The d-precoloring problem for k-degenerate graphs*

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Abstract

In this paper we deal with the d-Precoloring Extension problem, (d-PrExt), in various classes of graphs. The d-PrExt problem is the special case of precoloring extension problem where, for a fixed constant d, input instances are restricted to contain at most d precolored vertices for every available color. The goal is to decide if there exists an extension of given precoloring using only available colors or to find it.

We present a linear time algorithm for both, the decision and the search version of d-PrExt, in the following cases: (i) restricted to the class of k-degenerate graphs (hence also planar graphs) and with sufficiently large set S of available colors, and (ii) restricted to the class of partial k-trees (without any size restriction on S). We also study the following problem related to d-PrExt: given an instance of the d-PrExt problem which is extendable by colors of S, what is the minimum number of colors of S sufficient to use for precolorless vertices over all such extensions? We establish lower and upper bounds on this value for k-degenerate graphs and its various subclasses (e.g., planar graphs, outerplanar graphs) and prove tight results for the class of trees.

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1 Introduction

Let $G = (V, E)$ be a finite simple graph. For a set S of colors, a proper coloring (or S-coloring) of G is a mapping $f : V \to S$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$. An S-precoloring f' of G is any proper coloring $f' : C \to S$ of the subgraph of G induced by $C \subseteq V$. An S-precoloring of G is called a (d, S)-precoloring, if each color of S is used for at most d vertices. We say that an S-coloring f of G is an *extension* of an S-precoloring $f' : C \to S$, if $f(v) = f'(v)$ for each vertex $v \in C$. The subset $C \subseteq V$ is the set of *precolored* vertices and $V \setminus C$ is the set of *precolorless* vertices. The *admissible set* of colors for a precolorless vertex v is the set of all colors of S which do not occur on precolored neighbors of v.

A graph G is *k-degenerate* if every subgraph of G has a vertex of degree at most k. We can define the class of k-degenerate graphs also inductively as follows: a graph with at most $(k+1)$ vertices is a k-degenerate graph and recursively a k-degenerate graph with $(n + 1)$ vertices can be obtained from a k-degenerate graph with n vertices by adding a new vertex adjacent to at most k vertices. If we start with a $(k+1)$-clique and every new vertex has to be adjacent to all vertices of a k-clique, we obtain a k-tree introduced in [1]. A partial k-tree is a subgraph of a k-tree.

In this article we investigate the following problem:

d-Precoloring Extension Problem (d-PrExt), d is a fixed nonnegative integer
Instance: A set S of colors, a graph G, and a (d, S)-precoloring of G.
Question (decision): Is there an extension of the (d, S)-precoloring to the entire graph G (using only colors of S)?
Task (search): Find an extension to the entire graph G, if there is any.

The d-PrExt problem is a subproblem of the Precoloring Extension problem (PrExt). In the PrExt problem there is no restriction on the number of precolored vertices for any color of S. PrExt and d-PrExt arise in practical applications such as scheduling or VLSI-layout and were initiated in [2].

Obviously, 0-PrExt is the problem to decide whether the chromatic number $\chi(G)$ is at most $|S|$. Hence the Graph Coloring problem, which is NP-complete for general graphs ([8]), is a subproblem of the d-PrExt problem. However d-PrExt is NP-complete even for very restricted graph classes: e.g., 1-PrExt problem for bipartite graphs (and $|S| = 3$) [3], 1-PrExt problem for permutation graphs [12], and 2-PrExt problem for interval graphs [2]. On the other hand, PrExt can be solved in polynomial time for (a) split graphs, (b) complements of bipartite graphs [9], (c) for partial k-trees [11], [13] (if $|S|$ is a constant, in linear time [11]), and in linear time also for trees. Further research was continued in a series of papers, see [10], [14], and also [16] for a survey and references therein.

Summary of Results

In this paper we consider some combinatorial and algorithmic aspects of the d-PrExt problem restricted to the subclasses of k-degenerate graphs. They include some well studied classes of graphs such as partial k-trees, planar graphs (for $k = 5$), and outerplanar graphs (for $k = 2$).

In Section 2 we prove that if the answer to the d-PrExt problem for a k-degenerate graph is affirmative then there exists also an extension that uses at most $k(d+2)$ colors for all precolorless vertices independently of the size of S. On the other hand, we present (d, S)-precolored k-degenerate graphs that require to use at least $k(d+1)+1$ colors of S to extend the precoloring. The similar bounds are presented also for various subclasses of k-degenerate graphs (e.g., planar graphs, outerplanar graphs), with tight bounds for the class of trees.

In Section 3 we present a linear time algorithm for both, the decision and the search version of the d-PrExt problem in the following cases: (i) for k-degenerate graphs, if $|S| \geq 2k+1$, (ii) for planar graphs, if $|S| \geq 7$, (iii) for partial k-trees without any size restriction on S. The last result improves on the best known result for the d-PrExt problem restricted to the class of partial k-trees, with running time $O(|V||S|^{k+1})$ ([11]).

2 Structural Results

An interesting combinatorial question relevant to d-PrExt is the following: if there is an extension for every extendable (d, S)-precoloring to the entire graph G (using only colors of S), how many colors of S are necessary to the precolorless vertices of G?

Definition. *Define the d-chromatic number of a graph G, $\chi_d(G)$, as the minimum number m such that for every set of colors S and every extendable (d, S)-precoloring of G, there is an extension to the entire graph G that uses at most m colors of S for the precolorless vertices. For a class of graphs $\mathcal{G}$, let $\chi_d = \sup_{G \in \mathcal{G}} \chi_d(G)$ ($+\infty$, if $\chi_d(G)$ is unbounded).*

Firstly, we present the following lemma that expresses a property of k-degenerate graphs.

