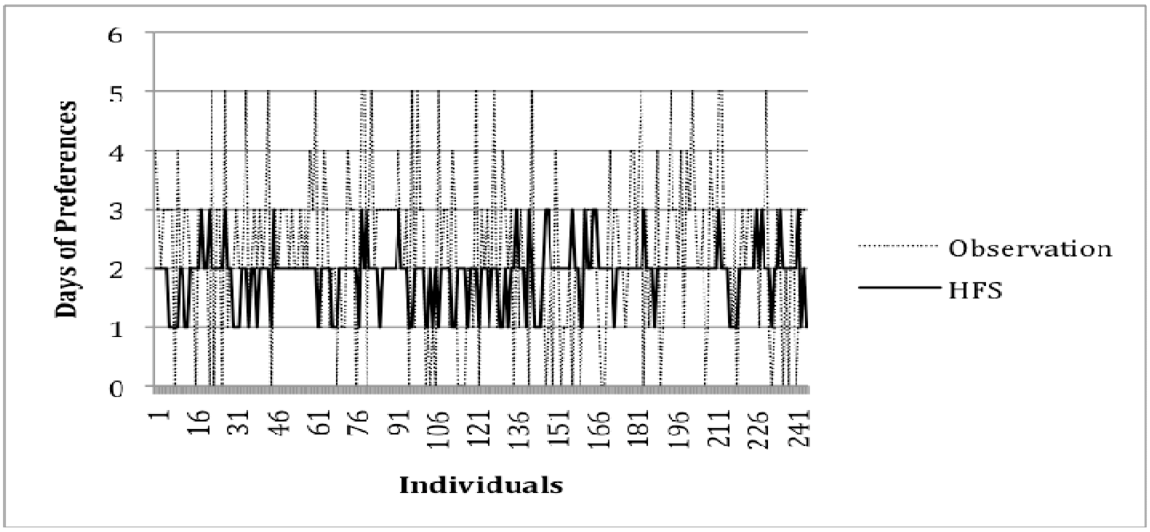


**Figure 21:** Simulation results for hierarchical fuzzy system in case study 1

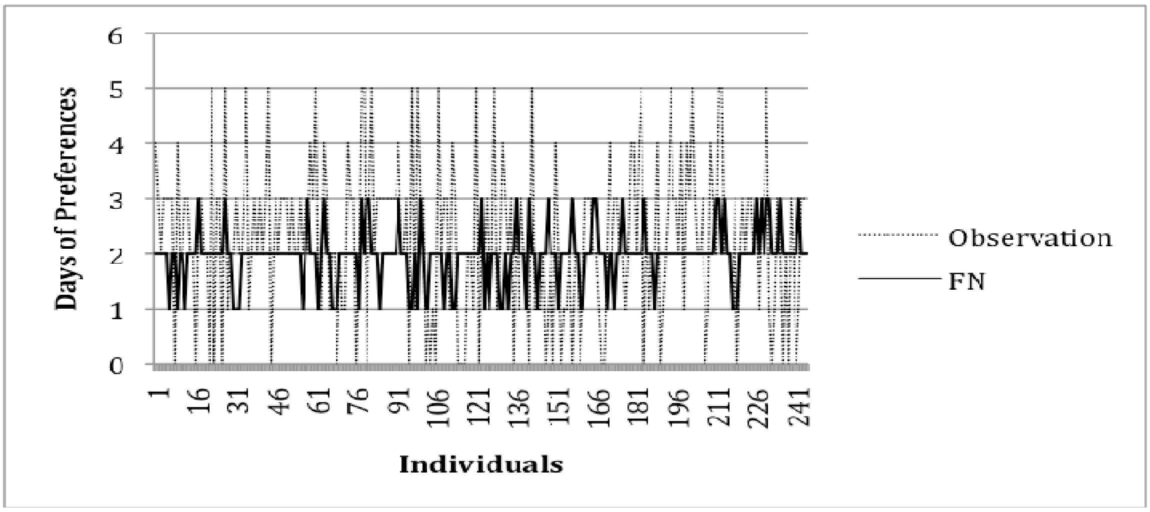


**Figure 22:** Simulation results for fuzzy network in case study 1

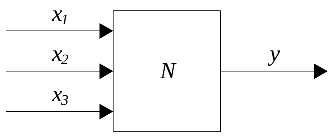


**Figure 23:** Standard fuzzy system for case study 2

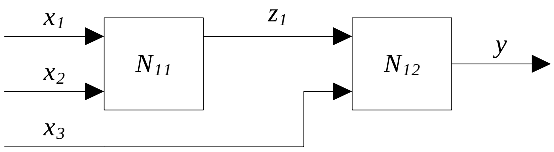


**Figure 24:** Hierarchical fuzzy system for case study 2

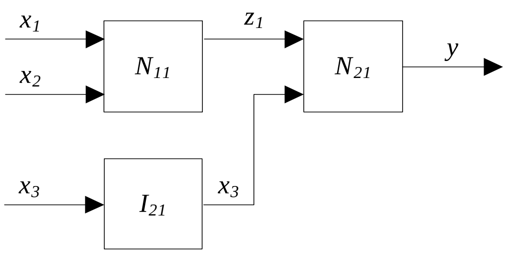


**Figure 25:** Fuzzy network for case study 2

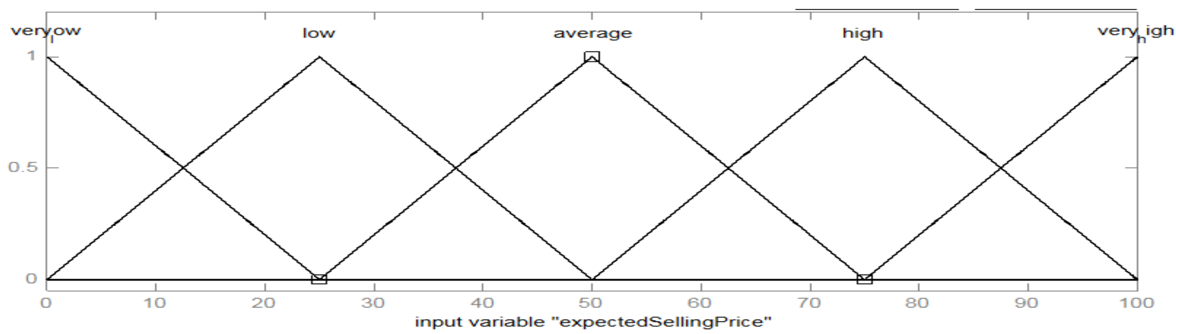


**Figure 26:** Linguistic terms for first input in case study 2

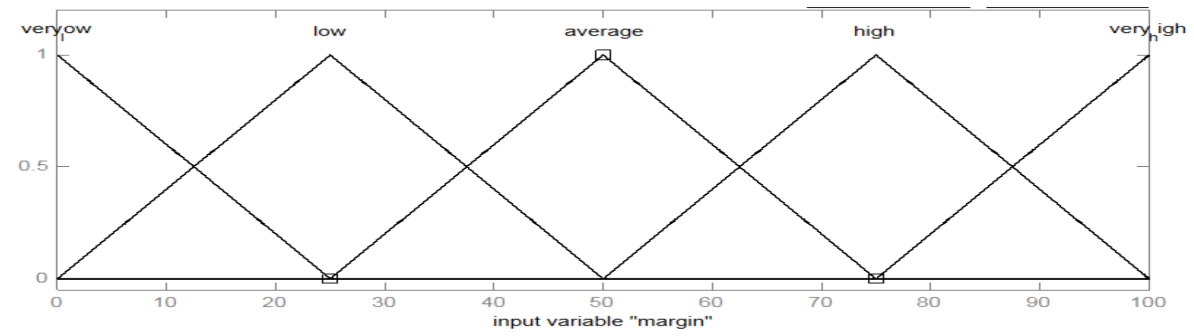


**Figure 27:** Linguistic terms for second input in case study 2

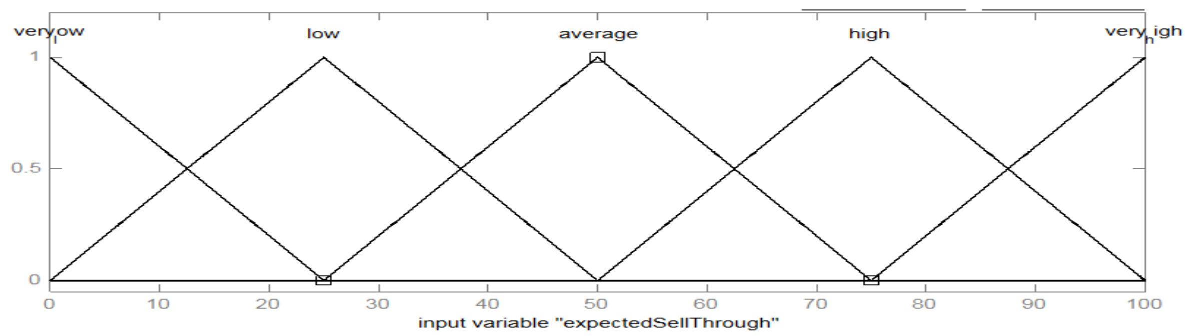


**Figure 28:** Linguistic terms for third input in case study 2

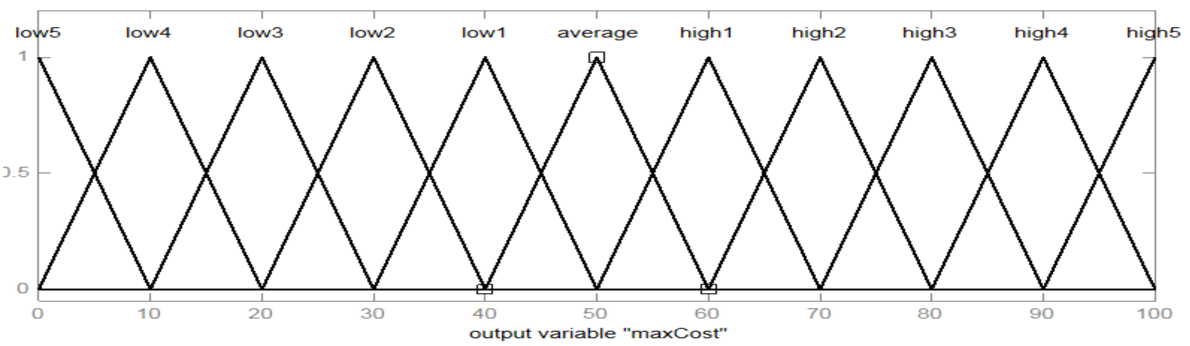


**Figure 29:** Linguistic terms for output in case study 2

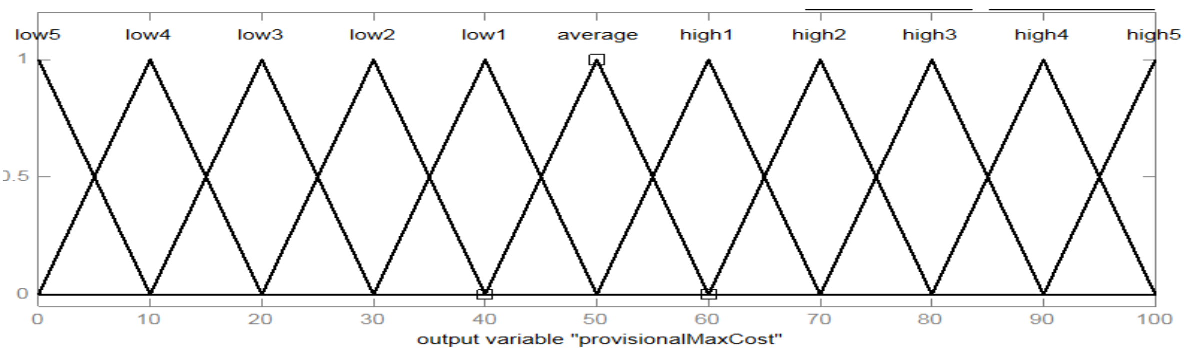


