A robust hierarchical nominal multicriteria classification method based on similarity and dissimilarity

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Abstract

Cat-SD (CATegorization by Similarity-Dissimilarity) is a multiple criteria decision aiding method for dealing with nominal classification problems (predefined and non-ordered categories). Actions are assessed according to multiple criteria and assigned to one or more categories. A set of reference actions is used to define each category. The assignment of an action to a given category depends on the comparison of the action to each reference set according to likeness thresholds. Distinct sets of criteria weights, interaction coefficients, and likeness thresholds can be defined per category. When applying Cat-SD to complex decision problems, may be useful to consider a hierarchy of criteria to give a more intelligible vision of the performances of the considered actions. We propose to apply Multiple Criteria Hierarchy Process to Cat-SD to take into account criteria structured in a hierarchical way. On the basis of the known deck of cards method, we also consider an imprecise elicitation of parameters permitting to consider interactions and antagonistic effects between criteria. The elicitation procedure we are proposing can be applied to any Electre method. With the purpose of exploring the assignments obtained by Cat-SD considering possible sets of parameters, we propose to apply the Stochastic Multicriteria Acceptability Analysis (SMAA). The SMAA methodology allows to draw statistical conclusions on the classification of the actions. The proposed method, SMAA-hCat-SD, helps the decision maker to check the effects of the variation of parameters on the classification at different levels of the hierarchy. We propose also a procedure, based on the concept of loss function, to get a deterministic classification fulfilling some requirements given by the decision maker and taking into account the hierarchy of criteria and the probabilistic assignments obtained through SMAA. Also this procedure can be applied to any classification Electre method. The application of the new proposal is shown through an example.

Keywords: Multiple criteria decision aiding, Hierarchy of criteria, Interaction effects, Deck of cards method, Robust optimization, Loss function, Deterministic classification.

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1. Introduction

In several decision situations, we face a classification problem involving the assessment of a set of actions (or alternatives), according to multiple criteria (usually conflicting), and their assignment to categories defined in a nominal way. In fact, the wide range of potential real-world applications in various areas (e.g., human resources management, finance, medicine, etc.) has motivated researchers to develop Multiple Criteria Decision Aiding (MCDA) methods for dealing with multiple criteria nominal classification problems. In this kind of classification problems, categories are predefined and no order exists among them (nominal categories). In opposition, in sorting problems (or ordinal classification problems), there is a preference order among the categories. In other fields, such as statistics and machine learning (ML), both terms discrimination and classification are used to refer to decision problems where the categories are defined a priori and there is no preferential order among them. The term *supervised classification problems* is usually used when the categories are previously defined, whereas *unsupervised classification problems* is used when there is no information about the categories and they are identified a posteriori (they are designed clusters) (Henriet, 2000; Perny, 1998). In clustering, the objective is to find such clusters, representing groups of actions with similar features. Recent proposals for handling classification problems are mainly based on operations research and artificial intelligence techniques (Doumpos and Zopounidis, 2002; Zopounidis and Doumpos, 2002). In fact, nominal classification has been addressed in an MCDA setting, but also in ML. The main difference between the MCDA setting and the standard nominal classification problems in ML is the role of criteria. Standard ML algorithms assume features (usually called attributes), whereas MCDA assumes criteria. In particular, criteria in MCDA have, in general, an increasing or a decreasing direction of preference that reveal the preferences of the Decision Maker (DM) on such criteria. On the contrary, features in ML have not any direction of preference and, instead, the relation between the values of the attributes and the preferences of the DM are discovered from data (Corrente et al., 2013).

In the literature, we can find proposals for nominal classification mainly using outranking-based procedures (Belacel, 2000; Henriet, 2000; Léger and Martel, 2002; Perny, 1998; Rigopoulos et al., 2010), rough set theory (Słowiński and Vanderpooten, 2000), and verbal decision analysis (Furems, 2013). The majority of existing MCDA nominal classification methods are based on outranking relations (see, for example, Belacel 2000; Perny 1998). While for choice, ranking and sorting problems outranking binary relations are acceptable, for nominal classification problems, they may be questionable. One may argue that in nominal classification the aim of the pairwise comparison should be to know whether two actions are similar and not if one action is preferred to the other. None of the current methods proposed a way to model preference information related to similarity concepts when comparing actions, neither to deal with criteria hierarchy and interactions between criteria. In addition, robustness concerns have not been considered, and it has been pointed out as an important issue in nominal classification (Zopounidis and Doumpos, 2002). The Cat-SD (CATegorization by Similarity-Dissimilarity) method has been recently proposed as a new MCDA method, covering some of these issues (Costa et al., 2018). This method allows to assign actions to nominal categories, based on similarity and dissimilarity between actions, using reference actions to define the categories. Multiple criteria and possible interactions in some pairs of criteria are considered. In Cat-SD, for each category, a particular set of preference parameters can be chosen (e.g., criteria weights and interaction coefficients), which means that distinct parameter sets can be defined for different categories. Thus, Cat-SD has been designed to model subjective judgments of the DM in pairwise comparison of actions in terms of similarity and dissimilarity between them. Then, likeness binary relations are constructed taking into account the preferences of the DM. Moreover, to the best of our knowledge, Cat-SD is the first MCDA nominal classification method that permits to model interactions between criteria. As stated in Costa et al. (2018), there are still aspects that need further research related to Cat-SD, namely considering a hierarchical structure of criteria and robustness analysis, while different vectors of parameter sets are taken into consideration. All these aspects are taken into account in the method we are proposing.

In several decision aiding scenarios, complex multiple criteria decision problems arise involving a great number of criteria for assessing actions (Belton and Stewart, 2002; Greco et al., 2016; Ishizaka and Nemery, 2013). The heterogeneity and the high number of criteria are the main reasons for the complexity of the decision problems. Structuring the criteria in a hierarchical way can be a useful approach for dealing with such decision problems. Multiple Criteria Hierarchy Process (MCHP) has been proposed to handle the decision problems in which the considered criteria are hierarchically structured (Corrente et al., 2012). In MCHP all criteria are not considered at the same level but they are grouped into subsets according to distinct points of view. In this way, the elicitation of preferences of the DM can be easier than considering a great number of heterogeneous criteria at the same level. To the best of our knowledge, there is no research work adopting such an approach to multiple criteria nominal classification methods. In this paper, we propose to apply MCHP to the Cat-SD method.

We introduce an adapted MCHP to handle the three types of interaction between criteria considered in Cat-SD: mutual-strengthening effect, mutual-weakening effect, and antagonistic effect (for more details on the meaning of these effects in case of outranking relations, see Figueira et al., 2009). Moreover, an imprecise elicitation of criteria weights is considered. For that, we adopt an extension of the Simos-Roy-Figueira (SRF) (Figueira and Roy, 2002) by considering imprecise preference information provided by the DM to assign values to the criteria weights (Corrente et al., 2017). To take into account interaction between criteria, we further extend this methodology obtaining a new version that can be applied to any Electre method considering such interaction.

The results provided by the Cat-SD method can include multiple assignments of an action, i.e., a given action can be assigned to several categories. It is interesting to know the robustness of the assignment of each action, considering then the robustness of the recommendations with respect to the assignment results. In this sense, to take into account all sets of weights and interaction coefficients compatible with the information provided by the DM, we propose to apply the Stochastic Multicriteria Acceptability Analysis (SMAA) (Lahdelma et al., 1998; Lahdelma and Salminen, 2010; Pelissari et al., 2019) to draw conclusions with respect to the assignments of each action. Application of MCHP and SMAA allows to obtain, for each action, the probability of its assignment to each category (or a set of categories), not only when considering the whole set of criteria, but also when considering a particular node in the hierarchy structure. Of course, this can be a relevant information for the DM. Finally, we propose a methodology to define a single classification taking into consideration the whole probabilistic information related to the imprecise elicitation of preference parameters. The procedure we propose is based on the concept of loss function and it has an autonomous interest that permits to apply this approach to any classification method, not only nominal but also ordinal.

We therefore aim to present a new method, in the sense of a more comprehensive framework, for dealing with these interrelated issues having the following main objectives:

1. To apply MCHP to the nominal classification method Cat-SD;
2. To use the imprecise SRF method for each category taking into account the hierarchy of criteria and the possible interactions between criteria; the method we propose has a general interest and can be applied to any outranking method considering interactions between criteria;
3. To apply SMAA to the hierarchical Cat-SD method by sampling several sets of parameters compatible with the preferences provided by the DM;

4. To propose a procedure that starting from the probabilistic assignments obtained by SMAA provides a final classification that fulfills some requirements given by the DM; the method we propose has a general interest and can be applied to any classification method, both nominal and ordinal.

It is worth to remark that the parameters elicitation is a fundamental step not only for our method but for all methods using an indirect preference information provided by the DM. The weights elicitation as well as the interaction coefficients elicitation can involve a certain difficulty and different methods have been proposed in literature to this aim. For instance, Figueira and Roy (2002) provides a method to elicit the weights and Figueira et al. (2009) presents an elicitation procedure for getting the values of the interaction effects (see also Bottero et al. 2015 and Costa et al. 2019b applying such a procedure to real-world cases).

This paper is organized as follows. Section 2 introduces the Cat-SD method. Section 3 is related to our proposal of applying MCHP to the Cat-SD method, in order to construct the hierarchical Cat-SD method, hCat-SD. Section 4 presents a way for dealing with imprecise information to determine the criteria weights when considering the hCat-SD method. Section 5 is devoted to the application of SMAA to the hCat-SD method, building the comprehensive method SMAA-hCat-SD. Section 6 proposes a procedure to obtain the final nominal classification results according to some requirements indicated by the DM. Section 7 provides a numerical example of application of the SMAA-hCat-SD method. Section 8 presents some concluding remarks and future lines of research.

2. The CAT-SD method

In this section, we briefly introduce the Cat-SD method (for more details, see Costa et al. 2018). This method deals with decision problems where categories are defined in a nominal way (they are not ordered). Each category is defined a priori and characterized by a set of reference actions. Each action is assessed on several criteria, and assigned to a category or a set of categories. The assignment of actions is based on the concepts of similarity and dissimilarity between two actions. The main notation, concepts and definitions are presented.

2.1. Main notation

In the Cat-SD method, the following notation is used:

- $A = \{a, b, \ldots\}$ is the set of actions (or alternatives) not necessarily known a priori;
- $G = \{g_1, ..., g_j, ..., g_n\}$ is the set of all criteria[1];
- $C = \{C_1, ..., C_h, ..., C_q, C_{q+1}\}$ is the set of nominal categories, where C_{q+1} is a dummy one considered to receive actions not assigned to the other categories;
- $B = \{B_1, ..., B_h, ..., B_q, B_{q+1}\}$ is the set of all sets of reference actions, where $B_{q+1} = \emptyset$;
- $\{b_{h1}, \ldots, b_{h\ell}, ..., b_{h|B_h|}\}$ is the set of (representative) reference actions chosen to define category C_h, for $h = 1, ..., q$;
- k_j^h is the weight of criterion g_j for category C_h, for $j = 1, ..., n$ and $h = 1, ..., q$;

[1]In the following, for the sake of simplicity and without loss of generality we shall write $g_j \in G$ or $j \in G$ interchangeably.

– $k_{j\ell}^h$ is a mutual-strengthening (or mutual-weakening) coefficient of the pair of criteria $\{g_j, g_\ell\}$, with $k_{j\ell}^h > 0$ (or $k_{j\ell}^h < 0$), for $h = 1, ..., q$;

– $k_{j|p}^h$ is an antagonistic coefficient for the ordered pair of criteria (g_j, g_p), with $k_{j|p}^h < 0$, for $h = 1, ..., q$;

– $k(C_h)$ is the set of all criteria weights and interaction coefficients of category C_h, for $h = 1, ..., q$;

– λ^h is a likeness threshold of category C_h, for $h = 1, ..., q$.

2.2. Modeling similarity-dissimilarity

Cat-SD is more focused on similarity between actions than on their dissimilarity, since likeness between actions is usually what counts most when categorizing actions. According to a given criterion, when an action a (the subject) is compared to an action b (the referent or the reference action), similarity-dissimilarity between them can be assessed. Indeed, the preferences of the DM with respect to the similarity-dissimilarity between the two actions on a criterion can be modeled through a function.

In what follows, let E_j denote the scale of criterion g_j, $j = 1, ..., n$ (generally bounded from below by $g_j^{\min}$ and from above by $g_j^{\max}$). Consider the difference of performances of actions a and b, $\Delta_j(a, b) = diff\,\{g_j(a), g_j(b)\}$. Let E_{Δ_j} denote the scale of such a difference. For ratio and interval scales, $diff\,\{g_j(a), g_j(b)\} = g_j(a) - g_j(b)$, and for ordinal scales, $diff\,\{g_j(a), g_j(b)\}$ corresponds to the number of performance levels between $g_j(a)$ and $g_j(b)$. Without loss of generality, we assume that criteria are to be maximized. A *per-criterion similarity-dissimilarity function* is a real-valued function $f_j : E_{\Delta_j} \to [-1, 1]$ such that:

1. f_j is a non-decreasing function of $\Delta_j(a, b)$, if $\Delta_j(a, b) \in [-diff\{g_j^{\max}, g_j^{\min}\}, 0]$;
2. f_j is a non-increasing function of $\Delta_j(a, b)$, if $\Delta_j(a, b) \in [0, diff\{g_j^{\max}, g_j^{\min}\}]$;
3. $f_j > 0$ iff criterion g_j contributes to similarity;
4. $f_j < 0$ iff criterion g_j contributes to dissimilarity.