Lemma 1 *Let $k \geq 1$, $M > k$ be fixed integers. Then every nonempty k-degenerate graph with n vertices contains less than $\frac{nk}{M-k}$ vertices with degree at least M.*

Proof. Due to the fact that any k-degenerate graph is a subgraph of a k-degenerate graph on the same vertex set with all vertices of degree at least k, it is sufficient to prove lemma for k-degenerate graphs with minimum degree k.

Let $G = (V, E)$ be a k-degenerate graph with n vertices and minimum degree k. If at least $\frac{nk}{M-k}$ vertices have degree at least M in G, then

$$\sum_{v \in V} d_G(v) \geq \frac{nk}{M-k} \cdot M + \left(n - \frac{nk}{M-k}\right) k = 2kn. \tag{1}$$

But the sum of vertex degrees in a k-degenerate graph G is bounded by

$$\sum_{v \in V} d_G(v) = k(k+1) + 2k(n-k-1) < 2kn, \tag{2}$$

using the inductive construction of k-degenerate graphs. Combining (1) and (2) we obtain a contradiction. □

Remark 1 *For a planar graph $G = (V, E)$ the inequality $|E| \leq 3|V| - 6$ is well known. From this one can prove better upper bound for the number of large degree vertices in planar graphs as follows from Lemma 1 for 5-degenerate graphs (hence also planar graphs). More precisely, a planar graph G contains at most $\frac{3(|V|-4)}{M-3}$ vertices of degree at least M, $M \geq 4$.*

In the following theorem we prove that for the class of k-degenerate graphs, the value χ_d can be bounded by a function of d and k. Note that, e.g., for the class of bipartite graphs, χ_d is unbounded.

Theorem 1 *Let $k \geq 1$, $d > 0$ be fixed integers. For any (d, S)-precoloring of a k-degenerate graph if there is an extension to the entire graph, then there is also an extension which uses at most $k(d+2)$ colors of S, i.e., $\chi_d \leq k(d+2)$ for the class of k-degenerate graphs.*

Proof. Suppose to the contrary that the theorem does not hold. The size of a (d, S)-precolored k-degenerate graph $G = (V, E)$ is defined as $|V| + |S|$. Let G with a (d, S)-precoloring g be a fixed counterexample of smallest size.

Then using Lemma 1 it follows that

$$\text{less than } \tfrac{|V|}{d+1} \text{ vertices have degree } \geq k(d+2). \tag{3}$$

Obviously, $|S| > k(d+2)$ and in the following we prove that

(a) every precolorless vertex in G has degree $\geq k(d+2)$,

(b) in every extension of g to an S-coloring of the entire graph G all colors of S are used for precolorless vertices.

To prove (a), we suppose that there is a precolorless vertex $v \in V$ of degree $< k(d+2)$. From the minimality of G, the (d, S)-precoloring g restricted to $G \setminus v$ can be extended using at most $k(d+2)$ colors to an S-coloring of the graph $G \setminus v$ and consequently also to the entire graph G. This gives a contradiction.

We prove also part (b) by contradiction. Suppose that there exists an extension of g to an S-coloring of the entire graph G which does not use (w.l.o.g.) color $l \in S$. Let C_l be the vertex set of G precolored with color l. Consider the restriction of the (d, S)-precoloring g to $G \setminus C_l$, which is obviously a $(d, S \setminus \{l\})$-precoloring. However from the minimality of our counterexample, the $(d, S \setminus \{l\})$-precoloring g restricted to $G \setminus C_l$ can be extended to an $(S \setminus \{l\})$-coloring of $G \setminus C_l$ using at most $k(d+2)$ colors for precolorless vertices. This extension is obviously also an extension of g to an S-coloring of G using at most $k(d+2)$ colors for precolorless vertices, a contradiction.

If we denote with L the set of precolorless vertices, then due to part (b) $|L| \geq |S|$ and obviously $|C| \leq |S|d$. It means, $|C| \leq |S|d \leq |L|d$ and clearly $|L| + |C| = |V|$. Therefore, $d|L| + |L| \geq |V|$ and hence the number of precolorless vertices is at least $\frac{|V|}{d+1}$. But then due to (a) at least $\frac{|V|}{d+1}$ vertices have degree at least $k(d+2)$, a contradiction with (3). $\square$

The previous theorem implies that $\chi_d \leq 2(d+2)$ for the class of outerplanar graphs and $\chi_d \leq 5(d+2)$ for the class of planar graphs. However, a stronger result can be proved for planar graphs using bound from Remark 1 in the proof of Theorem 1.

Theorem 2 *Let $d \geq 1$ be a fixed integer. For any (d, S)-precoloring of a planar graph if there is an extension to the entire graph, then there is also an extension which uses at most $3(d+2)$ colors of S, i.e., $\chi_d \leq 3(d+2)$ for the class of planar graphs.*

For $d = 0$, $\chi_0(G)$ is the chromatic number of a graph G. Hence $\chi_0 = 2$ for trees, $\chi_0 = 3$ for outerplanar graphs, and $\chi_0 = k+1$ for k-degenerate graphs and partial k-trees as follows from the inductive construction of these classes (see, e.g., [5] for partial k-trees). In case of planar graphs $\chi_0 = 4$ as follows from the Four Color Theorem [15]. In the following we establish the optimal value χ_d for trees and lower and upper bounds for some subclasses of k-degenerate graphs, for $d \geq 1$.

