Confidence levels q-rung orthopair fuzzy aggregation operators and its applications to MCDM problems

Bhagawati Prasad Joshi[1] and Alexander Gegov[2]
[1]Department of Applied Sciences, Seemant Institute of Technology, Pithoragarh 262 501, India
[2]School of Computing, University of Portsmouth, Portsmouth PO1 3HE, United Kingdom
Email: [1]bpjoshi.13march@gmail.com; [2]alexander.gegov@port.ac.uk

Abstract: The concept of q-rung orthopair fuzzy set (q-ROFS) is the extension of intuitionistic fuzzy set (IFS) in which the sum of the q^{th} power of the support for and the q^{th} power of the support against is bounded by one. Therefore, the q-ROFSs are an important way to express uncertain information in broader space, and they are superior to the IFSs and the Pythagorean fuzzy sets (PFSs). In this paper, the familiarity degree of the experts with the evaluated objects is incorporated to the initial assessments under q-rung orthopair fuzzy environment. For this, some aggregation operators are proposed to combine these two types of information. Their some important properties are also well proved. Furthermore, these developed operators are utilized in a multi criteria decision making approach and demonstrated with a real life problem of customers' choice. Then, the experimental results are compared with other existing methods to show its superiority over recent research works.

Keywords: Intuitionistic fuzzy set; Pythagorean fuzzy set; q-rung orthopair fuzzy set; MCDM problems; confidence levels.

1. Introduction

At present, multi-criteria decision making (MCDM) is a fast growing research field which provides the best possible option from the set of finite alternatives on the basis of certain criteria. But it is not possible to express the preferences more efficiently and precisely because of the complexity and various constraints subjected to the real world decision making problems. To cope with such situations, Zadeh (1965) introduced fuzzy sets (FSs) and which was further extended by Atanassov (1986) by introducing the concept of intuitionistic fuzzy set (IFS), which is characterized in such a way that the sum of the support for membership and support against membership is less than or equal to one. Due to this characteristic, IFS theory is one of the successful and powerful tools to deal with imprecise, vague and ambiguous information, and receives attention to many practitioners (Song and Chissom 1993; Chen 1996; De et al. 2001; Singh 2007; Joshi and Kumar 2012; Joshi and Kumar 2013; Joshi 2018; Joshi et al. 2018; Garg and Arora 2019; and et al.) to deal with real life situations. But, the aggregation of all the performances in dealing with real life problems is a very critical step to obtain decisions. Therefore, the aggregation operators obtain an important role during the information fusion process. For this, Xu and Yager (2006) presented the intuitionistic fuzzy weighted geometric (IFWG) operator and Xu (2007) presented intuitionistic fuzzy weighted average (IFWA) operator to add intuitionistic fuzzy numbers (IFNs). These are most basic and widely cited aggregation operators under intuitionistic fuzzy environment. Based on these operators, intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and intuitionistic fuzzy hybrid geometric (IFHG) presented by Xu and Yager (2006), intuitionistic fuzzy ordered weighted average (IFOWA) operator and intuitionistic fuzzy hybrid average (IFHA) operator presented by Xu (2007), generalized IFWA, generalized IFWG, generalized IFOWA, generalized IFOWG presented by Zhao et. al. (2010). Wang and Liu (2011) presented intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator and the intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) operator. Later on, Wang and Liu (2012) proposed intuitionistic fuzzy Einstein weighted averaging (IFEWA) operator and the intuitionistic fuzzy Einstein ordered weighted averaging (IFEOWA) operator. Next, Zhao and Wei (2013) proposed the intuitionistic fuzzy Einstein hybrid geometric (IFEHG) operator and intuitionistic fuzzy Einstein hybrid averaging (IFEHA) operator on the basis of Einstein sum and Einstein product. Many practitioners such as Xu (2010), Tan and Chen (2010), Xia and Chen (2011), Xia and Xu (2013), Yu (2013), Cagman and Karatas (2013), Yu (2014), Ma and Zeng (2014), Delia and Cagman (2015), Joshi and Kharayat (2016), and etc have investigated the MCDM problems under the different aspects of intuitionistic fuzzy environments. Furthermore, a comprehensive study of intuitionistic fuzzy aggregation operators to solve MCDM problem has been compiled by Yu (2015) and Xu and Zhao (2016).

The above studies are suitable under intuitionistic fuzzy environment only due to the limitation of the sum of membership grade and non-membership grade is bounded by one. But, in real life situations it is not always possible for the decision makers to provide preferences on this limitation. This type of situations is successfully handled with the Pythagorean fuzzy set (PFS) theory, proposed by Yager (2013) as an extension of IFS theory by relaxing the condition $0 \le t + f \le 1$ to $0 \le t^2 + f^2 \le 1$, where t and f represents the degrees of the satisfaction and dis-satisfaction of an object. This pioneering study is further studied by Yager and Abbasov

(2013) and revealed the relationship between the Pythagorean fuzzy numbers (PFNs) and the complex numbers. In order to combine PFNs, Yager (2014) presented a Pythagorean fuzzy weighted average (PFWA) operator, a Pythagorean fuzzy weighted geometric (PFWG) operator, a Pythagorean fuzzy weighted power average (PFWPA) operator and a Pythagorean fuzzy weighted power geometric average (PFWPG) operator. Zhang and Xu (2014) proposed an extended technique for order preference by similarity to ideal solution (TOPSIS) method under Pythagorean fuzzy environment. Peng and Yang (2015) proposed the division and subtraction for PFNs and proved their some properties. Peng and Yang (2016) considered the inter-dependency among the PFNs and presented a Pythagorean fuzzy Choquet integral average (PFCIA) operator and a Pythagorean fuzzy Choquet integral geometric (PFCIG) operator. Garg (2016a) presented some operators namely, Pythagorean fuzzy Einstein weighted averaging (PFEWA), Pythagorean fuzzy Einstein ordered weighted averaging (PFEOWA), generalized PFEWA, and generalized GPFEOWA on the basis of Einstein sum and Einstein product. Garg (2016b) proposed a novel correlation coefficient and weighted correlation coefficient formulation to measure the relationship between two PFSs. Garg (2017a) further extended these operators in geometric aspect. Garg (2017b) proposed a confidence Pythagorean fuzzy weighted averaging (CPFWA) operator and a confidence Pythagorean fuzzy ordered weighted averaging (CPFOWA) operator along with their some desired properties. Zhang et. al. (2017) extended the generalized Bonferroni mean to the Pythagorean fuzzy environment and introduced the generalized Pythagorean fuzzy Bonferroni mean and the generalized Pythagorean fuzzy Bonferroni geometric mean. Joshi (2019) presented some generalized Pythagorean fuzzy average aggregation operators by incorporating the concept of the generalized parameter to the Pythagorean fuzzy set theory.

With continuous complication of modeling human knowledge and the development of theory, Yager (2017) introduced a new concept called it q-rung orthopair fuzzy set (q-ROFS), in which the sum of the q^{th} power of the support for membership and the q^{th} power of the support against membership is bounded to one, and further proved that the q-ROFS is more general because IFS and PFS are all its special cases. We have to also note that as the rung q raises the space of acceptable orthopairs raises and thus provides the observers more liberty in expressing their belief in order to support for membership degree. Therefore, the q-ROFSs express a wider range of fuzzy information and are more flexible and more suitable tool to handle the uncertain environment. Yager and Alajlan (2017) discussed basic properties of these q-ROFSs and use these sets in knowledge representation. Recently, Liu and Wang (2018) proposed the q-rung orthopair fuzzy weighted averaging (q-ROFWA) operator and the q-rung orthopair fuzzy weighted geometric (q-ROFWG) operator, and develop some methods based on these operators to solve the MCDM problems. Joshi et al. (2018) introduced the concept of q-rung orthopair fuzzy sets (q-ROFSs) in which the sum of the q^{th} exponent of the support for membership and the q^{th} exponent of the support against membership is bounded by one. Some of its important operations such as: negation, union and intersection are also presented. Jun et al. (2019) proposed a family of q-rung orthopair fuzzy Muirhead mean operators for combining q-rung orthopair fuzzy information. Recently, Peng and Liu (2019) presented the information measures for q-ROFSs such as: distance measure, similarity measure, entropy, and inclusion measure,

Besides these surprisingly accomplishments under q-rung orthopair fuzzy environment, all of the existing efforts do not incorporate the familiarity degree in the information fusion step. The experts in a MCDM problem give performance of the alternatives on the basis of the mentioned criteria only i.e. the familiarity (called confidence levels) of experts with the evaluation objects is not included. So, it is must to incorporate the familiarity of observer in the original information under q-rung orthopair fuzzy environment. This type of shortcoming is focused in this study by incorporating the confidence levels of experts for their familiarity and awareness with the evaluated alternatives in the q-rung orthopair fuzzy information fusion step. To fuse these two types of information some confidence q-rung orthopair fuzzy aggregation operators namely, confidence q-rung orthopair fuzzy weighted average ($CFWA_q$), confidence q-rung orthopair fuzzy ordered weighted average ($CFWOA_q$), confidence q-rung orthopair fuzzy weighted geometric ($CFWG_q$), confidence q-rung orthopair fuzzy ordered weighted geometric ($CFOWG_q$) operators are proposed. Their some important properties are well established. These defined operators are capable to explain the real life situation more perceptibly with the help of their parameterizations property (confidence levels) under q-rung orthopair fuzzy environment.

Rest of the article is organised as follows. The coming section briefly reviews related to q-ROFSs. Then, some confidence q-rung orthopair fuzzy aggregation operators are developed in section 3. Section 4 provides a MCDM approach based on the developed operators and illustrated with a real life problem of customers' choice. The sensitivity analysis of proposed operators for different q-rung is discussed in section 5. Section 6 presents a detailed comparative analysis to show the feasibility and superiority of the proposed approach over the existing ones. In addition, some counter examples are also considered where existing methods fail but our approach can overcome their shortcomings. Finally, the conclusion of this study is summarized in section 7.

2.2. *q*-rung orthopair fuzzy set

Yager (2017) introduced the concept of *q*-rung orthopair fuzzy set (*q*-ROFS), as an extension of IFS theory and presented as follows:

Definition 1 (Yager; 2017). A *q*-ROFS *A* in the universal set *X* is defined as an object of the following form $A=\{\langle x, t_A(x), f_A(x)\rangle : x\in X\}$, where the functions $t_A : X \to [0,1]$ and $f_A : X \to [0,1]$ define the "support for membership" and the "support against membership" of the element $x \in X$ respectively, with the restriction $0 \le t_A(x)^q + f_A(x)^q \le 1, (q \ge 1)$. The degree of non-determinacy (uncertainty) for each element *x* of *X* in the *q*-ROFS *A*, is defined by $\pi_A(x) = \left(1 - t_A(x)^q - f_A(x)^q\right)^{1/q}$, where $\pi_A(x) \in [0,1]$.

For convenience, $\langle t_A(x), f_A(x)\rangle$ is called a *q*-rung orthopair fuzzy number (*q*-ROFN) and it can be written as $a = (t_a, f_a)$. Liu and Wang (2018) presented the following score and accuracy function to compare two *q*-ROFNs.