This function defines:

– A *per-criterion similarity function* $s_j(a, b) = f_j\big(\Delta_j(a, b)\big)$, if $f_j\big(\Delta_j(a, b)\big) > 0$, and $s_j(a, b) = 0$, otherwise;

– A *per-criterion dissimilarity function* $d_j(a, b) = f_j\big(\Delta_j(a, b)\big)$, if $f_j\big(\Delta_j(a, b)\big) < 0$, and $d_j(a, b) = 0$, otherwise.

Each f_j can be defined as a piecewise function by means of thresholds σ_1^j, σ_2^j, σ_3^j and σ_4^j as follows:

– For values of $\Delta_j(a, b)$ such that $|\Delta_j(a, b)| \leqslant \sigma_1^j$, we have $f_j\big(\Delta_j(a, b)\big) = 1$;

– For values of $\Delta_j(a, b)$ such that $\sigma_1^j \leqslant |\Delta_j(a, b)| \leqslant \sigma_2^j$, $f_j\big(\Delta_j(a, b)\big)$ is linear with $f_j\big(\Delta_j(a, b)\big) = 1$ if $|\Delta_j(a, b)| = \sigma_1^j$ and $f_j\big(\Delta_j(a, b)\big) = 0$ if $|\Delta_j(a, b)| = \sigma_2^j$;

– For values of $\Delta_j(a, b)$ such that $\sigma_2^j < |\Delta_j(a, b)| \leqslant \sigma_3^j$, we have $f_j\big(\Delta_j(a, b)\big) = 0$;

– For values of $\Delta_j(a, b)$ such that $\sigma_3^j \leqslant |\Delta_j(a, b)| \leqslant \sigma_4^j$, $f_j\big(\Delta_j(a, b)\big)$ is linear with $f_j\big(\Delta_j(a, b)\big) = 0$ if $|\Delta_j(a, b)| = \sigma_3^j$ and $f_j\big(\Delta_j(a, b)\big) = -1$ if $|\Delta_j(a, b)| = \sigma_4^j$;

– For values of $\Delta_j(a, b)$ such that $|\Delta_j(a, b)| > \sigma_4^j$, we have $f_j\big(\Delta_j(a, b)\big) = -1$.

The thresholds defining the function f_j can be induced with the following set of questions for the DM (possibly supported by the analyst):

– Which is the maximal difference σ_1^j between actions a and b on criterion g_j such that a and b can be considered absolutely similar with respect to the same criterion?

– Which is the minimal difference σ_2^j between actions a and b on criterion g_j such that a and b can be considered definitely not similar with respect to the same criterion?

– Which is the maximal difference σ_3^j between actions a and b on criterion g_j such that there is not any dissimilarity between a and b with respect to the same criterion?

– Which is the minimal difference σ_4^j between actions a and b on criterion g_j such that a and b can be considered absolutely dissimilar with respect to the same criterion?

Let us observe that an alternative procedure to elicit this kind of functions has been proposed in Costa et al. (2019a).

The CAT-SD method was designed to take into account interaction effects between pairs of criteria when computing likeness between two actions. In general, in real-world problems, the following three types of interactions between criteria can be considered (Figueira et al., 2009):

1. *Mutual-strengthening effect* between the criteria g_j and g_ℓ. This synergy effect between the two criteria, when both criteria are in favor of similarity between actions a and b, can be modeled through a positive coefficient $k_{j\ell}^h$ ($k_{j\ell}^h = k_{\ell j}^h$), which is added to the sum of the weights $k_j^h + k_\ell^h$;
2. *Mutual-weakening effect* between the criteria g_j and g_ℓ. This redundancy effect between the two criteria, when both criteria are in favor of similarity between actions a and b, can be modeled through a negative coefficient $k_{j\ell}^h$ ($k_{j\ell}^h = k_{\ell j}^h$), which is added to the sum of the weights $k_j^h + k_\ell^h$;
3. *Antagonistic effect* between the criteria g_j and g_p. This antagonistic effect exercised when criterion g_j is in favor of the similarity and criterion g_p is in favor of the dissimilarity between actions a and b, can be modeled through a negative coefficient $k_{j|p}^h$, which is added to the weight k_j^h (in general, $k_{j|p}^h$ is not equal to $k_{p|j}^h$ or, even more, one of the two antagonistic effects could not exist).

It should be remarked that distinct sets of weights and interaction coefficients, $k(C_h)$, can be defined among categories, $h = 1, ..., q$. For example, let us consider a problem in which some cars have to be assigned to categories "family car" and "sport car", and that criteria cost, safety, maximum speed and acceleration have to be taken into account. One can imagine that, on one hand, cost and safety are the most important criteria when assigning a car to the "family car" category, while, on the other hand, maximum speed and acceleration become the most important criteria in assigning a car to the "sport car" category.

To guarantee that the contribution of each criterion to the comprehensive similarity is not negative when considering the interaction effects, the following net flow condition has to be fulfilled (Figueira et al., 2009):

$$k_j^h \quad - \sum_{\{\{j,\ell\} \in M^h \,:\, k_{j\ell}^h < 0\}} |k_{j\ell}^h| \;-\; \sum_{(j,p) \in O^h} |k_{j|p}^h| \;\geqslant\; 0, \text{ for all } j \text{ and } h = 1, ..., q, \tag{1}$$

where

- M^h is the set of all pairs of criteria $\{j, \ell\}$ such that $f_j\big(\Delta_j(a,b)\big) > 0,\ f_\ell\big(\Delta_\ell(a,b)\big) > 0$, and there is mutual-weakening effect between them, for category $C_h,\ h = 1, ..., q$;
- O^h is the set of all ordered pairs of criteria (j, p) such that $f_j\big(\Delta_j(a,b)\big) > 0,\ f_p\big(\Delta_p(a,b)\big) < 0$, and g_p exercises an antagonistic effect on g_j, for category $C_h,\ h = 1, ..., q$.

Considering a similarity-dissimilarity function for each criterion, the set of criteria weights and the interaction coefficients defined for each category $C_h,\ h = 1, ..., q$, a comprehensive similarity aggregation function can be defined. Such a function measures the strength of the arguments in favor of likeness of action a with respect to action b. A *comprehensive similarity function* is a real-valued function $f^s : [0,1]^n \times [-1,0]^n \to [0,1]$ defined as follows:

$$s^h(a,b) = f^s\big(s_1(a,b), \ldots, s_j(a,b), \ldots, s_n(a,b), d_1(a,b), \ldots, d_j(a,b), \ldots, d_n(a,b), k(C_h)\big) =$$

$$= \frac{1}{K^h(a,b)} \left(\sum_{j \in G} k_j^h s_j(a,b) + \sum_{\{j,\ell\} \in M^h} s_j(a,b) s_\ell(a,b) k_{j\ell}^h + \sum_{(j,p) \in O^h} s_j(a,b) |d_p(a,b)| k_{j|p}^h \right) \quad (2)$$

and

$$K^h(a,b) = \sum_{j \in G} k_j^h + \sum_{\{j,\ell\} \in M^h} s_j(a,b) s_\ell(a,b) k_{j\ell}^h + \sum_{(j,p) \in O^h} s_j(a,b) |d_p(a,b)| k_{j|p}^h,$$

for $h = 1, ..., q$.

A comprehensive dissimilarity function, $d(a,b)$, can also be defined to measure the strength of the arguments in favor of dissimilarity between actions a and b, i.e., in opposition to likeness. The function considers only the dissimilarity values obtained from all per-criterion dissimilarity functions. A *comprehensive dissimilarity function* $d(a,b)$ can be defined for each $(a,b) \in A \times A$ through a real-valued function $f^d : [-1,0]^n \to [-1,0]$ as follows:

$$d(a,b) = f^d\big(d_1(a,b), \ldots, d_j(a,b), \ldots, d_n(a,b)\big) = \prod_{j=1}^{n} \big(1 + d_j(a,b)\big) - 1. \quad (3)$$

In order to calculate a likeness degree that aggregates similarity and dissimilarity, for each pair of actions (a,b) (a represents a given action and b a reference action), it is necessary to use an aggregation function. A *comprehensive likeness function* $\delta(a,b)$ can be defined for each $(a,b) \in A \times A$ through a real-valued function $f : [0,1] \times [-1,0] \to [0,1]$ as follows:

$$\delta(a,b) = f\big(s^h(a,b), d(a,b)\big) = s^h(a,b)\big(1 + d(a,b)\big). \quad (4)$$

Thus, it is possible to assess the degree of likeness of action a with respect to action b. $\delta(a,b)$ is called *likeness degree* between a and b.

A remark with respect to the dissimilarity $d(a,b)$ is in order. Indeed, differently from the similarity $s^h(a,b)$, we are considering the same dissimilarity $d(a,b)$ for each category. From a theoretical point of view, it is possible to define a dissimilarity $d^h(a,b)$ changing from one category to another. However, observe that the definition of the per criterion dissimilarity function $d_j(a,b)$ is related to the per-criterion similarity-dissimilarity function $f_j(\Delta_j(a,b))$. Consequently, the use of a dissimilarity function for each category would require the elicitation of a specific per-criterion similarity-dissimilarity function $f_j^h(\Delta_j(a,b))$ which would require a challenging cognitive burden for the DM that has to be taken into account in evaluating its possible adoption.

2.3. Relation between actions and reference actions

In order to assign actions to category C_h, $h = 1, \ldots, q$, each action has to be compared to each reference action, $b_{h\ell}$, $\ell = 1, ..., |B_h|$, computing the likeness degree, i.e., $\delta(a, b_{h\ell})$, between a and $b_{h\ell}$. A *likeness degree between the action a and the reference set* B_h can be defined as follows:

$$\delta(a, B_h) = \max_{\ell=1,\ldots,|B_h|} \{\delta(a, b_{h\ell})\}. \tag{5}$$

A *likeness threshold*, λ^h, can be chosen by the DM for each category C_h, $h = 1, ..., q$. This preference parameter is the minimum likeness degree considered necessary to say that an action a is similar to the set B_h, $h = 1, ..., q$, taking all criteria into account. It can be interpreted as a majority measure of likeness allowing an action to be assigned to the most adequate categories, if any. Then, λ^h takes a value within the range $[0.5, 1]$. A *likeness binary relation*, $S(\lambda^h)$, is defined as follows:

$$aS(\lambda^h)B_h \Leftrightarrow \delta(a, B_h) \geqslant \lambda^h. \tag{6}$$

2.4. Assignment procedure

The Cat-SD assignment procedure provides at least one category to which an action a can be assigned. Each category C_h, $h = 1, ..., q$, is defined to receive actions to be processed in an identical way, at least in a first step. Given $\lambda^h \in [0.5, 1]$, $h = 1, \ldots, q$, the *likeness assignment procedure* was designed for Cat-SD as follows:

i) Compare action a with set B_h, $h = 1, \ldots, q$;

ii) Identify $U = \{u \,:\, aS(\lambda^u)B_u\}$;

iii) Assign action a to category C_u, for all $u \in U$;

iv) If $U = \emptyset$, assign action a to category C_{q+1}.

The assignment of an action to a given category is independent from the assignment to another category. Accordingly, a given action a can be assigned to:

– A single category (including C_{q+1}), in the case of a being only suitable to one category C_h, $h = 1, ..., q$ (or any);

– A set of categories (excluding C_{q+1}), in the case of a being suitable for more than one category.

3. MCHP and the hierarchical CAT-SD method

In some real-world problems, criteria are not all at the same level but they can be structured in a hierarchical way as shown, for example, in Fig. 1. It is therefore possible to consider a root criterion $g_{\mathbf{0}}$, some macro-criteria descending from the root criterion and so on until the last level where the elementary criteria are placed.

The MCHP has been recently introduced in literature to deal with problems in which actions are evaluated on criteria structured in a hierarchical way (Corrente et al., 2012). The application of the MCHP permits to decompose the problem in small sub-problems giving to the DM the possibility to focus on a particular aspect of the problem at hand. In this way, the DM can provide information at partial level, that is considering a single criterion in the hierarchy and, at the same time, the DM can get information on the comparisons between alternatives taking into account

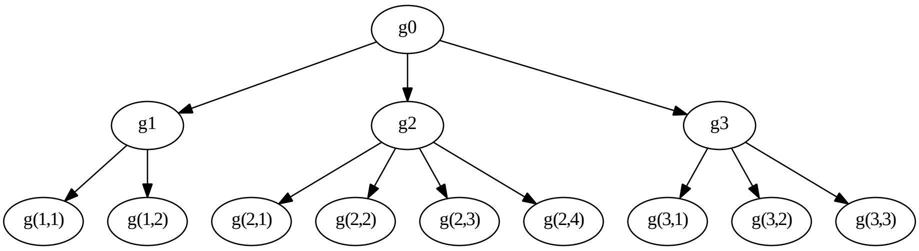

Figure 1: An example of criteria structured in a hierarchical way

the node on which he is more interested. In this section, we shall detail the extension of the CAT-SD method to the hierarchical case. Therefore, the MCHP and the CAT-SD method will be put together within a unified framework giving arise to the hCAT-SD method. To this aim, regarding the MCHP, we shall use the following notation:

- $\mathcal{G}$ is the set composed of all criteria in the hierarchy;
- $\mathcal{I}_{\mathcal{G}}$ is the set of the indices of criteria in $\mathcal{G}$;
- $EL \subseteq \mathcal{I}_{\mathcal{G}}$ is the set of the indices of elementary criteria;
- $g_{\mathbf{r}}$, with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, is a generic criterion in the hierarchy and it will be called *non-elementary criterion*;
- Given a non-elementary criterion $g_{\mathbf{r}}$, $E(g_r) \subseteq EL$ is the set of the indices of the elementary criteria descending from $g_{\mathbf{r}}$.