Theorem 3 *Let $d \geq 1$ be a fixed integer. Then*

(a) $\chi_d = d + 2$ *for the class of trees,*

(b) $\chi_1 \geq 4$ *and* $\chi_d \geq d + 4$, $d \geq 2$, *for the class of outerplanar graphs,*

(c) $\chi_d \geq 2d + 4$ *for the class of planar graphs,*

(d) $\chi_d \geq k(d+1) + 1$ *for the class of partial k-trees and k-degenerate graphs, $k \geq 1$.*

Proof. (a) Let $S = \{1, 2, \ldots, d+2\}$ be a set of colors. We will construct a (d, S)-precolored tree $T = (V, E)$ such that every extension to the entire tree T has to use all $d+2$ colors of S.

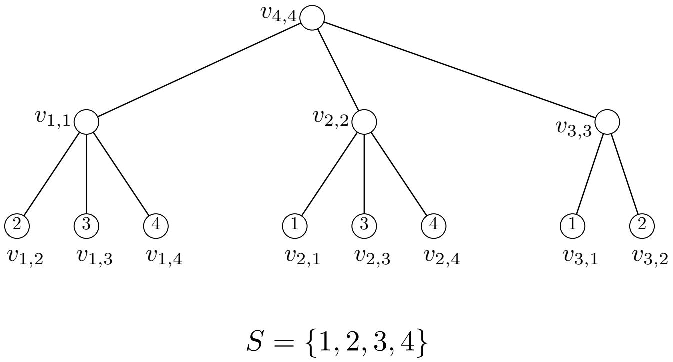


Figure 1: An example of precolored tree T for $d = 2$ with $\chi_2(T) \geq 4$.

Denote $C = \bigcup_{i\in\{1,\ldots,d\},j\in\{1,\ldots,d+2\},i\neq j}\{v_{i,j}\} \cup \bigcup_{j\in\{1,\ldots,d\}}\{v_{d+1,j}\}$ the set of precolored vertices,

$$V = \bigcup_{i\in\{1,\ldots,d+2\}} \{v_{i,i}\} \cup C, \text{ and}$$

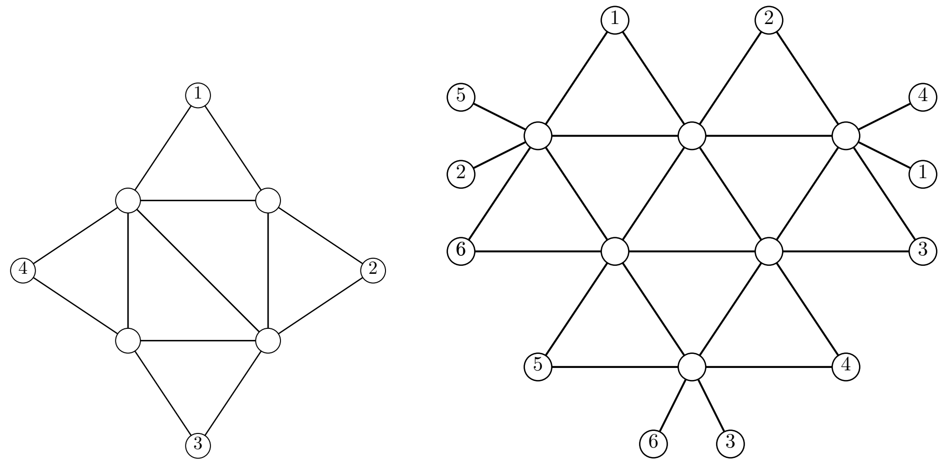
$$E = \bigcup_{i\in\{1,\ldots,d+1\}} \{\{v_{d+2,d+2}, v_{i,i}\}\} \cup \bigcup_{i\in\{1,\ldots,d\},j\in\{1,\ldots,d+2\},i\neq j} \{\{v_{i,i}, v_{i,j}\}\} \cup \bigcup_{j\in\{1,\ldots,d\}} \{\{v_{d+1,d+1}, v_{d+1,j}\}\}$$

(see Fig. 1).

Define a (d, S)-precoloring g of the subset C of V as follows: $g(v_{i,j}) = j$. Clearly each color j of S is used exactly for d precolored vertices. In any extension to an S-coloring of the entire T we have to color the precolorless vertex $v_{i,i}$ with color i, $i \in \{1, \ldots, d\}$. Furthermore we need two different colors for precolorless vertices $v_{d+1,d+1}$ and $v_{d+2,d+2}$. This together with the upper bound $\chi_d \leq d+2$ from Theorem 1 implies $\chi_d = d+2$ for trees.

(b) Examples of outerplanar graphs with given (d, S)-precoloring for values $d = 1$, $d = 2$, resp. $d = 3$, are given in Figures 2 and 3. It can be easily checked that they require to use all 4, 6, resp. 7 colors, for any extension to the entire graph.

For $d \geq 4$, let $S = \{1, 2, \ldots, d+4\}$ denote the set of available colors. We describe the construction of a (d, S)-precolored outerplanar graph G which requires $d+4$ colors for every extension to the entire G (see Fig. 4). We start with an induced subgraph G_L of



$S = \{1,2,3,4\}$ $\qquad$ $S = \{1,2,3,4,5,6\}$

Figure 2: Examples of precolored outerplanar graphs G for $d = 1$, resp. $d = 2$, with $\chi_1(G) \geq 4$, resp. $\chi_2(G) \geq 6$.