Definition 2 (Liu and Wang; 2018). Let $a = (t_a, f_a)$ and $b = (t_b, f_b)$ be two *q*-ROFNs, then $S(a) = t_a{}^q - f_a{}^q$, $S(b) = t_b{}^q - f_b{}^q$ are the score functions and $H(a) = t_a{}^q + f_a{}^q$, $H(b) = t_b{}^q + f_b{}^q$ are the accuracy functions of *a* and *b*. If $S(a) < S(b)$ then *a* is smaller than *b*, denoted by $a < b$, and if $S(a) = S(b)$, then if $H(a) < H(b)$ then *a* is smaller than *b*, denoted by $a < b$. If $H(a) = H(b)$ then *a* and *b* represent the same information, denoted by $a = b.$

For any two *q*-ROFNs $a = (t_a, f_a)$ and $b = (t_b, f_b)$, the following basic laws are defined:

1) $\bar{a} = (f_a, t_a)$.
2) $a \vee b = (\max\{t_a, t_b\}, \min\{f_a, f_b\})$.
3) $a \wedge b = (\min\{t_a, t_b\}, \max\{f_a, f_b\})$.
4) $a \oplus b = ((t_a{}^q + t_b{}^q - t_a{}^q t_b{}^q)^{1/q}, f_a f_b)$.
5) $a \otimes b = (t_a t_b, (f_a{}^q + f_b{}^q - f_a{}^q f_b{}^q)^{1/q})$.
6) $\lambda a = \left(\left(1-(1-t_a{}^q)^\lambda\right)^{1/q}, f_a{}^\lambda\right)$ for any $\lambda > 0$.
7) $a^\lambda = \left(t_a{}^\lambda, \left(1-(1-f_a{}^q)^\lambda\right)^{1/q}\right)$ for any $\lambda > 0$.

These laws are exercised by Liu and Wang (2018) and proved that the following relations are valid for $\lambda, \lambda_1, \lambda_2 > 0$:

1) $a \oplus b = b \oplus a$.
2) $a \otimes b = b \otimes a$.
3) $\lambda(a \oplus b) = \lambda a \oplus \lambda b$.
4) $(a \otimes b)^\lambda = a^\lambda \otimes b^\lambda$.
5) $\lambda_1 a + \lambda_2 a = (\lambda_1 + \lambda_2)a$.
6) $a^{\lambda_1} \otimes a^{\lambda_2} = a^{\lambda_1 + \lambda_2}$.

Based on the above laws and relations, Liu and Wang (2018) presented the following *q*-rung orthopair fuzzy weighted averaging (*q*-ROFWA) operator to combine *q*-ROFNs.

Definition 3 (Liu and Wang; 2018). Let $a_i = (t_{a_i}, f_{a_i}) (i = 1,2,...., n)$ be a collection of *n* *q*-ROFNs with weight vector $w = (w_1, w_2,...., w_n)^T$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then the *q*-rung orthopair fuzzy weighted averaging (*q*-ROFWA) operator is

$$q\text{-ROFWA}(a_1,a_2,...,a_n)=\left\langle\left(1-\prod_{i=1}^{n}(1-t_{a_i}{}^{q})^{w_i}\right)^{1/q},\prod_{i=1}^{n}f_{a_i}{}^{w_i}\right\rangle.$$

If the position of the collection of n q-ROFNs is also considered along with associated weight vector $\omega=(\omega_1,\omega_2,....,\omega_n)^T$ with $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$, then the q-rung orthopair fuzzy ordered weighted averaging (q-ROFOWA) operator can be defined as

$$q\text{-ROFOWA}(a_1,a_2,...,a_n)=\left\langle\left(1-\prod_{i=1}^{n}(1-t_{a_{\delta(i)}}{}^{q})^{\omega_i}\right)^{1/q},\prod_{i=1}^{n}(f_{a_{\delta(i)}})^{\omega_i}\right\rangle$$

where $a_{\delta(i)}=(t_{a_{\delta(i)}},f_{a_{\delta(i)}})(i=1,2,....,n)$ is a permutation in descending order.

3. *q*-rung orthopair fuzzy average aggregation operator under confidence levels

In general, all the existing efforts do not incorporate the confidence levels of experts for their familiarity and awareness with the evaluated alternatives in the fusion of q-rung orthopair fuzzy information. Therefore, a series of q-rung orthopair fuzzy averaging and geometric aggregation operators are proposed here by incorporating the confidence levels of experts with the evaluated options.

3.1. Confidence *q*-rung orthopair fuzzy weighted average operator

Definition 4. Let Ω be the collection of n q-ROFNs $a_i=(t_i,f_i)(i=1,2,....,n)$ and l_i be the confidence levels of a_i with $0\le l_i\le 1$. If $w=(w_1,w_2,....,w_n)^T$ is the weight vector of these q-ROFNs such that $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, then the mapping $\text{CFWA}_q:\Omega^n\to\Omega$ defined as:

$$\text{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle)=\overset{n}{\underset{i=1}{\oplus}}w_i(l_ia_i)=w_1(l_1a_1)\oplus w_2(l_2a_2)\oplus...\oplus w_n(l_na_n)$$

which is called the confidence q-rung orthopair fuzzy weighted average (CFWA_q) operator.

Theorem 1. Let $a_i=(t_i,f_i)(i=1,2,....,n)$ be a collection of n q-ROFNs with confidence level $l_i\in[0,1]$, and if $w=(w_1,w_2,....,w_n)^T$ be the weight vector of $a_i(i=1,2,....,n)$ such that $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, then their aggregated value by using the CFWA_q operator is also a q-ROFN and given by

$$\text{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle)=\left\langle\left(1-\prod_{i=1}^{n}(1-t_i{}^{q})^{l_iw_i}\right)^{1/q},\prod_{i=1}^{n}(f_i)^{l_iw_i}\right\rangle$$

Proof. The proof can be done by using mathematical induction on n.

For $n=2$, we have

$$\text{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle)=w_1(l_1a_1)\oplus w_2(l_2a_2)$$

On the basis of operation laws of q-ROFNs, we have

$$l_1a_1=\left\langle\left(1-(1-t_1{}^{q})^{l_1}\right)^{1/q},f_1^{\,l_1}\right\rangle=\langle\alpha_1,\beta_1\rangle$$

$$\Rightarrow w_1(l_1 a_1) = \left\langle \left(1-(1-\alpha_1{}^q)^{w_1}\right)^{1/q}, \beta_1{}^{w_1} \right\rangle = \left\langle \left(1-\left[1-\left\{\left(1-(1-t_1{}^q)^{l_1}\right)^{1/q}\right\}^q\right]^{w_1}\right)^{1/q}, \left(f_1{}^{l_1}\right)^{w_1} \right\rangle$$

$$= \left\langle \left(1-\left[1-\left(1-(1-t_1{}^q)^{l_1}\right)\right]^{w_1}\right)^{1/q}, \left(f_1{}^{l_1}\right)^{w_1} \right\rangle = \left\langle \left(1-(1-t_1{}^q)^{w_1 l_1}\right)^{1/q}, f_1{}^{w_1 l_1} \right\rangle$$

Similarly, we can write $w_2(l_2 a_2) = \left\langle \left(1-(1-t_2{}^q)^{w_2 l_2}\right)^{1/q}, f_2{}^{w_2 l_2} \right\rangle$

Then,

$$\mathrm{CFWA}_q(\langle a_1, l_1\rangle, \langle a_2, l_2\rangle) = w_1(l_1 a_1) \oplus w_2(l_2 a_2)$$

$$= \left\langle \left[\left\{\left(1-(1-t_1{}^q)^{w_1 l_1}\right)^{1/q}\right\}^q + \left\{\left(1-(1-t_1{}^q)^{w_1 l_1}\right)^{1/q}\right\}^q \right.\right.$$

$$\left.\left. - \left\{\left(1-(1-t_1{}^q)^{w_1 l_1}\right)^{1/q}\right\}^q \left\{\left(1-(1-t_1{}^q)^{w_1 l_1}\right)^{1/q}\right\}^q\right]^{1/q}, f_1{}^{w_1 l_1} f_2{}^{w_2 l_2} \right\rangle$$

$$\Rightarrow \mathrm{CFWA}_q(\langle a_1, l_1\rangle, \langle a_2, l_2\rangle) = \left\langle \left(1-\prod_{i=1}^{2}(1-t_i{}^q)^{l_i w_i}\right)^{1/q}, \prod_{i=1}^{2}(f_i)^{l_i w_i} \right\rangle.$$

Let the result holds for $n = k$ i.e.

$$\mathrm{CFWA}_q(\langle a_1, l_1\rangle, \langle a_2, l_2\rangle, ..., \langle a_k, l_k\rangle) = \left\langle \left(1-\prod_{i=1}^{k}(1-t_i{}^q)^{l_i w_i}\right)^{1/q}, \prod_{i=1}^{k}(f_i)^{l_i w_i} \right\rangle$$

Then, for $n = k+1$, and using the operational laws of q-ROFNs, we can write

$$\mathrm{CFWA}_q(\langle a_1, l_1\rangle, \langle a_2, l_2\rangle, ..., \langle a_k, l_k\rangle, \langle a_{k+1}, l_{k+1}\rangle) =$$

$$\mathrm{CFWA}_q(\langle a_1, l_1\rangle, \langle a_2, l_2\rangle, ..., \langle a_k, l_k\rangle) \oplus w_{k+1}(l_{k+1} a_{k+1})$$

$$= \left\langle \left(1-\prod_{i=1}^{k}(1-t_i{}^q)^{l_i w_i}\right)^{1/q}, \prod_{i=1}^{k}(f_i)^{l_i w_i} \right\rangle \oplus \left\langle \left(1-(1-t_{k+1}{}^q)^{w_{k+1} l_{k+1}}\right)^{1/q}, f_{k+1}{}^{w_{k+1} l_{k+1}} \right\rangle$$

$$= \left\langle \left(1-\prod_{i=1}^{k}(1-t_i{}^q)^{l_i w_i} + 1-(1-t_{k+1}{}^q)^{w_{k+1} l_{k+1}} - \left(1-\prod_{i=1}^{k}(1-t_i{}^q)^{l_i w_i}\right).\left(1-(1-t_{k+1}{}^q)^{w_{k+1} l_{k+1}}\right)\right)^{1/q} \right.$$

$$\left. , \prod_{i=1}^{k}(f_i)^{l_i w_i} \cdot f_{k+1}{}^{w_{k+1} l_{k+1}} \right\rangle$$

$$= \left\langle \left(1-\prod_{i=1}^{k+1}(1-t_i{}^q)^{l_i w_i}\right)^{1/q}, \prod_{i=1}^{k+1}(f_i)^{l_i w_i} \right\rangle$$

It confirms that for $n = k+1$, the result still holds. Therefore, the result is true for any number of q-ROFNs.

Next, in order to show that the aggregated value obtained by the CFWA_q operator is also a q-ROFN, the following analysis is carried out.

For every $i=1,2,...,n$, we have, $0\le t_i, f_i \le 1$ and $0\le 1-t_i^{\,q}\le 1$, which further implies that $0\le 1-t_i^{\,q}\le 1$. Therefore, $0\le \prod_{i=1}^{n}(1-t_i^{\,q})^{l_i w_i}\le 1 \Rightarrow 0\le \left(1-\prod_{i=1}^{n}(1-t_i^{\,q})^{l_i w_i}\right)^{1/q}\le 1$.

