Approximation Hardness of Optimization Problems in Intersection Graphs of d-dimensional Boxes

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Abstract

The Maximum Independent Set problem in d-box graphs, i.e., in the intersection graphs of axis-parallel rectangles in $\mathbb{R}^d$, is a challenge open problem. For any fixed $d \geq 2$ the problem is NP-hard and no approximation algorithm with ratio $o(\log^{d-1} n)$ is known. In some restricted cases, e.g., for d-boxes with bounded aspect ratio, a PTAS exists [17]. In this paper we prove APX-hardness (and hence non-existence of a PTAS, unless P = NP), of the Maximum Independent Set problem in d-box graphs for any fixed $d \geq 3$. We state also first explicit lower bound $\frac{443}{442}$ on efficient approximability in such case. Additionally, we provide a generic method how to prove APX-hardness for many NP-hard graph optimization problems in d-box graphs for any fixed $d \geq 3$. In 2-dimensional case we give a generic approach to NP-hardness results for these problems in highly restricted intersection graphs of axis-parallel unit squares (alternatively, in unit disk graphs).

1 Introduction

The *intersection graph* of a family of sets S_v, $v \in V$, is a graph with vertex set V such that u is adjacent to v if and only if $S_u \cap S_v \neq \emptyset$. The family $\{S_v, v \in V\}$ is an *intersection representation* of this graph. Geometrical models of intersection graphs deal with families of subsets of $\mathbb{R}^d$ with some geometric properties. In this paper we are mainly interested in families of axis-parallel d-dimensional boxes. Their intersection graphs are called *d-box intersection graphs*, or simply *d-box graphs*. Recall that a d-dimensional box (d-box) is a subset of $\mathbb{R}^d$ that is a Cartesian product of d intervals in $\mathbb{R}$. For convenience, terms an *interval* and a *rectangle* are used for 1-box and 2-box, respectively.

In this paper we describe a generic approach to approximation hardness results for many graph optimization problems as, e.g., Minimum Vertex Cover, Maximum Independent Set, in d-box intersection graphs for any fixed $d \geq 3$.

Overview. Many optimization problems like Maximum Clique, Maximum Independent Set, and Minimum (Vertex) Coloring are NP-hard for general graphs but solvable in polynomial time for interval graphs [20]. Many of them are known to be NP-hard already in 2-dimensional models of geometric intersection graphs (e.g., in unit disk graphs). In most cases the geometric restrictions allow us to obtain better approximation algorithms (or even in polynomial time solvability) for problems that are in general graphs extremely hard to approximate. On the other hand, these geometric restrictions make the task to achieve some hardness results more difficult.

Among basic NP-hard graph optimization problems in d-box graphs ($d \geq 2$), only Maximum Clique is known to be exactly solvable in polynomial time ([5], [29], [35]). Some optimization problems are known to be NP-hard in d-box intersection graphs for any fixed $d \geq 2$, for example Maximum Independent Set [18], [25], or Minimum Coloring [30].

The challenging open problem, the Maximum Independent Set problem (shortly, Max-IS), in d-box intersection graphs ($d \geq 2$) can be formulated as follows: for a given set $\mathcal{R}$ of n axis-parallel d-dimensional boxes, find a maximum cardinality subset $\mathcal{R}^* \subseteq \mathcal{R}$ of pairwise disjoint boxes. The problem has attracted an attention of many researchers ([1], [11], [12], [17], [23], [26], [33]) due to its applications in map labeling, data mining, VLSI design, image processing, and point location in d-dimensional Euclidean space. As the problem is NP-hard for any fixed $d \geq 2$ ([18], [25]), its approximability is intensively studied.

Let us describe briefly known approximability results for it, see [12] for more detailed overview. The earliest result was the shifting grid method based PTAS by Hochbaum and Maass [23] for the case of unit d-cubes. This method works for any collection of 'fat' objects in $\mathbb{R}^d$ of roughly the same size. Their scheme required $n^{O(k^d)}$ time to guarantee an approximation factor of $(1+\frac{1}{k})$. By applying dynamic programming along

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one of the dimensions, running time has been reduced to $n^{O(k^{d-1})}$ and generalized to objects, not necessarily fat, but whose projections to the last $(d-1)$ coordinates are fat and of roughly the same size. This was essentially established by Agarwal et al. [1] in their work on unit-height rectangles in the plane. To achieve $(1+\frac{1}{k})$-approximation they need time $O(n^{2k-1} + n\log n)$.