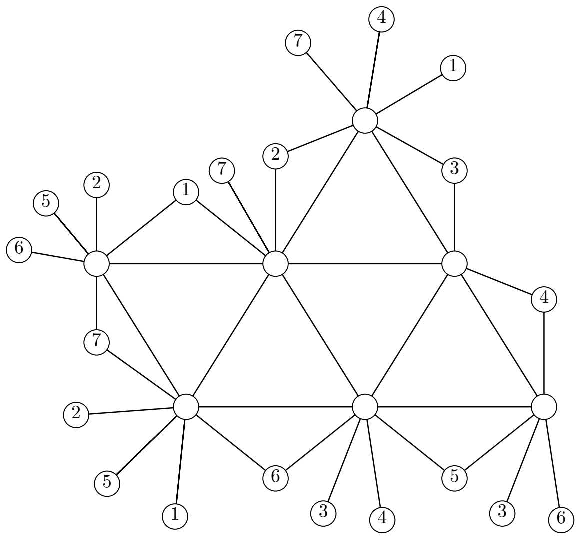
G created by the set of all precolorless vertices $L = \{v_1, \ldots, v_{d+4}\}$ in G. Denote $E_L := \bigcup_{i\in\{1,\ldots,d+2\}}\{\{v_i, v_{i+2}\}\} \cup \bigcup_{i\in\{1,\ldots,d+3\}}\{\{v_i, v_{i+1}\}\}$ the set of edges between precolorless vertices in G_L. Now to create the graph G, we add for each color of S exactly d precolored vertices to G_L and some edges in such way that in any extension each color $i \in S$ can be used only for precolorless vertices of a triangle in G. Then each color i can be used for at most one precolorless vertex. It means, if there exists an extension, then necessarily all colors of S have to be used.

This can be realized in the following way: for each color $i \in S$ we add to G_L a vertex u_i precolored with color i and edges

- $\{\{u_i, v_i\}, \{u_i, v_{i+2}\}\}$ for $i \in \{1, \ldots, d+2\}$,
- $\{\{u_{d+3}, v_{d+3}\}, \{u_{d+3}, v_{d+4}\}\}$ for $i = d+3$,
- $\{\{u_{d+4}, v_1\}, \{u_{d+4}, v_2\}\}$ for $i = d+4$.

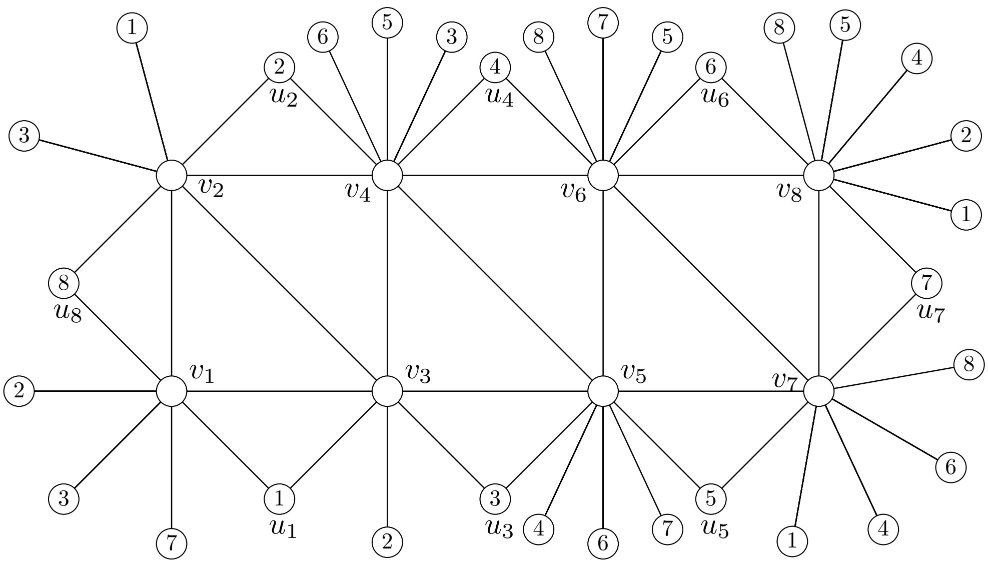
In this way color i is forbidden for coloring exactly two precolorless vertices of G_L. Furthermore for each color i we add the set C_i of $d-1$ precolored vertices. Each vertex $v \in C_i$ will be connected with only one precolorless vertex of L (a different vertex from C_i with a different vertex from L) in such way that color i can be used for coloring only the vertices of the following triangle in G

- $\{v_{i+3}, v_{i+4}, v_{i+5}\}$ for colors $i \in \{1, \ldots, d-1\}$,



$S = \{1, 2, 3, 4, 5, 6, 7\}$

Figure 3: An example of precolored outerplanar graph G for $d = 3$ with $\chi_3(G) \geq 7$.



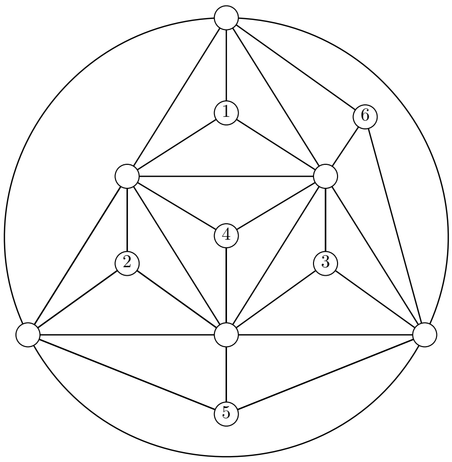
$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Figure 4: An example of precolored outerplanar graph G for $d = 4$ with $\chi_4(G) \geq 8$.