Also, for $0\le f_i \le 1$, we have $0\le \prod_{i=1}^{n}(f_i)^{l_i w_i}\le 1$

Now, $\left(\left(1-\prod_{i=1}^{n}(1-t_i^{\,q})^{l_i w_i}\right)^{1/q}\right)^q+\left(\prod_{i=1}^{n}(f_i)^{l_i w_i}\right)^q$

$$=1-\prod_{i=1}^{n}(1-t_i^{\,q})^{l_i w_i}+\prod_{i=1}^{n}(f_i^{\,q})^{l_i w_i}\le 1-\prod_{i=1}^{n}(f_i^{\,q})^{l_i w_i}+\prod_{i=1}^{n}(f_i^{\,q})^{l_i w_i}=1$$

Thus, $0\le \left(\left(1-\prod_{i=1}^{n}(1-t_i^{\,q})^{l_i w_i}\right)^{1/q}\right)^q+\left(\prod_{i=1}^{n}(f_i)^{l_i w_i}\right)^q\le 1$.

Therefore, the aggregated value obtained through the CFWA$_q$ operator is also a *q*-ROFN, which completes the proof.

NOTE: Especially, if $l_i=1$, for every $i=1,2,...,n$, then the CFWA$_q$ operator reduces to the *q*-ROFWA operator (Liu and Wang; 2018).

Example 1. Let $a_1=\langle(0.5,0.8),0.6\rangle$, $a_2=\langle(0.7,0.7),0.8\rangle$, $a_3=\langle(0.6,0.5),0.5\rangle$ and $a_4=\langle(0.6,0.7),0.9\rangle$ are four *q*-ROFNs along with their confidence level. If $w=(0.4,0.1,0.3,0.2)^T$ be their weight vector, then (suppose *q* = 4)

$$\left(1-\prod_{i=1}^{4}(1-t_i^{\,4})^{l_i w_i}\right)^{1/4}=\left(1-(1-0.5^4)^{0.6\times 0.4}\times(1-0.7^4)^{0.8\times 0.1}\times(1-0.6^4)^{0.5\times 0.3}\times(1-0.6^4)^{0.9\times 0.2}\right)^{1/4}$$

$=0.5316$. Also $\prod_{i=1}^{4}(f_i)^{l_i w_i}=0.8^{0.6\times 0.4}\times 0.7^{0.8\times 0.1}\times 0.5^{0.5\times 0.3}\times 0.7^{0.9\times 0.2}=0.7786$

Therefore, from theorem 1, we have

$$\mathrm{CFWA}_4(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\langle a_3,l_3\rangle,\langle a_4,l_4\rangle)=\left\langle\left(1-\prod_{i=1}^{4}(1-t_i^{\,4})^{l_i w_i}\right)^{1/4},\prod_{i=1}^{4}(f_i)^{l_i w_i}\right\rangle$$

$=(0.5316,0.7540)$.

Let $a_i=(t_i,f_i)(i=1,2,....,n)$ be a collection of *n* *q*-ROFN with confidence level l_i, and $w=(w_1,w_2,....,w_n)^T$ is the weight vector of $a_i(i=1,2,....,n)$ such that $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, then the proposed CFWA$_q$ operator satisfies the following properties:

Property 1. (Idempotency) If for every *i*, $\langle a_i,l_i\rangle=\langle a,l\rangle$ i.e. $t_i=t, f_i=f$ and $l_i=l$, then

$$\mathrm{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle)=la$$

Proof: If $\langle a_i,l_i\rangle=\langle a,l\rangle$, $(\forall i=1,2,....,n)$, then from theorem 1, we have

$$\mathrm{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\ldots,\langle a_n,l_n\rangle)=\left\langle\left(1-\prod_{i=1}^{n}(1-t^q)^{lw_i}\right)^{1/q},\prod_{i=1}^{n}(f)^{lw_i}\right\rangle$$

$$=\left\langle\left(1-(1-t^q)^{l\sum_{i=1}^{n}w_i}\right)^{1/q},f^{l\sum_{i=1}^{n}w_i}\right\rangle=\left\langle\left(1-(1-t^q)^l\right)^{1/q},f^l\right\rangle=la\ .$$

Property 2. (Boundary Condition) If $a_i^-=(t_{l_ia_i}^{\min},f_{l_ia_i}^{\max})$ and $a_i^+=(t_{l_ia_i}^{\max},f_{l_ia_i}^{\min})$, then for every w_i, $a_i^-\le \mathrm{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\ldots,\langle a_n,l_n\rangle)\le a_i^+$.

Proof: For every *i*, we have, $\min t_i\le t_i\le \max t_i$ which further implies that $1-(\max t_i)^q\le(1-t_i^q)\le 1-(\min t_i)^q$. Then for every *w*,

$$\prod_{i=1}^{n}(1-(\max t_i)^q)^{(\max l_i)w_i}\le\prod_{i=1}^{n}(1-t_i^q)^{l_iw_i}\le\prod_{i=1}^{n}(1-(\min t_i)^q)^{(\min l_i)w_i}$$

$$\Rightarrow(1-(\max t_i)^q)^{\max l_i\sum_{i=1}^{n}w_i}\le\prod_{i=1}^{n}(1-t_i^q)^{l_iw_i}\le(1-(\min t_i)^q)^{\min l_i\sum_{i=1}^{n}w_i}$$

$$\Rightarrow 1-(1-(\min t_i)^q)^{\min l_i}\le 1-\prod_{i=1}^{n}(1-t_i^q)^{l_iw_i}\le 1-(1-(\max t_i)^q)^{\max l_i}$$

$$\Rightarrow\left(1-(1-(\min t_i)^q)^{\min l_i}\right)^{1/q}\le\left(1-\prod_{i=1}^{n}(1-t_i^q)^{l_iw_i}\right)^{1/q}\le\left(1-(1-(\max t_i)^q)^{\max l_i}\right)^{1/q}$$

$$\Rightarrow t_{l_ia_i}^{\min}\le\left(1-\prod_{i=1}^{n}(1-t_i^q)^{l_iw_i}\right)^{1/q}\le t_{l_ia_i}^{\max}\ .$$

Furthermore, $\min f_i\le f_i\le\max f_i\Leftrightarrow(\min f_i)^{\min l_i}\le\prod_{i=1}^{n}(f)^{l_iw_i}\le(\max f_i)^{\max l_i}$

$$\Rightarrow f_{l_ia_i}^{\min}\le\prod_{i=1}^{n}(f)^{l_iw_i}\le f_{l_ia_i}^{\max}\ .$$

If $\mathrm{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\ldots,\langle a_n,l_n\rangle)=\alpha=(t_\alpha,f_\alpha)$, then from the above analysis we have $t_{l_ia_i}^{\min}\le t_\alpha\le t_{l_ia_i}^{\max}$ and $f_{l_ia_i}^{\min}\le f_\alpha\le f_{l_ia_i}^{\max}$. Thus, by definition of score function, we can conclude

$$a_i^-\le\mathrm{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\ldots,\langle a_n,l_n\rangle)\le a_i^+\ .$$

Property 3. (Monotonicity) Let $a_i{*}=(t_{a_i*},f_{a_i*})(i=1,2,\ldots,n)$ be a collection of *n* *q*-ORFNs such that $t_{a_i}\le t_{a_i*}$ and $f_{a_i}\ge f_{a_i*}$ for all *i*, then for every *w*

$$\mathrm{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\ldots,\langle a_n,l_n\rangle)\le\mathrm{CFWA}_q(\langle a_1{*},l_1\rangle,\langle a_2{*},l_2\rangle,\ldots,\langle a_n{*},l_n\rangle)\ .$$

Proof: Since, $t_{a_i}\le t_{a_i*}$ and $f_{a_i}\ge f_{a_i*}$ for all *i*, then

$$1-t_{a_i*} \leq 1-t_{a_i} \Rightarrow \prod_{i=1}^{n}(1-t_{a_i*}{}^{q})^{l_i w_i} \leq \prod_{i=1}^{n}(1-t_{a_i}{}^{q})^{l_i w_i}$$

$$\Rightarrow \left(1-\prod_{i=1}^{n}(1-t_{a_i}{}^{q})^{l_i w_i}\right)^{1/q} \leq \left(1-\prod_{i=1}^{n}(1-t_{a_i*}{}^{q})^{l_i w_i}\right)^{1/q}.$$

Also $\prod_{i=1}^{n}(f_{a_i})^{l_i w_i} \geq \prod_{i=1}^{n}(f_{a_i*})^{l_i w_i}$.

Therefore,

$$\Rightarrow \left(\left(1-\prod_{i=1}^{n}(1-t_{a_i}{}^{q})^{l_i w_i}\right)^{1/q}\right)^{q} - \left(\prod_{i=1}^{n}(f_{a_i})^{l_i w_i}\right)^{q}$$

$$\leq \left(\left(1-\prod_{i=1}^{n}(1-t_{a_i*}{}^{q})^{l_i w_i}\right)^{1/q}\right)^{q} - \left(\prod_{i=1}^{n}(f_{a_i*})^{l_i w_i}\right)^{q}$$

If $\text{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle) = (t_\alpha, f_\alpha) = \alpha$ and $\text{CFWA}_q(\langle a_1*,l_1\rangle,\langle a_2*,l_2\rangle,...,\langle a_n*,l_n\rangle) = \langle t_{\alpha*}, f_{\alpha*}\rangle = \alpha*$ then, we have $S(\alpha) \leq S(\alpha*)$. Now two cases exist:

1) If $S(\alpha) < S(\alpha*)$, using score function, we get $\text{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle) < \text{CFWA}_q(\langle a_1*,l_1\rangle,\langle a_2*,l_2\rangle,...,\langle a_n*,l_n\rangle)$.
2) If $S(\alpha) = S(\alpha*)$, we can find

$$\Rightarrow \left(\left(1-\prod_{i=1}^{n}(1-t_{a_i}{}^{q})^{l_i w_i}\right)^{1/q}\right)^{q} - \left(\prod_{i=1}^{n}(f_{a_i})^{l_i w_i}\right)^{q}$$

$$= \left(\left(1-\prod_{i=1}^{n}(1-t_{a_i*}{}^{q})^{l_i w_i}\right)^{1/q}\right)^{q} - \left(\prod_{i=1}^{n}(f_{a_i*})^{l_i w_i}\right)^{q}.$$

Since, we have $t_{a_i} \leq t_{a_i*}$ and $f_{a_i} \geq f_{a_i*}$ for all *i*, then

$$\Rightarrow \left(\left(1-\prod_{i=1}^{n}(1-t_{a_i}{}^{q})^{l_i w_i}\right)^{1/q}\right)^{q} = \left(\left(1-\prod_{i=1}^{n}(1-t_{a_i*}{}^{q})^{l_i w_i}\right)^{1/q}\right)^{q}$$

and $\left(\prod_{i=1}^{n}(f_{a_i})^{l_i w_i}\right)^{q} = \left(\prod_{i=1}^{n}(f_{a_i*})^{l_i w_i}\right)^{q}$.