**Figure 30:** Linguistic terms for connection in case study 2

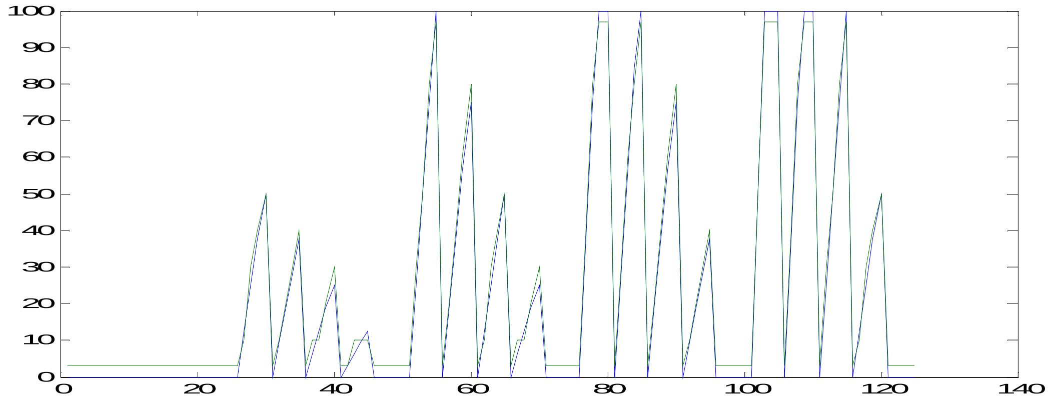


**Figure 31:** Simulation results for standard fuzzy system in case study 2

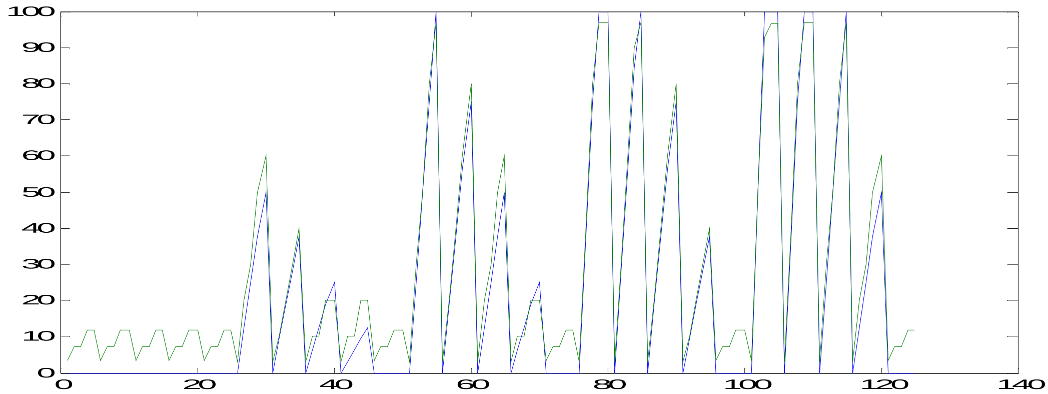


**Figure 32:** Simulation results for hierarchical fuzzy system in case study 2

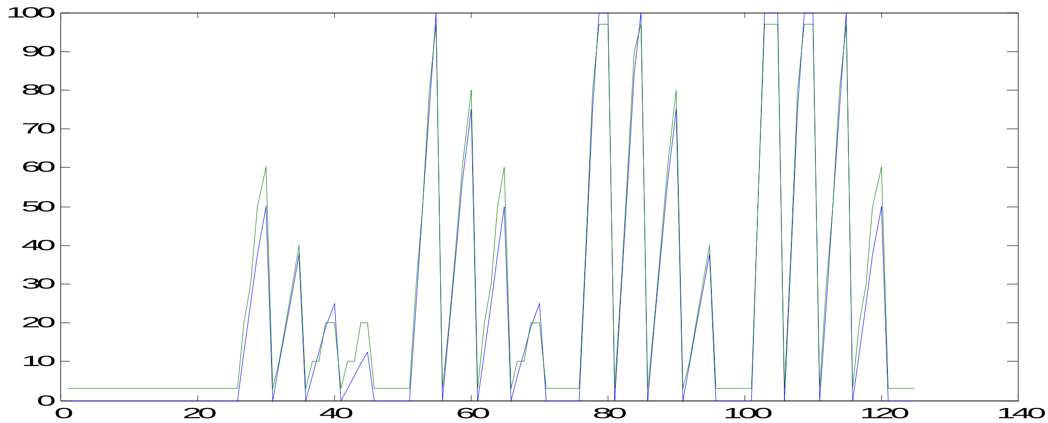


**Figure 33:** Simulation results for fuzzy network in case study 2

**Table 1:** Partial rule base for standard fuzzy system in case study 1

| Rule | x1 | x2 | x3 | x4 | x5 | x6 | x7 | y |
|---|---|---|---|---|---|---|---|---|
| ***1*** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| ***2*** | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 |
| ***3*** | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 2 |
| ***4*** | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 |
| ***5*** | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 1 |
| ***6*** | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 2 |
| ***7*** | 1 | 1 | 1 | 1 | 1 | 3 | 1 | 2 |