- $\{v_1, v_2, v_3\}$ for colors $i \in \{d, d+1, d+2\}$,
- $\{v_2, v_3, v_4\}$ for color $i = d+3$,
- $\{v_3, v_4, v_5\}$ for color $i = d+4$.

Clearly, this construction can be realized in outerplanar graphs because all vertices of C_i are of degree 1. It is also easy to see that there is an extension of the defined (d, S)-precoloring.

(c) An example of a planar graph with precoloring for $d = 1$ which requires 6 colors is given in Figure 5.



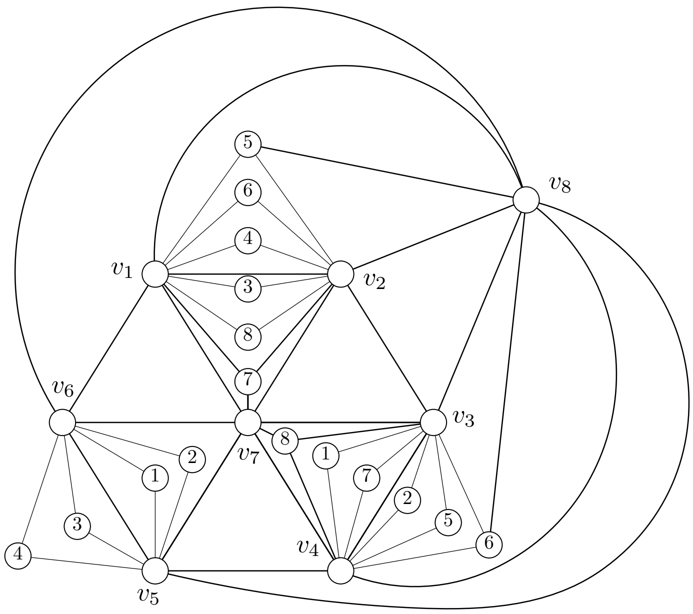
$S = \{1, 2, 3, 4, 5, 6\}$

Figure 5: An example of precolored planar graph G for $d = 1$ with $\chi_1(G) \geq 6$.

Now we show how to generalize the construction for $d \geq 2$ (see Figure 6 for $d = 2$). We define a (d, S)-precolored planar graph G with the property that every extension to the entire graph G has to use $2d + 4$ colors of S.

Let $S = \{1, 2, \ldots, 2d + 4\}$ be the set of colors and $L = \{v_1, \ldots, v_{2d+4}\}$ be the set of all precolorless vertices of G. The vertices $v_1, \ldots, v_{2d+2}$ will create a cycle (in this order). The vertex v_{2d+3} (resp. v_{2d+4}) will be inside (resp. outside) of the cycle and will be adjacent to all vertices $v_1, \ldots, v_{2d+2}$.

Now we describe the set of precolored vertices. For each edge $\{v_{2i+1}, v_{2i+2}\}$, $i \in \{0, \ldots, d-1\}$ we add the set of precolored vertices with all colors except $2i+1$, $2i+2$. It means that in any extension the precolorless vertices v_{2i+1} and v_{2i+2} can be colored only with colors $2i + 1$ and $2i + 2$. Furthermore, in case $i = 0$ or $i = 1$



$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Figure 6: An example of precolored planar graph G for $d = 2$ with $\chi_2(G) \geq 8$.

- the vertex precolored with color $2d + 3$ is adjacent to vertices of the triangle v_1, v_2, v_{2d+3} (the same for the vertex precolored with $2d + 4$ and the triangle v_3, v_4, v_{2d+3}),
- the vertex precolored with color $2d + 1$ (for the edge $\{v_1, v_2\}$) and the vertex precolored with color $2d + 2$ (for the edge $\{v_3, v_4\}$) will be adjacent to the vertex v_{2d+4}.

Until now we have used each color from $\{2d + 1, \ldots, 2d + 4\}$ for d precolored vertices, but each of other colors only for $d - 1$ precolored vertices. These other colors can be used to forbid colors $1, 2, \ldots, 2d$ for precolorless vertices v_{2d+1} and v_{2d+2}. Similarly as for $d = 2$, this construction can be done in the class of planar graphs.

Clearly, in any extension colors $2i + 1$, $2i + 2$ ($i \in \{0, \ldots, d - 1\}$) have to be used to color the precolorless vertices v_{2i+1} and v_{2i+2}. Furthermore we need 4 different colors for vertices v_{2d+1}, v_{2d+2}, v_{2d+3}, and v_{2d+4}.

