

LINEARISED HIGHER VARIATIONAL EQUATIONS

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ABSTRACT. This work explores the tensor and combinatorial constructs underlying the linearised higher-order variational equations LVE_ψ^k of a generic autonomous system along a particular solution ψ . The main result of this paper is a compact yet explicit and computationally amenable form for said variational systems and their monodromy matrices. Alternatively, the same methods are useful to retrieve, and sometimes simplify, systems satisfied by the coefficients of the Taylor expansion of a formal first integral for a given dynamical system. This is done in preparation for further results within Ziglin-Morales-Ramis theory, specifically those of a constructive nature.

1. Motivation and first definitions.

1.1. Introduction. *Integrability*, an informal word for reasonably simple solvability, is an important problem in Dynamical Systems. Its opposite phenomenon, and specifically low predictability with respect to time, is usually summarised under the term *chaos*. If the system is Hamiltonian, as are most problems in Mechanics, the “chaos vs solvability” disjunctive is doubly advantageous. On one hand, it is amenable to the techniques of Symplectic Geometry. On the other, theory and empirics yield the specific, thus observable integrability condition described in §1.3.

The introduction of the algebraic approach by Ziglin, Morales-Ruiz and Ramis produced hallmark contributions to the study of the integrability of Hamiltonian systems [6, 22, 23, 31], essentially couched on a study of the invariants of a given matrix group, associated to a linear system: the *first-order variational equations* introduced in 1.2. A second step forward was carried out by Morales-Ruiz, Ramis and Simó ([24]) in order to extend the preceding Galoisian framework to the groups of the higher-order variational equations along a particular solution.

The second step described above is the driving force behind this paper. A constructive version of the Morales-Ramis-Simó theorem was already started in [2] and tangentially tackled from another viewpoint in [5] (see §5) and the present work aims at expanding this effort by offering a closed-form expression for the linearised

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higher variationals. May the reader bear in mind that *nowhere from §2 onwards, except for §6, is the system required to be Hamiltonian.*

1.2. Dynamical systems and variational equations. In accordance with results described in §1.3 and thereafter, we need to observe the following convention outside of Sections 2 and 3: dependent and independent variables for all dynamical systems will be allowed to be *complex*. Any open set $T \subseteq \mathbb{P}_{\mathbb{C}}^1$ is an admissible domain for the time variable, embedded into the Riemann sphere to include $t = \infty$ as a valid singularity. Consider an autonomous holomorphic dynamical system:

$$\dot{z} = X(z), \quad \text{where } X : U \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n. \quad (\text{DS})$$

Conserved quantities and solution curve foliations are defined similarly to their real-valued counterparts. Indeed, a **first integral** of (DS) is a function $F : U \rightarrow \mathbb{C}$ constant along every solution of (DS). And for every $z \in U$, the unique solution $\varphi(t, z)$ of (DS) such that $\varphi(0, z) = z$ allows us to define a function $\varphi(\cdot, \cdot)$ in $n + 1$ variables called the **flow** of (DS). Clarifying preliminary comments are in order whenever a particular solution $\psi(t)$ is considered:

- a) partial derivatives $\frac{\partial^k}{\partial z^k} \varphi(t, \psi)$ are multilinear functions of increasing order (or multidimensional matrices, see e.g. [17]) and appear in the Taylor series of the flow along ψ :

$$\varphi(t, z) = \varphi(t, \psi) + \frac{\partial \varphi(t, \psi)}{\partial z} \{z - \psi\} + \frac{1}{2!} \frac{\partial^2 \varphi(t, \psi)}{\partial z^2} \{z - \psi\}^2 + \dots; \quad (1)$$

- b) each of these derivatives $\frac{\partial^k}{\partial z^k} \varphi(t, \psi)$ satisfies an echeloned set of differential systems, depending on the previous $k - 1$ partial derivatives and customarily called **variational equations or systems**. They are explicitly called **higher-order** whenever $k \geq 2$.
- c) variational system for $k = 1$ is *linear* and satisfied by the linear part of the flow along ψ :

$$\dot{Y}_1 = A_1 Y_1, \quad A_1(t) := X'(\psi) \in \text{Mat}_n(K), \quad (\text{VE}_{\psi})$$

$K = \mathbb{C}(\psi)$ being the smallest differential field containing $\mathbb{C}(t)$ and the solution.

- d) For $k \geq 2$, however, the system is not linear, yet a linearised version may be found. *The aim of the present paper is to do so with explicit formulae.*

1.3. Morales-Ramis-Ziglin theory and extensions. Heuristics of all results within the Ziglin-Morales-Ramis-Simó theoretical framework are firmly rooted in the following principle, expected to affect a widespread class of systems:

If general system (DS) is “integrable” in some reasonable sense, then the system satisfied by each of the partial derivatives of the flow at every particular solution ψ of (DS) must be also integrable in an accordingly reasonable sense.

Any attempt at ad-hoc formulations of this heuristic principle has an asset and a drawback:

- there *is* a valid integrability axiom for linear systems, e.g. (VE_ψ): the solvability of the Zariski identity component of the (linear algebraic) *differential Galois group* [22, 29];
- an explicit incarnation of this principle requires a clear notion of “integrability” for (DS).

The latter item is cleared in the Hamiltonian case by the *Liouville-Arnold Theorem* establishing a sufficient condition for a system to admit, at least locally, a new set of variables rendering it integrable by quadratures. Said condition is the hypothesis on H in the following:

Theorem 1.1 (Morales-Ruiz, Ramis, 2001). *Let X_H be an n -degree-of-freedom Hamiltonian system having n independent first integrals in pairwise involution, defined on a neighborhood of an integral curve ψ . Then, Galoisian identity component $\text{Gal}(\text{VE}_\psi)^\circ$ is an abelian group.* \square

See [23, Cor. 8] or [22, Th. 4.1] for a precise statement and a proof.

Theorem 1.2 (Morales-Ruiz, Ramis, Simó, 2005, [24, Th. 5]). *Let H be as in the previous theorem. Let G_k be the differential Galois group of the k -th variational equations VE_ψ^k , $k \geq 1$, and $G := \varprojlim G_k$ the formal differential Galois group (inverse limit of the groups) of X_H along ψ . Then, the identity components of the Galois groups G_k and G are abelian.* \square

Theorem 1.2 makes use of the language of jets, after proving non-linear VE_ψ^k equivalent to *any* consistent linearised completion. Efforts towards a constructive version of this main Theorem, as well as the line of study described in §5, are hampered by a lack of consensus on the explicit block structure of this completion. The present work, summarised in its main result (Proposition 4) aims at contributing to fill in this gap. Hence, outcomes will be restricted to symbolic calculus and bear no new results in the above theoretical framework.

Notation 1. Part of the conventions listed below were already introduced in [5].

1. The *modulus* $i = |\mathbf{i}|$ of a multi-index $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$ is the sum of its entries. Multi-index addition and subtraction are defined entrywise as usual.
2. *Multi-index order*: $(i_1, \dots, i_n) \leq (j_1, \dots, j_n)$ means $i_k \leq j_k$ for every $k \geq 1$.
3. *Standard lexicographic order*: $(i_1, \dots, i_n) <_{\text{lex}} (j_1, \dots, j_n)$ if $i_1 = j_1, \dots, i_{k-1} = j_{k-1}$ and $i_k < j_k$ for some $k \geq 1$.
4. Given complex analytic $F : U \subset \mathbb{C}^n \rightarrow \mathbb{C}$ we define the *lexicographically sifted differential of F of order m* as the row vector $F^{(m)}(\mathbf{x}) := \text{lex} \left(\frac{\partial^m F}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}(\mathbf{x}) \right)$, where $|\mathbf{i}| = m$ and entries are ordered as per $<_{\text{lex}}$ on multi-indices.
5. We define $d_{n,k} := \binom{n+k-1}{n-1}$, $D_{n,k} := \sum_{i=1}^k d_{n,i}$. It is easy to check there are $d_{n,k}$ k -ples of integers in $\{1, \dots, n\}$, and just as many homogeneous monomials of degree n in k variables.

Notation 2. Given integers $k_1, \dots, k_n \geq 0$, we define the usual multinomial coefficient as

$$\binom{k_1 + \dots + k_n}{k_1, \dots, k_n} := \binom{k_1 + \dots + k_n}{\mathbf{k}} := \frac{(k_1 + \dots + k_n)!}{k_1! k_2! \dots k_n!}.$$

For a multi-index $\mathbf{k} \in \mathbb{Z}_{\geq 0}^n$, define $\mathbf{k}! := k_1! \dots k_n!$. For any two such \mathbf{k}, \mathbf{j} , we define

$$\binom{\mathbf{k}}{\mathbf{p}} := \frac{k_1! k_2! \dots k_n!}{p_1! p_2! \dots p_n! (k_1 - p_1)! (k_2 - p_2)! \dots (k_n - p_n)!} = \binom{k_1}{p_1} \binom{k_2}{p_2} \dots \binom{k_n}{p_n}, \quad (2)$$

and the multi-index counterpart to the multinomial, $\binom{\mathbf{k}_1 + \dots + \mathbf{k}_m}{\mathbf{k}_1, \dots, \mathbf{k}_m} := \frac{(\mathbf{k}_1 + \dots + \mathbf{k}_m)!}{\mathbf{k}_1! \mathbf{k}_2! \dots \mathbf{k}_m!}$.

2. Symmetric products and powers of finite matrices.

2.1. Definition and properties. The compact formulation called for by (1) and Notation 1 (3) will be achieved through a product \odot that was already defined by other means by U. Bekbaev (e.g. [8, 9, 10, 11]) and will be systematised using basic categorical properties of the tensor product. Let K be a field and V a K -vector space. See [13, 14, 18] for details.

Definition 2.1. An r^{th} **symmetric tensor power** of V is a vector space S , together with a symmetric multilinear map $\varphi : V^r := V \times \dots \times V \rightarrow S$ satisfying the following *universal property*: for every vector space W and every symmetric multilinear map $f : V^r \rightarrow W$ there is a unique linear map $f_\odot : S \rightarrow W$ such that the following diagram commutes:

$$\begin{array}{ccc} V \times V \times \dots \times V & \xrightarrow{f} & W \\ \varphi \downarrow & \nearrow f_\odot & \\ S & & \end{array}$$

In other words, $\text{Hom}_K(S, W) \cong S(V^r, W)$ holds between the vector space of linear maps $S \rightarrow W$ and the vector space of symmetric multilinear maps $V^r \rightarrow W$.

Proposition 1. *Given any K -vector space V and any $r \in \mathbb{N}$,*

- a) *a symmetric power $(\text{Sym}^r V, \varphi)$ exists, unique up to isomorphism. We write $\mathbf{v}_1 \odot \dots \odot \mathbf{v}_r := \varphi(\mathbf{v}_1, \dots, \mathbf{v}_r)$, $\mathbf{v}^{\odot k} := \mathbf{v} \odot \dots \odot \mathbf{v}$ for any $\mathbf{v} \in V$, and $\mathbf{v}^{\odot \mathbf{p}} := \mathbf{v}_1^{\odot p_1} \odot \dots \odot \mathbf{v}_n^{\odot p_n}$, for any $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ and multi-index $\mathbf{p} \in \mathbb{Z}_{\geq 0}^n$.*
- b) *For any multilinear map $f : V^r \rightarrow W$, the linear map f_\odot induced by the universal property is defined on the generators of $\text{Sym}^r V$ as $f_\odot(\mathbf{v}_1 \odot \dots \odot \mathbf{v}_r) = f(\mathbf{v}_1, \dots, \mathbf{v}_r)$.*
- c) *If $\dim_K V = n < \infty$ then every basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of V induces a basis for $\text{Sym}^r V$:*

$$\{(\mathbf{e}_1 \odot \dots \odot \mathbf{e}_1) \odot (\mathbf{e}_2 \odot \dots \odot \mathbf{e}_2) \odot \dots \odot (\mathbf{e}_n \odot \dots \odot \mathbf{e}_n) : r_i \geq 0, |\mathbf{r}| = r\}; \quad (3)$$

hence, $\dim_K \text{Sym}^r V = d_{n,r}$. Conventions $\text{Sym}^1 V = V$ and $\text{Sym}^0 V = K$ arise naturally.

Hence, product \odot operates exactly like products of homogeneous polynomials in several variables.

Remark 1. Sym^r may also be defined in terms of the tensor power by $\text{Sym}^r V = \bigotimes^r V / \sim$ modulo the relation $\mathbf{v}_1 \otimes \dots \otimes \mathbf{v}_r \sim \mathbf{v}_{\sigma(1)} \otimes \dots \otimes \mathbf{v}_{\sigma(r)}$, $\sigma \in \mathfrak{S}_r$.