Given a non-elementary criterion $g_{\mathbf{r}}$, to perform the classification of the actions on $g_{\mathbf{r}}$, a *partial similarity function* $s^h_{\mathbf{r}}(a,b)$ can be defined for each $(a,b) \in A \times A$ through $f^s_{\mathbf{r}} : [0,1]^{|E(g_{\mathbf{r}})|} \times [-1,0]^{|E(g_{\mathbf{r}})|} \rightarrow [0,1]$, with $E(g_{\mathbf{r}}) = \{\mathbf{t}_1, \ldots, \mathbf{t}_r\}$, as follows:

$$s^h_{\mathbf{r}}(a,b) = f^s_{\mathbf{r}}\left(s_{\mathbf{t}_1}(a,b), \ldots, s_{\mathbf{t}_r}(a,b), d_{\mathbf{t}_1}(a,b), \ldots, d_{\mathbf{t}_r}(a,b), k(C_h)\right) =$$

$$= \frac{1}{K^h_{\mathbf{r}}(a,b)} \left(\sum_{\mathbf{t} \in E(g_{\mathbf{r}})} k^h_{\mathbf{t}} s_{\mathbf{t}}(a,b) + \sum_{\substack{\{\mathbf{t}_1,\mathbf{t}_2\} \in M^h: \\ \mathbf{t}_1,\mathbf{t}_2 \in E(g_{\mathbf{r}})}} s_{\mathbf{t}_1}(a,b) s_{\mathbf{t}_2}(a,b) k^h_{\mathbf{t}_1\mathbf{t}_2} + \sum_{\substack{(\mathbf{t}_1,\mathbf{t}_2) \in O^h: \\ \mathbf{t}_1,\mathbf{t}_2 \in E(g_{\mathbf{r}})}} s_{\mathbf{t}_1}(a,b) |d_{\mathbf{t}_2}(a,b)| k^h_{\mathbf{t}_1|\mathbf{t}_2} \right) \tag{7}$$

and

$$K^h_{\mathbf{r}}(a,b) = \sum_{\mathbf{t} \in E(g_{\mathbf{r}})} k^h_{\mathbf{t}} + \sum_{\substack{\{\mathbf{t}_1,\mathbf{t}_2\} \in M^h: \\ \mathbf{t}_1,\mathbf{t}_2 \in E(g_{\mathbf{r}})}} s_{\mathbf{t}_1}(a,b) s_{\mathbf{t}_2}(a,b) k^h_{\mathbf{t}_1\mathbf{t}_2} + \sum_{\substack{(\mathbf{t}_1,\mathbf{t}_2) \in O^h: \\ \mathbf{t}_1,\mathbf{t}_2 \in E(g_{\mathbf{r}})}} s_{\mathbf{t}_1}(a,b) |d_{\mathbf{t}_2}(a,b)| k^h_{\mathbf{t}_1|\mathbf{t}_2}.$$

In this way, the partial similarity function $s^h_{\mathbf{r}}(a,b)$ computes the similarity between the actions a and b taking into account the elementary criteria descending from $g_{\mathbf{r}}$ only.

As already done for the partial similarity function, the *partial dissimilarity function* $d_{\mathbf{r}}(a,b)$ can be defined for each non-elementary criterion $g_{\mathbf{r}}$ and for each $(a,b) \in A \times A$ through $f^d_{\mathbf{r}} : [-1,0]^{|E(g_r)|} \rightarrow [-1,0]$ as follows:

$$d_{\mathbf{r}}(a,b) = f_{\mathbf{r}}^{d}\left(d_{\mathbf{t}_1}(a,b),\ldots,d_{\mathbf{t}_r}(a,b)\right) = \prod_{\mathbf{t}\in E(g_{\mathbf{r}})} (1+d_{\mathbf{t}}(a,b)) - 1. \tag{8}$$

On the basis of the partial similarity and dissimilarity functions defined in eqs. (7) and (8), for each non-elementary criterion $g_{\mathbf{r}}$ a *partial likeness function* $\delta_{\mathbf{r}}(a,b)$ can be defined for each $(a,b) \in A \times A$ through $f_{\mathbf{r}} : [0,1] \times [-1,0] \rightarrow [0,1]$ as follows (also called likeness degree):

$$\delta_{\mathbf{r}}(a,b) = f_{\mathbf{r}}\left(s_{\mathbf{r}}^{h}(a,b), d_{\mathbf{r}}(a,b)\right) = s_{\mathbf{r}}^{h}(a,b)(1+d_{\mathbf{r}}(a,b)). \tag{9}$$

In order to assign the actions to the different categories on the non-elementary criterion $g_{\mathbf{r}}$, these have to be compared with the reference actions belonging to the reference set of the considered categories. Therefore, on the basis of eq. (9), the *partial likeness degree between action a and the reference set B_h* on $g_{\mathbf{r}}$ can be defined:

$$\delta_{\mathbf{r}}(a,B_h) = \max_{l=1,\ldots,|B_h|}\{\delta_{\mathbf{r}}(a,b_{hl})\}. \tag{10}$$

As a consequence, we say that a is alike to B_h on $g_{\mathbf{r}}$, and we write $aS_{\mathbf{r}}(\lambda_{\mathbf{r}}^{h})B_h$, iff $\delta_{\mathbf{r}}(a,B_h) \geqslant \lambda_{\mathbf{r}}^{h}$, where $\lambda_{\mathbf{r}}^{h} \in [0.5,1]$ is the likeness threshold. Pay attention to the fact that $\lambda_{\mathbf{r}}^{h}$ can be dependent on criterion $g_{\mathbf{r}}$ we are considering.

The *partial classification* of $a \in A$ on $g_{\mathbf{r}}$ is therefore performed following these steps:

i) Compare a with the set B_h on criterion $g_{\mathbf{r}}$, $h = 1,\ldots,q$,

ii) Identify $U_{\mathbf{r}} = \{u : aS_{\mathbf{r}}(\lambda_{\mathbf{r}}^{u})B_u\}$,

iii) Assign a to the category C_u for all $u \in U_{\mathbf{r}}$,

iv) If $U_{\mathbf{r}} = \emptyset$, assign a to C_{q+1}, being a fictitious category collecting all non-assigned actions.

The added value of the application of MCHP to the Cat-SD method is that one can get the classifications of the actions not only at comprehensive level, therefore considering simultaneously all criteria, but also at a partial level by considering a particular aspect of the problem only. In this way, the DM can have a deeper knowledge of the decision making problem he is dealing with.

4. The hierarchical and imprecise SRF method

As described in the previous section, the classification procedure used in the hCat-SD method is based on the knowledge of the weights of elementary criteria $g_{\mathbf{t}}$ ($k_{\mathbf{t}}$), the knowledge of the values representing the mutual-strengthening and mutual-weakening effects between elementary criteria $g_{\mathbf{t}_1}, g_{\mathbf{t}_2}$ ($k_{\mathbf{t}_1\mathbf{t}_2}$), and the knowledge of the values representing the antagonistic effect exercised from elementary criterion $g_{\mathbf{t}_2}$ over elementary criterion $g_{\mathbf{t}_1}$ $\left(k_{\mathbf{t}_1|\mathbf{t}_2}\right)$. Anyway, asking the DM to provide all these parameters is unreasonable for their huge number as well as for the cognitive burden related to the complexity of their meaning. Therefore, the application of an indirect technique is preferable in this case.

To get the weights of criteria involved in the decision problem at hand, in Figueira and Roy (2002) the SRF method was proposed. The procedure, known as deck of cards method, extended the proposal of Simos (Simos, 1990a,b) by permitting the DM to introduce the value z representing the ratio between the weight of the most important and the weight of the least important criteria. A further extension of the SRF method was recently introduced in Corrente et al. (2017), permitting the DM to provide imprecise information regarding both the number of cards that should be included between two successive subsets of criteria and the z-value introduced in the SRF method. The method was also applied to hierarchical structures of criteria. In the following, we shall briefly recall the main steps involved in the application of the SRF method to the set $\{g_{(\mathbf{r},1)},\ldots,g_{(\mathbf{r},n(\mathbf{r}))}\}$ composed of the immediate sub-criteria of the non-elementary criterion $g_{\mathbf{r}}$:

1. Rank the criteria from the least important $L_1^{\mathbf{r}}$, to the most important $L_v^{\mathbf{r}}$, where $v \leqslant n(\mathbf{r})$, with the possibility of some ex-aequo;
2. Define an interval $[low_s^{\mathbf{r}}, upp_s^{\mathbf{r}}]$ in which $e_s^{\mathbf{r}}$ can vary, where $e_s^{\mathbf{r}}$ is the number of blank cards to be included between $L_s^{\mathbf{r}}$ and $L_{s+1}^{\mathbf{r}}$, with $s=1,\ldots,v-1$. The greater the number of blank cards between $L_s^{\mathbf{r}}$ and $L_{s+1}^{\mathbf{r}}$, the more important are criteria in $L_{s+1}^{\mathbf{r}}$ with respect to criteria in $L_s^{\mathbf{r}}$;
3. Define an interval $\left[z_{low}^{\mathbf{r}}, z_{upp}^{\mathbf{r}}\right]$ in which $z^{\mathbf{r}}$ can vary, where $z^{\mathbf{r}}$ is the ratio between weights of criteria in $L_v^{\mathbf{r}}$ and criteria in $L_1^{\mathbf{r}}$.

Denoting by $K_{L_s^{\mathbf{r}}}$ the weight of a criterion in $L_s^{\mathbf{r}}$, with $s=1,\ldots,v$, and by $C_{\mathbf{r}}$ the importance of a blank card included between two successive subsets of criteria, the previous preference information is translated into the following set of linear constraints (see Corrente et al., 2017, for more details):

$$E_{\mathbf{r}}\left\{\begin{array}{l}\left.\begin{array}{l}K_{L_{s+1}^{\mathbf{r}}} \geqslant K_{L_s^{\mathbf{r}}} + (low_s^{\mathbf{r}}+1)\cdot C_{\mathbf{r}},\\ K_{L_{s+1}^{\mathbf{r}}} \leqslant K_{L_s^{\mathbf{r}}} + (upp_s^{\mathbf{r}}+1)\cdot C_{\mathbf{r}},\\ C_{\mathbf{r}} > 0,\end{array}\right\}\text{for all } s=1,\ldots,v-1,\\ z_{low}^{\mathbf{r}}\cdot K_{L_1^{\mathbf{r}}} - K_{L_v^{\mathbf{r}}} \leqslant 0,\\ K_{L_v^{\mathbf{r}}} - z_{upp}^{\mathbf{r}}\cdot K_{L_1^{\mathbf{r}}} \leqslant 0,\\ K_{L_1^{\mathbf{r}}} > 0.\end{array}\right.$$

Let us observe that constraints in $E_{\mathbf{r}}$ can be expressed in terms of the weights of elementary criteria assuming that, for each non-elementary criterion $g_{\mathbf{r}}$, $K_{\mathbf{r}} = \sum_{\mathbf{t}\in E(g_{\mathbf{r}})} k_{\mathbf{t}}$. Moreover, for each $s=1,\ldots,v$ and for each $g_{(\mathbf{r},j)} \in L_s^{\mathbf{r}}$, $K_{(\mathbf{r},j)} = K_{L_s^{\mathbf{r}}}$.

