

# Minimum 2SAT-DELETION: Inapproximability results and relations to Minimum Vertex Cover<sup>☆</sup>

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Received 26 September 2004; received in revised form 3 August 2005; accepted 28 April 2006

Available online 27 June 2006

## Abstract

The MINIMUM 2SAT-DELETION problem is to delete the minimum number of clauses in a 2SAT instance to make it satisfiable. It is one of the prototypes in the approximability hierarchy of minimization problems Khanna et al. [Constraint satisfaction: the approximability of minimization problems, Proceedings of the 12th Annual IEEE Conference on Computational Complexity, Ulm, Germany, 24–27 June, 1997, pp. 282–296], and its approximability is largely open. We prove a lower approximation bound of  $8\sqrt{5} - 15 \approx 2.88854$ , improving the previous bound of  $10\sqrt{5} - 21 \approx 1.36067$  by Dinur and Safra [The importance of being biased, Proceedings of the 34th Annual ACM Symposium on Theory of Computing (STOC), May 2002, pp. 33–42, also ECCC Report TR01-104, 2001]. For highly restricted instances with exactly four occurrences of every variable we provide a lower bound of  $\frac{3}{2}$ . Both inapproximability results apply to instances with no mixed clauses (the literals in every clause are both either negated, or unnegated).

We further prove that any  $k$ -approximation algorithm for the MINIMUM 2SAT-DELETION problem polynomially reduces to a  $(2 - 2/(k + 1))$ -approximation algorithm for the MINIMUM VERTEX COVER problem.

One ingredient of these improvements is our proof that the MINIMUM VERTEX COVER problem is hardest to approximate on graphs with perfect matching. More precisely, the problem to design a  $\rho$ -approximation algorithm for the MINIMUM VERTEX COVER on general graphs polynomially reduces to the same problem on graphs with perfect matching. This improves also on the results by Chen and Kanj [On approximating minimum vertex cover for graphs with perfect matching, Proceedings of the 11st ISAAC, Taipei, Taiwan, Lecture Notes in Computer Science, vol. 1969, Springer, Berlin, 2000, pp. 132–143].

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*Keywords:* Approximation hardness; 2SAT-Deletion; Vertex cover

## 1. Introduction

The theory of probabilistically checkable proofs (PCP) and subsequent improvements in PCP constructions have led for many optimization problems to optimal bounds on their efficient approximability, unless  $P = NP$ . However, in spite of a great deal of efforts, for several fundamental problems the tight bound on their approximability by a polynomial

<sup>☆</sup> A preliminary version of this paper (M. Chlebík, J. Chlebíková: On approximation hardness of the Minimum 2SAT-DELETION problem) was published in Proceedings of 29th Mathematical Foundations of Computer Science (MFCS) 2004, Prague, Czech Republic, August 22–27, 2004, Lecture Notes in Computer Science, vol. 3153, pp. 263–273, (J. Fiala et al., Eds.), ©Springer, Berlin, Heidelberg, 2004. With kind permission of Springer Science and Business Media.

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time algorithm is left open: *Is there a polynomial time approximation algorithm for the MINIMUM VERTEX COVER problem with a factor less than 2? Is there a polynomial time algorithm for the MINIMUM 2SAT-DELETION problem with a constant factor?*

Currently, the best lower bound on polynomial time approximability is the same for both problems MINIMUM VERTEX COVER and MINIMUM 2SAT-DELETION, namely  $10\sqrt{5} - 21 \approx 1.36067$ , due to Dinur and Safra [7]. For the MIN-2SAT-DELETION problem we can improve on this bound, even on highly restricted instances where every variable occurs in very small number of clauses. We show how two questions mentioned above are related to one another: the affirmative answer to the second question would imply the affirmative answer to the first one.

### 1.1. Definitions and preliminaries

Let us introduce the problems we will deal with and mention some known results about them. For the basic optimization terminology we refer the reader to Ausiello et al. [2].

Let  $G = (V, E)$  be a simple graph. A *matching* in the graph  $G = (V, E)$  is a subset of edges in  $E$  with no shared end vertices. A matching in  $G$  is *perfect* if each vertex of  $G$  is incident to an edge of this matching. For a set of vertices  $U \subseteq V$ , let  $G[U]$  denote the subgraph of  $G$  induced by  $U$ .

MINIMUM VERTEX COVER (shortly, MIN-VC)

*Instance:* A simple graph  $G = (V, E)$ .

*Feasible solution:* A vertex cover  $C$  for  $G$ , i.e., a subset  $C \subseteq V$  such that for each  $e \in E$ ,  $e \cap C \neq \emptyset$ .

*Objective function:* The cardinality  $|C|$  of the vertex cover  $C$ .