Generalizing in another direction, Erlebach et al. [17] obtained a PTAS for fat objects of possibly varying sizes, such as arbitrary d-cubes or bounded aspect ratio d-boxes. The running time of their algorithm ($n^{O(k^4)}$ for $d = 2$) has been improved to $n^{O(k^{d-1})}$ by Chan [12].

For *arbitrary* d-boxes, even for $d = 2$, the existence of a PTAS or a constant factor approximation algorithm, is largely open problem. As has been observed in several papers ([1], [26]), a logarithmic approximation factor is possible in this case. For example, Agarwal et al. [1] described $O(n \log n)$-time algorithm with factor at most $\lceil \log_2 n \rceil$. This generalizes to $\lceil \log_2 n \rceil^{d-1}$-approximation algorithms with $O(n \log_2^{d-1} n)$ running time, for general d. Nielsen [33] independently described algorithm with optimum-sensitive approximation factor $(1 + \log_2(is(\mathcal{R})))^{d-1}$, where $is(\mathcal{R})$ is the maximum number of independent boxes of $\mathcal{R}$. Currently, no polynomial time algorithm is known with $o(\log^{d-1} n)$-approximation factor, although Berman et al. [11] have observed that a $\log_2^{d-1} n$ bound can be reduced by arbitrary multiple constant. More precisely, a factor $\lceil \log_b n \rceil^{d-1}$ can be achieved in $n^{O(b^{d-1})}$ time, for any fixed integer $b \geq 2$. As it was already mentioned in [11], it remains open to understand the limits on the approximability of the Maximum Independent Set problem in intersection graphs of d-box graphs for $d \geq 2$.

Our results. In this paper we present the proof of APX-hardness (with some explicit lower bounds) for the Maximum Independent Set problem in axis-parallel d-dimensional boxes for any fixed $d \geq 3$. It follows, in particular, that the existence of a PTAS for the problem restricted to d-boxes with bounded aspect ratio [17] cannot be generalized (unless P = NP) to arbitrary d-boxes for any fixed $d \geq 3$.

We provide also a generic method how to prove approximation hardness results in d-box graphs that is applied simultaneously to several graph optimization problems such as covering, domination, and matching problems. The idea of our APX-hardness results is based on the proof of the following two facts:

- Many optimization problems are APX-hard even when restricted to suitable subdivisions of graphs, e.g., for a fixed $k \geq 0$, in graphs obtained by $2k$ (respectively, $3k$) subdivision of each edge from graphs of maximum degree 3 (Section 3).
- Each graph obtained from another one by at least 3-subdivision of each edge is an intersection graph of d-boxes for any fixed $d \geq 3$. Moreover, its representation can be provided in time polynomial in its size. (Section 2).

Both these results are interesting for their generality and can be of independent interest. The same applies to a newly introduced notion of a *shift-reduction* (Section 3) used in the proof of APX-hardness for many graph optimization problems in d-boxes ($d \geq 3$), e.g., Minimum Vertex Cover, Minimum (Independent) Dominating Set, and Maximum Induced Matching. The methods also allow to provide explicit lower bounds on efficient approximability. This is demonstrated on the Maximum Independent Set problem in d-box graphs ($d \geq 3$), for which we prove NP-hardness to achieve approximation factor of $1 + \frac{1}{442}$. Usually better approximation algorithms for optimization problems in d-box graphs need a representation of an input graph by d-boxes, not merely a graph. Our hardness results apply to this setting as well. Moreover, they apply to the highly restricted case, in which no point of $\mathbb{R}^d$ is simultaneously covered by more than two d-boxes, and each d-box intersects at most 3 others.

Our methods do not apply to the planar case in which there is still hope for a PTAS. All problems studied in the paper are NP-hard in 2-dimensions as well, even in very restricted setting of intersection graphs of axis-parallel unit squares (or unit disks). We provide a generic approach to NP-hardness results for many graph optimization problems in intersection graphs of axis-parallel unit squares (alternatively, in unit disk graphs).