Concerning the parameters $k_{\mathbf{t}_1\mathbf{t}_2}$ and $k_{\mathbf{t}_1|\mathbf{t}_2}$, with $\mathbf{t}_1,\mathbf{t}_2 \in EL$, the following constraints translate the preferences of the DM:

$$E_{int}\left\{\begin{array}{l}k_{\mathbf{t}_1\mathbf{t}_2} > 0 \ \text{ if } g_{\mathbf{t}_1} \text{ and } g_{\mathbf{t}_2} \text{ present a mutual-strengthening effect},\\ k_{\mathbf{t}_1\mathbf{t}_2} < 0 \ \text{ if } g_{\mathbf{t}_1} \text{ and } g_{\mathbf{t}_2} \text{ present a mutual-weakening effect},\\ k_{\mathbf{t}_1|\mathbf{t}_2} < 0 \ \text{ if } g_{\mathbf{t}_2} \text{ presents an antagonistic effect over } g_{\mathbf{t}_1}.\end{array}\right.$$

The following technical constraints have also to be satisfied:

$$(E_{Norm})\ \sum_{\mathbf{t}\in EL} k_{\mathbf{t}} + \sum_{\{\mathbf{t}_1,\mathbf{t}_2\}\subseteq EL} k_{\mathbf{t}_1\mathbf{t}_2} = 100,$$

$$(E_{Net})\ k_{\mathbf{t}_1} - \sum_{\substack{\{\mathbf{t}_1,\mathbf{t}_2\}\subseteq EL:\\ k_{\mathbf{t}_1\mathbf{t}_2}<0}} |k_{\mathbf{t}_1\mathbf{t}_2}| - \sum_{\mathbf{t}_3\in EL} |k_{\mathbf{t}_1|\mathbf{t}_3}| \geqslant 0 \text{ for all } \mathbf{t}_1 \in EL.$$

Let us observe that E_{Norm} is a technical constraint used only to put an upper bound on the coefficients. This will be useful in the sampling procedure that we will describe in Section 5. Anyway, if one uses the direct technique, that is the DM provides directly the values of the coefficients involved in the computations, then this constraint can be neglected. The space of the parameters involved in the hierarchical and imprecise SRF method is therefore defined by the constraints in the set:

$$E = \cup_{\mathbf{r}\in\mathcal{I}_{\mathcal{G}}\setminus EL} E_{\mathbf{r}} \cup E_{int} \cup E_{Norm} \cup E_{Net}.$$

To check if there exists at least one set of parameters compatible with the preferences provided by the DM, one has to solve the following LP problem:

$$\varepsilon^* = \max \varepsilon, \text{ subject to } E^{'} \tag{11}$$

where $E^{'}$ is obtained by E converting the strict inequality constraints in weak ones by using an auxiliary variable ε. For example, constraint $C_{\mathbf{r}} > 0$ is converted into $C_{\mathbf{r}} \geqslant \varepsilon$, while $k_{\mathbf{t}_1\mathbf{t}_2} < 0$ is converted into $k_{\mathbf{t}_1\mathbf{t}_2} \leqslant -\varepsilon$. If $E^{'}$ is feasible and $\varepsilon^* > 0$, then the space of parameters is not empty while, in the opposite case, the set of constraints $E^{'}$ is infeasible and the cause of the infeasibility can be checked by using one of the methods proposed by Mousseau et al. (2003b).

Let us observe that the hierarchical and imprecise SRF method involves the application of the imprecise SRF method to each node of the hierarchy. For example, if one deals with a hierarchical structure of criteria such that one shown in Fig. 1, the imprecise SRF method has to be applied at first to the set of criteria $\{g_{\mathbf{1}}, g_{\mathbf{2}}, g_{\mathbf{3}}\}$, and then to the three sets of elementary criteria $\{g_{(\mathbf{1},1)}, g_{(\mathbf{1},2)}\}$, $\{g_{(\mathbf{2},1)}, g_{(\mathbf{2},2)}, g_{(\mathbf{2},3)}, g_{(\mathbf{2},4)}\}$ and $\{g_{(\mathbf{3},1)}, g_{(\mathbf{3},2)}, g_{(\mathbf{3},3)}\}$.

4.1. Eliciting interaction and antagonistic coefficients with the SRF method

In Corrente et al. (2017) only the sign of the interaction coefficients and the presence of antagonistic coefficients were considered and coded with constraints in E_{int}. Instead it is possible to get more precise preference information from the DM by considering additional cards referred to pairs of criteria for which there is an interaction or an antagonistic effect. More precisely:

- in case of mutual-strengthening [mutual-weakening] effect between criteria g_i and g_j, a card will be associated to the the pair of criteria $\{g_i, g_j\}$ and the value $K(\{g_i, g_j\})$ assigned to that card will represent the importance of the two criteria together so that we have $K(\{g_i, g_j\}) = k_i + k_j + k_{ij}$ where $k_{ij} > 0$ [$k_{ij} < 0$] is a parameter used to represent the mutual-strengthening [mutual-weakening] effect between the two criteria at hand;
- in case of an antagonistic effect exercised by g_j over g_i, two cards will be associated to g_i, and they will be denoted by k_i and $k'_{i|j}$. The first (k_i) denotes the importance of g_i when the antagonistic effect is not taken into account. The second ($k'_{i|j}$) denotes, instead, the importance of g_i when g_j exercises the antagonistic effect over it and, consequently, $k'_{i|j} = k_i + k_{i|j}$ where $k_{i|j} < 0$ is a parameter representing the magnitude of the antagonistic effect.

In this way, applying the imprecise SRF method with the addition of these cards, the DM can provide more precise information not only regarding the type of interactions but also on its magnitude expressed by the eventual presence of blank cards between successive subsets of criteria.

In the following didactic example we shall show how the new procedure works. Suppose that there are four criteria: g_1, g_2, g_3 and g_4. Assume that there is:

- a mutual-strengthening effect between g_3 and g_4;
- a mutual-weakening effect between g_2 and g_4;
- an antagonistic effect exercised by g_3 over g_4.

To apply the SRF method, the DM is therefore provided with:

- a card for each criterion g_1, g_2, g_3 and g_4;
- a card for the pairs $\{g_3, g_4\}$ and $\{g_2, g_4\}$ of interacting criteria;

- a card representing criterion g_4 when g_3 exercises an antagonistic effect over it;
- a certain number of blank cards that can be used to represent the difference of importance between criteria, pairs of criteria or the criterion g_4 subject to the antagonistic effect exercised by g_3 over it.

Suppose the DM provides the following order of importance with respect to the criteria g_1, g_2, g_3 and g_4, the pairs of criteria $\{g_3, g_4\}$ and $\{g_2, g_4\}$ and the criterion g_4 when g_3 exercises an antagonistic effect over it which is denoted by $g'_{4|3}$ ($g_i \prec g_j$ means that "g_j is strictly more important than g_i"):

$$g_3 \prec g_1 \prec g'_{4|3} \prec g_4 \prec g_2 \prec \{g_3, g_4\} \prec \{g_2, g_4\}.$$

The DM added the number of blank cards among parenthesis to increase the difference of importance between successive subsets of criteria or pairs of criteria:

$$g_3 \ \ [1] \ \ g_1 \ \ [2] \ \ g'_{4|3} \ \ [0] \ \ g_4 \ \ [2] \ \ g_2 \ \ [0] \ \ \{g_3, g_4\} \ \ [2] \ \ \{g_2, g_4\}.$$

Let us note that not placing blank cards between two consecutive criteria or pairs of criteria does not mean that they have the same importance, but only that their difference is minimal. The number of units between g_3 and $\{g_2, g_4\}$ is $(1+1)+(2+1)+(0+1)+(2+1)+(0+1)+(2+1) = 13$. The DM declares that the pair of criteria $\{g_2, g_4\}$ is 20 times more important than g_3, that is, $z = 20$, so that, giving value 1 to g_3 and value 20 to $\{g_2, g_4\}$, we get the value of the unit (a single card): $u = \frac{20-1}{13} = 1.4615$. Consequently, considering the number of units separating two consecutive criteria, pairs of criteria and criterion g_4 under antagonistic effect, their importance is the following:

$$v(g_3) = 1, \ v(g_1) = 3.9231, \ v(g'_{4|3}) = 8.3077, \ v(g_4) = 9.7693, \ v(g_2) = 14.1539,$$

$$v(\{g_3, g_4\}) = 15.6154, \ v(\{g_2, g_4\}) = 20.$$

After normalization (E_{Norm}), we get

$$k_3 = 3.3592, \ k_1 = 13.1783, \ k'_{4|3} = 27.9070, \ k_4 = 32.8165, \ k_2 = 47.5452,$$

$$K(\{g_3, g_4\}) = 52.4548, \ K(\{g_2, g_4\}) = 67.1835$$

from which we obtain that:

- the mutual-strengthening coefficient of criteria g_3 and g_4 is $k_{34} = K(\{g_3, g_4\}) - k_3 - k_4 = 16.2791$,
- the mutual-weakening coefficient of criteria g_2 and g_4 is $k_{24} = K(\{g_2, g_4\}) - k_2 - k_4 = -13.1782$,
- the antagonistic coefficient of criterion g_3 over criterion g_4 is $k_{4|3} = k'_{4|3} - k_4 = -4.9096$.

Let us observe that constraints E_{Net} are satisfied, that is, $k_2 + k_{24} = 34.367 \geqslant 0$ and $k_4 + k_{24} + k_{4|3} = 29.4574 \geqslant 0$.

After the positive result of this last control, the weights $k_i, i = 1, 2, 3, 4$, the interaction coefficients k_{24} and k_{34}, and the antagonistic coefficient $k_{4|3}$ can be adopted and applied in a Cat-SD procedure, as well as in any Electre, or even in general, outranking method considering interaction and antagonistic effects between criteria. In Section 7 the new proposal further extended by coupling it with the imprecise SRF method is applied to the considered case study.

What to do if the required interaction between criteria stated by the DM is not obtained as the result of the application of the SRF method or if constraints in E_{Net} are not satisfied? In other words, in applying the SRF method as explained above, it is possible that $k_{ij} > 0$ even if the DM declared that there is a mutual-weakening effect between g_i and g_j or, conversely, that $k_{ij} < 0$ even if the DM declared that there is a mutual-strengthening effect between the same criteria. How to proceed? In this case the preference information provided by the DM applying the SRF method and stating the desired signs and types of interactions between criteria is not coherent with the adopted model. This is a situation that can be encountered in any MCDA elicitation procedure. For example, in UTA (Jacquet-Lagreze and Siskos, 1982), one of the most adopted methods to construct a value function on the basis of a certain number of preference pairwise comparison of reference alternatives, the value function is built by minimizing the sum of the errors, so that it is accepted that some piece of preference information cannot be exactly represented by the model. Of course, the preference comparisons not represented by the model have to be discussed with the DM, who has to eventually accept that the recommendation provided by the decision model can present some approximation and imprecision. In the same perspective, also for the modification of the deck of cards method we are proposing, one can use a procedure that looks for some minimal "modifications" of the preference information in terms of number of blank cards or z value, such that the required interaction between criteria and the constraints E_{Net} are satisfied. Let us first introduce the necessary notation:

- $MS = \{\{g_{j_1}, g_{j_2}\} \subseteq G$ for which there is mutual-strengthening effect between g_{j_1} and $g_{j_2}\}$,
- $MW = \{\{g_{j_1}, g_{j_2}\} \subseteq G$ for which there is mutual-weakening effect between g_{j_1} and $g_{j_2}\}$,
- $AN = \{(g_{j_1}, g_{j_2}), g_{j_1}, g_{j_2} \in G,$ for which g_{j_2} exercises an antagonistic effect over $g_{j_1}\}$,
- $L_1, \ldots, L_q \subseteq G \cup MW \cup AN$, $L_s \cap L_{s'} = \emptyset$ for all $s, s' = 1, \ldots, q$, $\bigcup_{s=1}^{r} L_s = G \cup MW \cup AN$ (that is, $\{L_1, \ldots, L_q\}$ is a partition of $G \cup MW \cup AN$) are the levels of criteria, pairs of interacting criteria or pairs of antagonistic criteria according to the order given by the DM (that is L_1 contains the criteria, pairs of interacting criteria and pairs of antagonistic criteria of the smallest level, L_2 contains the criteria, pairs of interacting criteria and pairs of antagonistic criteria of the second level, and so on),
- $e_r, r = 1, \ldots, q-1$, is the number of blank cards that the DM puts between the level L_r and the level L_{r+1},
- K_r and k_r, $r = 1, \ldots, q$, are the values given to the elements in L_r, before and after the normalization is applied, respectively,
- $\overline{K}_{j_1}$ and $\overline{k}_{j_1}$ are the non-normalized and normalized values assigned to criterion $g_{j_1} \in G$ when $g_{j_1} \in L_r$, that is, $\overline{K}_{j_1} = K_r$ and $\overline{k}_{j_1} = k_r$,
- $\overline{K}_{j_1 j_2}$ and $\overline{k}_{j_1 j_2}$ are the non-normalized and normalized values assigned to the pair of criteria $\{g_{j_1}, g_{j_2}\} \in MS \cup MW$ when $\{g_{j_1}, g_{j_2}\} \in L_r$, that is, $\overline{K}_{j_1 j_2} = K_r$ and $\overline{k}_{j_1 j_2} = k_r$,
- $\overline{K}_{j_1|j_2}$ and $\overline{k}_{j_1|j_2}$ are the non-normalized and normalized values assigned to the ordered pair of criteria $(g_{j_1}, g_{j_2}) \in AN$ when $(g_{j_1}, g_{j_2}) \in L_r$, that is, $\overline{K}_{j_1|j_2} = K_r$ and $\overline{k}_{j_1|j_2} = k_r$,
- z is the number of times that the value assigned to the elements in the highest level L_q is greater than the value assigned to the elements in the lowest level L_1, that is, $z = \frac{K_q}{K_1} = \frac{k_q}{k_1}$,

- $c = \dfrac{z-1}{\sum_{p=1}^{q-1}(e_p+1)}$ is the value of each card for the computation of the non-normalized weights $K_r, r = 1, \ldots, q$, that is:

  - $K_{r+1} = K_r + e_r \cdot c$, $r = 1, \ldots, q-1$, with $K_1 = 1$, or, equivalently,
  - $K_r = 1 + \sum_{p=1}^{r}(e_p + 1) \cdot c$,

- $e'_r, r = 1, \ldots, q-1$, denotes the corrected number of blank cards between levels L_r and L_{r+1} so that the required interaction and the constraints in E_{Net} are satisfied; analogously, K'_r, k'_r, $\overline{K}'_{j_1}$, $\overline{k}'_{j_1}$, $\overline{K}'_{j_1j_2}$, $\overline{k}'_{j_1j_2}$, z', c' denote the corrected values of K_r, k_r, $\overline{K}_{j_1}$, $\overline{k}_{j_1}$, $\overline{K}_{j_1j_2}$, $\overline{k}_{j_1j_2}$, z, c, respectively,

- δ is a positive value representing the maximal deviation of z' with respect to the original z value, that is $|z' - z| \leqslant \delta$,

- $\varepsilon_r^+, \varepsilon_r^-, r = 1, \ldots, q-1$, represent the number of cards to be added to or to be subtracted from e_r to obtain e'_r, that is, $e'_r = e_r + \varepsilon_r^+ - \varepsilon_r^-$; observe that $\varepsilon_r^- \leqslant e_r \;\; r = 1, \ldots, q-1$ so that $\overline{e}'_r \geqslant 0$,

- ε^* is a maximal threshold for $\varepsilon_r^+, \varepsilon_r^-$, $r = 1, \ldots, q-1$, that is, $\varepsilon_r^+ \leqslant \varepsilon^*$, and $\varepsilon_r^- \leqslant \varepsilon^*$, $r = 1, \ldots, q-1$,

- γ is a minimal positive threshold permitting to the constraints translating a certain interaction to be satisfied. For example, the mutual-strengthening effect between criteria g_{j_1} and g_{j_2}, is satisfied if
$$\overline{K}_{j_1j_2} - \overline{K}_{j_1} - \overline{K}_{j_2} \geqslant \gamma.$$

Let us observe that the normalized values $\overline{k}_{j_1}$, $\overline{k}_{j_1j_2}$, and $\overline{k}_{j_1|j_2}$, are obtained from the corresponding non-normalized values $\overline{K}_{j_1}, \overline{K}_{j_1j_2}, \overline{K}_{j_1|j_2}$ as follows

$$D = \sum_{g_j \in G} \overline{K}_j + \sum_{\{g_{j_1}, g_{j_2}\} \in MS \cup MW} \left(\overline{K}_{j_1j_2} - \overline{K}_{j_1} - \overline{K}_{j_2}\right),$$

$$\overline{k}_{j_1} = \frac{\overline{K}_{j_1}}{D}, \qquad \overline{k}_{j_1j_2} = \frac{\overline{K}_{j_1j_2}}{D}, \qquad \overline{K}_{j_1|j_2} = \frac{\overline{K}_{j_1|j_2}}{D}.$$

If the weights obtained as the result of the application of the deck of cards method with the blank cards and the z value specified by the DM cannot be accepted (some interactions or some antagonistic effects are not reflected by the obtained values as well as some net balance constraint is not satisfied), the minimal modifications in the number of blank cards and in the z value permitting to restore the exact information and satisfying all technical constraints can be obtained by solving the following optimization problem (P1):

$$min \sum_{r=1}^{q} \left(\varepsilon_r^+ + \varepsilon_r^-\right)$$

subject to

$$\begin{cases} e'_r = e_r + \varepsilon^+_r - \varepsilon^-_r, \quad r = 1,\ldots,q-1 \\ \varepsilon^+_r \leqslant \varepsilon^*, \quad \varepsilon^-_r \leqslant \varepsilon^*, \quad r = 1,\ldots,q-1 \\ \varepsilon^-_r \leqslant e_r \quad r = 1,\ldots,q-1 \\ z' - z \leqslant \delta \\ z - z' \leqslant \delta \\ K'_1 = 1 \\ c' = \dfrac{z'-1}{\sum\limits_{r=1}^{q-1}(e'_r + 1)} \\ K'_r = K'_{r-1} + (e'_r + 1)\cdot c', \quad r = 2,\ldots,q, \\ \overline{K}'_{j_1j_2} - \overline{K}'_{j_1} - \overline{K}'_{j_2} \geqslant \gamma, \ \text{ for all } \ \{g_{j_1}, g_{j_2}\} \in MS \\ \overline{K}'_{j_1j_2} - \overline{K}'_{j_1} - \overline{K}'_{j_2} \leqslant -\gamma, \ \text{ for all } \ \{g_{j_1}, g_{j_2}\} \in MW \\ m_{j_1j_2} = \overline{K}'_{j_1j_2} - \overline{K}'_{j_1} - \overline{K}'_{j_2} \ \text{ for all } \ \{g_{j_1}, g_{j_2}\} \in MW \\ m_{j_1|j_2} = \overline{K}'_{j_1} - \overline{K}'_{j_1|j_2} \ \text{ for all } \ (g_{j_1}, g_{j_2}) \in AN \\ \overline{K}'_{j_1} + \sum\limits_{\{g_{j_1},g_{j_2}\}\in MW} m_{j_1j_2} + \sum\limits_{(g_{j_1},g_{j_2})\in AN} m_{j_1|j_2} \geqslant 0, \ \text{ for all } g_{j_1} \in G \\ \text{(that is, the net balance constraints are satisfied)} \\ \varepsilon^+_r, \varepsilon^-_r \in \mathbb{N}_0. \end{cases}$$

Let us observe that the above optimization problem (P1) has integer variables and it is not linear. However, it can be solved using some heuristic approach such as the evolutionary algorithm supplied by the commercial solver Microsoft Excel, that we used in the didactic example shown in the e-appendix.

The above procedure can also be extended to the case in which the DM considers intervals for the number of blank cards to be added as well as for the value of z.

Let us again introduce the necessary notation:

- e^l_r and e^u_r, $r = 1,\ldots,q-1$, are the lower and upper bounds of the number of blank cards that the DM put between the level L_r and the level L_{r+1}, so that, if $e^l_r = e^u_r$ there is a precise number of blank cards between levels L_r and L_{r+1},
- z^l and z^u are the lower and upper bounds of the number of times that the value assigned to the elements in the highest level L_q is greater than the value assigned to the elements in the lowest level L_1, that is, $z^l \leqslant \frac{K_q}{K_1} = \frac{k_q}{k_1} \leqslant z^u$,
- $\overline{e}^l_r$ and $\overline{e}^u_r$, $r = 1,\ldots,q-1$, denotes the lower and the upper value for the number of corrected blank cards between L_r and L_{r+1}, so that the required interactions and the constraints in E_{Net} are satisfied,
- $\varepsilon^{l-}_r, r = 1,\ldots,q-1$ represents the number of cards to be subtracted from e^l_r to obtain $\overline{e}^l_r$; observe that $\varepsilon^{l-}_r \leqslant e^l_r, r = 1,\ldots,q-1$ so that $\overline{e}^l_r \geqslant 0$,
- $\varepsilon^{u+}_r, r = 1,\ldots,q-1$ represents the number of cards to be added to e^u_r to obtain $\overline{e}^u_r$,
- ε^* is a maximal threshold for ε^{l-}_r and ε^{u+}_r, $r = 1,\ldots,q-1$, that is, $\varepsilon^{l-}_r \leqslant \varepsilon^*$, and $\varepsilon^{u+}_r \leqslant \varepsilon^*$, $r = 1,\ldots,q-1$,

- $\overline{z}^l$ and $\overline{z}^u$, $\overline{z}^l \leqslant z^l$ and $\overline{z}^u \geqslant z^u$, represent the modified values of z^l and z^u, respectively, permitting that the interactions and the constraints in E_{Net} hold,
- δ is a positive value representing the maximal deviation of $\overline{z}^l$ and $\overline{z}^u$ with respect to the original values of z^l and z^u, that is $z^l - \overline{z}^l \leqslant \delta$ and $\overline{z}^u - z^u \leqslant \delta$,
- $(e_1, \ldots, e_{r-1}, z), e_r^l \leqslant e_r \leqslant e_r^u, r = 1, \ldots, r-1, z^l \leqslant z \leqslant z^u$, is a feasible configuration of blank cards and z-value,
- K_r and k_r, $r = 1, \ldots, q$, are the values given to the elements in L_r according to the feasible configuration $(e_1, \ldots, e_{r-1}, z)$, before and after the normalization is applied, respectively,
- $\overline{K}_{j_1}$, $\overline{k}_{j_1}$, $\overline{K}_{j_1,j_2}$, $\overline{k}_{j_1,j_2}$, $\overline{K}_{j_1|j_2}$, $\overline{k}_{j_1|j_2}$ and γ maintain their meaning.

To verify if there exists at least one feasible configuration of blank cards and z value $(e_1, \ldots, e_{q-1}, z), e_r^l \leqslant e_r \leqslant e_r^u, r = 1, \ldots, q-1, z^l \leqslant z \leqslant z^u$, compatible with the required interactions and the constraints in E_{Net}, the following optimization problem (P2) has to be solved

$$min \sum_{r=1}^{q} \left(\varepsilon_r^{u+} + \varepsilon_r^{l-} \right)$$

subject to

$$\begin{cases}
\overline{e}_r^l = e_r^l - \varepsilon_r^{l-}, \quad r = 1, \ldots, q-1 \\
\varepsilon_r^{l-} \leqslant e_r^l, \quad r = 1, \ldots, q-1 \\
\overline{e}_r^u = e_r^l + \varepsilon_r^{u+}, \quad r = 1, \ldots, q-1 \\
\varepsilon_r^{u+} \leqslant \varepsilon^*, \quad \varepsilon_r^{l-} \leqslant \varepsilon^*, \quad r = 1, \ldots, q \\
\overline{e}_r^l \leqslant e_r \leqslant \overline{e}_r^u, \quad r = 1, \ldots, q-1 \\
\overline{z}^u - z^u \leqslant \delta \\
z^l - \overline{z}^l \leqslant \delta \\
\overline{z}^l \leqslant z \leqslant \overline{z}^u \\
K_1 = 1 \\
c = \frac{z-1}{\sum\limits_{r=1}^{q-1} (e_r + 1)} \\
K_r = K_{r-1} + (e_r + 1) \cdot c, \quad r = 2, \ldots, q, \\
\overline{K}_{j_1 j_2} - \overline{K}_{j_1} - \overline{K}_{j_2} \geqslant \gamma, \quad \text{for all } \{g_{j_1}, g_{j_2}\} \in MS \\
\overline{K}'_{j_1 j_2} - \overline{K}_{j_1} - \overline{K}_{j_2} \leqslant -\gamma, \quad \text{for all } \{g_{j_1}, g_{j_2}\} \in MW \\
m_{j_1 j_2} = \overline{K}_{j_1 j_2} - \overline{K}_{j_1} - \overline{K}_{j_2} \quad \text{for all } \{g_{j_1}, g_{j_2}\} \in MW \\
m_{j_1|j_2} = \overline{K}_{j_1} - \overline{K}_{j_1|j_2} \quad \text{for all } (g_{j_1}, g_{j_2}) \in AN \\
\overline{K}'_{j_1} + \sum\limits_{\{g_{j_1}, g_{j_2}\} \in MW} m_{j_1 j_2} + \sum\limits_{(g_{j_1}, g_{j_2}) \in AN} m_{j_1|j_2} \geqslant 0, \text{ for all } g_{j_1} \in G \\
\text{(that is, the net balance constraints are satisfied)} \\
\varepsilon_r^+, \varepsilon_r^- \in \mathbb{N}_0.
\end{cases}$$

If the above optimization problem is solved with $min \sum_{r=1}^{q} \left(\varepsilon_r^{u+} + \varepsilon_r^{l-} \right) = 0$, then there exists at least one feasible configuration of blank cards and z value $(e_1, \ldots, e_{q-1}, z), e_r^l \leqslant e_r \leqslant e_r^u, r = 1, \ldots, q-1, z^l \leqslant z \leqslant z^u$, compatible with the required interactions and the constraints in E_{Net}.

Therefore the sampling procedure can be applied by maintaining the configuration of blank cards for which required interactions and constraints in E_{Net} hold. If the above optimization problem is solved with $min \sum_{r=1}^{q}\left(\varepsilon_r^{u+}+\varepsilon_r^{l-}\right)>0$, then the modified lower and upper bounds of blank cards $\overline{e}_r^l$ and $\overline{e}_r^u, r=1,\ldots,q-1$, as well as the modified lower and upper bounds $\overline{z}^l$ and $\overline{z}^u$ of z value have to be discussed with the DM, that can accept or not. In case the DM accepts, one can proceed as in the previous case. When the DM does not accept the new bounds $\overline{e}_r^l$ and $\overline{e}_r^u, r=1,\ldots,q-1$, and $\overline{z}^l$ and $\overline{z}^u$, the preference information has to be rediscussed with the DM that has to supply a configuration of cards and z-value, and a requirement on interaction between criteria being compatible between them. This is also the case, in which the above optimization problem does not admit solutions. An example of the proposed procedure is provided in the e-appendix.