Given any K -vector space W and two linear maps $f, g : V \rightarrow W$, define

$$h : V \times V \rightarrow \text{Sym}^2 W, \quad h(\mathbf{v}_1, \mathbf{v}_2) := \frac{1}{2} [f(\mathbf{v}_1) \odot g(\mathbf{v}_2) + f(\mathbf{v}_2) \odot g(\mathbf{v}_1)]. \quad (4)$$

Immediately bilinear and symmetric, it is granted a unique linear $h_\odot : \text{Sym}^2 V \rightarrow \text{Sym}^2 W$, $h_\odot(\mathbf{v}_1 \odot \mathbf{v}_2) := h(\mathbf{v}_1, \mathbf{v}_2)$, by the universal property. Write $f \odot g := h_\odot$. Then $f \odot g = g \odot f$ and $(f_1 \circ f) \odot (g_1 \circ g) = (f_1 \odot g_1) \circ (f \odot g)$ for any linear maps $f_1, g_1 : W \rightarrow W_1$. A similar construction applies to the symmetric product of $m \geq 3$

linear maps $f_i : V \rightarrow W$:

$$\begin{aligned} \text{Sym}^m V &\xrightarrow{f_1 \odot \cdots \odot f_m} \text{Sym}^m W \\ \mathbf{v}_1 \odot \cdots \odot \mathbf{v}_m &\longmapsto \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} f_1(\mathbf{v}_{\sigma(1)}) \odot \cdots \odot f_m(\mathbf{v}_{\sigma(m)}). \end{aligned} \quad (5)$$

Let us generalise the above symmetric product into one involving any two linear maps

$$f : \text{Sym}^{j_1} V \rightarrow \text{Sym}^{i_1} W, \quad g : \text{Sym}^{j_2} V \rightarrow \text{Sym}^{i_2} W, \quad j_1, j_2, i_1, i_2 \geq 0.$$

Assume V and W finite-dimensional, V having basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Defining the bilinear map $\varphi(\mathbf{u}_1, \mathbf{u}_2) := \mathbf{u}_1 \odot \mathbf{u}_2$, $\mathbf{u}_i \in \text{Sym}^{j_i} V$, we are interested in finding a bilinear function h in terms of f and g generalising (4), for which there is a unique linear h_\odot completing the diagram

$$\begin{array}{ccc} \text{Sym}^{j_1} V \times \text{Sym}^{j_2} V & \xrightarrow{h} & \text{Sym}^{i_1+i_2} W \\ \varphi \downarrow & \dashrightarrow^{h_\odot} & \\ \text{Sym}^{j_1+j_2} V & & \end{array} \quad (6)$$

We want h to yield coefficient 1 for all-round repeated vectors as in (4). Symmetric, multilinear $\tilde{h} : V^{\times j_1+j_2} \rightarrow \text{Sym}^{i_1+i_2} W$ is easier to define, generalising (4) and the example in [22, p. 155]: for any $\mathbf{u}_1, \dots, \mathbf{u}_{j_1+j_2} \in V$,

$$\tilde{h}(\mathbf{u}_1, \dots, \mathbf{u}_{j_1+j_2}) := \alpha_{j_1, j_2} \sum f(\mathbf{u}_{\sigma(1)} \odot \cdots \odot \mathbf{u}_{\sigma(j_1)}) \odot g(\mathbf{u}_{\sigma(j_1+1)} \odot \cdots \odot \mathbf{u}_{\sigma(j_1+j_2)}), \quad (7)$$

where $\alpha_{j_1, j_2} = \frac{1}{\binom{j_1+j_2}{j_1}}$ and the sum is taken over $\sigma \in S_{j_1, j_2}$ with

$$S_{j_1, j_2} := \{\sigma \in \mathfrak{S}_{j_1+j_2} : \sigma(1) < \cdots < \sigma(j_1) \text{ and } \sigma(j_1+1) < \cdots < \sigma(j_1+j_2)\}. \quad (8)$$

Define $(\varphi_1 \times \varphi_2)(\mathbf{u}_1, \dots, \mathbf{u}_{j_1+j_2}) = (\mathbf{u}_{i_1} \odot \cdots \odot \mathbf{u}_{i_{j_1}}, \mathbf{u}_{i_{j_1+1}} \odot \cdots \odot \mathbf{u}_{i_{j_1+j_2}})$, φ_i being the universal map of $\text{Sym}^{j_i} V$; we intend the diagram of functions involving the Cartesian product

$$\begin{array}{ccc} V^{\times j_1+j_2} & & \\ \varphi_1 \times \varphi_2 \downarrow & \searrow^{\tilde{h}} & \\ \text{Sym}^{j_1} V \times \text{Sym}^{j_2} V & \xrightarrow{h} & \text{Sym}^{i_1+i_2} W \end{array} \quad (9)$$

to commute. Let $\mathbf{u}_{i_1}, \dots, \mathbf{u}_{i_{j_1+j_2}} \in \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Split into copies of separate basis vectors: $\{\mathbf{u}_{i_1}, \dots, \mathbf{u}_{i_{j_1}}\} = \{\mathbf{e}_1, \mathbf{e}_1^{p_1}, \dots, \mathbf{e}_n, \mathbf{e}_n^{p_n}, \mathbf{e}_n\}$, $\{\mathbf{u}_{i_{j_1+1}}, \dots, \mathbf{u}_{i_{j_1+j_2}}\} = \{\mathbf{e}_1^{q_1}, \dots, \mathbf{e}_n^{q_n}\}$, with $|\mathbf{p}| = j_1$ and $|\mathbf{q}| = j_2$, and define $\mathbf{k} = \mathbf{p} + \mathbf{q}$. The expression of (7) in these basis elements is now an immediate consequence of basic combinatorics:

$$\tilde{h}(\mathbf{e}_1^{k_1}, \mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_n^{k_n}, \mathbf{e}_n) = \frac{1}{\binom{j_1+j_2}{j_1}} \sum_{|\mathbf{P}|=j_1, \mathbf{P} \leq \mathbf{k}} \left[\prod_{i=1}^n \binom{k_i}{P_i} \right] f(\mathbf{e}^{\odot \mathbf{P}}) \odot g(\mathbf{e}^{\odot \mathbf{k}-\mathbf{P}}),$$

leaving no option for (9) to commute but

$$h(\mathbf{e}^{\odot \mathbf{P}}, \mathbf{e}^{\odot \mathbf{Q}}) = \frac{1}{\binom{j_1+j_2}{j_1}} \sum_{|\mathbf{P}|=j_1, \mathbf{P} \leq \mathbf{p}+\mathbf{q}} \left[\prod_{i=1}^n \binom{p_i+q_i}{P_i} \right] f(\mathbf{e}^{\odot \mathbf{P}}) \odot g(\mathbf{e}^{\odot \mathbf{p}+\mathbf{q}-\mathbf{P}}).$$

Finally, the universal property on $(\text{Sym}^{j_1+j_2}V, \tilde{\varphi})$ yields a unique h_\odot such that $h_\odot \circ \tilde{\varphi} \equiv \tilde{h}$,

$$\begin{array}{ccc}
 & V^{\times j_1+j_2} & \\
 & \downarrow \varphi_1 \times \varphi_2 & \searrow \tilde{h} \\
 \tilde{\varphi} \swarrow & \text{Sym}^{j_1}V \times \text{Sym}^{j_2}V & \xrightarrow{h} \text{Sym}^{i_1+i_2}W \\
 & \downarrow \varphi & \nearrow h_\odot \\
 & \text{Sym}^{j_1+j_2}V &
 \end{array} \tag{10}$$

and $\varphi \circ (\varphi_1 \times \varphi_2) \equiv \tilde{\varphi}$. Fixing φ (and h) the uniqueness of h_\odot follows from construction: any other h_\bullet rendering (6) commutative would require the commutativity of the outer perimeter of (10), hence $h_\bullet \equiv h_\odot$. Hence all we need to do is express $f \odot g := h_\odot$ in terms of its action on base elements (3) to obtain a simple, explicit form.

Notation 3. When dealing with matrix sets, we will use super-indices and sub-indices:

1. The space of (i, j) -**matrices** $\text{Mat}_{m,n}^{i,j}(K)$ is either defined by its underlying set, i.e. all $d_{m,i} \times d_{n,j}$ matrices having entries in K , or as vector space $\text{Hom}_K(\text{Sym}^j K^n; \text{Sym}^i K^m)$.
2. It is clear from the above that $\text{Mat}_n^{0,0}(K)$ is the set of all scalars $\alpha \in K$ and $\text{Mat}_n^{0,k}(K)$ (resp. $\text{Mat}_n^{k,0}(K)$) is made up of all row (resp. column) vectors whose entries are indexed by $d_{n,k}$ lexicographically ordered k -tuples.
2. Reference to K may be dropped and notation may be abridged if dimensions are repeated or trivial, e.g. $\text{Mat}_n^{i,j} := \text{Mat}_{n,n}^{i,j}$, $\text{Mat}_{m,n}^i := \text{Mat}_{m,n}^{i,i}$, $\text{Mat}_n := \text{Mat}_n^1$, etcetera.

Checking product \odot defined below renders diagrams (6) and (10) commutative is immediate.

Definition 2.2 (Symmetric product of finite matrices). Let $A \in \text{Mat}_{m,n}^{i_1,j_1}(K)$, $B \in \text{Mat}_{m,n}^{i_2,j_2}(K)$, i.e. linear maps $A : \text{Sym}^{j_1}K^n \rightarrow \text{Sym}^{i_1}K^m$ and $B : \text{Sym}^{j_2}K^n \rightarrow \text{Sym}^{i_2}K^m$. Given any multi-index $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n$ such that $|\mathbf{k}| = k_1 + \dots + k_n = j_1 + j_2$, define $C := A \odot B \in \text{Mat}_{m,n}^{i_1+i_2,j_1+j_2}$ by

$$C \left(e_1^{\odot k_1} \dots e_n^{\odot k_n} \right) = \frac{1}{\binom{j_1+j_2}{j_1}} \sum_{\mathbf{p}} \binom{\mathbf{k}}{\mathbf{p}} A \left(e_1^{\odot p_1} \dots e_n^{\odot p_n} \right) \odot B \left(e_1^{\odot k_1-p_1} \dots e_n^{\odot k_n-p_n} \right), \tag{11}$$

notation abused by removing \odot to reduce space within basis elements (3), binomials as in (2) and summation taking place for specific multi-indices \mathbf{p} , namely those such that

$$|\mathbf{p}| = j_1 \quad \text{and} \quad 0 \leq p_i \leq k_i, \quad i = 1, \dots, n.$$

The following is a mere exercise in induction:

Lemma 2.3. Defining $\odot_{i=1}^r A_i$ recursively by $(\odot_{i=1}^{r-1} A_i) \odot A_r$ with $A_i \in \text{Mat}_{m,n}^{k_i,j_i}$,

$$(A_1 \odot \dots \odot A_r) e^{\odot \mathbf{k}} = \frac{1}{\binom{j_1+\dots+j_r}{j_1, j_2, \dots, j_r}} \sum_{\mathbf{p}_1, \dots, \mathbf{p}_r} \binom{\mathbf{k}}{\mathbf{p}_1, \dots, \mathbf{p}_r} \odot_{i=1}^r A_i e^{\odot \mathbf{p}_i}, \tag{12}$$

if $|\mathbf{k}| = j_1 + \dots + j_r$, sums obviously taken for $\mathbf{p}_1 + \dots + \mathbf{p}_r = \mathbf{k}$ and $|\mathbf{p}_i| = j_i$, for every $i = 1, \dots, r$. \square

Remarks 1.

1. For an equivalent “non-monic” formulation of (11) (i.e. one for which entry $_{1,1}$ need not have coefficient 1) using multi-indices in both columns *and* rows, see e.g. [8, 9, 10, 11].
2. Notation in Proposition 1 extends to matrices: $\text{Sym}^r A := A^{\odot r} := A \odot \dots \odot A$.
3. For square $A \in \text{Mat}_n^{1,1}$, powers $^{\odot r}$ according to (11) and (12) are obviously consistent with multiple product (5), hence equal to established definitions for group morphism $\text{Sym}^r : \text{GL}_n(V) \rightarrow \text{GL}_n(\text{Sym}^r(V))$ in multilinear algebra textbooks such as the expression in terms of the *permanent* of A (e.g. [13, Th. 9.2]), or $\frac{1}{r!} A \odot \dots \odot A$ in [2, 5, 7].

Example 1. Given matrices $A \in \text{Mat}_2^{1,1}(K)$ and $B \in \text{Mat}_2^{3,2}(K)$, we may write them as

$$A = (Ae_1 \mid Ae_2) = (a_{ij})_{i \leq 2, j \leq 2}, \quad B = (Be_1^{\odot 2} \mid Be_1 \odot e_2 \mid Be_2^{\odot 2}) = (b_{ij})_{i \leq 4, j \leq 3},$$

and it is immediate to check that the (4, 3) (hence four-column, five-row) matrix product

$$A \odot B = ((A \odot B)(e_1^{\odot 3}) \mid (A \odot B)(e_1^{\odot 2} \odot e_2) \mid (A \odot B)(e_1 \odot e_2^{\odot 2}) \mid (A \odot B)(e_2^{\odot 3})),$$

is equal to

$$\begin{pmatrix} a_{11}b_{11} & \frac{a_{12}b_{11}+2a_{11}b_{12}}{3} & \frac{2a_{12}b_{12}+a_{11}b_{13}}{3} & a_{12}b_{13} \\ a_{21}b_{11} + a_{11}b_{21} & M_2 & N_2 & a_{22}b_{13} + a_{12}b_{23} \\ a_{21}b_{21} + a_{11}b_{31} & M_3 & N_3 & a_{22}b_{23} + a_{12}b_{33} \\ a_{21}b_{31} + a_{11}b_{41} & M_4 & N_4 & a_{22}b_{33} + a_{12}b_{43} \\ a_{21}b_{41} & \frac{a_{22}b_{41}+2a_{21}b_{42}}{3} & \frac{2a_{22}b_{42}+a_{21}b_{43}}{3} & a_{22}b_{43} \end{pmatrix},$$

where M_i, N_i are defined by

$$M_i = \frac{2(a_{21}b_{i-1,2} + a_{11}b_{i,2}) + a_{22}b_{i-1,1} + a_{12}b_{i,1}}{3},$$

$$N_i = \frac{2(a_{22}b_{i-1,2} + a_{12}b_{i,2}) + a_{21}b_{i-1,3} + a_{11}b_{i,3}}{3}.$$

The following is straightforward to prove from either direct application of the universal property or the techniques used in [8, 10], and will not be delved into here:

Proposition 2. For any A, B, C , and whenever products make sense,

- a) $A \odot B = B \odot A$.
- b) $(A + B) \odot C = A \odot C + B \odot C$.
- c) $(A \odot B) \odot C = A \odot (B \odot C)$.
- d) $(\alpha A) \odot B = \alpha (A \odot B)$ for every $\alpha \in K$.
- e) If A is square and invertible, then $(A^{-1})^{\odot k} = (A^{\odot k})^{-1}$.
- f) $A \odot B = 0$ if and only if $A = 0$ or $B = 0$.
- g) If A is a square (1, 1)-matrix, then $A\mathbf{v}_1 \odot A\mathbf{v}_2 \odot \dots \odot A\mathbf{v}_m = A^{\odot m}\mathbf{v}_1 \odot \dots \odot \mathbf{v}_m$.
- h) If \mathbf{v} is a column vector, then $(A \odot B)\mathbf{v}^{\odot(p+q)} = (A\mathbf{v}^{\odot p}) \odot (B\mathbf{v}^{\odot q})$, $p, q \in \mathbb{Z}_{\geq 0}$.