| ***8*** | 1 | 1 | 1 | 1 | 1 | 3 | 2 | 2 |
| ***9*** | 1 | 1 | 1 | 1 | 1 | 3 | 3 | 3 |
| ***2179*** | 3 | 3 | 3 | 3 | 3 | 1 | 1 | 2 |
| ***2180*** | 3 | 3 | 3 | 3 | 3 | 1 | 2 | 3 |
| ***2181*** | 3 | 3 | 3 | 3 | 3 | 1 | 3 | 3 |
| ***2182*** | 3 | 3 | 3 | 3 | 3 | 2 | 1 | 2 |
| ***2183*** | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 3 |
| ***2184*** | 3 | 3 | 3 | 3 | 3 | 2 | 3 | 3 |
| ***2185*** | 3 | 3 | 3 | 3 | 3 | 3 | 1 | 2 |
| ***2186*** | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 3 |
| ***2187*** | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

**Table 2:** First rule base for hierarchical fuzzy system in case study 1

| Rules | x1 | x2 | Z12,13 |
|---|---|---|---|
| ***1*** | 1 | 1 | 1 |
| ***2*** | 1 | 2 | 1 |
| ***3*** | 1 | 3 | 2 |
| ***4*** | 2 | 1 | 1 |
| ***5*** | 2 | 2 | 2 |
| ***6*** | 2 | 3 | 3 |
| ***7*** | 3 | 1 | 2 |
| ***8*** | 3 | 2 | 3 |
| ***9*** | 3 | 3 | 3 |

**Table 3:** Second rule base for hierarchical fuzzy system in case study 1

| ***Rules*** | ***x3*** | ***x4*** | ***Z31,32*** |