Now, using accuracy function, we have

$$H(\alpha) = \left(\left(1-\prod_{i=1}^{n}(1-t_{a_i}{}^{q})^{l_i w_i}\right)^{1/q}\right)^{q} + \left(\prod_{i=1}^{n}(f_{a_i})^{l_i w_i}\right)^{q}$$

$$=\left(\left(\left(1-\prod_{i=1}^{n}(1-t_{a_i*}{}^{q})^{l_i w_i}\right)^{1/q}\right)^{q}+\left(\prod_{i=1}^{n}(f_{a_i*})^{l_i w_i}\right)^{q}\right)=H(\alpha*).$$

Thus, $\text{CFWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle)<\text{CFWA}_q(\langle a_1*,l_1\rangle,\langle a_2*,l_2\rangle,...,\langle a_n*,l_n\rangle)$.

3.2. Confidence *q*-rung orthopair fuzzy ordered weighted average operator

Definition 5. Let Ω be the collection of *n* *q*-ROFNs $a_i=(t_i,f_i)(i=1,2,....,n)$ and l_i be the confidence levels of a_i with $0\le l_i\le 1$. If $\omega=(\omega_1,\omega_2,....,\omega_n)^T$ is the associated weight vector of these *q*-ROFNs such that $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$, then the mapping $\text{CFOWA}_q:\Omega^n\to\Omega$ is called the confidence *q*-rung orthopair fuzzy ordered weighted average (CFOWA_q) operator and defined as:

$$\text{CFOWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle)=\omega_1(l_{\delta(1)}a_{\delta(1)})\oplus\omega_2(l_{\delta(2)}a_{\delta(2)})\oplus...\oplus\omega_n(l_{\delta(n)}a_{\delta(n)})$$

where $(\delta(1),\delta(2),...,\delta(n))$ is a permutation of $(1,2,\ldots,3)$ such that for any *i*, $a_{\delta(i-1)}\ge a_{\delta(i)}$.

Theorem 2. Let $a_i=(t_i,f_i)(i=1,2,....,n)$ be a collection of *n* *q*-ROFNs with confidence level l_i, then their aggregated value by using the CFOWA_q operator is also a *q*-ROFN and given by

$$\text{CFOWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle)=\left\langle\left(1-\prod_{i=1}^{n}(1-t_{\delta(i)}{}^{q})^{l_{\delta(i)}w_i}\right)^{1/q},\prod_{i=1}^{n}(f_{\delta(i)})^{l_{\delta(i)}w_i}\right\rangle.$$

Proof. The proof is similar to the Theorem 1.

Example 2. Let four *q*-ROFNs with confidence levels $a_1=\langle(0.6,0.5),0.6\rangle$, $a_2=\langle(0.8,0.5),0.8\rangle$, $a_3=\langle(0.7,0.4),0.8\rangle$ and $a_4=\langle(0.7,0.6),0.9\rangle$ and if $\omega=(0.35,0.3,0.2,0.15)^T$ be their associated weight vector, without loss of generality assume *q* = 4, then their respective scores are $S(a_1)=0.6^4-0.5^4=0.0671$, $S(a_2)=0.8^4-0.5^4=0.3471$, $S(a_3)=0.7^4-0.4^4=0.2145$ and $S(a_4)=0.7^4-0.6^4=0.1105$. So, $a_2>a_3>a_4>a_1$. Thus, $a_{\delta(1)}=a_2,a_{\delta(2)}=a_3,a_{\delta(3)}=a_4$ and $a_{\delta(4)}=a_1$. Therefore, we have

$$\left(1-\prod_{i=1}^{4}(1-t_{\delta(i)}{}^{4})^{l_{\delta(i)}w_i}\right)^{1/4}=\left(1-(1-0.8^4)^{0.6\times0.35}\times(1-0.7^4)^{0.8\times0.3}\times(1-0.7^4)^{0.9\times0.2}\times(1-0.6^4)^{0.6\times0.15}\right)^{1/4}$$

$=0.7004$. Also $\prod_{i=1}^{4}(f_{\delta(i)})^{l_{\delta(i)}w_i}=0.5^{0.6\times0.35}\times0.4^{0.8\times0.3}\times0.6^{0.9\times0.2}\times0.5^{0.6\times0.15}=0.5664.$

Thus, by theorem 2, we have

$$\text{CFOWA}_4(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\langle a_3,l_3\rangle,\langle a_4,l_4\rangle)=\left\langle\left(1-\prod_{i=1}^{4}(1-t_{\delta(i)}{}^{4})^{l_{\delta(i)}w_i}\right)^{1/4},\prod_{i=1}^{4}(f_{\delta(i)})^{l_{\delta(i)}w_i}\right\rangle$$

$=(0.7004,0.5664)$.

Similar to the CFWA_q operator, the CFOWA_q operator also satisfies the same property so these properties have been presented here without proof.

Property 4. The CFOWA$_q$ operator is

1) (Idempotent) If for every *i*, $\langle a_i, l_i\rangle = \langle a, l\rangle$ i.e. $t_i = t,\ f_i = f$ and $l_i = l$, then $\text{CFOWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle) = la$
2) (Bounded) If $a_i^- = (t_{l_i a_i}^{\min}, f_{l_i a_i}^{\max})$ and $a_i^+ = (t_{l_i a_i}^{\max}, f_{l_i a_i}^{\min})$, then for every ω_i, $a_i^- \le \text{CFOWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle) \le a_i^+$.
3) ((Monotonic) Let $a_i{*} = (t_{a_i*}, f_{a_i*})(i = 1,2,....,n)$ be a collection of *n* q-ORFNs such that $t_{a_i} \le t_{a_i*}$ and $f_{a_i} \ge f_{a_i*}$ for all *i*, then $\text{CFOWA}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle) \le \text{CFOWA}_q(\langle a_1{*},l_1\rangle,\langle a_2{*},l_2\rangle,...,\langle a_n{*},l_n\rangle)$.

3.3. Confidence *q*-rung orthopair fuzzy weighted geometric operator

Definition 6. Let $\langle a_i, l_i\rangle = \langle (t_i, f_i), l_i\rangle (i = 1,2,....,n)$ be the collection of *n* *q*-ROFNs with confidence level l_i such that $0 \le l_i \le 1$. If $w = (w_1, w_2,...., w_n)^T$ is their weight vector with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then the confidence *q*-rung orthopair fuzzy weighted geometric (CFWG$_q$) operator is given as:

$$\text{CFWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle) = \bigotimes_{i=1}^{n}\left(a_i^{l_i}\right)^{w_i} = \left(a_1^{l_1}\right)^{w_1} \otimes \left(a_2^{l_2}\right)^{w_2} \otimes ... \otimes \left(a_n^{l_n}\right)^{w_n}.$$

Theorem 3. Let $a_i = (t_i, f_i)(i = 1,2,....,n)$ be a collection of *n* *q*-ROFNs with confidence level $l_i \in [0,1]$, and if $w = (w_1, w_2,...., w_n)^T$ be the weight vector of $a_i (i = 1,2,....,n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then their aggregated value by using the CFWG$_q$ operator is also a *q*-ROFN and given by

$$\text{CFWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle) = \left\langle \prod_{i=1}^{n}(t_i)^{l_i w_i}, \left(1 - \prod_{i=1}^{n}(1 - f_i^{\,q})^{l_i w_i}\right)^{1/q} \right\rangle$$

Proof. The result is verified by using mathematical induction on *n*.

For $n = 2$, we have

$$\text{CFWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle) = \left(a_1^{l_1}\right)^{w_1} \otimes \left(a_2^{l_2}\right)^{w_2}$$

On the basis of operation laws of *q*-ROFNs, we have

$$a_1^{l_1} = \left\langle t_1^{l_1}, \left(1 - (1 - f_1^{\,q})^{l_1}\right)^{1/q} \right\rangle = \langle \alpha_1, \beta_1\rangle$$

$$\Rightarrow \left(a_1^{l_1}\right)^{w_1} = \left\langle \alpha_1^{w_1}, \left(1 - (1-\beta_1^{\,q})^{w_1}\right)^{1/q} \right\rangle = \left\langle \left(t_1^{l_1}\right)^{w_1}, \left(1 - \left[1 - \left\{\left(1 - (1 - f_1^{\,q})^{l_1}\right)^{1/q}\right\}^{q}\right]^{w_1}\right)^{1/q} \right\rangle$$

$$= \left\langle \left(t_1^{l_1}\right)^{w_1} \left(1 - \left[1 - \left(1 - (1 - f_1^{\,q})^{l_1}\right)\right]^{w_1}\right)^{1/q} \right\rangle = \left\langle t_1^{w_1 l_1}, \left(1 - (1 - f_1^{\,q})^{w_1 l_1}\right)^{1/q} \right\rangle$$

Similarly, we can write $\left(a_2^{l_2}\right)^{w_2} = \left\langle t_2^{w_2 l_2}, \left(1 - (1 - f_2^{\,q})^{w_2 l_2}\right)^{1/q} \right\rangle$

Then,

$$\mathrm{CFWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle)=\left(a_1^{\,l_1}\right)^{w_1}\otimes\left(a_2^{\,l_2}\right)^{w_2}$$

$$=\left\langle t_1^{\,w_1 l_1}t_2^{\,w_2 l_2},\left[\left\{\left(1-(1-f_1^{\,q})^{w_1 l_1}\right)^{1/q}\right\}^q+\left\{\left(1-(1-f_1^{\,q})^{w_1 l_1}\right)^{1/q}\right\}^q\right.\right.$$

$$\left.\left.-\left\{\left(1-(1-f_1^{\,q})^{w_1 l_1}\right)^{1/q}\right\}^q\left\{\left(1-(1-f_1^{\,q})^{w_1 l_1}\right)^{1/q}\right\}^q\right]^{1/q}\right\rangle$$

$$\Rightarrow \mathrm{CFWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle)=\left\langle \prod_{i=1}^{2}(t_i)^{l_i w_i},\left(1-\prod_{i=1}^{2}(1-f_i^{\,q})^{l_i w_i}\right)^{1/q}\right\rangle.$$

Let the result holds for $n=k$ i.e.

$$\mathrm{CFWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\ldots,\langle a_k,l_k\rangle)=\left\langle \prod_{i=1}^{k}(t_i)^{l_i w_i},\left(1-\prod_{i=1}^{k}(1-f_i^{\,q})^{l_i w_i}\right)^{1/q}\right\rangle$$

Then, for $n=k+1$, and using the operational laws of *q*-ROFNs, we can write

$$\mathrm{CFWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\ldots,\langle a_k,l_k\rangle,\langle a_{k+1},l_{k+1}\rangle)=$$

$$\mathrm{CFWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\ldots,\langle a_k,l_k\rangle)\otimes\left(a_{k+1}^{\,l_{k+1}}\right)^{w_{k+1}}$$

$$=\left\langle \prod_{i=1}^{k}(t_i)^{l_i w_i},\left(1-\prod_{i=1}^{k}(1-f_i^{\,q})^{l_i w_i}\right)^{1/q}\right\rangle\otimes\left\langle t_{k+1}^{\,w_{k+1}l_{k+1}},\left(1-(1-f_{k+1}^{\,q})^{w_{k+1}l_{k+1}}\right)^{1/q}\right\rangle$$

$$=\left\langle \prod_{i=1}^{k}(t_i)^{l_i w_i}.t_{k+1}^{\,w_{k+1}l_{k+1}},\left(1-\prod_{i=1}^{k}(1-f_i^{\,q})^{l_i w_i}+1-(1-f_{k+1}^{\,q})^{w_{k+1}l_{k+1}}\right.\right.$$

$$\left.\left.-\left(1-\prod_{i=1}^{k}(1-f_i^{\,q})^{l_i w_i}\right).\left(1-(1-f_{k+1}^{\,q})^{w_{k+1}l_{k+1}}\right)\right)^{1/q}\right\rangle$$

$$=\left\langle \prod_{i=1}^{k+1}(t_i)^{l_i w_i},\left(1-\prod_{i=1}^{k+1}(1-f_i^{\,q})^{l_i w_i}\right)^{1/q}\right\rangle$$

It confirms that for $n = k + 1$, the result still holds. Therefore, the result is true for any number of *q*-ROFNs.