Universal property on (7) and diagram (10) with different notation yields:

Lemma 2.4. For any two matrices $A \in \text{Mat}_n^{i,j}$ and $B \in \text{Mat}_n^{p,q}$ and $\mathbf{v}_1, \dots, \mathbf{v}_{j+q} \in V$, $(A \odot B)(\mathbf{v}_1 \odot \dots \odot \mathbf{v}_{j+q})$ is equal to

$$\frac{1}{\binom{j+q}{q}} \sum_{\sigma \in S_{j,q}} A(\mathbf{v}_{\sigma(1)} \odot \dots \odot \mathbf{v}_{\sigma(j)}) \odot B(\mathbf{v}_{\sigma(j+1)} \odot \dots \odot \mathbf{v}_{\sigma(j+q)}), \quad (13)$$

$S_{j,q}$ defined as in (8). \square

Lemma 2.5. a) ([8, 10]) For any $A \in \text{Mat}_n^{p,q}$, $B \in \text{Mat}_n^{q,r}$, and $\mathbf{v} \in \text{Sym}^j K^n$,

$$(\mathbf{v} \odot A)B = \mathbf{v} \odot (AB). \quad (14)$$

b) If \mathbf{e}_i are the columns of Id_n , $\sum_{m=1}^n (\mathbf{e}_m \odot \text{Id}_n^{\odot k-1}) (\mathbf{e}_m^T \odot \text{Id}_n^{\odot k-1}) = \text{Id}_n^{\odot k+1}$.

Proof. a) It suffices to prove it for basis elements of Sym^r : for any \mathbf{k} such that $|\mathbf{k}| = r$,

$$(\mathbf{v} \odot A)B\mathbf{e}^{\odot \mathbf{k}} = \mathbf{v} \odot (AB)\mathbf{e}^{\odot \mathbf{k}}. \quad (15)$$

But this is immediate from equation (13) or the definition (11) of \odot itself.

b) Using the previous item and the associative property in Proposition 2, and the fact that $\mathbf{e}_m \odot \mathbf{e}_m^T \in \text{Mat}_n^{2,2}$ is zero save for a 1 in position m,m ,

$$\sum_{m=1}^n (\mathbf{e}_m \odot \text{Id}_n^{\odot k-1}) (\mathbf{e}_m^T \odot \text{Id}_n^{\odot k-1}) = \left[\sum_{m=1}^n (\mathbf{e}_m \odot \mathbf{e}_m^T) \right] \odot \text{Id}_n^{\odot k-1} = \text{Id}_n^{\odot 2} \odot \text{Id}_n^{\odot k-1}. \quad \square$$

2.2. More properties of \odot . We need to generalise some of the properties in Proposition 2 for later purposes. Applying the universal property on (4) (with $V := \text{Sym}^k K^n$) or (7) (with $j_1 = j_2 = k$), followed by (11) and (16) for $m = 2$, as well as the universal property on (5) (with $\mathbf{v}_i := \odot_{i=1}^m A_i \mathbf{e}^{\odot \mathbf{p}_i}$) and prepending $\binom{[j]}{[j]}^{-1} \sum_{\mathbf{p}_1, \dots, \mathbf{p}_m} \binom{\mathbf{k}}{\mathbf{p}_1, \dots, \mathbf{p}_m}$ as in (12) for arbitrary m , we obtain

Lemma 2.6. Given square $A, B \in \text{Mat}_n^{k,k}$ and matrices $X_i \in \text{Mat}_n^{k,j_i}$, $i = 1, 2$,

$$(A \odot B)(X_1 \odot X_2) = \frac{1}{2} (AX_1 \odot BX_2 + BX_1 \odot AX_2), \quad (16)$$

and in general for any square $A_1, \dots, A_m \in \text{Mat}_n^{k,k}$ and any $X_i \in \text{Mat}_n^{k,j_i}$, $i = 1, \dots, m$,

$$\left(\bigodot_{i=1}^m A_i \right) \left(\bigodot_{i=1}^m X_i \right) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_k} \bigodot_{i=1}^m A_{\sigma(i)} X_i. \quad \square \quad (17)$$

Defining $B := \text{Id}_n^{\odot m-j}$ and $\mathbf{v}_i := X_i \mathbf{e}^{\odot \mathbf{p}_i}$ where $|\mathbf{p}_i| = q_i$ in equation (13), we have:

Lemma 2.7. Given $A \in \text{Mat}_n^{1,j}$ and X_1, \dots, X_m such that $X_i \in \text{Mat}_n^{1,q_i}$, $1 \leq j \leq m$,

$$\binom{m}{j} (A \odot \text{Id}_n^{\odot m-j}) \bigodot_{i=1}^m X_i = \sum_{1 \leq i_1 < \dots < i_j \leq m} [A(X_{i_1} \odot \dots \odot X_{i_j})] \odot \bigodot_{s \neq i_1, \dots, i_j} X_s. \quad \square \quad (18)$$

An immediate consequence of either Lemma 2.6 or Lemma 2.7 is

Corollary 1. Given a square matrix $A \in \text{Mat}_n^{1,1}$ and X_1, \dots, X_m such that $X_i \in \text{Mat}_n^{1,j_i}$,

$$(A \odot \text{Id}_n^{\odot m-1}) \left(\bigodot_{i=1}^m X_i \right) = \frac{1}{m} \sum_{i=1}^m (AX_i) \odot \left(X_1 \odot \dots \odot \widehat{X_i} \odot \dots \odot X_m \right). \quad \square \quad (19)$$

$(X\mathbf{v} \odot \text{Id}_n^{\odot r}) X^{\odot r} = X\mathbf{v} \odot X^{\odot r}$ in virtue of (14); applying this, (15), Proposition 2 and a detailed scrutiny of the effect on basis products $e^{\odot \star}$ yields:

Lemma 2.8. *Given square matrix $X \in \text{Mat}_n^{1,1}$, any vector $\mathbf{v} \in K^n$ and $r \geq 1$,*

$$(X\mathbf{v} \odot \text{Id}^{\odot r}) X^{\odot r} = X^{\odot r+1} (\mathbf{v} \odot \text{Id}^{\odot r}). \quad \square \quad (20)$$

If (K, ∂) is a *differential field* [29] and we extend derivation ∂ entrywise, $\partial(a_{i,j}) := (\partial a_{i,j})$, the Leibniz rule holds on *vector* products $\mathbf{x} \odot \mathbf{y}$ as trivially as it does for homogeneous polynomials in n variables in virtue of Proposition 1 or $\text{Sym}^{k_1+k_2}(V^*) \cong S^{k_1+k_2}(V, K)$; (11) implies:

Lemma 2.9. *For any given $X \in \text{Mat}_n^{k_1, j_1}(K)$ and $Y \in \text{Mat}_n^{k_2, j_2}(K)$,*

$$\partial(X \odot Y) = \partial(X) \odot Y + X \odot \partial(Y). \quad \square \quad (21)$$

Although the next result will be rendered academic by simplified expressions in §4.1, it is worth writing for the sake of clarifying certain routinely-appearing matrices a bit further. The proof is immediate from commutativity and (17), (19), Lemma 2.9, the distributive property and (19), as well as simple induction in (c):

Lemma 2.10. *Let (K, ∂) be a differential field.*

a) *If Y is a square $n \times n$ matrix having entries in K and $\partial Y = AY$, then*

$$\partial \text{Sym}^k Y = k \left(A \odot \text{Sym}^{k-1}(\text{Id}_n) \right) \text{Sym}^k Y. \quad (22)$$

b) *If $X \in \text{Mat}_n^{1, j_1}$ and $Y \in \text{Mat}_n^{1, j_2}$ satisfy systems $\partial X = AX + B_1$ and $\partial Y = AY + B_2$ with $A \in \text{Mat}_n^{1,1}$, $B_i \in \text{Mat}_n^{1, j_i}$, then symmetric product $X \odot Y$ satisfies linear system*

$$\partial(X \odot Y) = 2(A \odot \text{Id}_{d_{n,k}})(X \odot Y) + (B_1 \odot Y + B_2 \odot X). \quad (23)$$

c) *If $\partial X_i = AX_i + B_i$, $i = 1, \dots, m$, with $X_i, B_i \in \text{Mat}_n^{1, j_i}$, $A \in \text{Mat}_n^{1,1}$ then*

$$\partial \left(\bigodot_{i=1}^m X_i \right) = m \left(A \odot \text{Id}_{d_{n,k}}^{\odot m-1} \right) \bigodot_{i=1}^m X_i + \sum_{i=1}^m B_i \odot \left(\bigodot_{j \neq i} X_j \right). \quad \square \quad (24)$$

Remark 2. Albeit not explicitly as in (22), the matrix proven equal to $k(A \odot \text{Id}_n^{\odot k-1})$ has appeared in numerous references (e.g. [2, 3, 4, 5, 7]) whenever a differential equation for Sym^k arises, has been sometimes labelled sym^k and has been consistently called symmetric power *in the sense of Lie algebras*, its Lie group counterpart therein equal to \odot^k as defined in this paper.

3. Symmetric products and exponentials of infinite matrices. The next step towards a compact form to linearised higher variationals is assembling the matrix blocks alluded to in Lemma 2.10 and Remark 2 together into a single matrix. Again, we follow paths already trod with other aims and formulations, e.g. by Bekbaev in [10].

3.1. Products and exponentials. Of the myriad ways to note a set of infinite matrices, we may need one taking finite submatrix orders into account. Alternatively, of all the ways in which to write a K -algebra S , a need may arise to express it whenever possible $S = \text{Sym}(V) := \bigoplus_{k \geq 0} \text{Sym}^k(V)$ for a given vector space.

Notation 4. Let $\text{Mat}^{n,m}(K)$ denote the set of block matrices $A = (A_{i,j})_{i,j \geq 0}$ with $A_{i,j} : \text{Sym}^i K^m \rightarrow \text{Sym}^j K^n$, hence $A_{i,j} \in M_{d_{n,i} \times d_{m,j}}(K) = \text{Mat}_{n,m}^{i,j}(K)$:

$$A = \begin{pmatrix} \cdots & \vdots & \vdots & \vdots \\ \cdots & A_{2,2} & A_{2,1} & \leftarrow A_{2,0} \\ \cdots & A_{1,2} & A_{1,1} & \leftarrow A_{1,0} \\ \cdots & A_{0,2} & A_{0,1} & \leftarrow A_{0,0} \end{pmatrix}$$

We write $\text{Mat} := \text{Mat}^{n,n}$ if n is unambiguous. Conversely, $\text{Mat}_{n,m}^{i,j}$ is embedded in $\text{Mat}^{n,m}$ by identifying every matrix $A_{i,j}$ with an element of $\text{Mat}^{n,m}$ equal to 0 save for block $A_{i,j}$.

We define a product on $\text{Mat}^{n,m}$. For a formulation yielding the same results see [11, p. 2].

Definition 3.1. For any $A, B \in \text{Mat}^{n,m}(K)$, define $A \odot B = C \in \text{Mat}^{n,m}(K)$ by

$$C = (C_{i,j})_{i,j \geq 0}, \quad C_{i,j} = \sum_{0 \leq i_1 \leq i, 0 \leq j_1 \leq j} \binom{j}{j_1} A_{i_1, j_1} \odot B_{i-i_1, j-j_1}. \quad (25)$$

Same as always, \odot^k will stand for powers built with this product.

The following is immediate and part of it has already been mentioned before, e.g. [10]:

Lemma 3.2. $(\text{Mat}(K), +, \odot)$ is an integral domain, its identity element 1_{Mat} equal to zero save for block $(1_{\text{Mat}})_{0,0} = 1_K$. $\text{Mat}(K)$ is also a unital associative K -algebra with the usual product by scalars. \square

Definition 3.3. (See also [10]) for every matrix $A \in \text{Mat}^{n,m}$ we define the formal power series

$$\exp_{\odot} A := 1 + A^{\odot 1} + \frac{1}{2} A^{\odot 2} + \cdots = \sum_{i=0}^{\infty} \frac{1}{i!} A^{\odot i}.$$

Whenever $A = 0$ save for a finite distinguished submatrix $A_{j,k}$ (e.g. Examples 1 below or Lemma 3.6), the abuse of notation $\exp_{\odot} A_{j,k} = \exp_{\odot} A$ will be customary.

Commutativity of \odot renders the proof of the following similar to that of scalar exponentials:

Lemma 3.4.

- For every two $A, B \in \text{Mat}^{n,m}$, $\exp_{\odot}(A+B) = \exp_{\odot} A \odot \exp_{\odot} B$.
- For every $Y \in \text{Mat}^{n,m}$ and any derivation $\partial : K \rightarrow K$, $\partial \exp_{\odot} Y = (\partial Y) \odot \exp_{\odot} Y$.
- ([8, Corollary 3]) Given square matrices $A, B \in \text{Mat}_n^{1,1}$, $\exp_{\odot} AB = \exp_{\odot} A \exp_{\odot} B$.
- In particular, for every invertible square $A \in \text{Mat}_n^{1,1}$, $\exp_{\odot} A^{-1} = (\exp_{\odot} A)^{-1}$. \square

Examples 1.