5. SMAA and the SMAA-hCAT-SD method

As already stated in the previous section, the set of constraints E defines the space of vectors of parameters compatible with the preferences provided by the DM. Anyway, in general, if there exists one vector of parameters compatible with the preferences of the DM, then there exists more than one. Therefore, using only one of them could be considered arbitrary or meaningless. To avoid this choice, in this paper we shall apply the SMAA (see Lahdelma and Salminen 2010; Pelissari et al. 2019, for two surveys on SMAA; some recent extensions of the SMAA method have been presented in Arcidiacono et al. 2018; Corrente et al. 2017, 2019) which provides robust recommendations on the problem at hand taking into account all compatible vectors of parameters. In this section, we describe the application of SMAA to the hCat-SD method building, therefore, the SMAA-hCat-SD method. It starts from the sampling of several vectors of compatible parameters. Since the constraints in E define a convex space of parameters, one can use the Hit-And-Run (HAR) method to sample them (Smith, 1984; Tervonen et al., 2013; Van Valkenhoef et al., 2014). Of course, for each sampled vector of parameters, a classification of the actions at hand on the considered macro-criteria can be performed. Denoting by $\mathcal{K}$ the space of the vectors of parameters compatible with the preferences provided by the DM, for each $k\in\mathcal{K}$, $a\in A$, $g_{\mathbf{r}}$ and C_h, writing $a \underset{k,\mathbf{r}}{\longrightarrow} C_h$ we mean that alternative a is assigned to class C_h on criterion $g_{\mathbf{r}}$, considering the parameters in k. One can therefore define the set $\mathcal{K}_{\mathbf{r}}^h(a)\subseteq\mathcal{K}$ composed of the vectors of compatible parameters for which a is assigned to C_h on $g_{\mathbf{r}}$, that is $\mathcal{K}_{\mathbf{r}}^h(a)=\left\{k\in\mathcal{K}: a \underset{k,\mathbf{r}}{\longrightarrow} C_h\right\}$. As observed in Section 3, each action could be assigned to more than one category. Consequently, for each $\mathcal{C}\subseteq\{C_1,\ldots,C_q\}$, we can define also the set $\mathcal{K}_{\mathbf{r}}^{\mathcal{C}}(a)=\left\{k\in\mathcal{K}:\ \forall C_h\in\mathcal{C},\ a \underset{k,\mathbf{r}}{\longrightarrow} C_h\right\}$ composed of the vectors of compatible parameters for which a is assigned simultaneously to all categories in $\mathcal{C}$.

SMAA applied to the hCat-SD method permits therefore to calculate the approximate estimation of the probability with which an action is assigned to a single category (or a set of categories) on criterion $g_{\mathbf{r}}$, that is, $b_{\mathbf{r}}^h(a)=\frac{|\mathcal{K}_{\mathbf{r}}^h(a)|}{|\mathcal{K}|}$ and $b_{\mathbf{r}}^{\mathcal{C}}(a)=\frac{|\mathcal{K}_{\mathbf{r}}^{\mathcal{C}}(a)|}{|\mathcal{K}|}$.

In this way, it is possible to analyze not only the probability of the assignments when all elementary criteria are taken into account, but also when a particular macro-criterion is considered.

6. Additional requirements for the assignments

The two new aspects of the approach we are proposing with respect to the basic model presented in Costa et al. (2018) are the probabilistic nature of the classification and the hierarchy of criteria. Let us discuss their implications and their advantages. The idea of probabilistic classification has gained a great success in the domain of data mining and ML (see, for example, Taskar et al. 2001;

Williams and Barber 1998). The probabilistic aspect of the classification we are considering regards the imprecision related to the weights representing the importance of criteria, but, of course, other types of imprecision could be considered, such as values of other parameters of the model, as the likeness thresholds or the shape of the per-criterion similarity $s_j(a,b)$ through the function $f_j(\Delta_j(a,b))$.

The robustness concerns are taken into account through a probabilistic classification that gives, for each action, the probability to be assigned to a given category with respect to all non-elementary criteria in the hierarchy. The probability we compute expresses, the share of compatible vectors of parameters for which a given action is assigned to some categories with respect to some non-elementary criteria. SMAA has therefore the advantage of stating in a clear way that, on the basis of the preference information provided by the DM, one or several classifications are possible and, in this second case, how much one is more probable than the others. However, in general, for fulfilling his objectives, a DM needs one deterministic nominal classification. Consequently, there is a need to pass from the probabilistic classification to the deterministic classification in the most reasonable way and, in any case, taking into account the probabilistic classification supplied by SMAA.

In this perspective, let us remember some methodologies using the probabilistic results of SMAA to obtain precise non probabilistic recommendations in MCDA problems. Leskinen et al. (2006) suggests some procedures that supply a precise ranking on the basis of the probability $p(a,b)$ that an alternative a is preferred to another alternative b. Three examples of such rules are:

- the Copeland score that gives to each alternative a a value equal to the number of alternatives b such that $p(a,b) > 0.5$ minus the number of alternatives c such that $p(a,c) < 0.5$,
- the mean score that gives to each alternative a a value equal to the mean of $p(a,b)$ for all $b \neq a$,
- the Simpson rule that gives to each alternative a a value equal to the $\min_{b\neq a} p(a,b)$.

A similar approach is considered in Kadziński and Michalski (2016) which proposes different scoring procedures to summarize the probabilistic information got by the SMAA application. With the analogous objective of obtaining a precise ranking from the probability values $p(a,b)$ as well as from the probability $p(a,k)$ that an alternative $a \in A$ attains a given ranking position k, Vetschera (2017) proposes a different approach in which a function representing the quality of the obtained rank is optimized. More precisely, it is suggested to maximize the average probability of the assigned ranks or the sum of probability values $p(a,b)$ such that a precedes b in the considered ranking.

We propose here a procedure permitting to obtain a precise nominal classification on the basis of the probabilistic nominal classification $b^h_{\mathbf{r}}(a)$. Our proposal is similar to the approach of Vetschera (2017), however:

- instead of maximizing an objective function representing the quality of the proposed recommendation, we minimize an objective function representing the expected value of the sum of the possible regrets of assigning an action to class,
- instead of considering a ranking problem, we consider a nominal classification problem. Regarding this aspect we can say that, to the best of our knowledge, our procedure is the first one proposed in the literature which provides a deterministic nominal classification from probabilistic information.

More precisely, the procedure we are proposing aims to provide a deterministic nominal classification that:

1. Minimizes the error of misclassification taking into account the probabilistic information given by the application of the SMAA methodology;
2. Fulfills some prespecified requirements related to the cardinality of the considered categories (Mousseau et al., 2003a; Kadziński et al., 2015; Kadziński and Słowiński, 2013; Özpeynirci et al., 2018; Stal-Le Cardinal et al., 2011), such as:

   R1) At least s_h alternatives should be assigned to each category C_h, $h = 1, \ldots, q$;

   R2) At most s'_h alternatives should be assigned to each category C_h, $h = 1, \ldots, q$;

   R3) At most s'_{q+1} alternatives should be assigned to the category C_{q+1}, etc.

   Even if the introduced requirements could be considered "ad hoc", it is important to note that the successful application of any decision aiding procedure depends on the appropriate customization of the adopted formal model to the concrete decision problem at hand, so that many important points of the formal procedure depends on the context and must be ad hoc with respect to the specific problem. In general, we have to observe that the idea that all concepts in any discipline are in some form ad hoc is gaining more and more consensus (see, e.g., Casasanto and Lupyan 2015).

With respect to Point 1. above, a deterministic nominal classification will be obtained, for each non-elementary criterion $g_{\mathbf{r}}$, minimizing the following loss function (Gneiting and Raftery, 2007; Savage, 1971; Schervish, 1989)

$$L(\mathbf{y_r}) = \sum_{a\in A} \sum_{h=1}^{q+1} y^h_{a,\mathbf{r}} \sum_{k\neq h} b^k_{\mathbf{r}}(a) \tag{12}$$

where, for each $g_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, $\mathbf{y_r} = [\ y^h_{a,\mathbf{r}},\ \ a \in A,\ h = 1, ..., q+1]$ and $y^h_{a,\mathbf{r}} = 1$ if action a is assigned to category C_h with respect to $g_{\mathbf{r}}$, while $y^h_{a,\mathbf{r}} = 0$ otherwise.

Let us observe that, if a "true" classification could be observed ex-post (for example in case of a pathology classification) and for this "true" classification $b^k_r(a)$, $k = 1, \ldots, q+1$, would represent an ex-ante probability distribution, for each $a \in A$ and for each $h = 1, \ldots, q+1$, the quantity $\sum\limits_{k\neq h} b^k_r(a)$ in eq. (12) represents the probability of error made in assigning a to C_h w.r.t. $g_{\mathbf{r}}$ considering the probabilistic information given by the SMAA methodology. For example, considering a non-elementary criterion $g_{\mathbf{r}}$, let us assume that a could be assigned to only one between C_1, C_2, C_3 and C_4 with frequencies 10%, 20% and 65% and 5% respectively, where C_4 is the dummy category receiving actions not assigned to the other categories. Then, it is obvious that the probability of error made in assigning a to the considered categories is 90% (20% + 65% + 5%), 80% (10% + 65% + 5%), 35% (10% + 20% + 5%) and 95% (10% + 20% + 65%), respectively. Therefore, taking into account only a, the minimum value of $L(\mathbf{y_r})$ will be obtained when $y^1_{a,\mathbf{r}} = y^2_{a,\mathbf{r}} = y^4_{a,\mathbf{r}} = 0$ and $y^3_{a,\mathbf{r}} = 1$.
When a "true" classification does not exist (for instance, in the illustrative example we present in section 7, where some individuals have to be assigned to categories representing the specific job they could do) the above loss function maintains its interest and can be interpreted in terms of a measure of the overall uncertainty of the considered classification. Let us observe that, in general, the term error is used when the true classification is known and one would like to measure how far the obtained classification is from the true one. In this context, we have not a true classification and, consequently, the optimal value of the loss function, that we called error, can be considered as a measure of the overall uncertainty on the fact that the classification is the optimal one.

If the nominal classification is exclusive in the sense that for each vector of preference parameters each action a is assigned to a single category $C_h, h = 1, \ldots, q+1$, we would have that $\sum_{h=1}^{q+1} b_{\mathbf{r}}^h(a) = 1$. In this case, $\sum_{k \neq h} b_{\mathbf{r}}^k(a) = 1 - b_{\mathbf{r}}^h(a)$, so that equation (12) could be rewritten as

$$L(\mathbf{y_r}) = \sum_{a \in A} \sum_{h=1}^{q+1} y_{a,\mathbf{r}}^h (1 - b_{\mathbf{r}}^h(a)). \tag{13}$$

Moreover, since each action a has to be finally assigned to a single class, possibly the dummy one C_{q+1}, for each $a \in A$ we have $\sum_{h=1}^{q+1} y_{a,\mathbf{r}}^h = 1$, so that only $|A|$ of the $y_{a,\mathbf{r}}^h$ have a value of 1 and (13) becomes

$$L'(\mathbf{y_r}) = |A| - \sum_{a \in A} \sum_{h=1}^{q+1} y_{a,\mathbf{r}}^h b_{\mathbf{r}}^h(a). \tag{14}$$

Minimization of this loss function corresponds to the maximization of $G(\mathbf{y_r}) = \sum_{a \in A} \sum_{h=1}^{q+1} b_{\mathbf{r}}^h(a)$. In this perspective, one could imagine to maximize the conjoint probability of all assignments supposing that they are independent, that is, maximizing $\prod_{\substack{a \in A,\\ h=1,\ldots,q+1}} b_{\mathbf{r}}^h(a)$. This is equivalent to maximize $G'(\mathbf{y_r}) = \sum_{a \in A} \sum_{h=1}^{q+1} \log(b_{\mathbf{r}}^h(a))$ corresponding to the approach proposed by Vetschera (2017) for ranking problems. Observe that the nominal classification of Cat-SD we are considering is not exclusive, and, consequently the loss function $L(\mathbf{y_r})$ is not equivalent to the loss function $L'(\mathbf{y_r})$. However, in any case, maximization of $G(\mathbf{y_r})$ or $G'(\mathbf{y_r})$ can be considered as possible alternative approaches to the minimization of $L(\mathbf{y_r})$.

With respect to point 2. above, the considered requirements will be translated into linear constraints that should be respected while minimizing $L(\mathbf{y_r})$. For example, assuming that requirements **R1)**, **R2)** and **R3)** hold for each non-elementary criterion $g_{\mathbf{r}}$, they are translated into the constraints

C1) $\sum_{a \in A} y_{a,\mathbf{r}}^h \geqslant s_h$ for all $h = 1, \ldots, q$;

C2) $\sum_{a \in A} y_{a,\mathbf{r}}^h \leqslant s'_h$ for all $h = 1, \ldots, q$;

C3) $\sum_{a \in A} y_{a,\mathbf{r}}^{q+1} \leqslant s'_{q+1}$.

Moreover, to impose that each alternative is assigned exactly to one category (including the dummy one), the constraint $\sum_{h=1}^{q+1} y_{a,\mathbf{r}}^h = 1$ should be added for each alternative $a \in A$.