Let  $vc(G)$  stand for the cardinality of a minimum vertex cover for  $G$ . A half-integral vertex cover for  $G = (V, E)$  is a function  $x : V \rightarrow \{0, \frac{1}{2}, 1\}$  satisfying edge constraints  $x(u) + x(v) \geq 1$  for each edge  $\{u, v\} \in E$ . Let  $vc_*(G)$  stand for the weight of a minimum half-integral vertex cover for  $G$ , i.e., the minimum of  $w(x) := \sum_{u \in V} x(u)$  over all half-integral vertex covers  $x$ .

Clearly,  $vc_*(G) \leq vc(G)$ , as each vertex cover  $C$  define a half-integral vertex cover  $x : V \rightarrow \{0, 1\}$  with the property  $x(v) = 1$  iff  $v \in C$  and the weight equals to  $|C|$ . Further,  $vc_*(G) \leq \frac{1}{2}|V|$ , as the function  $x \equiv \frac{1}{2}$  on  $V$  is always a half-integral vertex cover for  $G$ .

The following theorem of Nemhauser and Trotter [16] is of the great importance for many problems related to the minimum vertex cover problems.

**Nemhauser–Trotter Theorem.** *There exists a polynomial time algorithm that partitions the vertex set  $V$  of a graph  $G$  into three subsets  $V_0, V_1, V_{1/2}$  such that  $V_0$  is an independent set,  $V_1$  is the set of all neighbors of  $V_0$ ,*

- (i)  $vc(G[V_{1/2}]) \geq vc_*(G[V_{1/2}]) = \frac{1}{2}|V_{1/2}|$ ; and
- (ii) *there exists some minimum vertex cover  $C$  for  $G$  such that  $V_1 \subseteq C \subseteq V_1 \cup V_{1/2}$  and  $C \cap V_{1/2}$  is a minimum vertex cover for  $G[V_{1/2}]$ .*

Having one such partition  $V = V_0 \cup V_1 \cup V_{1/2}$  fixed, it is easy to see that for any vertex cover  $C_{1/2}$  of  $G[V_{1/2}]$  the set  $C := C_{1/2} \cup V_1$  is a vertex cover of  $G$  such that

$$\frac{|C|}{vc(G)} \leq \frac{|C_{1/2}|}{vc(G[V_{1/2}])}.$$

Hence in designing a  $\rho$ -approximation algorithm for MIN-VC for  $G$  we only need to concentrate on  $\rho$ -approximating MIN-VC for the graph  $G[V_{1/2}]$ . It is important that  $G[V_{1/2}]$  satisfies  $vc_*(G[V_{1/2}]) = \frac{1}{2}|V_{1/2}|$ . There are several known characterizations of graphs with this property. For example, they are exactly the graphs satisfying  $|N(I)| \geq |I|$  for every independent set [16].

**Definition 1.** Consider a set of clauses  $C_1, C_2, \dots, C_N$  where each clause  $C_i$  is of the form  $l_1 \vee l_2$  and with a non-negative weight  $w_i$  associated with it. Each literal  $l_j$  is either one of Boolean variables  $x_1, x_2, \dots, x_n$ , or its negation. The goal of the MAX-2SAT problem is to assign Boolean values 0 and 1 to variables  $x_1, x_2, \dots, x_n$  so that the total weight of the satisfied clauses is maximized. For the complementary problem of *minimum unsatisfiability*,

MIN-2SAT-DELETION (also called MIN-2CNF-DELETION), the goal is to minimize the total weight of unsatisfied clauses over all assignments.

A common variant of the above is that all weights are the same but clauses can be repeated.

The MINIMUM 2SAT-DELETION problem is important as one of the prototypes in a complete classification of the approximability of minimization problems derived from Boolean constraint satisfaction [11]. While MAX-2SAT is approximable within a factor of 1.0638 [13] and it is NP-hard to approximate within a factor of 1.0476 [9], the approximability of the MIN-2SAT-DELETION problem is still widely open. The problem is known to be NP-hard to approximate to within a factor of 1.36067 [7] and, on the other hand, Agarwal et al. [1] presented recently an  $O(\sqrt{\log n})$ -approximation algorithm. The previously best known approximation factor was  $O(\log n \log \log n)$  by Klein et al. [12].