1. Let $A \in \text{Mat}(K)$ such that all blocks are zero except for ${}_{1,1}$, the reader can check that

$$\exp_{\odot} A_{1,1} = \exp_{\odot} \left(\begin{array}{c|c} A_{1,1} & 0 \\ \hline 0 & 0 \end{array} \right) = \text{diag}(\cdots, A_{1,1}^{\odot k}, \dots, A_{1,1}, 1).$$

2. If the only non-zero block in A is a row vector,

$$A = \left(\begin{array}{ccc|c} \cdots & 0 & 0 & \cdots \\ \cdots & 0 & \mathbf{x} & \cdots \end{array} \middle| \begin{array}{c} 0 \\ 0 \end{array} \right), \quad \mathbf{x} = (x_1, \dots, x_{d_{n,k}}) \in \text{Mat}_n^{0,k}(K),$$

the only non-zero block in $A^{\odot j}$ is $(A \odot A^{j-1})_{0,jk} = \binom{jk}{k} \cdot \binom{(j-1)k}{k} \cdots \binom{2k}{k} A_{0,k}^{\odot j}$

e.g. for $k = 1$, $\exp_{\odot} A = \sum_{j \geq 0} \mathbf{x}^{\odot j} = \left(\begin{array}{cccc|c} \cdots & 0 & 0 & 0 & 0 \\ \cdots & \mathbf{x}^{\odot 4} & \mathbf{x}^{\odot 3} & \mathbf{x}^{\odot 2} & \mathbf{x} \end{array} \middle| \begin{array}{c} 0 \\ 1 \end{array} \right).$

3. If the only non-zero block in A is column ${}_{0,k}$, the only one in $A^{\odot j}$ is ${}_{jk,0}$, obtained by switching rows and columns and expunging binomials from $(A \odot A^{j-1})_{0,jk}$ in **2**. For $k = 1$,

$$\exp_{\odot} \mathbf{x} = \exp_{\odot} \left(\begin{array}{c|c} 0 & \mathbf{x} \\ \hline 0 & 0 \end{array} \right) = \left(\begin{array}{c|c} \vdots & \vdots \\ 0 & \frac{1}{j!} \mathbf{x}^{\odot j} \\ \vdots & \vdots \\ 0 & \mathbf{x} \\ \hline 0 & 1 \end{array} \right).$$

Fourth example (27), i.e. matrices equal to 0 save for block row ${}_{1,k}$, deserves special attention.

Notation 5. For every set of indices $1 \leq i_1 \leq \cdots \leq i_r$ such that $\sum_{j=1}^r i_j = k$, c_{i_1, \dots, i_r}^k is defined as the amount of totally ordered partitions of a set of k elements among subsets of sizes i_1, \dots, i_r . We write $c_{\mathbf{i}}^k$ if $\mathbf{i} = (i_1, \dots, i_r)$ and omit super-index k if $|\mathbf{i}|$ is known beforehand.

Remarks 2.

1. $c_{i_1, \dots, i_j}^k = \#I_{1, \dots, k}^{i_1, \dots, i_j}$ in (43) below, $\sum_{|\mathbf{i}|=k} c_{i_1, \dots, i_j}^k = \{j\}^k$, the *Stirling number of the second type* ([1, §24.1.4]), and $\sum_{j=1}^k \sum_{|\mathbf{i}|=k} c_{i_1, \dots, i_j}^k = B_k$, the k^{th} *Bell number* [27, Vol 2, Ch. 3].
2. Since each subset of size i_s is supposed to be ordered, we must divide the total amount by the orders of the corresponding symmetric groups, hence the explicit formula:

$$c_{i_1, \dots, i_j}^k = \frac{\binom{k}{i_1 i_2 \dots i_j}}{n_1! \cdots n_m!}, \quad \begin{cases} (i_1, \dots, i_j) = (k_1^{n_1}, k_1, \dots, k_m^{n_m}, k_m), \\ 1 \leq k_1 < k_2 < \cdots < k_m. \end{cases} \quad (26)$$

Lemma 3.5. Let $Y \in \text{Mat}(K)$ equal to zero outside of block row ${}_{1,k}$, $k \geq 1$:

$$Y := \left(\begin{array}{ccc|c} \cdots & Y_3 & Y_2 & Y_1 \\ \hline 0 & 0 & 0 & 0 \end{array} \middle| \begin{array}{c} 0 \\ 0 \end{array} \right), \quad Y_i \in \text{Mat}_n^{1,i}. \quad (27)$$

Let $Z_{r,s}$, $s, r \geq 1$, be the corresponding block in $\exp_{\odot} Y$. Then,

- a) Row block r in $\exp_{\odot} Y$ is recursively obtained in terms of row blocks 1 and $r - 1$:

$$Z_{r,s} = \frac{1}{r} \sum_{j=1}^{s-r+1} \binom{s}{j} Y_j \odot Z_{r-1, s-j}. \quad (28)$$

In particular, $Z_{r,r} = Y_1^{\odot r}$ and $Z_{r,s} = 0_{d_{n,r}, d_{n,s}}$ whenever $r > s$.

b) For every $m, r \geq 1$ and any $\mathbf{v} \in K^n$,

$$(Y_1 \mathbf{v} \odot \text{Id}_n^{\odot r}) Z_{r,r} = Z_{r+1,r+1} (\mathbf{v} \odot \text{Id}_n^{\odot r}). \quad (29)$$

c) Using Notation 5 and (26), for every $s \geq r$

$$Z_{r,s} = \sum_{i_1 + \dots + i_r = s} c_{i_1, \dots, i_r}^s Y_{i_1} \odot Y_{i_2} \odot \dots \odot Y_{i_r}. \quad (30)$$

d) Let $A \in \text{Mat}(K)$ similar to Y , its horizontal strip not necessarily at level $1, *$:

$$A := \left(\begin{array}{cccc|c} \dots & A_3 & A_2 & A_1 & 0 \\ \dots & 0 & 0 & 0 & 0 \\ & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad A_t \in \text{Mat}_n^{p,t}.$$

For every $t, i \geq 1$ and $s \geq t + i$, the following factorization holds:

$$\sum_{j=t}^{s-i} \binom{s}{j} (A_t Z_{t,j}) \odot Z_{i,s-j} = \binom{t+i}{i} (A_t \odot \text{Id}_n^{\odot i}) Z_{t+i,s}. \quad (31)$$

e) If $Q \in \text{Mat}^n$ has only its square $1,1$ block different from zero, then $\exp_{\odot} QY = (\exp_{\odot} Q) (\exp_{\odot} Y)$.

Proof. a) Using (25) on $A = Y$, $B = Y^{\odot s-1}$, as well as the fact $Z_{i,j} = 0$ for $i > j$, (28) ensues.

b) Direct from (20) in Lemma 2.8.

c) By induction. For $s = 1$, r can only be equal to 1 in order to have a non-zero block and $Z_{1,1} = Y_1 = c_1^1 Y_1$. Assume (30) holds for all r smaller than or equal to $s - 1$. Summand redistribution renders $Z_{r,s} = \frac{1}{r} \sum_{j_1 + \dots + j_r = s} C_{j_1, \dots, j_r} Y_{j_1} \odot Y_{j_2} \odot \dots \odot Y_{j_r}$ where C_{j_1, \dots, j_r} splits into a certain sum, each of whose m terms is easily checked to be equal to $n_m c_{j_1, \dots, j_r}^s$, hence the coefficient of $Y_{j_1} \odot \dots \odot Y_{j_r}$ in equals $\frac{1}{r} \sum_{i=1}^m n_i c_{j_1, \dots, j_r}^s = c_{j_1, \dots, j_r}^s$.

d) the left-hand side in (31), expressed in terms of (30) and applying distributivity, equals

$$\sum_{j=t}^{s-i} \binom{s}{j} \sum_{m_1, \dots, m_t} \sum_{k_1, \dots, k_i} c_{m_1, \dots, m_t}^j c_{k_1, \dots, k_i}^{s-j} [A_t (Y_{m_1} \odot \dots \odot Y_{m_t})] \odot Y_{k_1} \odot \dots \odot Y_{k_i}. \quad (32)$$

A tedious exercise in counting index multiplicities and applying basic combinatorics allows us to apply Lemma 2.7 to A_t and $Y_{\odot \mathbf{m}} \odot Y_{\odot \mathbf{k}} := Y_{m_1} \odot \dots \odot Y_{m_t} \odot Y_{k_1} \odot \dots \odot Y_{k_i}$:

$$\binom{s}{j} c_{\mathbf{m} \mathbf{k}}^{s-j} [A_t (Y_{m_1} \odot \dots \odot Y_{m_t})] \odot Y_{k_1} \odot \dots \odot Y_{k_i} = \binom{i+t}{i} (A_t \odot \text{Id}_n^i) Y_{\odot \mathbf{m}} \odot Y_{\odot \mathbf{k}}. \quad (33)$$

The fact every summand in (32) fits the same profile as the left-hand side in (33) allows us to factor $\binom{i+t}{i} (A_t \odot \text{Id}_n^i)$ out of the whole sum, namely $Z_{i+t,s}$.

e) Replacing each factor Y_{i_j} by QY_{i_j} in (30) and applying Lemma 2 we obtain $\exp_{\odot} QY = (\tilde{Z}_{r,s})$ where $\tilde{Z}_{r,s} = \sum_{i_1 + \dots + i_r = s} Q^{\odot r} \odot c_{i_1, \dots, i_r}^s Y_{i_1} \odot Y_{i_2} \odot \dots \odot Y_{i_r} =$

$Q^{\odot r} Z_{r,s}$, hence matrix $\exp_{\odot} Y$ appears multiplied by $\text{diag}(\dots, Q^{\odot 2}, Q^{\odot 1}, 1) = \exp_{\odot} Q$. \square

Lemma 3.6. *Let A and Y be as in Lemma 3.5. Then,*

$$(A \exp_{\odot} Y) \odot \exp_{\odot} Y = (A \odot \exp_{\odot} \text{Id}_n) \exp_{\odot} Y. \quad (34)$$

Proof. Based on (25), $B := A \odot \exp_{\odot} \text{Id}_n \in \text{Mat}(K)$ is defined recursively by

$$B_1 = A_1, \quad B_k = \begin{pmatrix} \binom{k}{k-1} A_1 \odot \text{Id}_n^{\odot k-1} \\ \binom{k}{k-2} A_2 \odot \text{Id}_n^{\odot k-2} \\ \vdots \\ \binom{k}{0} A_k \end{pmatrix} \begin{array}{c} \boxed{\phantom{B_{k-1}}} \\ \\ \\ \boxed{B_{k-1}} \end{array}, \quad k \geq 2.$$

Let

$$\Phi_1 = Y_1, \quad \Phi_k = \begin{pmatrix} Z_{k,k} \\ Z_{k-1,k} \\ \vdots \\ Z_{1,k} \end{pmatrix} \begin{array}{c} \boxed{\phantom{\Phi_{k-1}}} \\ \\ \\ \boxed{\Phi_{k-1}} \end{array}, \quad k \geq 2,$$

be the matrix formed by the first k row and column blocks in $\exp_{\odot} Y$. Let M_r be the block row r of B , $A_k := (A_{1,k}, A_{1,k-1}, \dots, A_{1,1})$ the first k blocks in A and $Z^k := (Z_{k,k}, Z_{k-1,k}, \dots, Z_{1,k})^T$ the first block column in Φ_k . Given $s \geq 1$, block $1,s$ in $A \exp_{\odot} Y$ equals $A_s Z^s = \sum_{j=1}^s A_j Z_{j,s}$, hence for every $r = 1, \dots, k$ block r,k in $(A \exp_{\odot} Y) \odot \exp_{\odot} Y$ can be rewritten, in virtue of (31) with $p = 1$, $s = k$ and $i = r - 1$, as $\sum_{t=1}^{k-r+1} \binom{t+r-1}{r-1} (A_t \odot \text{Id}_n^{\odot r-1}) Z_{t+r-1,k} = M_r Z^k$. \square

3.2. Application to power series. Since polynomials and power series split into homogeneous components, Example 1(3) implies:

Lemma 3.7.

- a) *Let $F \in K[[\mathbf{x}]]$, $\mathbf{x} = (x_1, \dots, x_n)$, be a formal series. Then there exists a set of row blocks $M_F^{1,i} \in \text{Mat}_{m,n}^{1,i}(K)$, $i \geq 0$ such that F admits the expression $F(\mathbf{x}) = M_F \exp_{\odot} X$, where*

$$M_F := \left(\begin{array}{ccc|c} \dots & M_F^{1,2} & M_F^{1,1} & M_F^{1,0} \\ \dots & 0 & 0 & 0 \end{array} \right) \in \text{Mat}^{1,n}(K), \quad X := \left(\begin{array}{c|c} 0 & \mathbf{x} \\ \hline 0 & 0 \end{array} \right).$$

- b) *If $F = F_1 \times \dots \times F_m$ is a vector power series, adequate $M_F^{1,i} \in \text{Mat}_{m,n}^{1,i}(K)$ render*

$$\boxed{F(\mathbf{x}) = M_F \exp_{\odot} X} \quad \text{where } M_F := \left(\begin{array}{ccc|c} \dots & M_F^{1,2} & M_F^{1,1} & M_F^{1,0} \\ \dots & 0 & 0 & 0 \end{array} \right) \in \text{Mat}^{m,n}.$$

Following Definition 3.3, write $F(\mathbf{x}) = M_F \exp_{\odot} \mathbf{x}$ if it poses no clarity issue. \square

From the above Lemma it follows that every formal power series can be expressed in the form $M_F \exp_{\odot} \mathbf{x}$, where abusing notation once again

$$M_F = J_F + M_F^{1,0} := \left(\begin{array}{ccc|c} \dots & M_F^{1,2} & M_F^{1,1} & 0 \\ \dots & 0 & 0 & 0 \end{array} \right) + \left(\begin{array}{c|c} 0 & M_F^{1,0} \\ \hline 0 & 0 \end{array} \right). \quad (35)$$

In other words: M_F equals the sum of two matrices with easily computable \odot -exponentials: one following Example 1(3) (same as \mathbf{x}) and one following (27).

Lemma 3.4, Lemma 3.2 and the universal property of finite products \odot yield the following two results; see [8, 10] for a proof.