|---|---|---|---|
| ***10*** | 1 | 1 | 1 |
| ***11*** | 1 | 2 | 1 |
| ***12*** | 1 | 3 | 2 |
| ***13*** | 2 | 1 | 1 |
| ***14*** | 2 | 2 | 2 |
| ***15*** | 2 | 3 | 3 |
| ***16*** | 3 | 1 | 2 |
| ***17*** | 3 | 2 | 3 |
| ***18*** | 3 | 3 | 3 |

**Table 4:** Third rule base for hierarchical fuzzy system in case study 1

| ***Rules*** | ***x5*** | ***x6*** | ***Z41,32*** |
|---|---|---|---|
| ***19*** | 1 | 1 | 1 |
| ***20*** | 1 | 2 | 2 |
| ***21*** | 1 | 3 | 2 |
| ***22*** | 2 | 1 | 2 |
| ***23*** | 2 | 2 | 2 |
| ***24*** | 2 | 3 | 3 |
| ***25*** | 3 | 1 | 2 |
| ***26*** | 3 | 2 | 3 |
| ***27*** | 3 | 3 | 3 |

**Table 5:** Fourth rule base for hierarchical fuzzy system in case study 1

| ***Rules*** | ***Z31,32*** | ***Z41,32*** | ***Z32,13*** |
|---|---|---|---|
| ***28*** | 1 | 1 | 1 |
| ***29*** | 1 | 2 | 2 |
| ***30*** | 1 | 3 | 3 |
| ***31*** | 2 | 1 | 2 |
| ***32*** | 2 | 2 | 2 |
| ***33*** | 2 | 3 | 3 |
| ***34*** | 3 | 1 | 3 |
| ***35*** | 3 | 2 | 3 |
| ***36*** | 3 | 3 | 3 |