## 1.2. Summary of the results

In Section 2 we study the MINIMUM VERTEX COVER problem on graphs with perfect matching (MIN-VC-PM). It turns out that to solve MIN-VC exactly, or to approximate MIN-VC within a factor on general graphs, reduce to the corresponding problems on graphs with perfect matching. In particular, the problem to design a  $\rho$ -approximation algorithm for the MINIMUM VERTEX COVER on general graphs polynomially reduces to the same problem on graphs with perfect matching. Moreover, we observe that the NP-hard gap results of Dinur and Safra [7] for MIN-VC apply to MIN-VC-PM as well. Using this fact and the powerful reduction from MIN-VC-PM to MIN-2SAT-DELETION we can improve inapproximability results for MIN-2SAT-DELETION. More precisely, we prove that it is NP-hard to approximate MIN-2SAT-DELETION to within any constant factor less than  $8\sqrt{5} - 15 \approx 2.88854$ . We provide interesting lower bound also for small occurrence instances: it is NP-hard to approximate MIN-2SAT-DELETION to within any constant factor less than  $\frac{3}{2}$  on instances with exactly four occurrences of every variable. Both inapproximability results apply to instances with no mixed clauses (i.e., only clauses  $x \vee y$ ,  $\bar{x} \vee \bar{y}$ , with  $x$  and  $y$  distinct variables, are allowed).

We further prove that every  $k$ -approximation algorithm for the MIN-2SAT-DELETION problem polynomially reduces to a  $(2 - 2/(k + 1))$ -approximation algorithm for the MINIMUM VERTEX COVER problem. Combining these our results with the recent  $O(\sqrt{\log n})$ -approximation algorithm for MIN-2SAT-DELETION [1] we get an algorithm for MIN-VC with the same approximation factor  $2 - \Omega(1/\sqrt{\log n})$  as in the best known approximation algorithm for MIN-VC by Karakostas [10].

## 2. Reduction of MIN-VC to graphs with perfect matching

This section concentrates on the MINIMUM VERTEX COVER problem on graphs with perfect matching. We will show that the problems to solve MIN-VC exactly or to approximate MIN-VC to within a factor in general graphs reduce to the corresponding MIN-VC problems on graphs with perfect matching.

First, we define for a graph  $G$  the related bipartite graph  $G^b$  and recall known facts [3,16] that minimum half-integral vertex covers for  $G$  are generated by minimum vertex covers for  $G^b$ .

**Definition 2.** For a graph  $G=(V, E)$  we define the bipartite graph  $G^b=(V^b, E^b)$  as follows: there are two copies  $u^L$  and  $u^R$  of each vertex  $u \in V$  in  $G^b$ ,  $V^L := \{u^L : u \in V\}$ ,  $V^R := \{u^R : u \in V\}$ , and  $V^b := V^L \cup V^R$ . Each edge  $\{u, v\} \in E$  of  $G$  creates two edges in  $G^b$ , namely  $\{u^L, v^R\}$  and  $\{v^L, u^R\}$ . Hence  $E^b := \{\{u^L, v^R\}, \{v^L, u^R\} : \{u, v\} \in E\}$ .

For any set  $C \subseteq V^L \cup V^R$  we associate a map  $x_C : V \rightarrow \{0, \frac{1}{2}, 1\}$  in the following way:  $x_C(u) = \frac{1}{2}|C \cap \{u^L, u^R\}|$  for any  $u \in V$ . Clearly,  $w(x_C) = \frac{1}{2}|C|$  for any  $C \subseteq V^L \cup V^R$ .

**Lemma 3.** (i) If  $C$  is a vertex cover for  $G^b$  then  $x_C$  is a half-integral vertex cover for  $G$  of weight  $\frac{1}{2}|C|$ . In particular,  $vc_*(G) \leq \frac{1}{2}vc(G^b)$ .

(ii) If  $x : V \rightarrow \{0, \frac{1}{2}, 1\}$  is a half-integral vertex cover for  $G$  then there is a vertex cover  $C$  for  $G^b$  such that  $x_C = x$ . Hence  $\frac{1}{2}vc(G^b) \leq vc_*(G)$ .

(iii)  $vc_*(G) = \frac{1}{2}vc(G^b)$ .

**Proof.** (i) Being obvious, it is sufficient to show (ii) and the rest will trivially follow. Given  $x : V \rightarrow \{0, \frac{1}{2}, 1\}$  as required, let  $V_i^x := \{u \in V : x(u) = i\}$  for  $i \in \{0, \frac{1}{2}, 1\}$ . Obviously, there are no edges between  $V_0^x$  and  $V_{1/2}^x$  or within  $V_0^x$ . It follows that the set  $C$  defined by  $C := \bigcup_{u \in V_1^x} \{u^L, u^R\} \cup \bigcup_{u \in V_{1/2}^x} \{u^L\}$  is a vertex cover for  $G^b$  such that  $x_C = x$ .  $\square$

**Definition 4.** Given a graph  $G = (V, E)$  and consider the corresponding bipartite graph  $G^b = (V^b, E^b)$  as in Definition 2. By  $\tilde{G} = (\tilde{V}, \tilde{E})$  we denote a two-padding of  $G$ , namely the graph with  $\tilde{V} = V^b$  and

$$\tilde{E} = E^b \cup \{\{u^L, v^L\}, \{u^R, v^R\} : \{u, v\} \in E\}.$$

Clearly,  $\tilde{G}$  is obtained from  $G$  by the standard duplication of vertices. It is well known and easy to see that the optimization problems MIN-VC for  $G$  and  $\tilde{G}$ , respectively, are essentially equivalent. Namely, if  $C \subseteq V$  is a vertex cover in  $G$  then  $\bigcup_{u \in C} \{u^L, u^R\}$  is a vertex cover in  $\tilde{G}$  and, moreover, every inclusionwise minimal vertex cover of  $\tilde{G}$  is of the form  $\bigcup_{u \in C} \{u^L, u^R\}$  for some vertex cover  $C$  of  $G$ .