(d) We will construct a (d, S)-precolored partial k-tree G which requires to use $k(d + 1) + 1$ colors for any extension to the entire graph G. Let $S = \{1, 2, \ldots, k(d + 1) + 1\}$ be the set of colors. Take $d + 1$ copies of a k-clique, namely $G_1, \ldots, G_{d+1}$, which will contain all precolorless vertices of G. For each clique G_i, $i \in \{1, \ldots, d\}$ we add $kd + 1$ precolored vertices colored with all colors except $(i - 1) \cdot k + 1, (i - 1) \cdot k + 2, \ldots, i \cdot k$ and each of them will be adjacent to all vertices of G_i. It means, the admissible set for every precolorless vertex of G_i will contain k colors $(i-1)\cdot k+1, (i-1)\cdot k+2, \ldots, i\cdot k$. Now we have still kd colors $1, \ldots, kd$, each of them were used only for $d-1$ precolored vertices. These are used to color others kd precolored vertices, each adjacent to all precolorless vertices of G_{d+1}. To finish the construction of G, we add a new precolorless vertex v adjacent to all vertices of cliques G_i, $i \in \{1, \ldots, d+1\}$ (see Fig. 7). It is easy to see that G is a partial k-tree and every extension of the d-precoloring to the entire G uses all $k(d+1)+1$ colors of S. $\square$

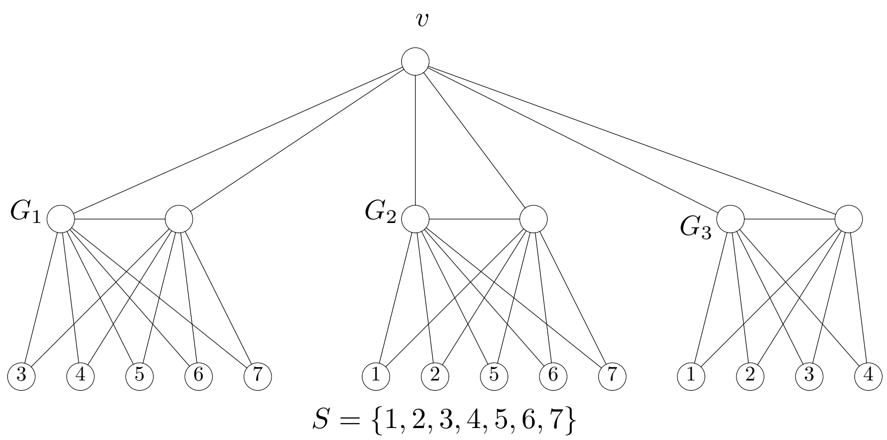


Figure 7: An example of precolored partial 2-tree G for $d = 2$ with $\chi_2(G) \geq 7$.

3 Algorithm Results

In this section we present a linear time algorithm for the d-PrExt problem and its search version (i) restricted to the class of k-degenerate graphs (and hence also planar graphs) with sufficiently large set S of available colors, (ii) restricted to the class of partial k-trees without any size restriction on the set of available colors S.

Our result is based on a *safe* reduction, which reduces a precolored k-degenerate graph to a bounded irreducible instance. The basic property of the safe reduction is to keep the same answer to PrExt problem in every step of the reduction.

Definition. *Let a set S of colors be fixed and $G = (V, E)$ be a graph with a fixed S-precoloring. For a precolorless vertex $v \in V$, let C_v be the set of colors occurring on precolored neighbors of v and let L_v be the set of precolorless neighbors of v. A vertex v is called safe, if v is precolorless and $|C_v| + |L_v| < |S|$. Deleting a safe vertex from G is called a safe reduction for the S-precolored graph G. An ordering $Q = [v_1, \ldots, v_m]$ of a subset of precolorless vertices from V is called a safe elimination scheme of G if for all i, $i \in \{1, \ldots, m\}$, v_i is safe in the graph $G \setminus \{v_1, \ldots, v_{i-1}\}$. An S-precolored graph is called safe-free (or irreducible) if it contains no safe vertex.*

Observation. Let $G = (V, E)$ be an instance of PREXT and $Q = [v_1, \ldots, v_m]$ be a safe elimination scheme of G. Then obviously the answer to the PREXT problem is the same for G as for any $G \setminus \{v_1, \ldots, v_i\}$, $i \in \{1, \ldots, m\}$.

First we prove the theorem describing crucial properties of safe-free k-degenerate instances for the d-PREXT problem. It is also easy to see that a similar theorem cannot hold for instances of the PREXT problem.

Theorem 4 *Let $d, k \geq 1$ be fixed integers and let G be a safe-free k-degenerate (d, S)-precolored graph, $|S| \geq 2k + 1$. Then the following hold:*

(a) G contains at most $\alpha(k, d)$ precolorless vertices, where $\alpha(k, d) = 2k^2d + kd - 1$,

(b) size of the admissible set of every precolorless vertex is at most $\alpha(k, d) - 1$.

Moreover, for $|S| \geq 2k(kd + 1)$, the number of precolorless vertices in G is at most kd and size of the admissible set of every precolorless vertex is at most $kd - 1$.

Proof. For a graph $G = (V, E)$ we denote C and L the set of precolored and precolorless vertices of G, respectively.

(a) According to the definition of a safe-free graph, every precolorless vertex of G has degree at least $|S|$, which together with Lemma 1 implies

$$|L| < \frac{|V|k}{|S| - k}. \tag{4}$$

Due to the fact that $|V| = |C| + |L|$ and the trivial bound $|C| \leq |S|d$, we can obtain from (4)

$$|L| < \frac{|C|k}{|S| - 2k} \leq \frac{|S|kd}{|S| - 2k}.$$

The function $g(|S|) := \frac{|S|kd}{|S|-2k}$ is decreasing for $|S| \geq 2k + 1$ and hence $|L| < 2k^2d + kd$. Moreover, $|L| < kd + 1$, if $|S| \geq 2k(kd + 1)$.