**Table 6:** Fifth rule base for hierarchical fuzzy system in case study 1

| ***Rules*** | ***Z12,13*** | ***Z32,13*** | ***x7*** | ***y*** |
|---|---|---|---|---|
| ***37*** | 1 | 1 | 1 | 1 |
| ***38*** | 1 | 1 | 2 | 1 |
| ***39*** | 1 | 1 | 3 | 2 |
| ***40*** | 1 | 2 | 1 | 1 |
| ***41*** | 1 | 2 | 2 | 1 |
| ***42*** | 1 | 2 | 3 | 2 |
| ***43*** | 1 | 3 | 1 | 2 |
| ***44*** | 1 | 3 | 2 | 2 |
| ***45*** | 1 | 3 | 3 | 3 |
| ***46*** | 2 | 1 | 1 | 1 |
| ***47*** | 2 | 1 | 2 | 1 |
| ***48*** | 2 | 1 | 3 | 2 |
| ***49*** | 2 | 2 | 1 | 1 |
| ***50*** | 2 | 2 | 2 | 2 |
| ***51*** | 2 | 2 | 3 | 2 |
| ***52*** | 2 | 3 | 1 | 2 |
| ***53*** | 2 | 3 | 2 | 2 |
| ***54*** | 2 | 3 | 3 | 3 |
| ***55*** | 3 | 1 | 1 | 1 |
| ***56*** | 3 | 1 | 2 | 1 |
| ***57*** | 3 | 1 | 3 | 2 |
| ***58*** | 3 | 2 | 1 | 1 |
| ***59*** | 3 | 2 | 2 | 2 |
| ***60*** | 3 | 2 | 3 | 3 |
| ***61*** | 3 | 3 | 1 | 2 |
| ***62*** | 3 | 3 | 2 | 3 |
| ***63*** | 3 | 3 | 3 | 3 |

**Table 7:** Partial rule base for fuzzy network in case study 1

| Rule | x1 | x2 | x3 | x4 | x5 | x6 | x7 | y |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 2 |
| 4 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 2 |
| 7 | 1 | 1 | 1 | 1 | 1 | 3 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 3 | 2 | 1 |
| 9 | 1 | 1 | 1 | 1 | 1 | 3 | 3 | 2 |
| 2179 | 3 | 3 | 3 | 3 | 3 | 1 | 1 | 2 |
| 2180 | 3 | 3 | 3 | 3 | 3 | 1 | 2 | 3 |
| 2181 | 3 | 3 | 3 | 3 | 3 | 1 | 3 | 3 |
| 2182 | 3 | 3 | 3 | 3 | 3 | 2 | 1 | 2 |
| 2183 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 3 |
| 2184 | 3 | 3 | 3 | 3 | 3 | 2 | 3 | 3 |
| 2185 | 3 | 3 | 3 | 3 | 3 | 3 | 1 | 2 |
| 2186 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 3 |
| 2187 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

**Table 8:** Comparative evaluation of three fuzzy models for case study 1

| Performance Indicator | Standard Fuzzy System | Hierarchical Fuzzy System | Fuzzy Network |
|---|---|---|---|
| Accuracy | 1.13 | 1.15 | 1.16 |
| Efficiency | 2187 | 63 | 2187 |
| Transparency | 8 | 0.72 | 0.72 |

**Table 9:** First part of rule base for standard fuzzy system in case study 2

| Rule | $x_1$ | $x_2$ | $x_3$ | $y$ | Rule | $x_1$ | $x_2$ | $x_3$ | $y$ | Rule | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 26 | 2 | 1 | 1 | 1 | 51 | 3 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 | 1 | 27 | 2 | 1 | 2 | 2 | 52 | 3 | 1 | 2 | 4 |
| 3 | 1 | 1 | 3 | 1 | 28 | 2 | 1 | 3 | 4 | 53 | 3 | 1 | 3 | 6 |
| 4 | 1 | 1 | 4 | 1 | 29 | 2 | 1 | 4 | 5 | 54 | 3 | 1 | 4 | 9 |
| 5 | 1 | 1 | 5 | 1 | 30 | 2 | 1 | 5 | 6 | 55 | 3 | 1 | 5 | 11 |
| 6 | 1 | 2 | 1 | 1 | 31 | 2 | 2 | 1 | 1 | 56 | 3 | 2 | 1 | 1 |
| 7 | 1 | 2 | 2 | 1 | 32 | 2 | 2 | 2 | 2 | 57 | 3 | 2 | 2 | 3 |
| 8 | 1 | 2 | 3 | 1 | 33 | 2 | 2 | 3 | 3 | 58 | 3 | 2 | 3 | 5 |
| 9 | 1 | 2 | 4 | 1 | 34 | 2 | 2 | 4 | 4 | 59 | 3 | 2 | 4 | 7 |
| 10 | 1 | 2 | 5 | 1 | 35 | 2 | 2 | 5 | 5 | 50 | 3 | 2 | 5 | 9 |
| 11 | 1 | 3 | 1 | 1 | 36 | 2 | 3 | 1 | 1 | 61 | 3 | 3 | 1 | 1 |
| 12 | 1 | 3 | 2 | 1 | 37 | 2 | 3 | 2 | 2 | 62 | 3 | 3 | 2 | 2 |
| 13 | 1 | 3 | 3 | 1 | 38 | 2 | 3 | 3 | 2 | 63 | 3 | 3 | 3 | 4 |
| 14 | 1 | 3 | 4 | 1 | 39 | 2 | 3 | 4 | 3 | 64 | 3 | 3 | 4 | 5 |
| 15 | 1 | 3 | 5 | 1 | 40 | 2 | 3 | 5 | 4 | 65 | 3 | 3 | 5 | 6 |
| 16 | 1 | 4 | 1 | 1 | 41 | 2 | 4 | 1 | 1 | 66 | 3 | 4 | 1 | 1 |
| 17 | 1 | 4 | 2 | 1 | 42 | 2 | 4 | 2 | 1 | 67 | 3 | 4 | 2 | 2 |
| 18 | 1 | 4 | 3 | 1 | 43 | 2 | 4 | 3 | 2 | 68 | 3 | 4 | 3 | 2 |
| 19 | 1 | 4 | 4 | 1 | 44 | 2 | 4 | 4 | 2 | 69 | 3 | 4 | 4 | 3 |
| 20 | 1 | 4 | 5 | 1 | 45 | 2 | 4 | 5 | 2 | 70 | 3 | 4 | 5 | 4 |
| 21 | 1 | 5 | 1 | 1 | 46 | 2 | 5 | 1 | 1 | 71 | 3 | 5 | 1 | 1 |
| 22 | 1 | 5 | 2 | 1 | 47 | 2 | 5 | 2 | 1 | 72 | 3 | 5 | 2 | 1 |
| 23 | 1 | 5 | 3 | 1 | 48 | 2 | 5 | 3 | 1 | 73 | 3 | 5 | 3 | 1 |
| 24 | 1 | 5 | 4 | 1 | 49 | 2 | 5 | 4 | 1 | 74 | 3 | 5 | 4 | 1 |
| 25 | 1 | 5 | 5 | 1 | 50 | 2 | 5 | 5 | 1 | 75 | 3 | 5 | 5 | 1 |