The relation between approximating the MIN-VC-PM problem and approximating the MIN-VC problem on general graphs was studied by Chen and Kanj [5]. They showed [5, Theorem 3.2] how any polynomial time  $(2 - 3\delta)$ -approximation algorithm for MIN-VC-PM ( $\delta \in (0, \frac{1}{4})$ ) could be used to design a polynomial time  $(2 - 2\delta)$ -approximation algorithm for MIN-VC on general graphs. We will now improve on this result showing that the problems MIN-VC and MIN-VC-PM are, in fact, equally hard to approximate.

**Theorem 5.** *For any fixed constant  $\rho \geq 1$ , if there is a polynomial  $\rho$ -approximation algorithm for MIN-VC-PM then there is a polynomial  $\rho$ -approximation algorithm for MIN-VC on general graphs.*

**Proof.** Using Nemhauser–Trotter Theorem, the MIN-VC problem for an arbitrary graph  $G = (V, E)$  reduces in approximation preserving way to the MIN-VC problem for its induced subgraph  $G[V_{1/2}]$  with the property that  $vc_*(G[V_{1/2}]) = \frac{1}{2}|V_{1/2}|$ . We now show that it follows that  $(G[V_{1/2}])^b$  (and, consequently,  $\tilde{G}[V_{1/2}]$ ) has a perfect matching. As  $(G[V_{1/2}])^b$  is a bipartite graph, due to König–Egerváry Theorem (see, e.g., [17]), size of a maximum matching is equal to  $vc((G[V_{1/2}])^b)$ . Using Lemma 3(iii) and  $vc_*(G[V_{1/2}]) = \frac{1}{2}|V_{1/2}|$  we obtain  $vc_*((G[V_{1/2}])^b) = 2vc_*(G[V_{1/2}]) = |V_{1/2}|$ . Hence  $(G[V_{1/2}])^b$  has a matching of size  $|V_{1/2}|$ , which is clearly a perfect matching. Thus,  $\tilde{G}[V_{1/2}]$  has a perfect matching.

Consequently, the MIN-VC problem for a graph  $G = (V, E)$  reduces in approximation preserving way to the MIN-VC problem for  $G[V_{1/2}]$  or, equivalently, for  $\tilde{G}[V_{1/2}]$ , thus to the MIN-VC-PM problem.  $\square$

Theorem 5 and its proof show that the MIN-VC problem is hardest to approximate on graphs with perfect matching. Any approximation hardness result for MIN-VC can be translated to the one for MIN-VC-PM, simply passing from an instance  $G = (V, E)$  to  $\tilde{G}[V_{1/2}]$  as above. Thus there is no surprise that “hard instances” produced in approximation hardness results by Håstad [9], and by Dinur and Safra [7], are graphs with perfect matching. (This fact is straightforward to see in [9]; in [7] it follows from the fact that the building blocks of their construction are vertex transitive graphs and any such graph with even number of vertices has a perfect matching.) The following theorem is just a reformulation of the NP-hard gap result by Dinur and Safra [7] to MIN-VC-PM.

**Theorem 6 (Dinur and Safra).** *Let  $p, q$  be constants such that  $(3 - \sqrt{5})/2 > p > q > \max\{p^2, 4p^3 - 3p^4\}$ . It is NP-hard for graphs  $G = (V, E)$  with perfect matching to distinguish between the following two cases:  $vc(G) < (1 - p)|V|$ , or  $vc(G) > (1 - q)|V|$ .*

We note that the theorem analogous to Theorem 5 is true for any class  $\mathcal{G}$  of graphs that is closed on operation of taking a two-padding of an induced subgraph. If the class  $\mathcal{G}$  of graphs does not have this property, but it is at least closed on the operation of taking an induced subgraph (e.g.,  $\mathcal{G} = \{\text{graphs with the maximum degree } \leq B\}$ , or  $\mathcal{G} = \{\text{everywhere } B\text{-sparse graphs}\}$ , for some constant  $B$ ), we can conclude that MIN-VC restricted to  $\mathcal{G}$  is as hard to approximate as MIN-VC restricted to  $\mathcal{G}'$ , where  $\mathcal{G}'$  consists of graphs  $G = (V, E)$  from  $\mathcal{G}$  with  $vc_*(G) = \frac{1}{2}|V|$ . In such restricted classes of graphs (for example, in graphs with the maximum degree  $\leq 3$ ) we do not know if MIN-VC-PM is as hard to approximate as MIN-VC, or easier.