Further to show that the aggregated value obtained by the CFWG_q operator is also a *q*-ROFN, the following part is considered here.

For every $i=1,2,\ldots,n$, we have, $0\le t_i,\ f_i\le 1$, and $0\le \prod_{i=1}^{n}(t_i)^{l_i w_i}\le 1$.

Also $0\le 1-f_i^{\,q}\le 1$. Therefore, $0\le \prod_{i=1}^{n}(1-f_i^{\,q})^{l_i w_i}\le 1 \Rightarrow 0\le\left(1-\prod_{i=1}^{n}(1-f_i^{\,q})^{l_i w_i}\right)^{1/q}\le 1$.

Now, $\left(\prod_{i=1}^{n}(t_i)^{l_i w_i}\right)^q+\left(\left(1-\prod_{i=1}^{n}(1-f_i^{\,q})^{l_i w_i}\right)^{1/q}\right)^q$

$$= \prod_{i=1}^{n} (t_i^q)^{l_i w_i} + 1 - \prod_{i=1}^{n} (1 - f_i^q)^{l_i w_i} \le \prod_{i=1}^{n} (t_i^q)^{l_i w_i} + 1 - \prod_{i=1}^{n} (t_i^q)^{l_i w_i} = 1$$

Thus, $0 \le \left(\prod_{i=1}^{n} (t_i)^{l_i w_i} \right)^q + \left(\left(1 - \prod_{i=1}^{n} (1 - f_i^q)^{l_i w_i} \right)^{1/q} \right)^q \le 1$.

Therefore, the combined value obtained through the CFWG$_q$ operator is also a q-ROFN, which completes the proof.

Example 3. Let $a_1 = \langle (0.5, 0.5), 0.6 \rangle$, $a_2 = \langle (0.7, 0.4), 0.8 \rangle$, $a_3 = \langle (0.6, 0.7), 0.9 \rangle$ and $a_4 = \langle (0.8, 0.3), 0.7 \rangle$ are four q-ROFNs with their confidence level. If $w = (0..25, 0.3, 0.15, 0.3)^T$ be their weight vector, then (suppose q = 4)

$\prod_{i=1}^{4} (t_i)^{l_i w_i} = 0.5^{0.6 \times 0.25} \times 0.7^{0.8 \times 0.3} \times 0.6^{0.9 \times 0.15} \times 0.8^{0.7 \times 0.3} = 0.7368$. Also

$$\left(1 - \prod_{i=1}^{4} (1 - f_i^4)^{l_i w_i} \right)^{1/4} = \left(1 - (1 - 0.5^4)^{0.6 \times 0.25} \times (1 - 0.4^4)^{0.8 \times 0.3} \times (1 - 0.7^4)^{0.9 \times 0.15} \times (1 - 0.3^4)^{0.7 \times 0.3} \right)^{1/4}$$

$= 0.4802$.

Therefore, from theorem 3, we have

$$\text{CFWG}_4(\langle a_1, l_1 \rangle, \langle a_2, l_2 \rangle, \langle a_3, l_3 \rangle, \langle a_4, l_4 \rangle) = \left\langle \prod_{i=1}^{4} (t_i)^{l_i w_i}, \left(1 - \prod_{i=1}^{4} (1 - f_i^4)^{l_i w_i} \right)^{1/4} \right\rangle$$

$= (0.7368, 0.4802)$.

Property 5. The CFWG$_q$ operator satisfies the following properties:

1) (Idempotent) If for every i, $\langle a_i, l_i \rangle = \langle a, l \rangle$ i.e. $t_i = t$, $f_i = f$ and $l_i = l$, then $\text{CFWG}_q(\langle a_1, l_1 \rangle, \langle a_2, l_2 \rangle, ..., \langle a_n, l_n \rangle) = a^l$.
2) (Bounded) If $a_i^- = (t_{l_i a_i}^{\min}, f_{l_i a_i}^{\max})$ and $a_i^+ = (t_{l_i a_i}^{\max}, f_{l_i a_i}^{\min})$, then for every ω_i, $a_i^- \le \text{CFWG}_q(\langle a_1, l_1 \rangle, \langle a_2, l_2 \rangle, ..., \langle a_n, l_n \rangle) \le a_i^+$.
3) ((Monotonic) Let $a_i^* = (t_{a_i^*}, f_{a_i^*}) (i = 1, 2, ..., n)$ be a collection of n q-ORFNs such that $t_{a_i} \le t_{a_i^*}$ and $f_{a_i} \ge f_{a_i^*}$ for all i, then $\text{CFWG}_q(\langle a_1, l_1 \rangle, \langle a_2, l_2 \rangle, ..., \langle a_n, l_n \rangle) \le \text{CFWG}_q(\langle a_1^*, l_1 \rangle, \langle a_2^*, l_2 \rangle, ..., \langle a_n^*, l_n \rangle)$.

3.4. Confidence *q*-rung orthopair fuzzy ordered weighted geometric operator

Definition 7. Let $\langle a_i, l_i \rangle = \langle (t_i, f_i), l_i \rangle (i = 1, 2, ..., n)$ be the collection of n q-ROFNs with confidence level l_i such that $0 \le l_i \le 1$. If $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is their associated weight vector with $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$, then the confidence q-rung orthopair fuzzy ordered weighted geometric (CFOWG$_q$) operator is given as:

$$\text{CFOWG}_q(\langle a_1, l_1 \rangle, \langle a_2, l_2 \rangle, ..., \langle a_n, l_n \rangle) = \left(a_{\delta(1)}^{\ l_{\delta(1)}} \right)^{w_1} \otimes \left(a_{\delta(2)}^{\ l_{\delta(2)}} \right)^{w_2} \otimes ... \otimes \left(a_{\delta(n)}^{\ l_{\delta(n)}} \right)^{w_n}.$$

Theorem 4. Let $a_i=(t_i,f_i)(i=1,2,....,n)$ be a collection of *n q*-ROFNs with confidence level $l_i\in[0,1]$, and if $w=(w_1,w_2,....,w_n)^T$ be the weight vector of $a_i(i=1,2,....,n)$ such that $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, then their aggregated value by using the CFOWG$_q$ operator is also a *q*-ROFN and given by

$$\text{CFOWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle)=\left\langle\prod_{i=1}^{n}(t_{\delta(1)})^{l_{\delta(1)}w_i},\left(1-\prod_{i=1}^{n}(1-f_{\delta(1)}{}^{q})^{l_{\delta(1)}w_i}\right)^{1/q}\right\rangle$$

where $(\delta(1),\delta(2),...,\delta(n))$ is a permutation of $(1,2,\ldots,3)$ such that for any *i*, $a_{\delta(i-1)}\geq a_{\delta(i)}$.

Proof. Proof is similar to theorem 3.

Example 4. Let four *q*-ROFNs with confidence levels $a_1=\langle(0.6,0.5),0.7\rangle$, $a_2=\langle(0.4,0.5),0.6\rangle$, $a_3=\langle(0.7,0.4),0.8\rangle$ and $a_4=\langle(0.7,0.6),0.7\rangle$ and if $\omega=(0.35,0.3,0.2,0.15)^T$ be their associated weight vector, without loss of generality assume *q* = 4, then their respective scores are $S(a_1)=0.6^4-0.5^4=0.0671$, $S(a_2)=0.4^4-0.5^4=-0.0369$, $S(a_3)=0.7^4-0.4^4=0.2145$ and $S(a_4)=0.7^4-0.6^4=0.1105$. So, $a_3>a_4>a_1>a_2$. Thus, $a_{\delta(1)}=a_3,a_{\delta(2)}=a_4,a_{\delta(3)}=a_1$ and $a_{\delta(4)}=a_2$. Therefore, we have

$$\prod_{i=1}^{4}(t_{\delta(i)})^{l_{\delta(i)}w_i}=0.7^{0.8\times0.35}\times0.7^{0.7\times0.3}\times0.6^{0.7\times0.2}\times0.4^{0.6\times0.15}=0.6943.$$ Also

$$\left(1-\prod_{i=1}^{4}(1-f_{\delta(i)}{}^{4})^{l_{\delta(i)}w_i}\right)^{1/4}=\left(1-(1-0.4^4)^{0.8\times0.35}\times(1-0.6^4)^{0.7\times0.3}\times(1-0.5^4)^{0.7\times0.2}\times(1-0.5^4)^{0.6\times0.15}\right)^{1/4}$$

$$=0.47606.$$

Thus, by theorem 4, we have

$$\text{CFOWG}_4(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,\langle a_3,l_3\rangle,\langle a_4,l_4\rangle)=\left\langle\prod_{i=1}^{4}(t_{\delta(i)})^{l_{\delta(i)}w_i},\left(1-\prod_{i=1}^{4}(1-f_{\delta(i)}{}^{4})^{l_{\delta(i)}w_i}\right)^{1/4}\right\rangle$$

$$=(0.6943,0.47606).$$

Similar to the CFWG$_q$ operator, the CFOWG$_q$ operator also satisfies the same property so presented here without proof.

Property 6. The CFOWG$_q$ operator is

1) (Idempotent) If for every *i*, $\langle a_i,l_i\rangle=\langle a,l\rangle$ i.e. $t_i=t,f_i=f$ and $l_i=l$, then $\text{CFOWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle)=a^l$
2) (Bounded) If $a_i^-=(t_{l_ia_i}^{\min},f_{l_ia_i}^{\max})$ and $a_i^+=(t_{l_ia_i}^{\max},f_{l_ia_i}^{\min})$, then for every ω_i, $a_i^-\leq\text{CFOWG}_q(\langle a_1,l_1\rangle,\langle a_2,l_2\rangle,...,\langle a_n,l_n\rangle)\leq a_i^+$.
3) ((Monotonic) Let $a_i*=(t_{a_i*},f_{a_i*})(i=1,2,....,n)$ be a collection of *n q*-ORFNs such that $t_{a_i}\leq t_{a_i*}$ and $f_{a_i}\geq f_{a_i*}$ for all *i*, then

$$\text{CFOWG}_q(\langle a_1, l_1\rangle, \langle a_2, l_2\rangle, ..., \langle a_n, l_n\rangle) \le \text{CFOWG}_q(\langle a_1*, l_1\rangle, \langle a_2*, l_2\rangle, ..., \langle a_n*, l_n\rangle).$$

4. An approach to MCDM problems under confidence levels

In this section, a decision making approach is proposed to solve MCDM problems on the basis of developed operators. A real life customers' choice problem is also considered to demonstrate the decision making approach effectively.