**Table 10:** Second part of rule base for standard fuzzy system in case study 2

| Rule | $x_1$ | $x_2$ | $x_3$ | $y$ | Rule | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|---|---|---|---|---|
| 76 | 4 | 1 | 1 | 1 | 101 | 5 | 1 | 1 | 1 |
| 77 | 4 | 1 | 2 | 5 | 102 | 5 | 1 | 2 | 6 |
| 78 | 4 | 1 | 3 | 9 | 103 | 5 | 1 | 3 | 11 |
| 79 | 4 | 1 | 4 | 11 | 104 | 5 | 1 | 4 | 11 |
| 80 | 4 | 1 | 5 | 11 | 105 | 5 | 1 | 5 | 11 |
| 81 | 4 | 2 | 1 | 1 | 106 | 5 | 2 | 1 | 1 |
| 82 | 4 | 2 | 2 | 4 | 107 | 5 | 2 | 2 | 5 |
| 83 | 4 | 2 | 3 | 7 | 108 | 5 | 2 | 3 | 9 |
| 84 | 4 | 2 | 4 | 9 | 109 | 5 | 2 | 4 | 11 |
| 85 | 4 | 2 | 5 | 11 | 110 | 5 | 2 | 5 | 11 |
| 86 | 4 | 3 | 1 | 1 | 111 | 5 | 3 | 1 | 1 |
| 97 | 4 | 3 | 2 | 3 | 112 | 5 | 3 | 2 | 4 |
| 88 | 4 | 3 | 3 | 5 | 113 | 5 | 3 | 3 | 6 |
| 89 | 4 | 3 | 4 | 7 | 114 | 5 | 3 | 4 | 9 |
| 90 | 4 | 3 | 5 | 9 | 115 | 5 | 3 | 5 | 11 |
| 91 | 4 | 4 | 1 | 1 | 116 | 5 | 4 | 1 | 1 |
| 92 | 4 | 4 | 2 | 2 | 117 | 5 | 4 | 2 | 2 |
| 93 | 4 | 4 | 3 | 3 | 118 | 5 | 4 | 3 | 4 |
| 94 | 4 | 4 | 4 | 4 | 119 | 5 | 4 | 4 | 5 |
| 95 | 4 | 4 | 5 | 5 | 120 | 5 | 4 | 5 | 6 |
| 96 | 4 | 5 | 1 | 1 | 121 | 5 | 5 | 1 | 1 |
| 97 | 4 | 5 | 2 | 1 | 122 | 5 | 5 | 2 | 1 |
| 98 | 4 | 5 | 3 | 1 | 123 | 5 | 5 | 3 | 1 |
| 99 | 4 | 5 | 4 | 1 | 124 | 5 | 5 | 4 | 1 |
| 100 | 4 | 5 | 5 | 1 | 125 | 5 | 5 | 5 | 1 |

**Table 11:** First rule base for hierarchical fuzzy system in case study 2

| Rule | $x_1$ | $x_2$ | $z_1$ | Rule | $x_1$ | $x_2$ | $z_1$ | Rule | $x_1$ | $x_2$ | $z_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 11 | 3 | 1 | 6 | 21 | 5 | 1 | 11 |
| 2 | 1 | 2 | 1 | 12 | 3 | 2 | 5 | 22 | 5 | 2 | 9 |
| 3 | 1 | 3 | 1 | 13 | 3 | 3 | 4 | 23 | 5 | 3 | 6 |
| 4 | 1 | 4 | 1 | 14 | 3 | 4 | 2 | 24 | 5 | 4 | 4 |
| 5 | 1 | 5 | 1 | 15 | 3 | 5 | 1 | 25 | 5 | 5 | 1 |
| 6 | 2 | 1 | 4 | 16 | 4 | 1 | 9 | - | - | - | - |
| 7 | 2 | 2 | 3 | 17 | 4 | 2 | 7 | - | - | - | - |
| 8 | 2 | 3 | 2 | 18 | 4 | 3 | 5 | - | - | - | - |
| 9 | 2 | 4 | 2 | 19 | 4 | 4 | 3 | - | - | - | - |
| 10 | 2 | 5 | 1 | 20 | 4 | 5 | 1 | - | - | - | - |