### 3. Relation of Min-VC to Min-2SAT-deletion

In [5] Chen and Kanj gave a polynomial time reduction (see Definition 7 below) from MIN-VC-PM to MAX-2SAT that also allowed them to design approximation algorithms for the MIN-VC-PM problem based on approximation algorithms for the MAX-2SAT problem. Our approximation preserving reduction from MIN-VC on general graphs to MIN-VC-PM described in the previous section could be combined with the reduction of Chen and Kanj to relate also MIN-VC to MAX-2SAT.

Let us note that results of [5] based on MAX-2SAT induce improvements over previously know algorithms for MIN-VC only on sparse graphs. Here we suggest that for general graphs it is more fruitful to study approximability of MIN-VC in terms of approximation factors for MIN-2SAT-DELETION rather than for MAX-2SAT.

We show how approximation algorithms for the MIN-2SAT-DELETION problem induce new approximation algorithms for the MIN-VC problem. And similarly, how known inapproximability results for MIN-VC imply new inapproximability results for MIN-2SAT-DELETION.

**Definition 7.** Let  $G = (V, E)$  be an instance of MIN-VC-PM and let  $M$  be a fixed perfect matching in  $G$ . Define an instance  $\mathcal{F}(G, M)$  of 2SAT as follows: the Boolean variable set is  $X_V := \{x_u : u \in V\}$ , and

$$\mathcal{F}(G, M) = \bigcup_{\{u,v\} \in E} \{(x_u \vee x_v)\} \cup \bigcup_{\{u,v\} \in M} \{(\bar{x}_u \vee \bar{x}_v)\}.$$

An assignment  $\sigma : X_V \rightarrow \{0, 1\}$  to variables is called *standard*, if all clauses with unnegated variables are satisfied by  $\sigma$ .

Hence  $\mathcal{F}(G, M)$  consists of  $N := |E| + \frac{1}{2}|V|$  clauses. All the clauses have exactly two (different) literals and are non-mixed, i.e., none of the clauses has both negated and unnegated literals. This variant of 2SAT is sometimes referred to as E2-NM-SAT (NM stands for non-mixed clauses). Obviously, an instance  $\mathcal{F} := \mathcal{F}(G, M)$  of E2-NM-SAT defined above has three additional properties:

- (P1) clauses are not repeated,
- (P2) each variable appears exactly once as negated, and
- (P3) if  $(\bar{a} \vee \bar{b}) \in \mathcal{F}$  for variables  $a$  and  $b$ , then  $(a \vee b) \in \mathcal{F}$  as well.

In what follows  $\mathcal{F}(G, M)$  is viewed as an instance of the MIN-2SAT-DELETION problem. Let  $\text{unsat}(\sigma)$  be the number of clauses of  $\mathcal{F}(G, M)$  that are unsatisfied by an assignment  $\sigma : X_V \rightarrow \{0, 1\}$ , and  $\text{OPT}(\mathcal{F}(G, M))$  be the minimum of  $\text{unsat}(\sigma)$  over all assignments  $\sigma : X_V \rightarrow \{0, 1\}$ . The following lemma is implicitly contained in Lemma 4.1 and Theorem 4.2 of [5].

**Lemma 8.** Let  $M$  be a perfect matching in a graph  $G = (V, E)$ , and the collection of clauses  $\mathcal{F}(G, M)$  with Boolean variables  $X_V$  be as in Definition 7. Then

- (i)  $\text{OPT}(\mathcal{F}(G, M)) = \text{vc}(G) - \frac{1}{2}|V|$ ,
- (ii) from any assignment  $\sigma : X_V \rightarrow \{0, 1\}$  a vertex cover  $C$  for  $G$  of cardinality at most  $(\frac{1}{2}|V| + \text{unsat}(\sigma))$  can be constructed in time  $O(|E|)$ .

**Proof.** If  $C$  is a vertex cover in  $G$ , let  $\sigma_C : X_V \rightarrow \{0, 1\}$  denote the following assignment:  $\sigma_C(x_u) = 1$  iff  $u \in C$ . The fact that  $C$  is the vertex cover means that  $\sigma_C$  is standard assignment. Clearly, standard assignments  $\sigma : X_V \rightarrow \{0, 1\}$  are in one-to-one correspondence with vertex covers in  $G$ .