Lemma 3.8. *Given power series $F = (F_1, \dots, F_m)$ and $G = (G_1, \dots, G_p)$ in n and m indeterminates, respectively,*

- a) *If $n = m$, $M_{FG} = M_F \odot M_G$.*
- b) *$\exp_{\odot} F(\mathbf{x}) = (\exp_{\odot} M_F)(\exp_{\odot} \mathbf{x})$.*
- c) *$M_{G \circ F} = M_G \exp_{\odot} M_F$.*
- d) *$\exp_{\odot} (M_G \exp_{\odot} M_F) = (\exp_{\odot} M_G)(\exp_{\odot} M_F)$. \square*

Corollary 2. *Let $F(\mathbf{x}) = (F_1, \dots, F_p)(x_1, \dots, x_n)$ be a vector power series, $\mathbf{y} = F(\mathbf{x})$ and*

$$\left. \begin{array}{l} \mathbf{X} = R_{x,X} \exp_{\odot} \mathbf{x} \in K^N, \\ \mathbf{Y} = S_{y,Y} \exp_{\odot} \mathbf{y} \in K^P, \end{array} \right\} \quad R_{x,X} \in \text{Mat}^{N,n}(K), \quad S_{y,Y} \in \text{Mat}^{P,p}(K),$$

be independent and dependent variable changes, which we assume admit formal inverse changes

$$\left. \begin{array}{l} \mathbf{x} = R_{X,x} \exp_{\odot} \mathbf{X}, \\ \mathbf{y} = S_{Y,y} \exp_{\odot} \mathbf{Y}, \end{array} \right\} \quad R_{X,x} \in \text{Mat}^{n,N}(K), \quad S_{Y,y} \in \text{Mat}^{p,P}(K).$$

Then, the expression of F in the new variables, written in that in those old, is

$$\boxed{M_{F,X,Y} = S_{y,Y} (\exp_{\odot} M_{F,x,y}) \exp_{\odot} R_{X,x}} \quad \text{where } \mathbf{y} = F(\mathbf{x}) = M_{F,x,y} \exp_{\odot} \mathbf{x}. \quad \square \quad (36)$$

As was hinted at in [10, p. 5], this result shows interesting light on the way finite-level transformations translate into transformations on $\text{Mat}^{n,m}$. For a linear transformation of the independent variables $\mathbf{x} = B\mathbf{X}$, however, basic properties of \exp_{\odot} are as useful as (36) in proving F admits the following expression in the new variable \mathbf{X} (mind the effect of the first matrix, equal to zero save for block $1,1$ which is equal to Id_n , on the second one):

$$F(\mathbf{X}) = \text{Id}_n (\exp_{\odot} M_F) (\exp_{\odot} B) \mathbf{X} = \left(J_F + M_F^{0,0} \right) (\exp_{\odot} B) \exp_{\odot} \mathbf{X}. \quad (37)$$

This will be applied to first integrals of dynamical systems in Section 5.

4. Higher-order variational equations.

4.1. Structure. Let us step back to what was said in §1.2. For each particular integral curve ψ of a given complex autonomous dynamical system (DS), the **variational system** VE_{ψ}^k for (DS) along ψ is satisfied by partial derivatives $\frac{\partial^k}{\partial \mathbf{z}^k} \varphi(t, \psi)$. Case $k = 1$ being trivial as shown in (VE $_{\psi}$), the situation of interest is $k > 1$. We will eschew formulations such as those in [24, eq (14)] in favour of the explicit formulae (38), (44), (LVE $_{\psi}$) and (VE $_{\psi}^k$) using Linear Algebra to express multilinear maps.

Notation 6. $K := \mathbb{C}(\psi)$, $A_i := X^{(i)}(\psi)$, $Y_i := \text{lex} \left(\frac{\partial^i}{\partial \mathbf{z}^i} \varphi(t, \psi) \right)$ and, per Lemma 3.5,

$$\Phi_1 = Y_1, \quad \Phi_k = \left(\begin{array}{c} Z_{k,k} \\ Z_{k-1,k} \\ \vdots \\ Z_{1,k} \end{array} \begin{array}{c} \boxed{\phantom{\Phi_{k-1}}} \\ \phantom{\Phi_{k-1}} \\ \phantom{\Phi_{k-1}} \\ \phantom{\Phi_{k-1}} \end{array} \right), \quad k \geq 2, \quad (38)$$

formed by the first k block rows and columns in $\Phi = \exp_{\odot} Y$. Define $A, Y \in \text{Mat}(K)$ as in Lemma 3.5 with the above A_i, Y_i as blocks. Denote the canonical basis on K^n (meaning the set of columns of Id_n) by $\{e_1, \dots, e_n\}$.

Lemma 4.1. *In the hypotheses described in Notation 6, let $k \geq 1$ and $m = 1, \dots, n$. Then,*

$$Y_k = \sum_{j=1}^n \frac{\partial Y_{k-1}}{\partial z_j} \left(e_j^T \odot \text{Id}_n^{\odot k-1} \right), \quad (39)$$

$$\frac{\partial}{\partial z_m} Y_k = Y_{k+1} \left(e_m \odot \text{Id}_n^{\odot k} \right), \quad (40)$$

$$\frac{\partial}{\partial z_m} Z_{r,k} = Z_{r,k+1} \left(e_m \odot \text{Id}_n^{\odot k} \right) - (Y_1 e_m \odot \text{Id}_n^{\odot r-1}) Z_{r-1,k}, \quad r \leq k, \quad (41)$$

$$\frac{\partial}{\partial z_m} A_k = A_{k+1} \left(Y_1 e_m \odot \text{Id}_n^{\odot k} \right). \quad (42)$$

Proof. We will explicitly prove (40); (39) is an immediate consequence of Lemma 2.5 and (40). We have, for every given ordered multi-index $\mathbf{i} = (i_1, \dots, i_k)$,

$$\frac{\partial Y_k}{\partial z_m} e_{i_1} \odot \dots \odot e_{i_k} = \frac{\partial}{\partial z_m} \frac{\partial^k \varphi}{\partial z_{i_1} \partial z_{i_2} \dots \partial z_{i_k}} = Y_{k+1} e_m \odot e_{i_1} \odot \dots \odot e_{i_k}.$$

The right-hand side in (40) is equal to this expression, too, by simple application of the same principle as in (15). The effect of $\frac{\partial}{\partial z}$ on A_j is clear as well: chain rule implies

$$\frac{\partial A_k}{\partial z_m} e_{i_1} \odot \dots \odot e_{i_k} = \sum_{r=1}^n \frac{\partial^{k+1} X}{\partial z_{i_1} \partial z_{i_2} \dots \partial z_{i_k} \partial z_r} \frac{\partial \varphi_r}{\partial z_m} = \sum_{r=1}^n A_{k+1} \left(e_r \odot e^{\odot \mathbf{i}} \right) \frac{\partial \varphi_r}{\partial z_m},$$

which is equal, again using (15) in order to obtain $\frac{\partial \varphi_r}{\partial z_m} e_r \odot e^{\odot \mathbf{i}} = \left(\frac{\partial \varphi_r}{\partial z_m} e_r \odot \text{Id}_n^{\odot r} \right) e^{\odot \mathbf{i}}$, to

$$A_{k+1} \sum_{r=1}^n \left(e_r \odot e^{\odot \mathbf{i}} \right) \frac{\partial \varphi_r}{\partial z_m} = A_{k+1} \sum_{r=1}^n \left(\frac{\partial \varphi_r}{\partial z_m} e_r \odot e^{\odot \mathbf{i}} \right) = A_{k+1} \sum_{r=1}^n \left(\frac{\partial \varphi_r}{\partial z_m} e_r \odot \text{Id}_n^{\odot r} \right) e^{\odot \mathbf{i}},$$

hence to $A_{k+1} \left(\frac{\partial \varphi}{\partial z_m} \odot \text{Id}_n^{\odot k} \right) e^{\odot \mathbf{i}} = A_{k+1} \left(Y_1 e_m \odot \text{Id}_n^{\odot k} \right) e^{\odot \mathbf{i}}$.

The reader can check (41) and (42). For instance the latter is obtained by induction over k using derivation of (28), (40) and Leibniz rule (21), as well as application of (31) with $i = 1, t = r - 2, s = k, A_t = Y_1 e_m \odot \text{Id}_n^{\odot r-2}$ and $p = r - 1$, use of (20) and the fact $Z_{r-2,r-2} = Y_1^{\odot r-2}$. \square

Proposition 3 (First explicit version of non-linearised VE_{ψ}^k). *In the above hypotheses,*

$$\dot{Y} = A \exp_{\odot} Y; \quad (\text{VE}_{\psi})$$

in other words, for every $k \geq 1$,

$$\frac{d}{dt} Y_k = \sum_{j=1}^k A_j Z_{k,j} = \sum_{j=1}^k A_j \sum_{i_1 + \dots + i_j = k} c_{i_1, \dots, i_j}^k Y_{i_1} \odot Y_{i_2} \odot \dots \odot Y_{i_j}. \quad (\text{VE}_{\psi}^k)$$

Proof. Assume VE_{ψ}^{k-1} can be expressed as $\frac{d}{dt} Y_{k-1} = \sum_{j=1}^{k-1} A_j Z_{j,k-1}$. The entries in Y_{k-1} are partial derivatives of $\varphi(t, \mathbf{z})$, hence $\frac{d}{dt} \equiv \frac{\partial}{\partial t}$ on every entry, Schwarz Lemma applies and derivation of (39) yields $\frac{d}{dt} Y_k = \sum_{m=1}^n \frac{\partial}{\partial z_m}$

$\frac{\partial Y_{k-1}}{\partial t} \left(\mathbf{e}_m^T \odot \text{Id}_n^{\odot k-1} \right)$; induction hypothesis and Leibniz rule render $\frac{d}{dt} Y_k = \sum_{m=1}^n \left[\sum_{p=1}^{k-1} \frac{\partial A_p}{\partial z_m} Z_{p,k-1} + A_p \frac{\partial Z_{p,k-1}}{\partial z_m} \right] \left(\mathbf{e}_m^T \odot \text{Id}_n^{\odot k-1} \right)$; equations (41) and (42) imply this is equal to $S_1 + S_2 - S_3$, where

$$\begin{aligned} S_1 &= \sum_{m=1}^n \sum_{p=1}^{k-1} A_{p+1} \left(Y_1 \mathbf{e}_m \odot \text{Id}_n^{\odot p} \right) Z_{p,k-1} \left(\mathbf{e}_m^T \odot \text{Id}_n^{\odot k-1} \right), \\ S_2 &= \sum_{m=1}^n \sum_{p=1}^{k-1} A_p Z_{p,k} \left(\mathbf{e}_m \odot \text{Id}_n^{\odot k-1} \right) \left(\mathbf{e}_m^T \odot \text{Id}_n^{\odot k-1} \right), \\ S_3 &= \sum_{m=1}^n \sum_{p=1}^{k-1} A_p \left(Y_1 \mathbf{e}_m \odot \text{Id}_n^{\odot p-1} \right) Z_{p-1,k-1} \left(\mathbf{e}_m^T \odot \text{Id}_n^{\odot k-1} \right). \end{aligned}$$

Sum swapping in $\sum_m \sum_p$ and Lemma 2.5 (b) imply $S_2 = \sum_{p=1}^{k-1} A_p Z_{p,k}$; (29), Lemma 2.5 (b) and Proposition 2 (b) render $S_1 - S_3$ equal to missing summand $A_k Z_{k,k}$ in S_2 . \square

Corollary 3 (Second explicit version of non-linearised VE_ψ^k). *Let $\varphi(t, \boldsymbol{\psi}) = (\varphi_i)_i$ be the flow of (DS) along $\boldsymbol{\psi}$. Given $k \geq 1$, $\mathbf{N} \in \mathbb{N}^k, r = 1, \dots, k$ and $0 \leq m_1 \leq \dots \leq m_r$, define:*

- a) *the set $\mathbf{S}^{\mathbf{m}}$ of $\sigma \in \mathfrak{S}_k$ such that $\sigma(1, \dots, k) = (\mathbf{i}_1, \dots, \mathbf{i}_r)$, $\mathbf{i}_j = (i_{j,1}, \dots, i_{j,m_j})$ and $i_{j,s} < i_{j,s+1}$ for every j, s and $m_j = m_{j+1}$ implies $i_{j,1} < i_{j+1,1}$;*
- b) *the index-ordered partitions of \mathbf{N} in subsets of sizes $0 \leq m_1 \leq \dots \leq m_r$:*

$$I_{\mathbf{N}}^{\mathbf{m}} := \left\{ (N_{\sigma(1)}, \dots, N_{\sigma(k)}) = (\mathbf{K}_1, \dots, \mathbf{K}_r) : \mathbf{K}_i \in \mathbb{N}^{m_i} \text{ and } \sigma \in \mathbf{S}^{\mathbf{m}} \right\}; \quad (43)$$

- c) *and, using abridged notation \sum_{j_1, \dots, j_r} to denote $\sum_{j_1=1}^n \sum_{j_2=1}^n \dots \sum_{j_r=1}^n$,*

$$T_{N_1, \dots, N_k}^{m_1, \dots, m_r} := \sum_{(\mathbf{K}_1, \dots, \mathbf{K}_r) \in I_{\mathbf{N}}^{\mathbf{m}}} \sum_{j_1, \dots, j_r} \frac{\partial^r X_i}{\partial z_{j_1} \dots \partial z_{j_r}} \frac{\partial^{m_1} \varphi_{j_1}}{\partial \mathbf{z}_{\mathbf{K}_1}} \dots \frac{\partial^{m_r} \varphi_{j_r}}{\partial \mathbf{z}_{\mathbf{K}_r}}.$$

Then, the order- k variational equation along $\boldsymbol{\psi} = \{\boldsymbol{\psi}(t)\}$ is summarised in the following:

$$\frac{d}{dt} \frac{\partial^k \varphi_i}{\partial z_{N_1} \partial z_{N_2} \dots \partial z_{N_k}} = \sum_{r=1}^k \sum_{m_1, \dots, m_r} T_{N_1, \dots, N_k}^{m_1, \dots, m_r}, \quad i, N_1, \dots, N_k \in \{1, \dots, n\},$$

indices in \sum_{m_1, \dots, m_r} constrained by $m_1 \leq \dots \leq m_r$ and $\sum_i^r m_i = k$. \square

(VE_ψ^k) in Proposition 3 effectively settles the entries for lower n rows in $A_{\text{LVE}_\psi^k}$ and the first n columns in Φ_k . Let us now find the rest of the matrices.