**Table 12:** Second rule base for hierarchical fuzzy system in case study 2

| Rule | $z_1$ | $x_3$ | $y$ | Rule | $z_1$ | $x_3$ | $y$ | Rule | $z_1$ | $x_3$ | $y$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 21 | 5 | 1 | 1 | 41 | 9 | 1 | 1 |
| 2 | 1 | 2 | 1 | 22 | 5 | 2 | 3 | 42 | 9 | 2 | 5 |
| 3 | 1 | 3 | 1 | 23 | 5 | 3 | 5 | 43 | 9 | 3 | 9 |
| 4 | 1 | 4 | 1 | 24 | 5 | 4 | 7 | 44 | 9 | 4 | 11 |
| 5 | 1 | 5 | 1 | 25 | 5 | 5 | 9 | 45 | 9 | 5 | 11 |
| 6 | 2 | 1 | 1 | 26 | 6 | 1 | 1 | 46 | 10 | 1 | 1 |
| 7 | 2 | 2 | 2 | 27 | 6 | 2 | 4 | 47 | 10 | 2 | 6 |
| 8 | 2 | 3 | 2 | 28 | 6 | 3 | 6 | 48 | 10 | 3 | 10 |
| 9 | 2 | 4 | 3 | 29 | 6 | 4 | 9 | 49 | 10 | 4 | 11 |
| 10 | 2 | 5 | 3 | 30 | 6 | 5 | 11 | 50 | 10 | 5 | 11 |
| 11 | 3 | 1 | 1 | 31 | 7 | 1 | 1 | 51 | 11 | 1 | 1 |
| 12 | 3 | 2 | 2 | 32 | 7 | 2 | 4 | 52 | 11 | 2 | 6 |
| 13 | 3 | 3 | 3 | 33 | 7 | 3 | 7 | 53 | 11 | 3 | 11 |
| 14 | 3 | 4 | 4 | 34 | 7 | 4 | 10 | 54 | 11 | 4 | 11 |
| 15 | 3 | 5 | 5 | 35 | 7 | 5 | 11 | 55 | 11 | 5 | 11 |
| 16 | 4 | 1 | 1 | 36 | 8 | 1 | 1 | - | - | - | - |
| 17 | 4 | 2 | 3 | 37 | 8 | 2 | 5 | - | - | - | - |
| 18 | 4 | 3 | 4 | 38 | 8 | 3 | 8 | - | - | - | - |
| 19 | 4 | 4 | 6 | 39 | 8 | 4 | 11 | - | - | - | - |
| 20 | 4 | 5 | 7 | 40 | 8 | 5 | 11 | - | - | - | - |