It is clear that for a standard assignment  $\sigma_C$ , the edges in the perfect matching  $M$  are partitioned into two sets  $M_1$  and  $M_2$  such that each edge  $\{u, v\}$  in  $M_i$  has exactly  $i$  endpoints in  $C$ ,  $i = 1, 2$ . Thus  $|C| = |M_1| + 2|M_2|$ , and clearly  $\text{unsat}(\sigma_C) = |M_2| = |C| - (|M_1| + |M_2|) = |C| - \frac{1}{2}|V|$ . Consequently,  $\text{OPT}(\mathcal{F}(G, M)) \leq \text{vc}(G) - \frac{1}{2}|V|$ .

Let now an assignment  $\sigma : X_V \rightarrow \{0, 1\}$  be given. We can modify  $\sigma$  (in time  $O(|E|)$ ) to the standard assignment  $\sigma_C$  such that  $\text{unsat}(\sigma_C) \leq \text{unsat}(\sigma)$ . In other words, we can provide in time  $O(|E|)$  a vertex cover  $C$  with  $\text{unsat}(\sigma_C) = |C| - \frac{1}{2}|V| \leq \text{unsat}(\sigma)$ .



We will process edges  $\{u, v\} \in E$ , one after another, as follows: if  $\sigma(x_u) = \sigma(x_v) = 0$  for an edge  $\{u, v\} \in E$ , then we modify  $\sigma$  at exactly one endpoint of  $\{u, v\}$ , say  $u$ , setting  $\sigma(x_u) = 1$  instead. This change does not increase the value  $\text{unsat}(\sigma)$ : the unsatisfied clause  $(x_u \vee x_v)$  becomes satisfied, and at most one satisfied clause can become unsatisfied, namely the one containing the literal  $\bar{x}_u$ . Having this done, one after another, for all edges  $\{u, v\} \in E$ , the resulting assignment will be standard, hence of the form  $\sigma_C$  for a vertex cover  $C$  of  $G$ , and  $\text{unsat}(\sigma_C) \leq \text{unsat}(\sigma)$ .

In particular, it follows (taking  $\sigma$  with  $\text{unsat}(\sigma) = \text{OPT}(\mathcal{F}(G, M))$ ) that  $\text{OPT}(\mathcal{F}(G, M)) = \text{vc}(G) - \frac{1}{2}|V|$ .  $\square$

Using the presented reduction from MIN-VC-PM to special instances of MIN-2SAT-DELETION we can obtain from Theorem 6:

**Theorem 9.** *Let  $p, q$  be constants such that  $(3 - \sqrt{5})/2 > p > q > \max\{p^2, 4p^3 - 3p^4\}$ . It is NP-hard for instances  $\mathcal{F}$  of MIN-2SAT-DELETION to distinguish between two cases:  $\text{OPT}(\mathcal{F}) < (\frac{1}{2} - p)|X|$  or  $\text{OPT}(\mathcal{F}) > (\frac{1}{2} - q)|X|$ , where  $X$  is the set of Boolean variables of  $\mathcal{F}$ . Consequently, it is NP-hard to approximate the MIN-2SAT-DELETION problem to within any constant approximation factor less than  $8\sqrt{5} - 15 \approx 2.88854$ . The same NP-hardness result applies to instances with no mixed clauses and satisfying conditions (P1)–(P3).*

**Proof.** The  $((\frac{1}{2} - p)|X|, (\frac{1}{2} - q)|X|)$ -gap result follows directly from the reduction  $G \mapsto \mathcal{F}(G, M)$  using Theorem 6 and Lemma 8. Hence inapproximability to within  $(1 - 2q)/(1 - 2p)$  follows, for any  $p, q$  satisfying our assumptions. Notice that for  $p \in (\frac{1}{3}, 1)$ ,  $\max\{p^2, 4p^3 - 3p^4\} = 4p^3 - 3p^4$ , and as  $q$  can approach  $4p^3 - 3p^4$  (from above) and  $p$  can approach  $(3 - \sqrt{5})/2$  (from below),  $(1 - 2q)/(1 - 2p)$  can approach  $8\sqrt{5} - 15$  (from below). Hence NP-hardness to approximate the problem to within any constant factor less than  $8\sqrt{5} - 15$  follows.  $\square$

Assume now that we have an approximation algorithm  $\mathcal{A}$  for the MIN-2SAT-DELETION problem. The polynomial time reduction from Definition 7 suggests an approximation algorithm (based on  $\mathcal{A}$ ) for MIN-VC-PM and, consequently, for the MINIMUM VERTEX COVER problem.