4.1. MCDM approach under confidence levels

Consider a decision making problem, in which the set of alternatives $\{A_1, A_2, ..., A_m\}$ is estimated on the basis of the set of criteria $\{c_1, c_2, ..., c_n\}$. Let $w = (w_1, w_2, ..., w_n)^T$ is weight vector of the criterion set with $w_j > 0,\ j = 1,2,.., n$ and $\sum_{j=1}^{n} w_i = 1$. Consider a group of decision makers/experts $\{D_1, D_2, ..., D_p\}$ with weight vectors $\xi = (\xi_1, \xi_2, ..., \xi_p)^T$ satisfying $\xi_k > 0, k = 1,2,.., p$ and $\sum_{k=1}^{p} \xi_i = 1$, which provide their individual assessment of each alternative against each criterion in the form of *q*-ROFNs and is denoted by $(a_{ij}^k)_{m\times n} = (t_{ij}^k, f_{ij}^k)_{m\times n}$, where in view of the expert D_k, t_{ij}^k and f_{ij}^k indicate the degree that the alternative a_i support for and support against the criteria c_j respectively and $0 \le (t_{ij}^k)^q + (f_{ij}^k)^q \le 1, \forall i = 1,2,...,m$, $\forall j = 1,2,...,n$ and $\forall k = 1,2,..., p$. In order to incorporate the notion of confidence levels, the experts simultaneously also provide the degrees that they are familiar with the evaluated alternatives and assign the confidence levels $l_{ij}^k (0 \le l_{ij}^k \le 1)$. To facilitate the developed operators in group decision making problems effectively, the following steps are carried out:

Step 1: Establish the individual expert judgment related to each alternative against the mentioned criterion set in the form of *q*-ROFNs along with their confidence levels and then construct the corresponding expert's judgment matrix $[A^k]_{m\times n} = \langle (t_{ij}^k, f_{ij}^k), l_{ij}^k\rangle$.

Step 2: Utilize the following form of the CFWA$_q$ operator (or the CFWG$_q$ operator) to combine all individual expert judgment matrix $[A^k]_{m\times n} = \langle (t_{ij}^k, f_{ij}^k), l_{ij}^k\rangle$ into a collective expert judgment matrix $[A]_{m\times n}$.

$$a_{ij} = \text{CFWA}_q(\langle a_{ij}^1, l_{ij}^1\rangle, \langle a_{ij}^2, l_{ij}^2\rangle, ..., \langle a_{ij}^p, l_{ij}^p\rangle) = \left\langle \left(1 - \prod_{k=1}^{p} (1 - (t_{ij}^k)^q)^{l_{ij}^k \xi_i}\right)^{1/q}, \prod_{k=1}^{p} (f_{ij}^k)^{l_{ij}^k \xi_i} \right\rangle$$

or $$a_{ij} = \text{CFWG}_q(\langle a_{ij}^1, l_{ij}^1\rangle, \langle a_{ij}^2, l_{ij}^2\rangle, ..., \langle a_{ij}^p, l_{ij}^p\rangle) = \left\langle \prod_{k=1}^{p} (t_{ij}^k)^{l_{ij}^k \xi_i}, \left(1 - \prod_{k=1}^{p} (1 - (f_{ij}^k)^q)^{l_{ij}^k \xi_i}\right)^{1/q} \right\rangle$$

Step 3: Aggregate the performance of each alternative of the matrix $[A]_{m\times n}$ row wise to obtain the overall performance, by utilizing the following form of the *q*-ROFWA operator (or the *q*-ROFWG operator) for A_i $(i = 1,2,...,m)$ and is denoted by a_i.

$$a_i = q\text{-ROFWA}(a_{i1}, a_{i2}, ..., a_{in}) = \left\langle \left(1 - \prod_{j=1}^{n} (1 - t_{ij}{}^q)^{w_i}\right)^{1/q}, \prod_{j=1}^{n} (f_{ij})^{w_i} \right\rangle$$

or $$a_i = q\text{-ROFWG}(a_{i1}, a_{i2}, ..., a_{in}) = \left\langle \prod_{j=1}^{n} (t_{ij})^{w_i}, \left(1 - \prod_{j=1}^{n} (1 - f_{ij}{}^q)^{w_i}\right)^{1/q} \right\rangle.$$

Then, calculate score of each aggregated value $a_i (i = 1,2,..., m)$ and rank the aggregated values in descending order.

4.2. Real life application to the customers' choice problem

To apply the developed approach effectively, a decision making problem with customers' choice to purchase a laptop from four different options $\{A_1, A_2, A_3, A_4\}$ on the basis of the parameters $\{c_1, c_2, c_3, c_4\}$ where $c_j\,(j=1,2,3,4)$ stands for "processor", "system memory", "screen size" and "hard size" respectively. Let $w=(0.25,0.28,0.22,0.25)^T$ be the weight vector of the parametric set. Let, three decision makers/experts $\{D_1, D_2, D_3\}$ with weight vector $\xi=(0.4,0.3,0.3)^T$ provide their individual assessment in the form of q-ROFNs $\langle (t_{ij}^k, f_{ij}^k), l_{ij}^k \rangle (i=1,2,3,4, j=1,2,3,4)$ for each option and the corresponding assessments are presented in Table 1, Table 2 and Table 3 respectively.

Then, the steps of the presented approach are executed to find the best suitable option and are demonstrated as follows:

Step 1: The individual experts' assessment matrix $[A^k]_{4\times 4} = \langle (t_{ij}^k, f_{ij}^k), l_{ij}^k \rangle (k=1,2,3)$ related to each option on the basis of the parametric set has been collected and presented in Table 1, Table 2 and Table 3 respectively.

4.2.1. Based on confidence average aggregation operator

Step 2: Utilize the CFWA$_q$ operator to combine all individual expert judgment matrix $[A^k]_{4\times 4} = \langle (t_{ij}^k, f_{ij}^k), l_{ij}^k \rangle$ into a collective one and corresponding combined expert judgment matrix $[A]_{4\times 4}$ is summarized in Table 4 (without loss of generality we can assume $q=4$).

Step 3: Now, by utilizing the q-ROFWA operator to aggregate the performance of each alternative of the matrix $[A]_{4\times 4}$ row wise for A_i $(i=1,2,\ldots,m)$ and are summarized in Table 5 (without loss of generality we can assume $q=4$). Finally, rank them on the calculate scores for each aggregated value $a_i\,(i=1,2,\ldots,m)$ [see Table 5].

4.2.2. Based on confidence geometric aggregation operator

Step 2: Utilize the CFWG$_q$ operator to add all individual expert judgment matrix and corresponding combined expert judgment matrix $[A]_{4\times 4}$ is presented in Table 6 (without loss of generality we can assume $q=4$).

Step 3: Then, apply the q-ROFWG operator to aggregate the performance of each alternative and rank them. All the corresponding results are depicted in Table 7 (without loss of generality we can assume $q=4$).

5. Sensitivity analysis

Here, an investigation has been performed by the proposed CFWA$_q$ operator to analyze the discrepancy in the scores and the rankings of the alternatives with the flexibility and sensitivity of the parameter q. The corresponding results are summarized in Table 8.

Table 8 clearly indicates that different score values are found corresponding to different values of the parameter q in the CFWA$_q$ operator. These discrepancies in score values did not make any effect on the ranking of the mentioned alternatives corresponding to the different values of q under consideration. Furthermore, scores of the overall combined values are relatively large when q is relatively small i.e. from 2 to 5, and scores become smaller in the increase of q. Thus, the approach of decision makers is more optimistic when q is from 2 to 5, and the pessimistic nature of decision makers reflects when q is large. Generally, different experts may fix different value to q as per their requirements.

Table 1: q-rung orthopair fuzzy expert $"D_1"$ assessment matrix $[A^1]_{4\times4}$

	c_1	c_2	c_3	c_4
A_1	$\langle(0.6,0.5),0.92\rangle$	$\langle(0.8,0.2),0.88\rangle$	$\langle(0.5,0.7),0.79\rangle$	$\langle(0.8,0.3),0.91\rangle$
A_2	$\langle(0.6,0.6),0.8\rangle$	$\langle(0.8,0.4),0.78\rangle$	$\langle(0.6,0.5),0.81\rangle$	$\langle(0.7,0.4),0.90\rangle$
A_3	$\langle(0.7,0.4),0.89\rangle$	$\langle(0.8,0.3),0.94\rangle$	$\langle(0.6,0.5),0.91\rangle$	$\langle(0.8,0.2),0.93\rangle$
A_4	$\langle(0.7,0.5),0.81\rangle$	$\langle(0.4,0.8),0.75\rangle$	$\langle(0.5,0.7),0.76\rangle$	$\langle(0.5,0.6),0.84\rangle$

Table 2: q-rung orthopair fuzzy expert $"D_2"$ assessment matrix $[A^2]_{4\times4}$

	c_1	c_2	c_3	c_4
A_1	$\langle(0.7,0.5),0.9\rangle$	$\langle(0.8,0.3),0.86\rangle$	$\langle(0.6,0.7),0.78\rangle$	$\langle(0.7,0.3),0.89\rangle$
A_2	$\langle(0.6,0.5),0.78\rangle$	$\langle(0.7,0.5),0.79\rangle$	$\langle(0.5,0.5),0.8\rangle$	$\langle(0.6,0.4),0.91\rangle$
A_3	$\langle(0.8,0.4),0.9\rangle$	$\langle(0.8,0.4),0.93\rangle$	$\langle(0.7,0.6),0.88\rangle$	$\langle(0.8,0.3),0.91\rangle$
A_4	$\langle(0.6,0.5),0.79\rangle$	$\langle(0.4,0.7),0.73\rangle$	$\langle(0.6,0.6),0.78\rangle$	$\langle(0.5,0.7),0.81\rangle$

Table 3: q-rung orthopair fuzzy expert $"D_3"$ assessment matrix $[A^3]_{4\times4}$

	c_1	c_2	c_3	c_4
A_1	$\langle(0.6,0.6),0.91\rangle$	$\langle(0.7,0.3),0.89\rangle$	$\langle(0.6,0.4),0.8\rangle$	$\langle(0.7,0.3),0.87\rangle$
A_2	$\langle(0.7,0.6),0.81\rangle$	$\langle(0.6,0.5),0.88\rangle$	$\langle(0.6,0.6),0.84\rangle$	$\langle(0.6,0.4),0.90\rangle$
A_3	$\langle(0.8,0.4),0.91\rangle$	$\langle(0.9,0.2),0.93\rangle$	$\langle(0.6,0.5),0.81\rangle$	$\langle(0.6,0.3),0.86\rangle$
A_4	$\langle(0.6,0.5),0.81\rangle$	$\langle(0.5,0.6),0.78\rangle$	$\langle(0.6,0.6),0.80\rangle$	$\langle(0.5,0.5),0.83\rangle$

Table 4: q-rung orthopair fuzzy combined expert assessment matrix $[A]_{4\times4}$ using CFWA$_q$ operator

	c_1	c_2	c_3	c_4
A_1	(0.6228,0.5589)	(0.7557,0.3016)	(0.5360,0.6596)	(0.7312,0.3416)
A_2	(0.6047,0.6377)	(0.6925,0.5309)	(0.5492,0.5947)	(0.6332,0.4371)
A_3	(0.7514,0.4387)	(0.8298,0.3143)	(0.6171,0.5737)	(0.7480,0.2899)
A_4	(0.62581,0.6610)	(0.4096,0.7675)	(0.5349,0.7042)	(0.4776,0.6499)

Table 5: Aggregated values for each option and their scores using q-ROFWA operator