**Table 13:** First part of rule base for fuzzy network in case study 2

| Rule | $x_1$ | $x_2$ | $x_3$ | $y$ | Rule | $x_1$ | $x_2$ | $x_3$ | $y$ | Rule | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 26 | 2 | 1 | 1 | 1 | 51 | 3 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 | 1 | 27 | 2 | 1 | 2 | 3 | 52 | 3 | 1 | 2 | 4 |
| 3 | 1 | 1 | 3 | 1 | 28 | 2 | 1 | 3 | 4 | 53 | 3 | 1 | 3 | 6 |
| 4 | 1 | 1 | 4 | 1 | 29 | 2 | 1 | 4 | 6 | 54 | 3 | 1 | 4 | 9 |
| 5 | 1 | 1 | 5 | 1 | 30 | 2 | 1 | 5 | 7 | 55 | 3 | 1 | 5 | 11 |
| 6 | 1 | 2 | 1 | 1 | 31 | 2 | 2 | 1 | 1 | 56 | 3 | 2 | 1 | 1 |
| 7 | 1 | 2 | 2 | 1 | 32 | 2 | 2 | 2 | 2 | 57 | 3 | 2 | 2 | 3 |
| 8 | 1 | 2 | 3 | 1 | 33 | 2 | 2 | 3 | 3 | 58 | 3 | 2 | 3 | 5 |
| 9 | 1 | 2 | 4 | 1 | 34 | 2 | 2 | 4 | 4 | 59 | 3 | 2 | 4 | 7 |
| 10 | 1 | 2 | 5 | 1 | 35 | 2 | 2 | 5 | 5 | 50 | 3 | 2 | 5 | 9 |
| 11 | 1 | 3 | 1 | 1 | 36 | 2 | 3 | 1 | 1 | 61 | 3 | 3 | 1 | 1 |
| 12 | 1 | 3 | 2 | 1 | 37 | 2 | 3 | 2 | 2 | 62 | 3 | 3 | 2 | 3 |
| 13 | 1 | 3 | 3 | 1 | 38 | 2 | 3 | 3 | 2 | 63 | 3 | 3 | 3 | 4 |
| 14 | 1 | 3 | 4 | 1 | 39 | 2 | 3 | 4 | 3 | 64 | 3 | 3 | 4 | 6 |
| 15 | 1 | 3 | 5 | 1 | 40 | 2 | 3 | 5 | 3 | 65 | 3 | 3 | 5 | 7 |
| 16 | 1 | 4 | 1 | 1 | 41 | 2 | 4 | 1 | 1 | 66 | 3 | 4 | 1 | 1 |
| 17 | 1 | 4 | 2 | 1 | 42 | 2 | 4 | 2 | 2 | 67 | 3 | 4 | 2 | 2 |
| 18 | 1 | 4 | 3 | 1 | 43 | 2 | 4 | 3 | 2 | 68 | 3 | 4 | 3 | 2 |
| 19 | 1 | 4 | 4 | 1 | 44 | 2 | 4 | 4 | 3 | 69 | 3 | 4 | 4 | 3 |
| 20 | 1 | 4 | 5 | 1 | 45 | 2 | 4 | 5 | 3 | 70 | 3 | 4 | 5 | 3 |
| 21 | 1 | 5 | 1 | 1 | 46 | 2 | 5 | 1 | 1 | 71 | 3 | 5 | 1 | 1 |
| 22 | 1 | 5 | 2 | 1 | 47 | 2 | 5 | 2 | 1 | 72 | 3 | 5 | 2 | 1 |
| 23 | 1 | 5 | 3 | 1 | 48 | 2 | 5 | 3 | 1 | 73 | 3 | 5 | 3 | 1 |
| 24 | 1 | 5 | 4 | 1 | 49 | 2 | 5 | 4 | 1 | 74 | 3 | 5 | 4 | 1 |
| 25 | 1 | 5 | 5 | 1 | 50 | 2 | 5 | 5 | 1 | 75 | 3 | 5 | 5 | 1 |

**Table 14:** Second part of rule base for fuzzy network in case study 2

| Rule | $x_1$ | $x_2$ | $x_3$ | $y$ | Rule | $x_1$ | $x_2$ | $x_3$ | $y$ |
|---|---|---|---|---|---|---|---|---|---|
| 76 | 4 | 1 | 1 | 1 | 101 | 5 | 1 | 1 | 1 |
| 77 | 4 | 1 | 2 | 5 | 102 | 5 | 1 | 2 | 6 |
| 78 | 4 | 1 | 3 | 9 | 103 | 5 | 1 | 3 | 11 |
| 79 | 4 | 1 | 4 | 11 | 104 | 5 | 1 | 4 | 11 |
| 80 | 4 | 1 | 5 | 11 | 105 | 5 | 1 | 5 | 11 |
| 81 | 4 | 2 | 1 | 1 | 106 | 5 | 2 | 1 | 1 |
| 82 | 4 | 2 | 2 | 4 | 107 | 5 | 2 | 2 | 5 |
| 83 | 4 | 2 | 3 | 7 | 108 | 5 | 2 | 3 | 9 |
| 84 | 4 | 2 | 4 | 10 | 109 | 5 | 2 | 4 | 11 |
| 85 | 4 | 2 | 5 | 11 | 110 | 5 | 2 | 5 | 11 |
| 86 | 4 | 3 | 1 | 1 | 111 | 5 | 3 | 1 | 1 |
| 97 | 4 | 3 | 2 | 3 | 112 | 5 | 3 | 2 | 4 |
| 88 | 4 | 3 | 3 | 5 | 113 | 5 | 3 | 3 | 6 |
| 89 | 4 | 3 | 4 | 7 | 114 | 5 | 3 | 4 | 9 |
| 90 | 4 | 3 | 5 | 9 | 115 | 5 | 3 | 5 | 11 |
| 91 | 4 | 4 | 1 | 1 | 116 | 5 | 4 | 1 | 1 |
| 92 | 4 | 4 | 2 | 2 | 117 | 5 | 4 | 2 | 3 |
| 93 | 4 | 4 | 3 | 3 | 118 | 5 | 4 | 3 | 4 |
| 94 | 4 | 4 | 4 | 4 | 119 | 5 | 4 | 4 | 6 |
| 95 | 4 | 4 | 5 | 5 | 120 | 5 | 4 | 5 | 7 |
| 96 | 4 | 5 | 1 | 1 | 121 | 5 | 5 | 1 | 1 |
| 97 | 4 | 5 | 2 | 1 | 122 | 5 | 5 | 2 | 1 |
| 98 | 4 | 5 | 3 | 1 | 123 | 5 | 5 | 3 | 1 |
| 99 | 4 | 5 | 4 | 1 | 124 | 5 | 5 | 4 | 1 |
| 100 | 4 | 5 | 5 | 1 | 125 | 5 | 5 | 5 | 1 |

**Table 15:** Comparative evaluation of three fuzzy models for case study 2

| Performance indicator | Standard fuzzy system | Hierarchical fuzzy system | Fuzzy network |
|---|---|---|---|
| Accuracy | 2.86 | 5.57 | 3.64 |
| Efficiency | 125 | 80 | 125 |
| Transparency | 4 | 1.33 | 1.33 |