(b) For every precolorless vertex v the sum of the number of distinct colors occurring on its precolored neighbors together with the number of its precolorless neighbors is at least $|S|$. Due to part (a) every precolorless vertex has less than $\frac{|S|kd}{|S|-2k} - 1$ precolorless neighbors. This means that for every precolorless vertex more than $|S| - \frac{|S|kd}{|S|-2k} + 1$ distinct colors occur on its precolored neighbors and only less than $\frac{|S|kd}{|S|-2k} - 1$ colors belong to the admissible set of v. □

Remark 2 *We can obtain a stronger result for planar graphs using Remark* 1 *in the proof of the previous theorem: any safe-free* (d, S)*-precolored planar graph,* $|S| \geq 7$*, contains at most* $21d - 12$ *precolorless vertices and the admissible set of every precolorless vertex is at most* $21d - 13$.

For a k-degenerate graph G, a safe-elimination scheme Q and a safe-free subgraph $G \setminus Q$ can be found in linear time, as follows from the following theorem.

Theorem 5 *Let* $G = (V, E)$ *be a* k*-degenerate* S*-precolored graph. Then a safe-elimination scheme* Q *with a safe-free subgraph* $G \setminus Q$ *can be constructed in linear time* $O(|V|)$.

Proof. Suppose that for every precolorless vertex $v \in V$ the set N_v of its neighbors is given. For every precolorless vertex v we create the sets L_v and C_v of precolorless neighbors and colors used for precolored neighbors of v, respectively. Furthermore, we add to a set W all safe vertices of G. This first step together requires $O(|E|)$ time. In second step we can apply the following loop until the set W is empty: (i) Choose any vertex $w \in W$ and remove it from G at the end of the ordering Q. (ii) For every precolorless neighbor u of w delete w from the set L_u and if u becomes safe in $G := G \setminus v$, add u into the set W.

The following pseudocode describe the algorithm:

$Q := \emptyset$; $W := \emptyset$;
{`Step 1:`}
for every precolorless vertex $v \in V$ **do**
  begin $L_v := \emptyset$; $C_v := \emptyset$;
    for every neighbor u of v **do**
      if u is a precolorless vertex **then** $L_v := L_v \cup \{u\}$
      else $C_v := C_v \cup color(u)$;
    if $|L_v| + |C_v| < |S|$ **then** $W := W \cup \{v\}$;
  end;
{`Step 2:`}
while $W \neq \emptyset$ **do**
  begin choose $w \in W$; $W := W \setminus \{w\}$;
    $Q := Q[w]$; {add w at the end of Q}
    for every precolorless neighbor u of w **do**
      begin $L_u := L_u \setminus \{w\}$;
        if $|L_u| + |C_u| < |S|$ **then** $W := W \cup \{u\}$;
      end;
  end;

In the **while** loop every precolorless vertex appears at most once in W and the **for** loop deletes always one edge of G. Hence, this step can be done also in time $O(|E|)$.

Due to the fact that $|E| \leq k|V|$ for k-degenerate graphs, the total time complexity of the algorithm is $O(|V|)$. □

The assumption $|S| \geq k+1$ is sufficient to obtain a linear time algorithm for the decision and search version of the 0-PrExt problem for k-degenerate graphs, as it follows simply from its construction. In the following theorem we prove that similar results hold also for the d-PrExt problem, $d \geq 1$.

Theorem 6 *Let $d,\ k \geq 1$ be fixed integers and let G be a (d,S)-precolored k-degenerate graph, $|S| \geq 2k+1$. Then the d-PrExt problem and its search version can be solved in linear time $O(|V|)$.*

Proof. We suppose that $|S| < |V|$, otherwise the problem is trivial.

(a) Using the algorithm from Theorem 5 we can reduce in $O(|V|)$ time a k-degenerate (d,S)-precolored graph G into a safe-free subgraph $G \setminus Q$ (with a safe elimination scheme Q) such that the answer to the d-PrExt problem for G and the safe-free subgraph $G \setminus Q$ are the same. Assuming $|S| \geq 2k+1$, a safe-free (d,S)-precolored subgraph $G \setminus Q$ contains at most $\alpha(k,d)$ precolorless vertices and the size of the admissible set of any of them is at most $\alpha(k,d)-1$. (If the reduced safe-free graph is empty, the answer is always YES.)

(b) For every precolorless vertex $v \in G \setminus Q$ we construct the admissible set of colors $S \setminus C_v$. This requires at most $\alpha(k,d)|S|$ operations, hence $O(|V|)$ time due to the assumption $|S| < |V|$.

(c) In last step we generate all possible assignments for precolorless vertices such that color of every precolorless vertex is chosen from its admissible set. The answer is affirmative if and only if there exists at least one assignment corresponding to a (proper) coloring of the subgraph $G \setminus Q$. This requires at most $\alpha(kd)^{\alpha(k,d)-1}$ operations, it means a constant time independent of $|S|$ and $|V|$.

Hence, going through all steps (a)–(c), the algorithm for d-PrExt requires linear time $O(|V|)$.