Let us observe that more than one deterministic nominal classification can restore the same value of the loss function $L(\mathbf{y_r})$. Denoting by $\mathbf{y}_{\mathbf{r}}^*$ the binary vector obtained as a solution of the minimization of eq. (12) and by $z_{\mathbf{r}}^*$ the number of 1s in $\mathbf{y}_{\mathbf{r}}^*$, one can check for the existence of another deterministic nominal classification respecting the provided requirements and having the same value $L(\mathbf{y}_{\mathbf{r}}^*)$ by minimizing eq. (12) subject to the constraints translating the considered requirements with the addition of the following ones:

$$L(\mathbf{y}_{\mathbf{r}}) = L(\mathbf{y}_{\mathbf{r}}^*),$$
$$\sum_{y_{a,\mathbf{r}}^h \in \mathbf{y}_{\mathbf{r}}^* :\, y_{a,\mathbf{r}}^h = 1} y_{a,\mathbf{r}}^h \leqslant z_{\mathbf{r}}^* - 1.$$

The first constraint is used to avoid a deterioration of the optimal value of the loss function previously found, while the second one avoids to obtain, again, the deterministic nominal classification previously obtained. If the LP problem is feasible, then another nominal classification is obtained, otherwise, the previously found is unique. By proceeding in an iterative way, it is therefore possible to obtain all the deterministic nominal classifications minimizing the misclassification error and respecting all the considered requirements.

Let us conclude this section observing that without any additional requirements provided by the DM, the nominal classification got minimizing the loss function in eq. (12) is obtained in a straightforward way considering the class assignment probabilities of SMAA. Indeed, for all $a \in A$, the solution of the programming problem will give back $y_{a,\mathbf{r}}^h = 1$ iff $b_{\mathbf{r}}^h = \max_k b_{\mathbf{r}}^k$, that is, the class to which the alternative a has been more frequently assigned taking into account all the compatible models.

7. Illustrative example

In this section, we shall apply the SMAA-hCat-SD method presented in the previous sections extending the numerical example presented in Costa et al. (2018).

Seven soldiers $(a_1, \ldots, a_7)$ have to be assigned to five categories $(C_1, \ldots, C_5)$: snipers (C_1), breachers (C_2), communications operators (C_3), heavy weapons operators (C_4), and non-assigned candidates (C_5). Their evaluation is performed considering several criteria structured in a hierarchical way as shown in Fig. 2.

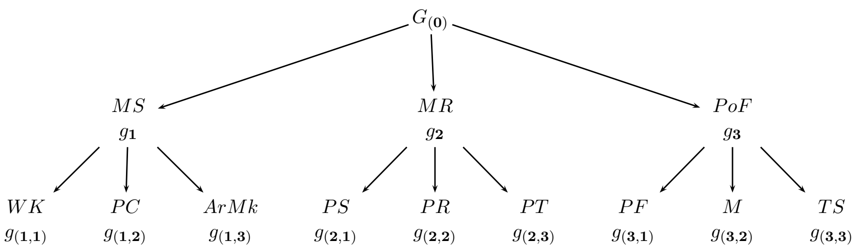

Figure 2: Hierarchical structure of criteria considered in the case study

The hierarchy of criteria is composed of three macro-criteria that are Mental Sharpness (MS), Mental Resilience (MR) and Physical and other Features (PoF). Each of these macro-criteria has three elementary criteria descending from them. In particular, World Knowledge (WK), Paragraph Comprehension (PC) and Arithmetic reasoning and Mathematics knowledge ($ArMk$) descend from MS; Performance Strategies (PS), Psychological Resilience (PR) and Personality Traits (PT) descend from MR; finally, Physical Fitness (PF), Motivation (M) and Teamwork Skills (TS) are sub-criteria of PoF. The description of the nine considered elementary criteria is given in Table 1.

The performance of the seven soldiers on the nine elementary criteria is given in Table 2.

Each reference set B_h is composed of one reference action only. Their performances are provided in Table 3.

Table 1: Description of the elementary criteria

Macro-criterion	Elementary criterion	Elementary criterion description
MS	WK	Identification of word synonyms and right definition of words in a given context
	PC	Identification of the meaning of texts
	$ArMk$	Solving arithmetic problems and knowledge of mathematics principles (algebra and geometry)
MR	PS	Goal setting, self-talk, and emotional control
	PR	Acceptance of life situations, and ability for dealing with cognitive challenges and threats
	PT	Character traits such as adaptability, dutifulness, social orientation, self-reliance, stress tolerance, vigilance, and impulsivity
PoF	PF	Physical ability with respect to aerobic fitness and strength
	M	Self motivation, persistence, and dedication
	TS	Communication skills and camaraderie

Table 2: Performance of the considered soldiers on the elementary criteria at hand

Soldier	$g_{(\mathbf{1},1)}$	$g_{(\mathbf{1},2)}$	$g_{(\mathbf{1},3)}$	$g_{(\mathbf{2},1)}$	$g_{(\mathbf{2},2)}$	$g_{(\mathbf{2},3)}$	$g_{(\mathbf{3},1)}$	$g_{(\mathbf{3},2)}$	$g_{(\mathbf{3},3)}$
a_1	75	75	90	3	4	4	740	6	4
a_2	67	80	73	3	3	3	760	5	6
a_3	60	70	70	4	3	3	770	5	6
a_4	80	90	75	2	3	3	880	4	5
a_5	65	65	70	3	2	3	870	6	6
a_6	70	75	85	4	3	4	750	5	4
a_7	75	70	70	4	3	3	710	5	6
Function	f_2	f_2	f_2	f_3	f_3	f_3	f_1	f_3	f_3

Table 3: Performance of the reference soldiers on the elementary criteria at hand

Reference set	**Reference action**	$g_{(\mathbf{1},1)}$	$g_{(\mathbf{1},2)}$	$g_{(\mathbf{1},3)}$	$g_{(\mathbf{2},1)}$	$g_{(\mathbf{2},2)}$	$g_{(\mathbf{2},3)}$	$g_{(\mathbf{3},1)}$	$g_{(\mathbf{3},2)}$	$g_{(\mathbf{3},3)}$
B_1	b_{11}	80	75	85	4	4	4	700	6	4
B_2	b_{21}	70	70	75	3	3	3	800	6	6
B_3	b_{31}	80	90	85	2	2	3	950	4	4
B_4	b_{41}	60	65	65	3	3	3	700	5	6

The interested reader is deferred to the e-appendix for the formulation and representation of the per-criterion similarity-dissimilarity functions f_1, f_2 and f_3.

To get the weights of the elementary criteria and the interaction coefficients, the hierarchical and imprecise SRF method has been applied for each category. Anyway, since the DM provided some information regarding interactions and antagonistic effects between few elementary criteria, we had to adapt the imprecise and hierarchical SRF method as described in the following lines.

Let us suppose that the DM provided the following information:

1. There is a mutual-strengthening effect between $ArMk$ and PR;
2. There is a mutual-weakening effect between PF and TS;
3. There is an antagonistic effect exercised by PF over PS.

Each of the previous three pieces of preference information implies a small modification in the application of the imprecise SRF method:

- In consequence of the first piece of preference information, a mutual-strengthening effect between MS and MR exists too. Therefore, in applying the SRF method at the first level, that is the level composed of criteria $\{MS, MR, PoF\}$, the DM is provided with an additional card with the name of the two criteria MS and MR on, to consider their importance together. Then, the SRF method will be applied to the set composed now of 4 cards $\{MS, MR, \{MS, MR\}, PoF\}$. From a technical point of view, in addition to the weights

K_{MS}, K_{MR}, and K_{PoF} representing the importance of criteria MS, MR and PoF, respectively, we shall take into account also the weight $K(\{MS, MR\})$. In consequence of the mutual-strengthening effect between $ArMk$ and PR, we have that

$$K(\{MS, MR\}) = K_{MS} + K_{MR} + k_{ArMk,PR},$$

where $k_{ArMk,PR} > 0$ represents, indeed, the value of this effect. Of course, $K(\{MS, MR\}) > K_{MS}$ and $K(\{MS, MR\}) > K_{MR}$;

- Since elementary criteria PF and TS descend from the same macro-criterion PoF, the mutual-weakening effect between them is considered adding another card for the pair $\{PF, TS\}$ to take into account their importance together. The imprecise SRF method will be therefore applied to the set $\{PF, M, TS, \{PF, TS\}\}$. The weight of the pair of criteria $\{PF, TS\}$, that is $K(\{PF, TS\})$, will be such that

$$K(\{PF, TS\}) = k_{PF} + k_{TS} + k_{PF,TS}$$

where, $k_{PF,TS} < 0$ is a parameter representing the mutual-weakening effect between them; of course, in consequence of the net flow condition (1), $K(PF, TS) > k_{PF}$ and $K(PF, TS) > k_{TS}$;

- Finally, in consequence of the antagonistic effect exercised by PF over PS, the original weight of PS will be reduced. The DM is therefore asked to apply the SRF method to the subset of criteria $\{PS, PS^{'}, PR, PT\}$, where $K(PS^{'})$ is the importance of criterion PS when PF is exercising its antagonistic effect over it. Consequently, we have

$$K(PS^{'}) = k_{PS} + k_{PS|PF}$$

where $k_{PS|PF} < 0$ represents the magnitude of the antagonistic effect. In this way, if the DM, for example, in applying the SRF method will order $PS^{'}$ after PR, then this means that PS is more important than PR even if there is another criterion (PF) opposing to it.

As a consequence of the description above, the hierarchical and imprecise SRF method is therefore applied to the sets of criteria $\{MS, MR, \{MS, MR\}, PoF\}$, $\{PF, M, TS, \{PF, TS\}\}$, $\{PS, PS^{'}, PR, PT\}$ and $\{WK, PC, ArMK\}$ for each of the four categories. Table 4 summarizes the preference information provided by the DM. For example, considering the category C_1, the DM states that PoF is less important than MR that is less important than MS that, in turn, is less important than $\{MS, MR\}$. The number of blank cards to be inserted between $\{MS, MR\}$ and MS belongs to the interval $[2, 3]$; the number of blank cards between MS and MR varies in the interval $[1, 2]$, while there is one blank card between MR and PoF. Finally, the ratio between the weight of $\{MS, MR\}$ and the weight of PoF belongs to the interval $[4, 6]$. Other information contained in the Table can be analogously explained.

Introducing all the constraints translating the preference information provided by the DM, we solved the LP problem (11) obtaining $\varepsilon^* > 0$. Therefore, there exists at least one set of parameters compatible with the preferences provided by the DM and, consequently, we applied the HAR method to sample 100,000 sets of compatible parameters for each of the four categories.[2]
Considering the likeness thresholds for each category shown in Table 5, and assuming that they are the same for each non-elementary criterion $g_{\mathbf{r}}$, we applied the hCat-SD method for each sampled vector of compatible parameters. Therefore, we were able to compute the probability of assigning each soldier to the considered categories reported in Table 6.

[2]The interested reader is deferred to the e-appendix for more details on the steps performed to get the considered assignments.

Table 4: Data used in the hierarchical and imprecise SRF

	C_1			C_2			C_3			C_4		
Rank	Criterion	No. blank cards	z	Criterion	No. blank cards	z	Criterion	No. blank cards	z	Criterion	No. blank cards	z
1	$\{MR, MS\}$	$[2, 3]$	$[4, 6]$	$\{MS, MR\}$	$[2, 3]$	6	PoF	1	$[4, 6]$	$\{MS, MR\}$	$[1, 2]$	9
2	MS	$[1, 2]$		MR	1		$\{MS, MR\}$	$[2, 3]$		MR	1	
3	MR	1		MS	$[1, 2]$		MR	$[2, 3]$		MS	$[1, 2]$	
4	PoF			PoF			MS			PoF		
1	$ArMk$	$[0, 2]$	3	$ArMk$	0	$[3, 5]$	WK, $ArMk$	1	$[2, 4]$	$ArMk$	$[1, 2]$	4
2	PC	$[0, 1]$		PC	$[1, 2]$		PC			PC, WK		
3	WK			WK								
1	PT	1	$[2, 3]$	PT	$[0, 1]$	$[3, 4]$	PS, PT	1	$[2, 4]$	PT	$[0, 1]$	$[2, 4]$
2	PR	$[1, 2]$		PR	0		PR	$[0, 1]$		PR	0	
3	PS	$[0, 1]$		PS	$[0, 1]$		PS'			PS	$[0, 1]$	
4	PS'			PS'						PS'		
1	$\{PF, TS\}$	$[1, 2]$	$[4, 6]$	$\{PF, TS\}$	1	$[3, 6]$	$\{PF, TS\}$	$[0, 1]$	$[3, 4]$	$\{PF, TS\}$	$[1, 3]$	$[3, 5]$
2	TS	$[1, 2]$		TS	$[2, 3]$		PF, M	$[1, 2]$		M, TS	$[1, 2]$	
3	PF	$[1, 2]$		M	$[1, 2]$		TS			PF		
3	M			PF								

Table 5: Likeness threshold for the four categories

	$h = 1$	$h = 2$	$h = 3$	$h = 4$
$\lambda_{\mathbf{r}}^h$	0.65	0.60	0.65	0.60

Table 6: Probability of assignments expressed in percentage

(a) Comprehensive level

Soldier	C_1	C_2	C_3	C_4	$\{C_2, C_4\}$	C_5
a_1	100	0	0	0	0	0
a_2	0	0	0	0	100	0
a_3	0	0	0	0	100	0
a_4	0	0	100	0	0	0
a_5	0	100	0	0	0	0
a_6	100	0	0	0	0	0
a_7	0	0	0	0	100	0