We also contribute to another independent set problem connected with sets of d-boxes, introduced by Bafna et al. [4]. The problem is the Maximum Weighted Independent Set problem (Max-w-IS) in proper d-union graphs, i.e., in graphs that can be expressed as the union of d proper interval graphs. For $d = 2$ the problem is quite well understood. For $d \geq 3$ we improve significantly the upper bounds on approximability of this problem (see [4], [9], [10], [6]). Namely, the Max-w-IS problem in proper d-union graphs, $d \geq 2$, can be approximated within $(d + \frac{1}{2})$ in polynomial time, and within $(2d - 1)$ in nearly linear time.

Definitions and notations. All graphs in this paper are simple. For a graph $G = (V, E)$ and a nonnegative integer k, a *k-subdivision of an edge* $e \in E$ is the operation replacing the edge $e = \{u, v\}$ by a path with endvertices u, v, and k new internal vertices. A

k-*subdivision of* G, denoted by $\mathrm{div}_k(G)$, is a graph obtained from G by k-subdividing of each edge e of G. For an integer $B \geq 3$, let $\mathcal{G}_B$ denote the set of graphs of maximum degree at most B (without isolated vertices). Notice that a graph $G = (V, E)$ from $\mathcal{G}_B$ has $|E| \leq \frac{|V|}{2}B$ edges, its k-subdivision $\mathrm{div}_k(G)$ has $|E|k + |V| \leq \frac{|V|}{2}(Bk + 2)$ vertices, and $|E|(k + 1) \leq \frac{|V|}{2}B(k + 1)$ edges. Moreover, equalities hold for B-regular graphs.

A graph $G = (V, E)$ is a d-*box* graph if it is an intersection graph of a family of axis-parallel d-dimensional boxes (so called d-boxes). It means, a d-box R_v can be assigned for each vertex $v \in V$ such that $\{u, v\} \in E$ if and only if the boxes R_u and R_v intersect.

In a graph $G = (V, E)$ a vertex $v \in V$ is said to *cover* itself, all edges incident with v, and all vertices adjacent to v. An edge $\{u, v\}$ is said to *cover* itself, vertices u and v, and all edges incident with u or v. A *vertex cover* is a subset C of V that covers E, an *edge cover* is a subset S of E that covers V, a *dominating set* is a subset D of V that covers V, and an *edge dominating set* is a subset S of E that covers E. Two elements of $V \cup E$ are *independent* if neither covers the other. A subset I of V is a *strong independent set* in G, if for $u, v \in I$, $u \neq v$, implies $\mathrm{dist}_G(u, v) > 2$. A strong independent set is also called 2-packing, or 2-independent set by some authors.

The goal of the Maximum Independent Set problem, resp. Maximum Strong Independent Set (Max-SIS), is to find an independent set, resp. a strong independent set, of maximum cardinality in G; let $is(G)$, resp. $sis(G)$, denote its cardinality. The Minimum Vertex Cover problem (Min-VC), resp. Maximum Minimal Vertex Cover (Max-Minl-VC), is looking for a vertex cover of minimum cardinality, resp. a minimal vertex cover of maximum cardinality, in G. Denote $vc(G)$ the optimum value for Min-VC. Similarly, Maximum Minimal Edge Cover (Max-Minl-EC) is looking for a minimal edge cover of maximum cardinality in G. The problems Minimum Dominating Set (Min-DS), Minimum Independent Dominating Set (Min-IDS), resp. Minimum Edge Dominating Set (Min-EDS), ask for a dominating set, an independent dominating set, resp. an edge dominating set, of minimum size in G. Let $ds(G)$, $ids(G)$, and $eds(G)$ stand for the corresponding minima for Min-DS, Min-IDS, and Min-EDS in G.