**Theorem 10.** *Given an algorithm that approximates the solution of the MIN-2SAT-DELETION problem within an approximation factor  $f(n, N) \geq 1$  on instances with  $n$  variables and  $N$  clauses, all non-mixed and satisfying (P1)–(P3). (Here  $f : \mathbb{N}_+^2 \rightarrow \langle 1, \infty \rangle$  is a function separately non-decreasing in every variable.) It can be reduced to the one that approximates MIN-VC-PM (respectively, MIN-VC) on instances with  $n$  vertices and  $m$  edges within  $2 - 2/(f(n, m + \frac{n}{2}) + 1)$  (respectively,  $2 - 2/(f(2n, 4m + n) + 1)$ ).*

**Proof.** Let  $G = (V, E)$  be a graph with  $n := |V|$  and  $m := |E|$ . We can assume that  $G$  has a perfect matching. Otherwise we can work with a graph  $\tilde{G}[V_{1/2}]$ , that is a graph with  $n' \leq 2n$  vertices,  $m' \leq 4m$  edges, and with a perfect matching. Let  $\mathcal{A}$  be an  $f$ -approximation algorithm for MIN-2SAT-DELETION, and consider the following algorithm:

*Step 1:* Construct a perfect matching  $M$  in  $G$ ,

*Step 2:* Construct the corresponding instance  $\mathcal{F}(G, M)$  of the MIN-2SAT-DELETION problem, with  $n$  variables and  $N := m + n/2$  clauses,

*Step 3:* Applying the algorithm  $\mathcal{A}$  to  $\mathcal{F}(G, M)$  construct an assignment  $\sigma : X_V \rightarrow \{0, 1\}$  that approximates the optimal solution for  $\mathcal{F}(G, M)$  within  $f := f(n, N)$ ,

*Step 4:* Construct a vertex cover  $C$  in  $G$  of cardinality at most  $(\frac{1}{2}|V| + \text{unsat}(\sigma))$ , according to Lemma 8.

*Step 5:* Return a vertex cover  $C$  of  $G$ .

Our aim is to show that the algorithm returns a vertex cover  $C$  of  $G$ , with the property  $|C| \leq \text{vc}(G)(2 - 2/(f + 1))$ . The assumptions on  $\mathcal{A}$  guarantee that the assignment  $\sigma$  provided in Step 3 satisfies

$$\text{unsat}(\sigma) \leq \text{OPT}(\mathcal{F}(G, M)) \cdot f. \tag{1}$$

If  $\text{unsat}(\sigma) \geq \frac{1}{2}|V|$ , we conclude from (1) and from Lemma 8 that

$$\text{vc}(G) = \frac{1}{2}|V| + \text{OPT}(\mathcal{F}(G, M)) \geq \frac{1}{2}|V| + \frac{1}{2} \frac{|V|}{f}.$$

As clearly  $|C| \leq |V|$ ,  $|C|/\text{vc}(G) \leq 2 - 2/(f + 1)$  easily follows.

If  $\text{unsat}(\sigma) = \frac{1}{2}|V| \cdot t$  for  $t \in (0, 1)$ , we conclude similarly that

$$vc(G) \geq \frac{1}{2}|V| + \frac{1}{2}|V| \frac{t}{f},$$

but now we use better estimate on  $|C|$  from Step 4,  $|C| \leq \frac{1}{2}|V| + \frac{1}{2}|V|t$ . Hence

$$\frac{|C|}{vc(G)} \leq \frac{1+t}{1+(t/f)} = \frac{(1+t)f}{f+t}.$$

The right-hand side as a function of  $t \in (0, 1)$  achieves its maximum at  $t = 1$  (we use that  $f \geq 1$  for this argument), hence

$$\frac{|C|}{vc(G)} \leq \frac{2f}{f+1} = 2 - \frac{2}{f+1}. \quad \square$$

**Corollary 11.** *To approximate MIN-2SAT-DELETION within a constant  $k$  is at least as difficult (up to polynomial reduction between problems) as to approximate MIN-VC within a factor  $2 - 2/(k+1)$ .*

If we apply Theorem 10 to the  $O(\sqrt{\log n})$ -approximation algorithm given for MIN-2SAT-DELETION by Agarwal et al. [1], we obtain an algorithm for MIN-VC with the same approximation factor  $(2 - \Omega(1/\sqrt{\log n}))$  as was obtained by Karakostas [10]. An improvement to  $o(\sqrt{\log n})$  of the above factor for MIN-2SAT-DELETION (at least on instances with non-mixed clauses, satisfying (P1)–(P3)) would improve on currently the best polynomial time approximation factor for the MIN-VC problem.

Theorem 10 and its proof can be modified to deal with situations when the quality of approximation of the algorithm  $\mathcal{A}$  for MIN-2SAT-DELETION is measured with different parameters. Consider an algorithm  $\mathcal{A}$  for MIN-2SAT-DELETION that is *robust on almost-satisfiable instances*. That means, for an instance  $\mathcal{F}$  of MIN-2SAT-DELETION, whose optimum assignment leaves only  $\varepsilon$  fraction of clauses unsatisfied,  $\mathcal{A}$  finds an assignment that leaves at most  $g(\varepsilon)$  fraction of clauses of  $\mathcal{F}$  unsatisfied, where  $g : (0, 1) \rightarrow (0, 1)$  is a function with  $\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = g(0) = 0$ .