Proposition 4 (Explicit version of LVE_ψ^k). *Still following Notation 6, the infinite system*

$$\boxed{\dot{X} = A_{\text{LVE}_\psi^k} X}, \quad A_{\text{LVE}_\psi^k} := A \odot \exp_{\odot} \text{Id}_n, \quad (\text{LVE}_\psi)$$

has $\Phi := \exp_{\odot} Y$ as a solution matrix. Hence, for every $k \geq 1$,

a) the lower-triangular recursive $D_{n,k} \times D_{n,k}$ form for LVE_ψ^k is $\dot{Y} = A_{\text{LVE}_\psi^k} Y$, its system matrix being obtained from the first k row and column blocks of A_{LVE_ψ} :

$$A_{\text{LVE}_\psi^k} = \begin{pmatrix} \binom{k}{k-1} A_1 \odot \text{Id}_n^{\odot k-1} & & & & \\ \binom{k}{k-2} A_2 \odot \text{Id}_n^{\odot k-2} & & & & \\ \vdots & & & & \\ \binom{k}{0} A_k & & & & \\ & & & & \boxed{A_{\text{LVE}_\psi^{k-1}}} \end{pmatrix}, \quad (44)$$

b) and the principal fundamental matrix for LVE_ψ^k is Φ_k from $\exp_\odot Y$ in Notation 6.

Proof. (34) in 3.6, (VE_ψ) in Proposition 3, and item (b) in Lemma 3.4 imply

$$\overline{\exp_\odot Y} = \dot{Y} \odot \exp_\odot Y = (A \exp_\odot Y) \odot \exp_\odot Y = (A \odot \exp_\odot \text{Id}_n) \exp_\odot Y.$$

The rest follows from Lemma 3.5. \square

Example 2. For instance, for $k = 5$ we have

$$A_{\text{LVE}_\psi^5} = \begin{pmatrix} 5A_1 \odot \text{Id}_n^{\odot 4} & & & & & & \\ 10A_2 \odot \text{Id}_n^{\odot 3} & 4A_1 \odot \text{Id}_n^{\odot 3} & & & & & \\ 10A_3 \odot \text{Id}_n^{\odot 2} & 6A_2 \odot \text{Id}_n^{\odot 2} & 3A_1 \odot \text{Id}_n^{\odot 2} & & & & \\ 5A_4 \odot \text{Id}_n & 4A_3 \odot \text{Id}_n & 3A_2 \odot \text{Id}_n & 2A_1 \odot \text{Id}_n & & & \\ A_5 & A_4 & A_3 & A_2 & A_1 & & \end{pmatrix},$$

and, using any of the equivalent (28), (30), the principal fundamental matrix Φ_5 is

$$\begin{pmatrix} Y_1^{\odot 5} & & & & & & \\ 10Y_1^{\odot 3} \odot Y_2 & Y_1^{\odot 4} & & & & & \\ 10Y_1^{\odot 2} \odot Y_3 + 15Y_1 \odot Y_2^{\odot 2} & 6Y_1^{\odot 2} \odot Y_2 & Y_1^{\odot 3} & & & & \\ 10Y_2 \odot Y_3 + 5Y_1 \odot Y_4 & 4Y_1 \odot Y_3 + 3Y_2 \odot Y_2 & 3Y_1 \odot Y_2 & Y_1^{\odot 2} & & & \\ Y_5 & Y_4 & Y_3 & Y_2 & Y_1 & & \end{pmatrix}, \quad (45)$$

hence (VE_ψ^k) for $k = 5$ is the lowest row in $A_{\text{LVE}_\psi^5}$ times the leftmost column in Φ_5 .

4.2. Explicit solution and monodromy matrices for LVE_ψ^k . Let $T \subseteq \mathbb{P}_\mathbb{C}^1$ be the domain for time variable t in (DS) and $\gamma \subset T$ a closed path based at $t_0 \in T$. Assume $k = 1$. If Y_1 is a fundamental matrix of first-order (VE_ψ) , analytic continuation along γ yields $Y_1(t_0) \xrightarrow[\text{cont}]{\gamma} Y_1(t_0) \cdot M_{1,\gamma}$, $M_{1,\gamma}$ being the *monodromy matrix* ([32]) of (VE_ψ) . Assume $Y_1 := \Phi_1$ is the principal fundamental matrix for (VE_ψ) , any other solution matrix Ψ_1 recovered from $\Psi_1 = Y_1 \Psi_1(t_0)$. The non-linearised second-order equation, after Proposition 3, is

$$\dot{Y}_2 = A_1 Y_2 + A_2 \cdot \text{Sym}^2(Y_1). \quad (\text{VE}_\psi^2)$$

Following Proposition 4, linearised completion LVE_ψ^2 has principal fundamental matrix

$$\Phi_2 = \begin{pmatrix} Y_1^{\odot 2} & \\ Y_2 & Y_1 \end{pmatrix}.$$

A particular solution $Y_2 = Y_1 \int Y_1^{-1} A_2 \text{Sym}^2(Y_1)$ of (VE_ψ^2) is found via variation of constants, which becomes a contour integral whenever time is taken along path γ :

$$Y_2 \xrightarrow[\text{cont}]{\gamma} Q_{1,2,\gamma} := M_{1,\gamma} \int_\gamma Y_1^{-1} A_2 \text{Sym}^2(Y_1), \quad (46)$$

hence

$$\text{Id}_n = \Phi_2(t_0) \xrightarrow[\text{cont}]{\gamma} \begin{pmatrix} Y_1^{\odot 2}(t_0) M_{1,\gamma}^{\odot 2} & 0 \\ Y_1(t_0) Q_{1,2,\gamma} & Y_1(t_0) M_{1,\gamma} \end{pmatrix} = \Phi_2(t_0) \begin{pmatrix} M_{1,\gamma}^{\odot 2} & 0 \\ Q_{1,2,\gamma} & M_{1,\gamma} \end{pmatrix},$$

and $[\gamma] \mapsto M_{i,\gamma}$ is a group morphism $\pi_1(T, t_0) \rightarrow \text{GL}_{D_{n,i}}(\mathbb{C})$, hence for *any* fundamental matrix

$$\Psi_2(t_0) \xrightarrow[\text{cont}]{\gamma} \Psi_2(t_0) \begin{pmatrix} M_{1,\gamma}^{\odot 2} & 0 \\ Q_{1,2,\gamma} & M_{1,\gamma} \end{pmatrix};$$

therefore the monodromy of LVE_ψ^2 along γ will be

$$M_{2,\gamma} := \begin{pmatrix} M_{1,\gamma}^{\odot 2} & 0 \\ Q_{1,2,\gamma} & M_{1,\gamma} \end{pmatrix} = \begin{pmatrix} M_{1,\gamma}^{\odot 2} & 0 \\ M_{1,\gamma} \int_\gamma Y_1^{-1} A_2 Y_1^{\odot 2} & M_{1,\gamma} \end{pmatrix}. \quad (47)$$

Assume $k = 3$. The principal fundamental matrix of LVE_ψ^3 consists of the lower right 3×3 -block of (45) and all solution matrices can be expressed $\Psi_3 = \Phi_3 C$. Let us now find a solution to

$$\dot{Y}_3 = A_1 Y_3 + 3A_2 Y_1 \odot Y_2 + A_3 \text{Sym}^3(Y_1), \quad (48)$$

Same as before, variation of constants on (48) yields another contour integral if $\tau \in \gamma$:

$$Y_3 \xrightarrow[\text{cont}]{\gamma} Q_{1,3,\gamma} := M_{1,\gamma} \int_\gamma Y_1^{-1} (3A_2 Y_1 \odot Y_2 + A_3 \text{Sym}^3(Y_1)) d\tau. \quad (49)$$

The remaining term of our monodromy matrix is a direct consequence of analytic continuation:

$$0 = 3Y_1(t_0) \odot Y_2(t_0) \xrightarrow[\text{cont}]{\gamma} 3M_{1,\gamma} \odot Q_{1,2,\gamma} = 3M_{1,\gamma} \odot \left(M_{1,\gamma} \int_\gamma Y_1^{-1} A_2 Y_1^{\odot 2} \right).$$

Our monodromy matrix is

$$M_{3,\gamma} := \begin{pmatrix} M_{1,\gamma}^{\odot 3} & & & & \\ 3M_{1,\gamma} \odot Q_{1,2,\gamma} & M_{1,\gamma}^{\odot 2} & & & \\ Q_{1,3,\gamma} & Q_{1,2,\gamma} & M_{1,\gamma} & & \\ & & & & \end{pmatrix} = \begin{pmatrix} M_{1,\gamma}^{\odot 3} & & & & \\ 3M_{1,\gamma} \odot Q_{1,2,\gamma} & & & & \\ Q_{1,3,\gamma} & & & & \\ & & & & M_{2,\gamma} \end{pmatrix}. \quad (50)$$

The pattern is clear now. Assume we have computed solutions Y_1, \dots, Y_{k-1} and performed continuation up to $k-1$:

$$\Phi_{k-1} \xrightarrow[\text{cont}]{\gamma} \Phi_{k-1} M_{k-1,\gamma} := \Phi_{k-1} \begin{pmatrix} Q_{k-1,k-1,\gamma} & & & & \\ Q_{k-2,k-1,\gamma} & Q_{k-2,k-2,\gamma} & & & \\ \vdots & \vdots & \ddots & & \\ Q_{2,k-1,\gamma} & Q_{2,k-2,\gamma} & \cdots & Q_{2,2,\gamma} & \\ Q_{1,k-1,\gamma} & Q_{1,k-2,\gamma} & \cdots & Q_{1,2,\gamma} & Q_{1,1,\gamma} \end{pmatrix},$$

where

$$Q_{r,s,\gamma} := \sum_{i_1+\dots+i_r=s} c_{i_1,\dots,i_r}^s Q_{1,i_1,\gamma} \odot Q_{1,i_2,\gamma} \odot \cdots \odot Q_{1,i_r,\gamma}, \quad s \geq r \geq 2. \quad (51)$$

Then, the fundamental matrix for (LVE_ψ^k) will be expressed in the form (38), its lower left block Y_k being computable in terms of the blocks $Z_{2,k}, \dots, Z_{k,k}$ above it (all of which involve Y_1, \dots, Y_{k-1}) in virtue of (VE_ψ^k) : $Y_k = Y_1 V_k$, which is continued into $Q_{1,k,\gamma} := M_{1,\gamma} \int_\gamma V_k$, where $\overline{V}_k = Y_1^{-1} \sum_{j=2}^k A_j Z_{j,k}$. Upper terms $Z_{2,k}, \dots, Z_{k,k}$ are continued into $Q_{2,k}, \dots, Q_{k,k}$ as in (51), s replaced by k . It is clear we have proven the following:

Lemma 4.2. *The monodromy matrix Φ_k of LVE_ψ^k along closed path γ is composed by the first k row and column blocks in*

$$\exp_\odot Q_\gamma := \exp_\odot \left(\begin{array}{ccc|c} \cdots & 0 & 0 & 0 \\ \cdots & Q_{1,2,\gamma} & Q_{1,1,\gamma} & 0 \\ \cdots & 0 & 0 & 0 \end{array} \right), \quad (52)$$

where $Q_{1,1,\gamma} := M_{1,\gamma}$, blocks above the bottom row are computed according to (51) and

$$Q_{1,s,\gamma} := M_{1,\gamma} \int_\gamma Y_1^{-1} \sum_{j=2}^s A_j Z_{j,s}, \quad 2 \leq s \leq k. \quad \square \quad (53)$$

Hence it is clear the computation of a monodromy matrix follows a block order such as the one below, blocks in the bottom row requiring quadratures:

$$\begin{array}{cccc} \ddots & & & \\ \cdots & \boxed{7} & & \\ \cdots & \boxed{8} & \boxed{4} & \\ \cdots & \boxed{9} & \boxed{5} & \boxed{2} \\ \cdots & \boxed{10} & \boxed{6} & \boxed{3} & \boxed{1} \end{array} \quad (54)$$

Computing the monodromy matrix is concomitant to computing the fundamental matrix, i.e. said bottom-row quadratures must be both indefinite (yielding terms $Z_{1,s}$ to be used in the computation of $Z_{j,s}$ in (53)) and contour integrals. See §6 for an example.

We assume there are two generators $[\gamma], [\tilde{\gamma}] \in \pi_1(T; t_0)$, yielding two different matrices:

$$\gamma \longleftrightarrow Q_\gamma, \quad \tilde{\gamma} \longleftrightarrow Q_{\tilde{\gamma}}.$$

Commutativity of monodromy matrices now admits simple, compact formulation:

Proposition 5. *Two monodromy matrices $M_{k,\gamma}$ and $M_{k,\tilde{\gamma}}$ for LVE_ψ^k commute if, and only if, their previous blocks $M_{k-1,\gamma}, M_{k-1,\tilde{\gamma}}$ commute and the additional properties hold*

$$\sum_{j=r}^k Q_{r,j,\gamma} Q_{j,k,\tilde{\gamma}} = \sum_{j=r}^k Q_{r,j,\tilde{\gamma}} Q_{j,k,\gamma}, \quad \text{for every } r = 1, \dots, k-1,$$

matrices defined as in (51) and (53). □

Remarks 3.

- a) The monodromy group of a linear system is contained in its differential Galois group (e.g. [29]). The motivation for the above Lemma and Proposition is to capitalise on this fact. This may in turn be a step towards future constructive incarnations of the Morales-Ramis-Simó Theorem 1.2. The main obstacle implementing Proposition 5, symbolico-computational issues aside, is the incertitude on whether $M_{k,\gamma}$ and $M_{k,\tilde{\gamma}}$ belong to the Zariski identity component $\text{Gal} \left(\text{LVE}_\psi^k \right)^\circ$; a sufficient condition for arbitrary order is fulfilment at order 1, $M_{1,\gamma}, M_{1,\tilde{\gamma}} \in \text{Gal}(\text{VE}_\psi)^\circ$, itself an open problem in general.
- b) All disquisitions and results on the variational jet in [20, 21] are referred to the lower n -row strip for commutators of these monodromies. More specifically:
 - what is called *jet* therein is lower strip Y in principal fundamental matrix $\Phi = \exp_\odot Y$ for infinite system (LVE_ψ), and we will use this terminology in the following Section;

- morphism properties imply monodromy matrices along path commutators equal matrix commutators: $M_{k,\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1} = M_{k,\gamma_2}^{-1}M_{k,\gamma_1}^{-1}M_{k,\gamma_2}M_{k,\gamma_1}$;
- hence, “jet commutation” in [20, 21] amounts to lower strip $Q_{k,\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1}$ (that is Y after passage along $\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1$) equalling Id_n .