A *matching* is a subset M of E whose elements are pairwise independent. A matching M is *induced* if for each edge $e = \{u, v\} \in E$, $u, v \in V(M)$ implies $e \in M$. A subset T of $V \cup E$ whose elements are pairwise independent is said to be a *total matching* for G. The Minimum Maximal Matching problem (Min-Maxl-Match) asks to find a maximal matching of minimum cardinality in G. A maximal matching corresponds an independent edge dominating set, hence Min-Maxl-Match corresponds to Minimum Independent Edge Dominating Set. The objective of the Maximum Induced Matching problem (Max-Induced-Match), resp. Maximum Total Matching (Max-Total-Match), is to find a maximum induced matching in G, resp. a maximum total matching in G. Let $im(G)$ denote the cardinality of a maximum induced matching in G.

Given a spanning forest F for G, an edge $\{u, v\}$ of F is a *pendant* edge for F if the degree of u or v in F is 1. The goal of the Maximum Spanning Forest problem (Max-SF) is to find a spanning forest of G with the maximum number of pendant edges among all spanning forests in G.

For the basic optimization terminology we refer the reader to Ausiello et al. [3]. For any NPO optimization problem P, I_P is the set of instances of P, $\mathrm{sol}_P(x)$ is the set of feasible solutions of $x \in I_P$, and $m_P(x, y)$ is the value of feasible solution y, for every pair $x \in I_P$ and $y \in \mathrm{sol}_P(x)$. Let P and Q be two NPO problems and f be a polynomial time computable function that maps instances of P to instances of Q. Then f is said to be an L-*reduction* from P to Q, if there are constants α, $\beta \in (0, \infty)$ and a polynomial time computable function g such that for every $x \in I_P$ (1) $\mathrm{OPT}_Q(f(x)) \leq \alpha \mathrm{OPT}_P(x)$, (2) for every $y' \in \mathrm{sol}_Q(f(x))$, $g(x, y') \in \mathrm{sol}_P(x)$ so that $|\mathrm{OPT}_P(x) - m_P(x, g(x, y'))| \leq \beta |OPT_Q(f(x)) - m_Q(f(x), y')|$. To show APX-completeness of a problem $P \in$ APX it is enough to show that there is an L-reduction from some APX-complete problem to P. This enables us to prove APX-completeness by means of the easier-to-use L-reducibility.

2 Intersection Graphs of Axis-Parallel Boxes

Roberts [37] proved that every graph can be realized as an intersection graph of axis-parallel d-dimensional boxes for some d depending on the graph. For any fixed $d \geq 2$ the recognition of d-box graphs is NP-hard ([28], [39]), and hence the reconstruction of their representation by d-boxes as well. In this section we prove that highly non-trivial subclasses of general graphs are d-box graphs for $d = 3$ (and hence for any $d \geq 3$).

Theorem 2.1. *Let us given a graph $G = (V, E)$ and for each edge $e \in E$ an integer $s(e) \geq 3$. Let G' denote a graph obtained from G by $s(e)$-subdivision of each edge e, i.e., replacing $e = \{u, v\}$ by a path with endvertices u, v, and $s(e)$ new internal vertices (all paths are pairwise* *internally disjoint). For any fixed integer $d \geq 3$, the graph G' can be realized as the intersection graph of a set of axis-parallel d-dimensional boxes. Moreover, such representation can be done in time polynomial in size of G and $\sum_e s(e)$.*

Sketch of the proof. We can assume that $V = \{1, 2, \ldots, n\}$ and denote $N = |V| + \sum_e s(e)$. Any edge $e = \{i, j\}$ (with $i < j$ for the definiteness) is replaced in G' by a path i, A_e^1, A_e^2, ..., $A_e^{s(e)}$, j. We suppose that $d = 3$ and describe the representation of G' as an intersection graph of a set $\{R_1, R_2, \ldots, R_N\}$ of axis-parallel boxes in $\mathbb{R}^3$. For any $d > 3$ one can take simply $\{R_i \times [0,1]^{d-3} : i = 1, 2, \ldots, N\}$ as the corresponding set of boxes in $\mathbb{R}^d$, representing G'.

First, we fix a sufficiently large integer C and put $I := [0, C]$. (The value of C can be easily given explicitly, along with the other constants chosen later in the proof, depending polynomially on N.) The box I^3 will contain all boxes which we will construct.