Option	Aggregated Values	Scores	Developed ranking
a_1	$(0.6884, 0.4312)$	0.1900	$A_3 > A_1 > A_2 > A_4$
a_2	$(0.6329, 0.5349)$	0.0785	Thus A_3 is the best
a_3	$(0.7594, 0.3822)$	0.3112	choice for customers
a_4	$(0.5293, 0.6959)$	-0.1561	

Table 6: q-rung orthopair fuzzy combined expert assessment matrix $[A]_{4\times 4}$ using CFWG$_q$ operator

	c_1	c_2	c_3	c_4
A_1	$(0.6545, 0.5250)$	$(0.7934, 0.2635)$	$(0.6305, 0.6176)$	$(0.7637, 0.2915)$
A_2	$(0.6909, 0.5090)$	$(0.7490, 0.4461)$	$(0.6309, 0.5123)$	$(0.6664, 0.3900)$
A_3	$(0.7802, 0.3896)$	$(0.8389, 0.3214)$	$(0.6674, 0.5192)$	$(0.7590, 0.2646)$
A_4	$(0.6971, 0.6537)$	$(0.5284, 0.6862)$	$(0.6358, 0.6102)$	$(0.5633, 0.5902)$

Table 7: Aggregated values for each option and their scores using q-ROFWG operator

Option	Aggregated Values	Scores	Developed ranking
a_1	$(0.7120, 0.4862)$	0.2012	$A_3 > A_1 > A_2 > A_4$
a_2	$(0.6865, 0.4705)$	0.1731	Thus A_3 is the best
a_3	$(0.7641, 0.4030)$	0.3144	choice for customers
a_4	$(0.5993, 0.6417)$	-0.0405	

Table 8: Ranking using CFWA$_q$ operator for different values of q

q	$S(a_1)$	$S(a_2)$	$S(a_3)$	$S(a_4)$	Developed Ranking
2	0.2598	0.084114	0.408133	-0.24416	$A_3 > A_1 > A_2 > A_4$
3	0.233666	0.088431	0.37102	-0.2057	$A_3 > A_1 > A_2 > A_4$
4	0.190079	0.078595	0.311283	-0.15611	$A_3 > A_1 > A_2 > A_4$
5	0.148163	0.064584	0.253662	-0.12083	$A_3 > A_1 > A_2 > A_4$
10	0.038285	0.016974	0.089175	-0.02325	$A_3 > A_1 > A_2 > A_4$
15	0.010747	0.004293	0.035368	-0.004	$A_3 > A_1 > A_2 > A_4$

Table 9: Comparison with some existing methods

Method	**Operator used**	$S(a_1)$	$S(a_2)$	$S(a_3)$	$S(a_4)$	**Developed Ranking**
Xu and Yager (2006)	IFWG	0.2819	0.1607	0.3850	-0.1365	$A_3 > A_1 > A_2 > A_4$
Xu (2007)	IFWA	0.3125	0.1781	0.4163	-0.0982	$A_3 > A_1 > A_2 > A_4$
Wang and Liu (2012)	IFEWA	0.3020	0.1746	0.4099	-0.1050	$A_3 > A_1 > A_2 > A_4$
Yager (2014)	PFWA	0.3426	0.2043	0.4067	-0.1098	$A_3 > A_1 > A_2 > A_4$
Garg (2016a)	PFEWA	0.3325	0.1991	0.3896	-0.1195	$A_3 > A_1 > A_2 > A_4$
Liu and Wang (2018)	*q*-ROFWA (*q*=3)	0.2922	0.1786	0.4142	-0.0924	$A_3 > A_1 > A_2 > A_4$
Liu and Wang (2018)	*q*-ROFWA (*q*=5)	0.1746	0.2767	0.4142	-0.0488	$A_3 > A_1 > A_2 > A_4$
Proposed	CFWA_q (*q*=3)	0.2336	0.0884	0.3710	-0.2057	$A_3 > A_1 > A_2 > A_4$
	CFWG_q (*q*=3)	0.2630	0.2258	0.3879	-0.0317	$A_3 > A_1 > A_2 > A_4$

6. Comparative analysis and superiority of presented approach over existing methods

The above analysis finds the ranking $A_3 > A_1 > A_2 > A_4$ under different values of *q*. If the proposed approach is compared with the existing methods by assuming that all the decision makers/experts are taken to be definitely familiar i.e. $l_{ij}^k = 1$ for all *i*, *j* and *k* with the objects to be evaluated, then obtained results are summarized in Table 9. Therefore, the presented approach provides the same best suitable alternative as obtained by different existing average aggregation operators which verifies the proposed approach is practical and feasible. In order to show the supremacy of the presented approach over the existing operators the following analysis is taken in consideration.

In some practical decisions, the IFNs have a disadvantage that the sum of the membership and non-membership is not more than 1, i.e., $t + f \leq 1$. The methods proposed by Xu and Yager (2006), Xu (2007) and Wang and Liu (2012) involved simple calculation but its scope of application is very narrow, it can only handle the real life problems expressed under intuitionistic fuzzy environment, however, the assessment (0.6, 0.5) provided by expert in above customers' choice problem cannot fully express by these methods, so it will easily cause the distortion of the information. But the PFNs are more superior than the IFNs, because they require membership and non-membership must meet to $t^2 + f^2 \leq 1$. The methods given by Yager (2014) and Garg (2016a) can address only the practical problems under Pythagorean fuzzy environment. However, the assessment (0.8, 0.7) cannot fully express by these methods as $0.8^2 + 0.7^2 > 1$, so it will also cause the misrepresentation of the information. The method introduced by Liu and Wang (2018) has been developed in the assumptions that all the experts are 100% familiar with the evaluated objects. But these types of limitations are not fully met in dealing with practical problems. In such cases proposed approach is superior to other recent research works.

Furthermore, some contradictory examples are considered under the environment of *q*-ROFSs, where the existing operators are unable to find the best alternative while the presented approach can overcome their shortcoming.

Example 5. Let consider a MCDM problem in which two alternatives A_1 and A_2 are evaluated on the basis of three criteria c_1, c_2 and c_3 with weight vector $w = (0.32, 0.35, 0.33)^T$. Let an expert is asked to rank them and he gives his assessment in the form of *q*-ROFNs as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{ccc} \quad c_1 \quad & \quad c_2 \quad & \quad c_3 \quad \end{array} \\ \begin{bmatrix} (0.502084, 0.497916) & (0.63907, 0.36093) & (0.273324, 0.726676) \\ (0.671957, 0.719462) & (0.259944, 0.528584) & (0.49145, 0.339313) \end{bmatrix} \end{array}$$

In order to find the best alternative, if the q-ROFWA operator is utilized for $q=1$ (operator reduced in IFWA operator) we get score values for alternatives as $S(A_1)=-0.008$ and $S(A_2)=-0.008$ respectively. Thus, we fail to find the ranking as both alternatives have same score values. This happens because of the reality that the q-ROFWA operator pays no attention to familiarity of the expert with the alternatives.

In other side, if we employed the proposed CFWA$_q$ operator for $q=1$ by counting the confidence levels of the expert as $l = \begin{array}{c} A_1 \\ A_2 \end{array}\begin{bmatrix} 0.91 & 0.88 & 0.75 \\ 0.74 & 0.84 & 0.89 \end{bmatrix}$ with the alternatives, we find scores $S(A_1)=-0.10207$ and $S(A_2)=-0.13466$ for alternatives A_1 and A_2 respectively. Therefore, the ranking is $A_1 > A_2$.

Example 6. Assume a decision making problem consisting of two different alternatives namely A_1 and A_2. Let these alternatives are evaluated against the criteria c_1, c_2 and c_3 whose weight vector is $w=(0.32,0.35,0.33)^T$. Let the decision maker provides his assessment in the form of q-ROFNs as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{ccc} \quad c_1 \quad & \quad c_2 \quad & \quad c_3 \quad \end{array} \\ \begin{bmatrix} (0.631051, 0.601775) & (0.780268, 0.391182) & (0.727329, 0.470992) \\ (0.73482, 0.46004) & (0.6094674, 0.62855) & (0.794008, 0.369551) \end{bmatrix} \end{array}$$

If the q-ROFWA operator is utilized for $q=2$ (operator reduced in PFWA operator) we get score values as $S(A_1)=0.47736$ and $S(A_2)=0.47736$ respectively. Thus, we can't find the best alternative as both have same scores. This happens because of the fact that the q-ROFWA operator for $q=2$ (or PFWA operator) did not take attentions to the familiarity of the expert with the evaluated objects. If the proposed CFWA$_q$ operator for $q=2$ is applied here by adding the confidence levels of the expert as $l = \begin{array}{c} A_1 \\ A_2 \end{array}\begin{bmatrix} 0.91 & 0.88 & 0.75 \\ 0.74 & 0.84 & 0.89 \end{bmatrix}$ towards the evaluated objects, we find the scores $S(A_1)=0.176372$ and $S(A_2)=0.164526$ for alternatives A_1 and A_2 respectively. Thus, the ranking is $A_1 > A_2$.

Example 7. Consider a decision making problem having two different alternatives namely A_1 and A_2, which are estimated against the criteria c_1, c_2 and c_3 whose weight vector is $w=(0.32,0.35,0.33)^T$. Let the decision maker provides his assessment in the form of q-ROFNs and presented as

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{ccc} \quad c_1 \quad & \quad c_2 \quad & \quad c_3 \quad \end{array} \\ \begin{bmatrix} (0.634348, 0.497916) & (0.545183, 0.391182) & (0.66758, 0.418213) \\ (0.560212, 0.358232) & (0.656693, 0.439574) & (0.628765, 0.50855) \end{bmatrix} \end{array}$$

Now applying the q-ROFWA operator for $q=3$, we obtain scores as $S(A_1)=0.158072$ and $S(A_2)=0.158072$ respectively. Thus, we are unable to find the best alternative on the basis of q-ROFWA operator. This is because of the statement that the q-ROFWA operator did not notice the confidence levels of expert.

On the other side, if the proposed CFWA$_q$ operator is employed for $q=3$ by adding together the confidence levels of the expert as $l = \begin{matrix} A_1 \\ A_2 \end{matrix}\begin{bmatrix} 0.91 & 0.88 & 0.75 \\ 0.74 & 0.84 & 0.89 \end{bmatrix}$ towards the alternatives, we obtain the scores $S(A_1)=0.084029$ and $S(A_2)=0.074836$. Thus, A_1 is the best suitable alternative.

Therefore, the presented analysis discussed so far has the following advantages over the existing methods.

1) The presented approach suggests a broader range of imprecise and vague information in that environment where the sum of q^{th} power of support for membership degree and q^{th} power of support against membership degree is more than one. Due to this characteristic, the generalized theory can deal not only incomplete data but also the indeterminate and inconsistent data, which exist commonly in real world situations. Therefore, the scope of application of the developed operators is broader than the existing methods for solving and designing the real life situations.
2) All the existing methods under *q*-rung orthopair fuzzy environment have been developed in the assumptions that all the experts are 100% familiar with the evaluated objects. But these types of limitations are not fully met in dealing with real life problems. In other side, proposed approach considered the situation where the experts are not fully familiar with evaluated objects. A comparison analysis (Table 9) conducted and provided the same ranking as obtained by different existing methods which verifies the proposed approach is practical and feasible. If we do not consider the confidence levels of experts for the familiarity with the evaluated objects, then our presented operators are reduced to the existing *q*-rung orthopair fuzzy aggregation operators.
3) Further, the superiority of the presented study over the existing ones is provided with the help of some contradictory examples under the environment of *q*-ROFSs. In these situations existing operators are unable to find the best alternative while the presented approach can overcome their shortcoming and offered the best alternative. Table 8 analyzed the variation in the scores and the rankings of the alternatives with the flexibility and sensitivity of the parameter *q*.
4) As both IFS and for PFS are special cases of *q*-ROFS. Therefore, proposed operators are more general because some of the existing operators for IFS and for PFS are special cases of the developed operators. Therefore, the proposed aggregation operators are better than the existing aggregation operators for IFNs and PFNs. Thus, they are more general and more suitable to solve MCDM problems more precisely.