Although [20, 21] clearly benefit from the use of automatic differentiation techniques (see also [19]), it may be argued that expressions such as those in (LVE_ψ) provide for a fuller control of the general structure of the whole variational complex when it comes to symbolic computations, as well as a further check aid for the aforementioned techniques. See §6.1 for an example. See also [28] for a recent application to the Friedmann-Robertson-Walker Hamiltonian arising from Cosmology.

5. First integrals and higher-order variational equations. Let $F : U \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a holomorphic function and $\psi : I \subset \mathbb{C} \rightarrow U$. Firstly, the flow $\varphi(t, \mathbf{z})$ of X admits, at least formally, Taylor expansion (1) along ψ which is expressible as

$$\varphi(t, \psi + \xi) = \psi + Y_1 \xi + \frac{1}{2} Y_2 \xi^{\odot 2} + \dots = \psi + J_\psi \exp_{\odot} \xi, \quad (55)$$

where J_ψ is the jet for flow $\varphi(t, \cdot)$ along ψ , displayed as Y in (27) and defined in Notation 6 – that is, the matrix whose \odot -exponential Φ is a solution matrix for (LVE_ψ) . Secondly, the Taylor series of F along ψ can be written, cfr. [5, Lemma 2] and Notation 1,

$$F(\mathbf{y} + \psi) = F(\psi) + \sum_{m=1}^{\infty} \frac{1}{m!} \langle F^{(m)}(\psi), \text{Sym}^m \mathbf{y} \rangle. \quad (56)$$

Basic scrutiny of Example 1(3), Lemma 3.7 and (35) trivially implies (56) can be expressed as $F(\mathbf{y} + \psi) = M_F^\psi \exp_{\odot} \mathbf{y}$, where

$$M_F^\psi = J_F^\psi + F^{(0)}(\psi) := \left(\begin{array}{ccc|c} \dots & 0 & 0 & 0 \\ \dots & F^{(2)}(\psi) & F^{(1)}(\psi) & F^{(0)}(\psi) \\ \dots & 0 & 0 & 0 \end{array} \right) \in \text{Mat}^{1,n}(K),$$

i.e. J_F^ψ is the jet or horizontal strip of lex-sifted partial derivatives of F at ψ .

Definition 5.1. We call

$$\boxed{\dot{X} = A_{\text{LVE}_\psi^*} X}, \quad A_{\text{LVE}_\psi^*} := -(A \odot \exp_{\odot} \text{Id}_n)^T, \quad (\text{LVE}_\psi^*)$$

the **adjoint** or **dual** variational system of (DS) along ψ . Same as in (LVE_ψ) and all throughout 4.1, consideration of finite subsystems, namely the lowest $D_{n,k} \times D_{n,k}$ block, leads to specific notation $(\text{LVE}_\psi^k)^*$.

The following is immediate upon derivation of equation $\Phi_k \Phi_k^{-1} = \text{Id}_{D_{n,k}}$:

Lemma 5.2. $(\Phi_k^{-1})^T$ is a principal fundamental matrix of $(\text{LVE}_\psi^k)^*$, $k \geq 1$.

Hence, $\lim_k (\Phi_k^{-1})^T$, is a solution to (LVE_ψ^*) . \square

The following was proven in [24] and recounted in [5, Lemma 7], and may now be expressed in a simple, compact fashion:

Lemma 5.3. Let F and ψ be a holomorphic first integral and a non-constant solution of (DS) respectively. Let $V := J_F^T$ be the transposed jet of F along ψ . Then, V is a solution of (LVE_ψ^*) .

Proof. Let us recall formal expansion (55) and $F(\mathbf{y}) = J_F^\psi \exp_\odot \mathbf{y}$ for every $\mathbf{y} \in K^n$. Let $\phi = \varphi(t, \psi + \xi)$. We have, using Lemma 3.8,

$$F(\phi) = F(\psi + J_\psi \exp_\odot \xi) = M_F^\psi \exp_\odot (J_\psi \exp_\odot \xi) = \left(M_F^\psi \exp_\odot J_\psi \right) \exp_\odot \xi,$$

and $F(\phi)$ is supposed to be constant, hence applying (LVE $_\psi$) and Lemma 3.8

$$\begin{aligned} 0 &= \overline{\left(M_F^\psi \exp_\odot J_\psi \right)} \exp_\odot \xi = \left(\overline{M_F^\psi} + M_F^\psi A_{\text{LVE}_\psi} \right) \exp_\odot J_\psi \exp_\odot \xi \\ &= \left(\overline{M_F^\psi} + M_F^\psi A_{\text{LVE}_\psi} \right) \exp_\odot (\phi - \psi), \end{aligned}$$

hence $\overline{M_F^\psi} + M_F^\psi A_{\text{LVE}_\psi} = 0$ leading us to the final result after transposing both sides. \square

Compound the jet of field X , i.e. A in Notation 6 and Proposition 4, with a $1,0$ term A_0 , equal to $X^{(0)} = X(\psi) = \dot{\psi}$:

$$\widehat{A} := \left(\begin{array}{ccc|c} \cdots & 0 & 0 & 0 \\ \cdots & A_2 & A_1 & A_0 \\ \cdots & 0 & 0 & 0 \end{array} \right), \quad A_i := X^{(i)}(\psi) \in \text{Mat}_n^{1,i}(K).$$

It is easy to check, via possibilities offered on i_1 and j_1 in (25), that the symmetric product of \widehat{A} with $\exp_\odot \text{Id}_n$ adds only a relatively minor addendum to A_{LVE_ψ} , namely a superdiagonal of blocks $\binom{i}{i} A_0 \odot \text{Id}_n^{\odot i} \in \text{Mat}_n^{i+1,i}$, $i \geq 1$, effectively rendering it block-Hessenberg:

$$\widehat{A}_{\text{LVE}_\psi} := \widehat{A} \odot \exp_\odot \text{Id}_n = \lim_k \widehat{A}_{\text{LVE}_\psi^k},$$

where, isolating $A_{\text{LVE}_\psi^k}$ within $\widehat{A}_{\text{LVE}_\psi^k}$ by means of a solid line,

$$\begin{aligned} \widehat{A}_{\text{LVE}_\psi^k} &:= \left(\begin{array}{c|ccc} A_0 \odot \text{Id}_n^{\odot k} & & & \\ \hline \binom{k}{k-1} A_1 \odot \text{Id}_n^{\odot k-1} & \ddots & & \\ & \vdots & \ddots & A_0 \odot \text{Id}_n^{\odot 2} \\ \binom{k}{1} A_{k-1} \odot \text{Id}_n & \cdots & 2A_1 \odot \text{Id}_n & A_0 \odot \text{Id}_n \\ \hline A_k & \cdots & A_2 & A_1 \\ 0 & \cdots & 0 & 0 \end{array} \right) \quad (57) \\ &= \left(\begin{array}{c|c} \binom{k}{k} X^{(0)}(\psi) \odot \text{Id}_n^{\odot k} \\ \binom{k}{k-1} X^{(1)}(\psi) \odot \text{Id}_n^{\odot k-1} \\ \binom{k}{k-2} X^{(2)}(\psi) \odot \text{Id}_n^{\odot k-2} \\ \vdots \\ \binom{k}{0} X^{(k)}(\psi) \odot \text{Id}_n^{\odot 0} \\ \hline \widehat{A}_{\text{LVE}_\psi^{k-1}} \end{array} \right). \end{aligned}$$

Using the M_k - \mathcal{M}_k notation in [5], it is immediate to check that

$$M_k^T = \text{Id}_n^{\odot k-1} \odot \dot{\psi} = \text{Id}_n^{\odot k-1} \odot X(\psi), \quad \widehat{A}_{\text{LVE}_\psi^k} = \mathcal{M}_{k-1}^T \text{ for every } k \geq 1.$$

A result in [5] using said notation is easier to prove in this setting. Indeed, the same reasoning underlying (40) applies to row $F^{(k)}$ and $\frac{\partial}{\partial z_m} F^{(k)} = F^{(k+1)} \left(\mathbf{e}_m \odot \text{Id}_n^{\odot k} \right)$;

following Lemma 2.5,

$$\overline{F^{(k)}} = F^{(k+1)} \sum_{m=1}^n \left(\mathbf{e}_m \odot \text{Id}_n^{\odot k} \right) \dot{\psi}_m = F^{(k+1)} \left(\dot{\psi} \odot \text{Id}_n^{\odot k} \right) = F^{(k+1)} \left(A_0 \odot \text{Id}_n^{\odot k} \right),$$

implying $\overline{\left(F^{(k)} \right)^T} = \left(A_0 \odot \text{Id}_n^{\odot k} \right)^T \left(F^{(k+1)} \right)^T$; placing all terms on one side, and observing Lemma 5.3 and the transpose of expression (57), we obtain:

Proposition 6 ([5, Th. 12]). *Let F , ψ , V as in Lemma 5.3. Then $\widehat{A}_{\text{LVE}_\psi}^T V = 0$. \square*

Hence, blocks in $\mathbf{V}_1, (\mathbf{V}_2, \mathbf{V}_1)^T, (\mathbf{V}_3, \mathbf{V}_2, \mathbf{V}_1)^T, \dots$ having all entries in the base field K and satisfying both equations in Proposition 6 and 5.3 are candidates for jet blocks $F^{(1)}, F^{(2)}, \dots$ of a formal first integral. These blocks belonging to the intersection of $\ker \widehat{A}_{\text{LVE}_\psi}^T$ and the solution subspace $\text{Sol}_K \left(\text{LVE}_\psi^k \right)^*$ were called *admissible* solutions of the order- k adjoint system in [5].

This takes us back to the end of Section 3.2. Consider *gauge transformation* ([2, 5, 6, 22]) $\mathbf{x} = P\mathbf{X}$ transforming linear system $\dot{\xi} = A_1\xi$ into equivalent

$$\dot{\Xi} = P[A_1]\Xi := \left(P^{-1}A_1P - P^{-1}\dot{P} \right) \Xi.$$

Using notation $Y_i = PX_i$, $J_\psi = PX$ and item (e) in Lemma 3.5, we recover the result already seen in previous references, summarised in the extension of gauge transformations to higher dimensions via $P^{\odot k}$:

$$\exp_{\odot}(X) = \exp_{\odot}(P^{-1}J_\psi) = \exp_{\odot}P^{-1} \exp_{\odot}J_\psi = \text{diag} \left(\dots, (P^{-1})^{\odot 2}, P^{-1}, 1 \right) \exp_{\odot}J_\psi,$$

and very simple application of properties seen so far extends the general structure of the gauge transformation to $\Psi = \exp_{\odot}P^{-1} \exp_{\odot}J_\psi$:

$$\dot{\Psi} = P[A_{\text{LVE}_\psi}] \Psi := \left(\exp_{\odot}P^{-1}A_{\text{LVE}_\psi} \exp_{\odot}P + \left(\overline{P^{-1}} \odot \exp_{\odot}P^{-1} \right) \exp_{\odot}P \right) \Psi. \quad (58)$$

Second summand $\left(\overline{P^{-1}} \odot \exp_{\odot}P^{-1} \right) \exp_{\odot}P$ can be simplified into:

$$\text{diag} \left(\dots, k \left[\overline{P^{-1}} \odot (P^{-1})^{\odot k-1} \right] P^{\odot k}, \dots, 2 \left(\overline{P^{-1}} \odot P^{-1} \right) P^{\odot 2}, -P^{-1}\dot{P}, 0 \right), \quad (59)$$

with $\overline{P^{-1}} = -P^{-1}\dot{P}P^{-1}$. The above gauge transformation can be seen as the effect of transformation $\mathbf{z} = P\mathbf{Z}$ on the jet of (DS). Given a first integral F of the latter, we may always assume $F(\psi) = 0$, which implies $M_F^{1,0} = 0$ and, as seen in (37) or in Lemma 3.4,

$$F_P(\mathbf{Z}) = J_F(\exp_{\odot}P) \exp_{\odot}\mathbf{Z}.$$

The jet of this formal series is

$$J_{F_P} = J_F(\exp_{\odot}P) = \begin{pmatrix} \dots & F^{(0)}(\psi)P^{\odot 3} & F^{(2)}(\psi)P^{\odot 2} & F^{(1)}(\psi)P \\ \dots & 0 & 0 & 0 \end{pmatrix} \in \text{Mat}^{1,n},$$

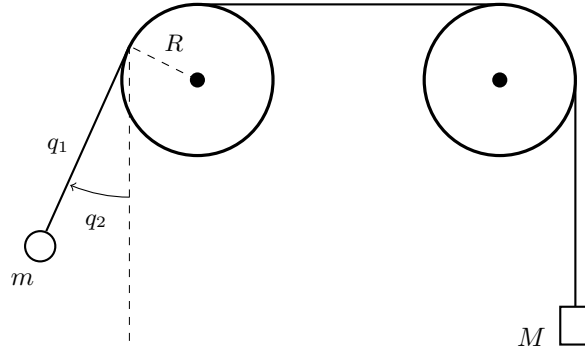
and applying (58), Lemmas 5.3 and 6, and identity $(P^{-1})^{\odot k} \dot{P}^{\odot k} = -\overline{(P^{-1})^{\odot k}} P^{\odot k}$, we have just proven the following:

Proposition 7. *The transposed jet $V_P := J_{F_P}^T$ in the new variables must satisfy*

$$\dot{V}_P = -P [A_{\text{LVE}_\psi}]^T V_P, \quad \widehat{A}_{\text{LVE}_\psi}^T (\exp_{\odot} P^{-1})^T V_P = 0. \quad \square \quad (60)$$

The key importance in practical examples resides in the choice of the particular solution ψ and the reduction matrix P , in order to render (60) easier (or more convenient) to solve than its unreduced counterparts, Lemma 5.3 and Proposition 6: see also [3, 4].