Next, we choose integers $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < C$ that are well separated one from another (say, nearly equidistributed). The box $R_i := [a_i, b_i]^3$ will be our representation of the vertex $i \in V$, for each $i \in \{1, 2, \ldots, n\}$. Let $N(i)$ denote the set of neighbors of a vertex i in G, $i \in \{1, 2, \ldots, n\}$. The notation $U(x)$ is used for the unit interval $[x, x + 1]$.

Now we describe boxes representing new vertices, i.e., vertices of G' that are not vertices of G. For each fixed vertex $i \in V$ we choose in $[a_i, b_i)$ well separated integers $a_{i,j}$ for each $j \in N(i)$. Let us consider now one fixed edge $e = \{i, j\} \in E$ with $i < j$ and define boxes R_e^1, R_e^2, ..., $R_e^{s(e)}$ representing vertices A_e^1, A_e^2, ..., $A_e^{s(e)}$, respectively. Let us define first $R_e^1 := I \times U(a_{i,j}) \times U(a_{i,j})$, $R_e^{s(e)} := U(a_{j,i}) \times I \times U(a_{j,i})$, and consider a box $S_e := U(a_{j,i}) \times U(a_{i,j}) \times I$.

If $s(e) = 3$, one can take $R_e^2 := S_e$ to represent the vertex A_e^2 of the path i, A_e^1, A_e^2, A_e^3, j. If $s(e) \geq 4$, then the vertices A_e^2, A_e^3, ..., $A_e^{s(e)-1}$ will be realized by subboxes R_e^2, R_e^3, ..., $R_e^{s(e)-1}$ of S_e of the form $R_e^l := U(a_{j,i}) \times U(a_{i,j}) \times [c_l, d_l]$, $l = 2, 3, \ldots, s(e) - 1$, where $c_2 = 0$, $d_{s(e)-1} = C$, $a_e + 1 < c_3 < d_2$, $c_{s(e)-1} < d_{s(e)-2} < a_{j,i}$ and, if $s(e) \geq 5$, $c_{l+1} < d_l < c_{l+2}$ whenever $2 \leq l \leq s(e) - 3$. One can easily check that the intersection graph of the set $\{R_1, R_2, \ldots, R_n\} \cup \bigcup_{e \in E} \{R_e^1, R_e^2, \ldots, R_e^{s(e)}\}$ of axis-parallel boxes in $\mathbb{R}^3$ is (isomorphic to) G'. Moreover, the construction described above has time complexity polynomial in size of G and $\sum_e s(e)$. □

Remark 2.1. *The graph G' from the Theorem 2.1 is of girth at least 9. In any representation of G' by axis-parallel d-boxes no point of $\mathbb{R}^d$ is simultaneously covered by more than two boxes. In planar case, a $4K$-approximation algorithm is known for* Max-w-IS *in such case, where K is the maximum number of rectangles that simultaneously cover a point in plane* [31].

2.1 Intersection Graphs of Sets of Axis Parallel Lines.

For $d \geq 3$, d-box intersection graphs are from topological point of view as complex as general graphs. Topological obstructions do not allow us to apply similar constructions in planar case. Also the complexity of intersection graphs of axis-parallel lines in dimensions 2 and 3 significantly differs one from another. In $\mathbb{R}^2$ these graphs are exactly the complete bipartite graphs, on which classical optimization problems are easily solvable. On the other hand, for intersection graphs of sets of axis-parallel lines in $\mathbb{R}^3$ the situation is the same as in case of axis-parallel boxes. It can be proved similarly as in Theorem 2.1 that any higher subdivisions of general graphs (at least 5 subdivision of each edge) can be realized as intersection graphs of sets of axis-parallel lines in $\mathbb{R}^3$.

3 APX-hardness of Graph Problems in Certain Subdivisions of Graphs

Let $\mathcal{P}$ be a set of the problems Max-IS, Min-VC, Min-DS, Min-IDS, Min-EDS, Max-Induced-Match, and Max-SIS. Our aim now is to show APX-completeness for each problem from $\mathcal{P}$ when restricted to suitable subdivisions of graphs. First, we show that certain subdivision operations are in fact L-reductions that self-reduce any problem from $\mathcal{P}$ when restricted to $\mathcal{G}_B$ (except Min-IDS).