(b) Mental Sharpness (MS)

Soldier	C_1	C_2	C_3	C_4	$\{C_1, C_3\}$	$\{C_2, C_4\}$	C_5
a_1	0	0	0	0	100	0	0
a_2	0	100	0	0	0	0	0
a_3	0	0	0	0	0	100	0
a_4	0	0	0	0	0	0	100
a_5	0	0	0	0	0	100	0
a_6	100	0	0	0	0	0	0
a_7	0	0	0	0	0	100	0

(c) Mental Resilience (MR)

Soldier	C_1	C_2	C_3	C_4	$\{C_2, C_4\}$	$\{C_2, C_3, C_4\}$	C_5
a_1	100	0	0	0	0	0	0
a_2	0	0	0	0	100	0	0
a_3	0	0	0	0	100	0	0
a_4	0	0	0	0	0	100	0
a_5	0	0	0	0	100	0	0
a_6	100	0	0	0	0	0	0
a_7	0	0	0	0	100	0	0

(d) Physical and other Features (PoF)

Soldier	C_1	C_2	C_3	C_4	$\{C_2, C_4\}$	C_5
a_1	100	0	0	0	0	0
a_2	0	0	0	0	100	0
a_3	0	0	0	0	100	0
a_4	0	0	97.269	0	0	2.731
a_5	0	100	0	0	0	0
a_6	100	0	0	0	0	0
a_7	0	0	0	100	0	0

Looking at Tables 6(a)-6(d) one can observe that the results are quite stable, that is, the frequency of assignment is very close to 100% in almost all cases. This is due to the fact that the preference information provided by the DM was quite precise and, consequently, the space of vector of parameters compatible with this information was quite narrow. However, one can observe the following:

– At comprehensive level (Table 6(a)), all candidates are assigned to at least one category. In particular, a_1 and a_6 are surely suitable to be snipers (C_1), a_5 can be assigned with certainty to the breachers (C_2), a_3 is surely suitable to be a communication operator (C_3), while the other three candidates, that is a_2, a_3 and a_7, can be included among breachers or heavy weapons operators ($\{C_2, C_4\}$);

– With respect to MS, only two candidates can be assigned with certainty to a unique category. In particular, a_2 is always assigned to breachers (C_2) and a_6 is always assigned to snipers (C_1); regarding the remaining candidates, a_1 can cover both snipers and communications operators ($\{C_1, C_3\}$), a_3, a_5 and a_7 can be included in breachers and heavy weapons operator simultaneously ($\{C_2, C_4\}$); finally, a_4 is not idoneous to any of the considered categories;

- On MR, all candidates are assigned with certainty to at least one category. a_1 and a_6 are idoneous to be included in the snipers category (C_1); a_4 has evaluations such that he can be included in all categories apart from snipers one ($\{C_2, C_3, C_4\}$); finally, all the other candidates (a_2, a_3, a_5 and a_7) can be breachers or heavy weapons operators ($\{C_2, C_4\}$);

- Considering PoF, there is a better distribution of the candidates among the different categories: a_1 and a_6 are assigned with certainty to the snipers (C_1); a_5 is surely assigned to the breachers (C_2); a_4 is included among the communication operators (C_3) with a frequency of the 97.269%, while he is not assigned to any category in the remaining cases; a_7 is certainly idoneous to be included in the heavy weapons operators category (C_4). The remaining two candidates, that is a_2 and a_3, can be assigned to the breachers and heavy weapons operators categories ($\{C_2, C_4\}$).

To conclude this section, we shall show how the classification procedure described in Section 6 can be applied to this problem to get a deterministic nominal classification taking into account the results obtained by using the SMAA methodology and the following additional requirements that are specified by the DM for each non-elementary criterion $g_{\mathbf{r}}$:

R1) At least one soldier should be assigned to each C_h, $h = 1, \ldots, 4$;

R2) At most two soldiers should be assigned to each C_h, $h = 1, \ldots, 4$;

R3) At most two soldiers should not be assigned (at most two soldiers should be assigned to the dummy category C_5).

Taking into account the SMAA results given in tables 6(a)-6(d), one deterministic nominal classification can be obtained for each non-elementary criterion. Anyway, in the following we shall explain in detail how to get the deterministic nominal classification at comprehensive level, that is considering $g_{\mathbf{0}}$.

Looking at Table 6(a) we observe that a_2, a_3 and a_7 can be always simultaneously assigned to categories C_2 and C_4. Therefore, since we would like to consider a nominal classification assigning soldiers to one among $C_1 - C_4$ or to the dummy category C_5 we rewrite the table 6(a) as shown in Table 7.

Table 7: Frequencies of assignments at comprehensive level

Soldier	C_1	C_2	C_3	C_4	C_5
a_1	100	0	0	0	0
a_2	0	100	0	100	0
a_3	0	100	0	100	0
a_4	0	0	100	0	0
a_5	0	100	0	0	0
a_6	100	0	0	0	0
a_7	0	100	0	100	0

Considering that, in this case, $A = \{a_1, \ldots, a_7\}$, a deterministic nominal classification taking into account the probabilistic information given by the SMAA methodology and the requirements provided by the DM, one has to solve the following optimization problem where all variables are binary and constraints $[C1] - [C3]$ translate the requirements provided by the DM:

$$
\left.\begin{array}{ll}
\min L(\mathbf{y_0}) = \sum_{a\in A}\sum_{h=1}^{5} y_{a,\mathbf{0}}^h \sum_{k\neq h} b_{\mathbf{0}}^k(a), \text{ subject to} & \\
\text{for each } h=1,\ldots,4, \ \sum_{a\in A} y_{a,\mathbf{0}}^h \geqslant 1 & [C1] \\
\text{for each } h=1,\ldots,4, \ \sum_{a\in A} y_{a,\mathbf{0}}^h \leqslant 2 & [C2] \\
\sum_{a\in A} y_{a,\mathbf{0}}^5 \leqslant 2 & [C3] \\
\text{for each } a\in A, \ \sum_{h=1}^{5} y_{a,\mathbf{0}}^h = 1 & \\
y_{a,\mathbf{0}}^h \in \{0,1\}, \ \forall a\in A, \ \forall h=1,\ldots,5. &
\end{array}\right\} E^{LF} \tag{15}
$$

Solving the problem (15), we get $y_{1,\mathbf{0}}^{1,*} = y_{2,\mathbf{0}}^{4,*} = y_{3,\mathbf{0}}^{2,*} = y_{4,\mathbf{0}}^{3,*} = y_{5,\mathbf{0}}^{2,*} = y_{6,\mathbf{0}}^{1,*} = y_{7,\mathbf{0}}^{4,*} = 1$, while all the other binary variables are equal to zero. This means that the deterministic nominal classification shown in the first column of Table 8 is therefore obtained.

Table 8: Deterministic nominal classifications obtained at comprehensive level

Soldier	$1st$	$2nd$	$3rd$
a_1	C_1	C_1	C_1
a_2	C_4	C_2	C_4
a_3	C_2	C_4	C_4
a_4	C_3	C_3	C_3
a_5	C_2	C_2	C_2
a_6	C_1	C_1	C_1
a_7	C_4	C_4	C_2

To check for the existence of another deterministic nominal classification, considering that the optimal value of the loss function previously found is $L_{\mathbf{y_0^*}} = 300$, one has to solve the same problem (15) with the addition of the constraints

$$
\begin{array}{ll}
L(\mathbf{y_0}) = 300 & [C1'] \\
y_{1,\mathbf{0}}^1 + y_{2,\mathbf{0}}^4 + y_{3,\mathbf{0}}^2 + y_{4,\mathbf{0}}^3 + y_{5,\mathbf{0}}^2 + y_{6,\mathbf{0}}^1 + y_{7,\mathbf{0}}^4 \leqslant 6 & [C2']
\end{array}
$$

where $[C1']$ imposes that the optimal value of the loss function should not be deteriorated, while $[C2']$ ensures that the previous solution of the problem is not found anymore. Proceeding in this way, one gets $y_{1,\mathbf{0}}^{1,*} = y_{2,\mathbf{0}}^{2,*} = y_{3,\mathbf{0}}^{4,*} = y_{4,\mathbf{0}}^{3,*} = y_{5,\mathbf{0}}^{2,*} = y_{6,\mathbf{0}}^{1,*} = y_{7,\mathbf{0}}^{4,*} = 1$ that provides the deterministic nominal classification shown in the second column of Table 8. Analogously, we find only another deterministic nominal classification summarizing the results obtained by the application of the SMAA methodology and compatible with the requirements provided by the DM that is shown in the last column of Table 8.

A similar procedure can be used to obtain the deterministic nominal classifications w.r.t. each of the three macro-criteria. We will not give the detail of the computations in these cases but the obtained classifications are shown in Tables 9(a)-9(c).

Table 9: Deterministic nominal classification at partial level

(a) Mental Sharpness (MS)

Soldier	$1st$	$2nd$	$3rd$
a_1	C_3	C_3	C_3
a_2	C_2	C_2	C_2
a_3	C_2	C_4	C_4
a_4	C_5	C_5	C_5
a_5	C_4	C_4	C_2
a_6	C_1	C_1	C_1
a_7	C_4	C_2	C_4

(b) Mental Resilience (MR)

Soldier	$1st$	$2nd$	$3rd$	$4th$	$5th$	$6th$
a_1	C_1	C_1	C_1	C_1	C_1	C_1
a_2	C_2	C_4	C_2	C_4	C_4	C_2
a_3	C_2	C_2	C_4	C_4	C_2	C_4
a_4	C_3	C_3	C_3	C_3	C_3	C_3
a_5	C_4	C_4	C_4	C_2	C_2	C_2
a_6	C_1	C_1	C_1	C_1	C_1	C_1
a_7	C_4	C_2	C_2	C_2	C_4	C_4

(c) Physical and other Features (PoF)

Soldier	$1st$	$2nd$
a_1	C_1	C_1
a_2	C_4	C_2
a_3	C_2	C_4
a_4	C_3	C_3
a_5	C_2	C_2
a_6	C_1	C_1
a_7	C_4	C_4

8. Conclusions

In this paper, we proposed a comprehensive method extending a recently proposed nominal classification method, namely, the Cat-SD method. Firstly, we applied MCHP to the Cat-SD method. Thus, we have introduced the hierarchical Cat-SD, hCat-SD. The hierarchical decomposition of a complex multiple criteria nominal classification problem is then possible when applying Cat-SD. Secondly, interactions and antagonistic effects between criteria structured in a hierarchical way were handled in our method. Then, to elicit the values of the criteria weights as well as the interactions and antagonistic coefficients used in the hCat-SD method, we proposed a new development of the hierarchical and imprecise SRF method. We applied SMAA to the hCat-SD method with the aim of obtaining the probablity with which an action is assigned to a category (or categories) at a comprehensive level and at a macro-criterion level. Finally, considering the concept of loss function, we proposed a procedure that starting from the probabilistic assignments obtained by SMAA provides a final classification that fulfills some requirements given by the DM. Putting together all these aspects, we therefore built the SMAA-hCat-SD method. We presented a numerical example to illustrate the application of SMAA-hCat-SD.

The proposed method gives to the DM the possibility:

- To structure the set of criteria in a hierarchical way (logical subsets of criteria can be created in the hierarchy);
- To provide imprecise information for obtaining the criteria weights as well as the interaction and antagonistic coefficients by using the imprecise SRF method;
- To analyze, for several sets of compatible parameters, the probability of the assignment results provided by the Cat-SD, considering all criteria or one macro-criterion only;
- To obtain a final assignment that takes into account robustness concerns as represented by the probabilistic classification provided by SMAA.

Several advantages can be underlined with respect to the application of the proposed method. The main features can be stated as follows:

1. In situations in which the DM has to handle a great number of criteria to assess actions, adopting hCat-SD is a more adequate approach than applying Cat-SD considering all criteria at the same level;
2. For the elicitation of the criteria weights and interaction and antagonistic coefficients, it is easier for the DM thinking about a small number of related criteria than a large number;

3. Besides the possibility of eliciting criteria weights for subsets of criteria, our method gives to the DM the possibility to provide imprecise information during the process of determining them;
4. Applying SMAA to the hCat-SD, the DM can better understand the decision problem at hand exploring it more in deep.

To sum up, in this work we have considered robustness concerns by taking into account the set of all vectors of weights and interaction and antagonistic coefficients compatible with the preference information provided by the DM, while taking advantage of the hierarchical structure of criteria. Let us remark that:

– The extension of the SRF method to elicit weights of criteria as well as interaction and antagonistic coefficients can be applied to all Electre methods and, in general, to all outranking methods;

– The procedure permitting to pass from the probabilistic classification provided by SMAA to the final assignment can be applied to other probabilistic versions of classification methods, also ordinal, such as Electre Tri and its variants.

Future research can rely on applying the SMAA-hCat-SD method to real-world nominal classification problems. Extending the method to group decision making is also an interesting direction of research.

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