6. Example. The dynamics of the *Swinging Atwood Machine (SAM)*, summarised in the diagram below,



are governed by Hamiltonian

$$\mathcal{H} = \frac{1}{2} \left[\frac{p_1^2}{M_t} + \frac{(p_2 + R p_1)^2}{m q_1^2} \right] + g q_1 (M - m \cos q_2) - g R (M q_2 - m \sin q_2),$$

where $M_t = M + m + 2I_p/R^2$ and I_p is the pulley inertial momentum. We know the following:

Theorem 6.1 ([26, Th. 7.5]). *For every physically consistent value of the parameters, regardless of I_p and R , \mathcal{H} is meromorphically non-integrable.*

Consider *SAM* without massive pulleys, i.e. the limit case $I_p = R = 0$ and $M_t = M + m$:

$$\dot{z} = X_{\mathcal{H}_w}(z) := J \nabla \mathcal{H}_w(z), \quad \mathcal{H}_w = \frac{1}{2} \left(\frac{p_1^2}{M+m} + \frac{p_2^2}{m q_1^2} \right) + g q_1 (M - m \cos q_2). \quad (61)$$

Theorem 6.2. *Define $\mu := \frac{M}{m}$ and $\mu_p := \frac{p(p+1)}{p(p+1)-4}$, $p \in \mathbb{Z}$.*

1. ([15, Th. 1]) *If $M > m$ and $\mu \neq \mu_p$ for every $p \geq 2$, then $X_{\mathcal{H}_w}$ is non-integrable.*
2. ([30, (16)]) *For $\mu = \mu_2 = 3$, (61) is integrable with additional first integral:*

$$I = q_1^2 q_2 \left(q_1 c - \frac{q_1 q_2}{2} s \right) + g q_1^2 s c^2 = g q_1^2 c^2 s + p_2 \frac{p_1 q_1 c - 2 p_2 s}{4 m^2 q_1}. \quad (62)$$

3. ([20, Theorem 4]) *Degenerate cases μ_p , $p \geq 2$ in item 1 are non-integrable. \square*

Canonical transformation $(q_1, q_2, p_1, p_2) = (Q_1, \arccos Q_2, P_1, -P_2 \sqrt{1 - Q_2^2})$ on \mathcal{H}_w yields

$$H = g Q_1 (M - m Q_2) + \frac{1}{2} \left(\frac{P_1^2}{M+m} - \frac{P_2^2 (Q_2^2 - 1)}{m Q_1^2} \right). \quad (63)$$

Let us apply our formulation to the variational systems for H . The non-meromorphic nature of the canonical transformation, along with other related issues, precludes us from extending the conclusions of §6.1 to Hamiltonian (61). This will be the subject of further upcoming work.

6.1. Monodromy matrices and integrability. Consider the particular solution $\psi = (Q_1, Q_2, P_1, P_2)$ defined by

$$\psi(t) = \left(-\frac{g(t-1)t}{2}, -1, -\frac{g(m+M)(2t-1)}{2}, \frac{g^2 m(t-1)t(C_1 - 2C_1 t + t^2)}{4(C_1 - t)} \right). \quad (64)$$

In the forthcoming calculations, any value of C_1 different from 0 or 1 will ensure the presence of logarithms in the fundamental matrix, and any value of C_1 different from 1/2 will avoid division by zero. Choose, for instance, $C_1 = 1/3$. Gauge transformation $\mathbf{x} = P\mathbf{X}$,

$$P := \frac{1}{\sqrt{M+m}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{2(M+m)(1-3t)^2}{9gm(t-1)^2 t^2} & 0 & 0 \\ 0 & \frac{(M+m)(1+t)(3t-1)}{9(t-1)^2 t^2} & -M-m & 0 \\ -\frac{gm(1+t)}{2(3t-1)} & -\frac{gF(t)}{90(1-3t)^2(t-1)t} & 0 & \frac{9gm(t-1)^2 t^2}{2(1-3t)^2} \end{pmatrix},$$

and $F = 15m(t-1)t(3t-1)^3 + M(1+t(16+15t(-9+t(37+27(-2+t)t))))$ transforms (\mathbf{VE}_ψ) into

$$\dot{\mathbf{X}} = P[A_1]\mathbf{X} := \begin{pmatrix} 0 & -\frac{1}{9} \left(\frac{4}{(t-1)^2} - \frac{1}{t^2} \right) & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{8M(1-3t)}{405m(t-1)^4 t^4} & \frac{1}{9} \left(\frac{4}{(t-1)^2} - \frac{1}{t^2} \right) & 0 \end{pmatrix} \mathbf{X}. \quad (65)$$

Defining

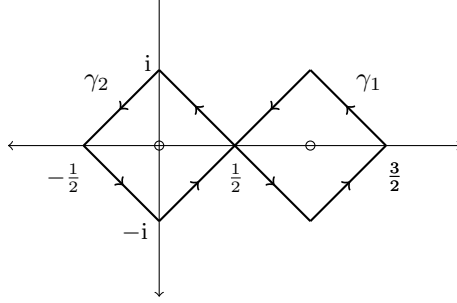
$$\begin{aligned} G(t) &= \frac{2}{3}(2t-1)[1+5(t-1)t(6(t-1)t-1)] \\ &+ t[3+5t(3-2t(11+3t(2t-5)))] - 20(t-1)^3 t^3 \ln \frac{t-1}{t}, \end{aligned}$$

the principal fundamental matrix for (65) is $\Psi(t) := \tilde{\Psi}(t) \cdot (\tilde{\Psi}(1/2))^{-1}$, where

$$\tilde{\Psi} := e^{\int P[A_1]} = \begin{pmatrix} 1 & \frac{1+3t}{9(t-1)t} & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{4MG(t)}{405m(t-1)^3 t^3} & -\frac{1+3t}{9(t-1)t} & 1 \end{pmatrix},$$

because $P[A_1]$ commutes with $\int P[A_1]$ – an interesting topic for further study is the possible relationship between said commutation and the level of *reduction* of the matrix in the sense of [3], but this is not central to our present work. Our intention

is to perform analytical continuation along paths homotopic to γ_1, γ_2 shown below:



Along γ_i , all terms outside of the diagonal in Ψ vanish except for term $(M_{1,\gamma_i})_{4,2}$ which can be obtained from (65) by assuming $X_2 = 1, X_3 = 0$, which yields $\frac{8M(3t-1)}{405m(t-1)^4 t^4} + \dot{X}_4 = 0$ and thus

$$(M_{1,\gamma_1})_{4,2} = - \int_{\gamma_1} \frac{8M(3t-1)}{405m(t-1)^4 t^4} = \frac{32iM\pi}{81m}, \quad (M_{1,\gamma_2})_{4,2} = - \frac{32iM\pi}{81m}. \quad (66)$$

Order-2,3 monodromies are given by (46), (49) respectively. The only change therein is replacing A_1, A_2, A_3 by their gauge transforms given by (58), (59) i.e. the lower row block in $P \left[A_{\text{LVE}_\psi^3} \right]$, equal to

$$\left(\begin{array}{cc} 3 \left[Q^{\odot 3} (A_1 \odot \text{Id}_n^{\odot 2}) + \dot{Q} \odot Q^{\odot 2} \right] P^{\odot 3} & \\ 3Q^{\odot 2} (A_2 \odot \text{Id}_n) P^{\odot 3} & 2 \left[Q^{\odot 2} (A_1 \odot \text{Id}_n) + \dot{Q} \odot Q \right] P^{\odot 2} \\ P^{-1} A_3 P^{\odot 3} & P^{-1} A_2 P^{\odot 2} \quad P[A_1] \end{array} \right),$$

where $Q := P^{-1}$. Let $Y_1 = \Psi$. Following order (54), we compute (46), (49) and $M_{3,\gamma_1}, M_{3,\gamma_2}$. Let us check whether our third-order monodromy matrices commute. We have $C := M_{3,\gamma_1} M_{3,\gamma_2} - M_{3,\gamma_2} M_{3,\gamma_1}$ equal to zero except for the following terms:

$$C_{34,5} = - \frac{\left(\frac{25600}{85293} - \frac{8960}{85293} i \right) M^2 \pi^2}{g^2 m^2 (M+m)}, \quad C_{34,11} = \frac{6815744 M^3 \pi^3 i}{2187 g^2 m^3 (M+m)}. \quad (67)$$

This, coupled with the fact that $M_{k,\gamma_i} \in \text{Gal} \left(\text{LVE}_\psi^k \right)^\circ$ for every k (since M_{1,γ_i} , being unipotent, belong to $\text{Gal}(\text{VE}_\psi)^\circ$ and fundamental matrices Ψ_k are obtained from quadratures), allows us to complement the non-integrability proof in [20] using linearised variational equations on (63).

Remarks 4.

1. Same reasoning can be applied to pre-gauge monodromies, although calculations are more cumbersome; defining $P_k := \text{diag} (P^{\odot k}, \dots, P)$, we have

$$\Psi_k \xrightarrow[\text{cont}]{\gamma_i} \Psi_k M_{k,\gamma_i} \Rightarrow \tilde{\Psi}_k := P_k \Psi_k (P_k \Psi_k)_{t=1/2}^{-1} \xrightarrow[\text{cont}]{\gamma_i} P_k \Psi_k M_{k,\gamma_i} (P_k \Psi_k)_{t=1/2}^{-1},$$

thereby rendering monodromy $\tilde{M}_{k,\gamma_i} = (P_k \Psi_k)_{t=1/2} M_{k,\gamma_i} (P_k \Psi_k)_{t=1/2}^{-1}$. The reader may check that \tilde{M}_{1,γ_i} have the same structure as M_{1,γ_i} , albeit with $\pm \frac{ig^2 m M \pi}{2(M+m)}$ in lieu of (66), and the non-zero terms in $\tilde{C} := \tilde{M}_{3,\gamma_1} \tilde{M}_{3,\gamma_2} - \tilde{M}_{3,\gamma_2} \tilde{M}_{3,\gamma_1}$ share the same indices as (67):

$$\tilde{C}_{34,5} = \frac{\left(\frac{50}{117} - \frac{35i}{234} \right) gm M^2 \pi^2}{(M+m)^2}, \quad \tilde{C}_{34,11} = \frac{4992 g^2 m M^3 \pi^3 i}{(m+M)^3}. \quad (68)$$

2. Although we used monodromies, we can also use the presence of $\ln(3t-1)$ in Y_3 and Picard-Vessiot extension $\mathbb{C}(t)(\ln t, \ln(t-1), \ln(3t-1)) \mid \mathbb{C}(t)(\ln t, \ln(t-1))$ to glean the structure of a generic Galois group matrix on the fundamental matrix Ψ_3 or its pre-gauge counterpart $P_3\Psi_3$; see e.g. [25].

We therefore have the following result:

Theorem 6.3. *H in (63) is not meromorphically integrable for any $M, m > 0$. \square*

6.2. Formal first integrals and admissible solutions. Let us now try to apply gauge transforms to the adjoint system for the same Hamiltonian. We have a particular solution $\psi = \left(-\frac{g}{2}(t-1)t, -1, -\frac{g}{2}(M+m)(2t-1), -\frac{g^2 m}{4}(t-1)t^2\right)$, which corresponds to special case $C_1 = 0$ in (64), and gauge transformation $\mathbf{x} = P\mathbf{X}$ with

$$P := \begin{pmatrix} \frac{1-t}{\sqrt{M+m}} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{M+m}}{gm(t-1)^2} & 0 & 0 \\ -\sqrt{M+m} & \frac{\sqrt{M+m}}{(t-1)^2} & \frac{\sqrt{M+m}}{1-t} & 0 \\ \frac{gm(1-t)}{\sqrt{M+m}} & \frac{g(3m-M)}{12(t-1)\sqrt{M+m}} - \frac{gt}{4}\sqrt{M+m} & 0 & \frac{gm(t-1)^2}{\sqrt{M+m}} \end{pmatrix},$$

transforms (VE $_{\psi}$) into the parameter-free, simplified system $\dot{\mathbf{X}} = P[A_1]\mathbf{X}$ where

$$P[A_1] = \begin{pmatrix} 0 & -\frac{1}{(t-1)^3} & \frac{1}{(t-1)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(t-1)^3} & 0 \end{pmatrix}.$$

The principal fundamental matrix for this system is

$$\Psi := \begin{pmatrix} 1 & \frac{1}{2}\left(\frac{1}{(t-1)^2} - 1\right) & \frac{t}{(1-t)} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{(t-2)t}{2(t-1)^2} & 1 \end{pmatrix},$$

and $P[A_{\text{LVE}^3_{\psi}}]$ is computed as in §6.1. In the following table, the first two columns correspond to order k and total solution space dimension $D_{4,k}$. The latter two display the dimension of the subspace of rational solutions (i.e. those having their entries in the base field $K = \mathbb{C}(t)$), and the subspace of those among the former satisfying (60), respectively:

k	$\dim_{\mathbb{C}} \text{Sol}P[\text{LVE}^k_{\psi}]^*$	$\dim_{\mathbb{C}} \text{Sol}_K P[\text{LVE}^k_{\psi}]^*$	$\dim_{\mathbb{C}} \text{Sol}_{\text{adm}}(P[\text{LVE}^k_{\psi}]^*)$
1	4	4	3
2	14	14	9
3	34	32	17

Using solution (64) in §6.1 and the other P given in (65), however, $\dim_{\mathbb{C}} \text{Sol}_K P[\text{LVE}^k_{\psi}]^*$ is considerably reduced and we obtain the following table, displaying

lower bounds on the amount of admissible solutions:

k	$\dim_{\mathbb{C}} \text{Sol}P [\text{LVE}_{\psi}^k]^*$	$\dim_{\mathbb{C}} \text{Sol}_{KP} [\text{LVE}_{\psi}^k]^*$	$\dim_{\mathbb{C}} \text{Sol}_{\text{adm}} \left(P [\text{LVE}_{\psi}^k]^* \right)$
1	4	3	2
2	14	9	5
3	34	19	9

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