Lemma 3.1. *Let a graph $G = (V, E)$ and an edge $e \in E$ be given. Let G' be a graph obtained from G by 2-subdivision of the edge e, i.e., replacing $e = \{u, v\}$ by a path u, u', v', v with new vertices u' and v'. Then $vc(G') = vc(G) + 1$ and $is(G') = is(G) + 1$.*

Applying iteratively the steps of the proof of Lemma 3.1 we obtain the following

Theorem 3.1. *Let us given a graph $G = (V, E)$ and for each edge $e \in E$, let $s(e)$ be a nonnegative integer. Let G' be a graph obtained from G by $2s(e)$-subdivision of each edge e, i.e., replacing $e = \{u, v\}$ by a path with endvertices u, v, and $2s(e)$ new vertices (the paths are pairwise internally disjoint). Let P be either the problem* Min-VC, *or* Max-IS. *Then*

(A) $\mathrm{OPT}_P(G') = \mathrm{OPT}_P(G) + \sum_e s(e)$;

(B) *every $y \in \mathrm{sol}_P(G)$ can be transformed in polynomial time (in size of G and $\sum_e s(e)$) to $y' \in \mathrm{sol}_P(G')$ such that $|y'| = |y| + \sum_e s(e)$;*

(C) *every $y' \in \mathrm{sol}_P(G')$ can be transformed in polynomial time to $y \in \mathrm{sol}_P(G)$ such that $|y'| - \sum_e s(e) \le |y|$ if P is* Max *problem; respectively $|y| \le |y'| - \sum_e s(e)$ if P is* Min *problem.*

Similarly as *is* and *vc*, several other graph parameters behave well under suitable subdivision operations. We will demonstrate that at least for *ds*, *eds*, *im*, and *sis*.

Lemma 3.2. *Let a graph $G = (V, E)$ and an edge $e \in E$ be given. Let G' be a graph obtained from G by 3-subdivision of the edge e, i.e., replacing $e = \{u, v\}$ by a path u, u', w, v', v with new vertices u', w, and v'. Then (i) $ds(G') = ds(G) + 1$, (ii) $eds(G') = eds(G) + 1$, (iii) $im(G') = im(G) + 1$, and (iv) $sis(G') = sis(G) + 1$.*

Using the steps of the proof of previous lemma we obtain the following

Theorem 3.2. *Let us given a graph $G = (V, E)$ and for each edge $e \in E$, let $s(e)$ be a nonnegative integer. Let G' be a graph obtained from G by a $3s(e)$-subdivision of each edge e, i.e., replacing $e = \{u, v\}$ by a path with endvertices u, v, and $3s(e)$ new vertices (the paths are pairwise internally disjoint). If P is one of the problems* Min-DS, Min-EDS, Max-Induced-Match, *or* Max-SIS, *then the conditions (A)–(C) from Theorem 3.1 hold. Moreover, if $s(e) > 0$ for each $e \in E$, then $ids(G') = ds(G')$ and every dominating set D in G can be transformed in polynomial time to an independent dominating set D' in G' with $|D'| = |D| + \sum_e s(e)$.*

Remark 3.1. *If $s(e)$ is an odd integer for each edge $e \in E$ in Theorem 3.2, then the graph G' is bipartite.*

Now let us firstly consider the problem Max-IS. For any graph $G = (V, E)$ and fixed integer $k \ge 0$ we obtain by Theorem 3.1

$$is(\mathrm{div}_{2k}(G)) = is(G) + |E|k. \tag{3.1}$$

To see that div_{2k} is an L-reduction for Max-IS restricted to the set $\mathcal{G}_B$, we have to check that for some constant α we have $is(\mathrm{div}_{2k}(G)) \le \alpha is(G)$ for every $G \in \mathcal{G}_B$. The key point is that for some positive constant c (depending on B) one can easily prove the lower bound for the optimum value of the form $is(G) \ge c|V|$ for every $G = (V, E) \in \mathcal{G}_B$. Namely, $is(G) \ge \frac{|V|}{B+1}$. Now, recalling that $|E| \le \frac{|V|}{2}B$, it can be easy verified that the choice $\alpha := 1 + B\frac{k}{2c}$ will do. The second condition from the definition of L-reduction is satisfied with $\beta = 1$ by Theorem 3.1. Hence div_{2k} is an L-reduction that self-reduces Max-IS restricted to $\mathcal{G}_B$.