Zwick's efficient algorithm [18] has this robustness property with  $g(\varepsilon) = 5\varepsilon^{1/3}$ . An interesting question is whether such algorithms exist with  $g(\varepsilon) = o(\varepsilon^{1/2})$ , and if yes, how far one can go beyond this bound. One can easily check (along the lines of the proof of Theorem 10) that any robust algorithm with  $g(\varepsilon) = O(\varepsilon^t)$ ,  $t \in (0, 1)$ , provides a  $(2 - c \cdot (1 + \bar{d})^{1-1/t})$ -approximation algorithm for MIN-VC-PM on graphs with average degree  $\bar{d} := 2|E|/|V|$  (here  $c \in (0, 1)$  is an absolute constant). For any  $t > \frac{1}{2}$  the existence of a robust algorithm with  $g(\varepsilon) = O(\varepsilon^t)$  would significantly improve (for large  $\bar{d}$ ) on currently the best approximation factor  $(2 - 5/(2\bar{d} + 3))$  on polynomial time approximation for MIN-VC-PM on graphs with average degree  $\bar{d}$  [8].

### 3.1. Bounded occurrence instances of 2SAT problems

One can obtain inapproximability results also for bounded occurrence instances of MIN-2SAT-DELETION using inapproximability results for MIN-VC on bounded degree graphs. We will show this on restricted instances with exactly four occurrences of each variable.

**Theorem 12.** *It is NP-hard to approximate the MIN-2SAT-DELETION problem within any constant approximation factor less than  $\frac{3}{2}$  on instances with exactly four occurrences of each variable, no mixed clauses, and satisfying conditions (P1)–(P3).*

**Proof.** One can check that instances produced in [6] to achieve the inapproximability results for the MIN-VC problem in cubic graphs, have a perfect matching. It is proven there that it is NP-hard for a cubic graph  $G$  with  $n$  vertices and having a perfect matching to distinguish the case  $vc(G) < (\frac{1}{2} + 2\delta + \varepsilon)n$  from the case  $vc(G) > (\frac{1}{2} + 3\delta - \varepsilon)n$  where  $\delta > 0$  is a positive constant related to parameters of an amplifier used in the construction, and  $\varepsilon \in (0, \delta/2)$  can be arbitrarily small independently of  $\delta$ . The instance  $\mathcal{F}(G, M)$  that corresponds to such  $G$  (and to an arbitrary matching  $M$  in  $G$ ) has  $n$  variables,  $2n$  clauses (all non-mixed and satisfying (P1)–(P3)) with exactly four occurrences of each

variable. Due to Lemma 8, the corresponding NP-hard problem is to distinguish  $\text{OPT}(\mathcal{F}(G, M)) < (2\delta + \varepsilon)n$  from  $\text{OPT}(\mathcal{F}(G, M)) > (3\delta - \varepsilon)n$ .  $\square$

In this way one can derive inapproximability results also for the complementary MAX-2SAT satisfiability problem with exactly four occurrences of each variable. The corresponding NP-hard problem is to decide whether the optimum is greater than  $(2 - 2\delta - \varepsilon)n$ , or less than  $(2 - 3\delta + \varepsilon)n$ . Now the inapproximability factor  $\approx (2 - 2\delta)/(2 - 3\delta)$  depends crucially on parameters of an amplifier (hidden in  $\delta$ ) used in that hardness result for cubic graphs. From estimates of [6], the inapproximability to within  $1 + \frac{1}{385}$  follows. It is worse than the recent hardness factor  $1 + \frac{1}{268}$  obtained for this problem in [4], but on the other hand, it applies to instances with no mixed clauses.

### 3.2. Concluding remarks

It has been conjectured by several authors that it is NP-hard to approximate MIN-VC within any constant factor less than 2. By Theorem 10 and Corollary 11 this would imply also NP-hardness of approximating MIN-2SAT-DELETION within any constant factor.

The methods used in this paper show that in order to prove NP-hardness to approximate MIN-2SAT-DELETION within (any fixed constant)  $k$ , it suffices to provide instances  $G = (V, E)$  with a perfect matching for which it is NP-hard to distinguish  $vc(G) < \frac{1}{2}|V| + \varepsilon(G)$  from  $vc(G) > \frac{1}{2}|V| + k \cdot \varepsilon(G)$ , for some efficiently computable function  $\varepsilon = \varepsilon(G) > 0$ . On the other hand, to prove that it is NP-hard to approximate MIN-VC within any constant factor smaller than 2 requires to show, for arbitrarily small constant  $\varepsilon > 0$ , NP-hardness to distinguish instances with  $vc(G) < (\frac{1}{2} + \varepsilon)|V|$  from those with  $vc(G) > (1 - \varepsilon)|V|$ .

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