7. Conclusions

The IFS theory and PFS theory are more suitable tools to express information under uncertain environment in MCDM problems. But, Yager (2017) pointed out that the *q*-ROFS is more general than the IFS and PFS. It is also notable that as the rung *q* increases the space of acceptable orthopairs increases and thus gives experts more freedom in expressing their belief about membership grade. Based on these advantages, some aggregation operators were proposed by different authors to add *q*-ROFNs. But these existing *q*-rung ortopairs aggregation operators are developed by assuming experts are surely familiar with evaluated objects i.e. all experts provided their assessment of the different alternative at the same level of confidence. This type of situations partially fulfilled in modelling real world problems. For this, the present study offers a series of confidence averaging and confidence geometric aggregation operator by incorporating the confidence levels of experts during evaluation step under *q*-rung orthopair fuzzy environment. Their some important properties are well established. These defined operators are capable to explain the real life situation more perceptibly with the help of experts' confidence levels during evaluation and will resemble the much more real situations under *q*-rung orthopair fuzzy environment. Finally, a detailed discussion has been carried out to illustrate the applicability and superiority of presented approach over the existing ones.

Due to the broader space of acceptance of *q*-ROFS, we will make effort in future to apply the concept of *q*-ROFS to solve the real life problems such as fuzzy cluster analysis, uncertain programming and pattern recognition, and so on. In addition, we will also focus on developing some new aggregation operators for *q*-ROFNs.

References

1) Zadeh, L.A., "Fuzzy sets," Information and Control 8(3), pp. 338–356, 1965.
2) Atanassov, K.T., "Intuitionistic fuzzy sets," Fuzzy Sets and Systems 20, pp. 87–96, 1986.
3) Song, Q. and Chissom., B., "Fuzzy Time Series and Its Models," Fuzzy Sets and Systems 54, pp. 269–277, 1993.
4) Chen, S.M., "Forecasting Enrollments based on Fuzzy Time Series," Fuzzy Sets and Systems 81, pp. 311–319, 1996.
5) De, S.K., Biswas, R. and Roy, A.K., "An application of intuitionistic fuzzy sets in medical diagnosis" Fuzzy Sets and Systems 117(2), pp. 209-213, 2001.
6) Singh, S.R., "A simple Method of Forecasting based on Fuzzy Time Series," Applied Mathematics and Computation 186, pp. 330–339, 2007.
7) Joshi, B.P. and Kumar, S., "Intuitionistic fuzzy sets based method for fuzzy time series forecasting," Cybernetics and Systems 43(1), pp. 34-47, 2012.
8) Joshi, B.P. and Kumar, S., "A Computational method for fuzzy time series forecasting based on difference parameters," International Journal of Modeling, Simulation and Scientific Computing 4(1), pp. 1-12, 2013.
9) Joshi, B.P., Kumar, A., Singh, A., Bhatt, P.K. and Bharti, B.K., "Intuitionistic fuzzy parameterized fuzzy soft set theory and its application" Journal of Intelligent & Fuzzy Systems 35(5), pp. 5217-5223, 2018.
10) Joshi, B.P., "Moderator intuitionistic fuzzy sets with applications in multi-criteria decision-making" Granular Computing 3(1), pp. 61-73, 2018.
11) Garg, H. and Arora, R., "Generalized intuitionistic fuzzy soft power aggregation operator based on *t*-norm and their application in multicriteria decision-making," International Journal of Intelligent Systems 34 (2), pp. 215-246, 2019.
12) Xu, Z.S. and Yager, R.R., "Some geometric aggregation operators based on intuitionistic fuzzy sets," International Journal of General System 35, pp. 417-433, 2006.
13) Xu, Z.S., "Intuitionistic fuzzy aggregation operators," IEEE Transactions on Fuzzy Systems 15, pp. 1179–1187, 2007.
14) Zhao, H., Xu, Z.S., Ni, M. and Liu, S., "Generalized aggregation operators for intuitionistic fuzzy sets," International Journal of Intelligent Systems 25, pp. 1–30, 2010.
15) Wang, W.Z. and Liu, X.W., "Intuitionistic fuzzy geometric aggregation operators based on Einstein operations," International Journal of Intelligent Systems 26, pp. 1049–1075, 2011.
16) Wang, W. and Liu, X., "Intuitionistic Fuzzy Information Aggregation Using Einstein Operations," IEEE Transactions on Fuzzy Systems 20(5), pp. 923-938, 2012.
17) Zhao, X. and Wei, G., "Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making," Knowledge-Based Systems 37, pp. 472–479, 2013.
18) Xu, Z.S., "Choquet integrals of weighted intuitionistic fuzzy information," Information Sciences 18, pp. 726–736, 2010.
19) Tan, C.Q. and Chen, X.H., "Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making," Expert Systems with Applications 37, pp. 149–157, 2010.
20) Xia, M.M., Xu, Z.S. and Chen, N., "Induced aggregation under confidence levels," International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 19, pp. 201–227, 2011.
21) Xia, M.M. and Xu, Z.S., "Group decision making based on intuitionistic multiplicative aggregation operators," Applied Mathematical Modelling 37, pp. 5120–5133, 2013.
22) Yu, D.J., "Intuitionistic fuzzy information aggregation and its application on multi-criteria decision-making," Journal of Industrial and Production Engineering 30, pp. 281–290, 2013.
23) Cagman, N. and Karatas, S., I"ntuitionistic fuzzy soft set theory and its decision making," Journal of Intelligent & Fuzzy Systems 24(4), pp. 829-836, 2013.
24) Yu, D., "Intuitionistic fuzzy information aggregation under confidence levels," Applied Soft Computing 19, pp. 147–160, 2014.
25) Ma, Z. and Zeng, S., "Confidence Intuitionistic Fuzzy Hybrid Weighted Operator and its Application in Multi-Criteria Decision Making," Journal of Discrete Mathematical Sciences and Cryptography 17, pp. 529-538, 2014.
26) Delia, I. and Cagman, N., "Intuitionistic fuzzy parameterized soft set theory and its decision making," Applied Soft Computing 28, pp. 109–113, 2015.
27) Joshi, B.P. and Kharayat, P.S., "Moderator Intuitionistic Fuzzy Sets and Application in Medical Diagnosis," Satapathy S., Raju K., Mandal J., Bhateja V. (eds) Proceedings of the Second International

Conference on Computer and Communication Technologies, Advances in Intelligent Systems and Computing 380, 2016. Springer, New Delhi. DOI: 10.1007/978-81-322-2523-2_16
28) Yu, D., "A scientometrics review on aggregation operator research," Scientometrics 105(1), pp. 115–133, 2015.
29) Xu, Z. and Zhao, N., "Information fusion for intuitionistic fuzzy decision making: an overview," Information Fusion 28, pp.10–23, 2016.
30) Yager, R. R., "Pythagorean fuzzy subsets," Proc Joint IFSA World Congress and NAFIPS, Annual Meeting, Edmonton, Canada; June 24–28, pp 57–61, 2013. DOI: 10.1109/IFSA-NAFIPS.2013.6608375
31) Yager, R. R. and Abbasov, A. M., "Pythagorean membeship grades, complex numbers and decision making," International Journal of Intelligent Systems 28, pp. 436–452, 2013.
32) Yager, R. R., "Pythagorean membership grades in multicriteria decision making," IEEE Transactions on Fuzzy Systems 22, pp. 958–965, 2014.
33) Zhang, X. L. and Xu, Z. S. "Extension of TOPSIS to multi-criteria decision making with pythagorean fuzzy sets," International Journal of Intelligent Systems 29, pp. 1061–1078, 2014.
34) Peng, X. and Yang, Y., "Some results for Pythagorean fuzzy sets," International Journal of Intelligent Systems 30(11), pp. 1133–1160, 2015.
35) Peng, X. and Yang, Y., "Pythagorean Fuzzy Choquet Integral Based MABAC Method for Multiple Attribute Group Decision Making," International Journal of Intelligent Systems 31(10), pp. 989–1020, 2016.
36) Garg, H., "A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making," International Journal of Intelligent Systems 31(9), pp. 886–920, 2016a.
37) Garg, H., "A novel correlation coefficients between Pythagorean fuzzy sets and its applications to decision-making processes," International Journal of Intelligent Systems 31(12), pp. 1234–1253, 2016b.
38) Garg, H., "Generalized pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multicriteria decision-making process," International Journal of Intelligent Systems 32(6), pp. 597-630, 2017a. DOI:10.1002/int.21860.
39) Garg, H., "Confidence levels based Pythagorean fuzzy aggregation operators and its application to decision-making process," Computational and Mathematical Organization Theory 23(4), pp. 546-571, 2017b. doi: 10.1007/s10588-017-9242-8.
40) Zhang, R., Wang, J., Zhu, X., Xia, M. and Yu, M., "Some Generalized Pythagorean Fuzzy Bonferroni Mean Aggregation Operators with Their Application to Multiattribute Group Decision-Making," Complexity, pp. 1-16. 2017. DOI:10.1155/2017/5937376
41) Joshi, B.P., "Pythagorean fuzzy average aggregation operators based on generalized and group-generalized parameter with application in MCDM problems," International Journal of Intelligent Systems 34 (5), pp. 895-919, 2019.
42) Yager, R.R., "Generalized Orthopair Fuzzy Sets," IEEE Transactions on Fuzzy Systems 25(5), pp. 1222-1230, 2017. DOI 10.1109/TFUZZ.2016.2604005
43) Yager, R.R. and Alajlan, N., "Approximate Reasoning with Generalized Orthopair Fuzzy Sets," Information Fusion, 2017. DOI: 10.1016/j.inffus.2017.02.005
44) Liu, P. and Wang, P., "Some *q*-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making," International Journal of Intelligent Systems 33(2), pp. 259–280, 2018. DOI: 10.1002/int.21927
45) Joshi, B.P., Singh, A., Bhatt, P.K. and Vaisla, K.S., "Interval valued *q*-rung orthopair fuzzy sets and their properties," Journal of Intelligent & Fuzzy Systems 35(5), pp. 5225-5230, 2018.
46) Jun, W., Runtong, Z., Xiaomin, Z., Zhen, Z, Xiaopu, S. and Weizi, L., "Some *q*-rung orthopair fuzzy Muirhead means with their application to multi-attribute group decision making" Journal of Intelligent & Fuzzy Systems 36(2), pp. 1599-1614, 2019.
47) Peng, X. and Liu, L., "Information measures for *q*-rung orthopair fuzzy sets," International Journal of Intelligent Systems 34 (8), pp. 1795-1834, 2019.