Similarly, we can prove that div_{2k}, resp. div_{3k}, when restricted to $\mathcal{G}_B$, is an L-reduction that self-reduces Min-VC, resp. Min-DS, Min-EDS, Max-Induced-Match, and Max-SIS. Moreover, div_{3k} for $k > 0$, when restricted to $\mathcal{G}_B$, reduces Min-DS to Min-IDS in the same way as well. The second condition from the definition of an L-reduction is satisfied with $\beta = 1$ by Theorems 3.1 and 3.2. The existence of α for an L-reduction is again based on the lower bound $\mathrm{OPT}_P(G) \ge c|V_G|$ (with a positive constant c depending on the problem P and on B, but independent of G within the class $\mathcal{G}_B$) works for each problem from $\mathcal{P}$. To comment briefly on those lower bounds, let $G = (V, E) \in \mathcal{G}_B$ be fixed. Easy counting arguments show that $is(G) \ge ids(G) \ge ds(G) \ge \frac{|V|}{B+1}$, $vc(G) \ge \frac{|V|}{B+1}$, $eds(G) \ge \frac{|V|}{2B}$, $sis(G) \ge \frac{|V|}{B^2+1}$, and $im(G) \ge \frac{|V|}{4B^2-4B+2}$. (Let us note that for some of these results our restriction to graphs $\mathcal{G}_B$ without isolated vertices is crucial.)

One can use these lower bounds to prove that the problems are in APX when restricted to $\mathcal{G}_B$. For any of minimization problems above any feasible solution approximates within a constant. For maximization problems above, the lower bounds given apply to any inclusionwise maximal independent set (strong independent set, and induced matching, respectively), that provide a constant factor approximation in this case. Simple greedy methods in bounded degree graphs provide constant factor approximations. E.g., any inclusionwise maximal independent set is a $(B + 1)$-approximation for any of problems Max-IS, Min-DS, or Min-IDS, when restricted to graphs of degree at most B. Additionally, each problem from $\mathcal{P}$ is well known to be APX-hard when restricted to $\mathcal{G}_3$ (graphs of degree at most 3), in most cases a proof for 3-regular graphs is given as well (see [2], [16], [19], [21], [32], and reference therein). Moreover, explicit lower bounds on their efficient approximability are known for several of them ([13], [14], [15]). Hence, the considered problems are APX-complete when restricted to $\mathcal{G}_3$ (or even in 3-regular graphs). Due to properties of L-reduction the following theorems hold

Theorem 3.3. *For any fixed integer $k \ge 0$ the restrictions of problems* Max-IS *and* Min-VC *to $2k$-subdivisions of 3-regular graphs are* APX*-complete.*

Theorem 3.4. *For any fixed integer $k \ge 0$ the restrictions of problems* Min-DS, Min-EDS, Max-Induced-Match, Max-SIS, *and* Min-IDS *to $3k$-subdivisions of graphs of maximum degree 3 are* APX*-complete.*

Remark 3.2. *If k is odd then Theorem 3.4 claims* APX*-completeness results in bipartite graphs of maximum degree 3 and girth at least $9k + 3$.*

For later applications to the MAX-IS problem in d-box graphs we now formulate some explicit NP-hard gap type results for this problem restricted to certain subdivisions of graphs. We will use the corresponding NP-hard gap results for MAX-IS in B-regular graphs, $B = 3, 4$ proved in [14]. For any $\varepsilon > 0$ it is NP-hard to decide for a B-regular graph $G = (V, E)$ of whether $is(G) < \frac{|V|}{2}\Big(1 - 3\delta_B - \varepsilon\Big)$ or $is(G) > \frac{|V|}{2}\Big(1 - 2\delta_B + \varepsilon\Big)$ where $\delta_3 \approx 0.0103305$ and $\delta_4 \approx 0.020242915$. Using the formula (3.1) we see that this translates to the following NP-hardness result for 4-subdivision of B-regular graphs: for any $\varepsilon > 0$ it is NP-hard to decide of whether $is(\mathrm{div}_4(G)) < \frac{|V|}{2}(1 + 2B - 3\delta_B - \varepsilon)$, or $is(\mathrm{div}_4(G)) > \frac{|V|}{2}(1 + 2B - 2\delta_B + \varepsilon)$. Consequently, approximation within any constant smaller than $1 + \frac{\delta_B}{1+2B-3\delta_B}$ is NP-hard. In particular, we have proved the following