For the search version of d-PrExt, we fix one coloring of the precolorless vertices of $G \setminus Q$ according (c) (when the elimination scheme Q contains all vertices, we start with an empty graph). An extension for the entire graph G can be easily found using the safe elimination scheme $Q = [v_1, \ldots, v_m]$ in linear time $O(|V|)$. Denote $G_0 = G$ and $G_i = G \setminus \{v_1, \ldots, v_i\}$ for $i \in \{1, \ldots, m\}$. We use a boolean vector p of length $|S|$ for storing an information which colors were used for neighbors of v_i: $p[j] =$ TRUE if and only if there is a neighbor of v_i colored by color j. Initially, $p[j] :=$ FALSE for all $j \in S$. Suppose that we have colored the graph G_i, $i \in \{1, \ldots, m\}$. It means all neighbors of v_i in G_{i-1} are colored and less than $|S|$ colors occur on all its neighbors. Hence there is color of S which can be used for v_i to obtain G_{i-1} colored as well. In $d_{G_{i-1}}(v_i)$ steps we can store in the vector p colors used for neighbors of v_i and just checking at most $d_{G_{i-1}}(v_i)+1$ first items in p we have to find one unused color for v_i of S. Then again in $d_{G_{i-1}}(v_i)$ steps we return the vector p into initialization position. The number of such steps for all vertices in G can be bounded by $\sum_{1\le i\le m}(3d_{G_{i-1}}(v_i)+1) \le \sum_{1\le i\le m}(3d_G(v_i)+1) < 7|E|$. The rest follows from the fact that the number of edges in a k-degenerate graph is at most $k|V|$. $\square$

Recall that for planar graphs 0-PrExt is NP-complete for $|S|=3$ even for graphs with maximum degree 4 ([8]), the search version of 0-PrExt can be solved in quadratic time for $|S|=4$ ([15]) and in linear time for $|S|\ge 5$ ([4], [7]). The previous theorem implies a linear time algorithm for both version of d-PrExt for planar graphs in case $|S|\ge 11$. However, in the same way as previous theorem (and using Remark 2) a stronger result can be proved in such case:

Theorem 7 *Let $d\ge 1$ be a fixed integer and let G be a (d,S)-precolored planar graph, $|S|\ge 7$. Then the d-PrExt problem and its search version can be solved in linear time $O(|V|)$.*

Linear time algorithms for many intractable problems over the class of partial k-trees follow from a general result: all graph properties definable in monadic second-order logic with quantification over vertex and edge sets can be decided in linear time for partial k-trees [6]. However, the PrExt problem does not appear to be directly expressible by these means. It is an open problem, whether the property "a precoloring of a graph is extendable to all vertices" is an EMS-property when a set of colors is a part of the input. Even though we do not know the answer to the previous problem, we have derived a linear time algorithm for the d-PrExt problem and its search version for partial k-trees.

Theorem 8 *Let $d,k\ge 1$ be fixed integers and let G be a (d,S)-precolored partial k-tree. Then the d-PrExt problem and its search version can be solved in linear time $O(|V|)$.*

Proof. For $|S|\ge 2k+1$, the theorem follows directly from Theorem 6. For $|S|<2k+1$, we apply a linear time algorithm for constant number of colors for PrExt and its search version from [11, Theorem 2.4]. $\square$

Conclusion

The following natural question arises: Is there a linear time algorithm to extend the (d,S)-precoloring with bounded number of colors of S for the precolorless vertices in the class of k-degenerate graphs?

Other interesting questions are to tighten the bounds for χ_d for any pair of $k\ge 2$ and $d\ge 1$ and subclasses of k-degenerate graphs or to show the complexity of the d-PrExt problem for planar graphs with $|S|\in\{4,5,6\}$.

References

[1] S. Arnborg and A. Proskurowski: *Recognition of partial k-trees*, Congressus Numerantium **47** (1985), 69–75.

[2] M. Biró, M. Hujter and Z. Tuza: *Precoloring extension. I. Interval graphs*, Discrete Math. **100** (1992), 267–279.

[3] H. L. Bodlaender, K. Jansen and G. Woeginger: *Scheduling of incompatible jobs*, Discrete Applied Mathematics **55** (1994), 219–232.

[4] N. Chiba, T. Nishizeki and N. Saito: *A linear 5-coloring algorithm of planar graphs*, Journal of Algorithms **2(4)** (1981), 317–327.

[5] J. Chlebíková: *Partial k-trees with maximum chromatic number*, Discrete Mathematics **259**(1-3) (2002), 269–276.

[6] B. Courcelle: *The monadic second order logic of graphs I: Recognizable sets of finite graphs*, Information and Comput. **85** (1990), 12–75.

[7] G. N. Frederickson: *On linear-time algorithms for five-coloring planar graphs*, Information Processing Letters **19(5)** (1984), 219–224.

[8] M. R. Garey and D. S. Johnson: *Computers and Intractability: A Guide to the Theory of NP-completeness*, Freeman, San Francisco, 1979.

[9] M. Hujter and Z. Tuza: *Precoloring extension. II. Graphs classes related to bipartite graphs*, Acta Math. Univ. Comenianae **LXII**(1) (1993), 1–11.

[10] M. Hujter and Z. Tuza: *Precoloring extension. III. Classes of perfect graphs*, Combin. Probab. Comput. **5**(1996), 35–56.

[11] K. Jansen and P. Scheffler: *Generalized coloring for tree-like graphs*, Discrete Applied Mathematics **75** (1997), 135–155.

[12] K. Jansen: *The optimum cost chromatic partition problem*, Algorithms and Complexity, Proc. CIAC 3 (G. Bongiovanni et al., eds.), Lecture Notes in Computer Science **1203**, 1997, 25–36.

[13] J. Kratochvíl: *Precoloring extension with fixed color bound*, Acta Math. Univ. Comenianae **LXII**(2) (1993), 139–153.

[14] J. Kratochvíl and A. Sebö: *Coloring precolored perfect graphs*, J. Graph Theory **25** (1997), 207–215.

[15] N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas: *The Four Color Theorem*, J. Combin. Theory Ser. **B70** (1997), 2–44.

[16] Zs. Tuza: *Graph colorings with local constraints – a survey*, Discussiones Mathematicae Graph Theory **17**(2) (1997), 161–228.