THEOREM 3.5. *It is* NP*-hard to approximate* MAX-IS *in* 4*-subdivision of* 3*-regular graphs within* $1 + \frac{1}{675}$*, and in* 4*-subdivision of* 4*-regular graphs within* $1 + \frac{1}{442}$.

3.1 Extension of Hardness Results to Other Problems. In order to prove the version of Theorems 3.1 and 3.2 for some other problems, we introduce the notion of a *shift-reduction*. First, for any NPO problem P we define the *sign* σ_P of P to be $+1$ if the goal of P is maximum, and -1 if the goal is minimum.

DEFINITION 3.1. *Let P and Q be two* NPO *problems and f be a polynomial time computable function that maps instances of P to instances of Q. We say that f is a shift reduction from P to Q if there are polynomial time computable functions φ, h, and g such that*

(1) $\varphi : I_P \to \mathbb{R}$,

(2) *for every $x \in I_P$ and every $y \in \mathrm{sol}_P(x)$, $h(x,y) \in \mathrm{sol}_Q(f(x))$ so that $\sigma_Q m_Q(f(x), h(x,y)) - \sigma_P m_P(x,y) \geq \varphi(x)$,*

(3) *for every $x \in I_P$ and every $y' \in \mathrm{sol}_Q(f(x))$, $g(x,y') \in \mathrm{sol}_P(x)$ so that $\sigma_Q m_Q(f(x), y') - \sigma_P m_P(x, g(x,y')) \leq \varphi(x)$.*

In such case f is called a shift for the ordered pair (P,Q) with a difference φ. If both P and Q are restrictions of the same problem P_0 to instance sets $\mathcal{G}$ and $\mathcal{G}'$, respectively, we say simply that f is a shift for P_0 restricted to $\mathcal{G}$.

Clearly, (2) implies $\sigma_Q \mathrm{OPT}_Q(f(x)) - \sigma_P \mathrm{OPT}_P(x) \geq \varphi(x)$, and (3) implies the opposite inequality. Hence $\sigma_Q \mathrm{OPT}_Q(f(x)) - \sigma_P \mathrm{OPT}_P(x) = \varphi(x)$, and with this equality in hand one can rewrite inequalities in (2) and (3) in the form

(2′) $|\mathrm{OPT}_Q(f(x)) - m_Q(f(x), h(x,y))| \leq |\mathrm{OPT}_P(x) - m_P(x,y)|$, $x \in I_P$, $y \in \mathrm{sol}_P(x)$,

(3′) $|\mathrm{OPT}_P(x) - m_P(x, g(x,y'))| \leq |\mathrm{OPT}_Q(f(x)) - m_Q(f(x), y')|$, $x \in I_P$, $y' \in \mathrm{sol}_Q(f(x))$.

Observation. Shift-reductions compose as follows: If f_1 is a shift for (P,Q) with difference φ_1, and f_2 is a shift for (Q,R) with difference φ_2, then $f_2 \circ f_1$ is a shift for (P,R) with difference $\varphi = \varphi_1 + \varphi_2 \circ f_1$.

Many reductions between pairs of optimization problems in the literature are shift-reductions, and proofs given there show exactly the properties (2) and (3) from Definition 3.1 for them. Here we are focused on the fact that for a lot of graph problems certain subdivision operations on graphs are shifts that reduce the problem to itself, or to another known problem.

Theorem 3.1 (resp., Theorem 3.2) says, in particular, that operation div_{2k} (resp., div_{3k}) is a shift for all problems considered in these theorems. However, in some applications it is useful to consider subdivisions that are not uniform, but rather variable. Let us describe such more general subdivision operations more formally.

Let $\mathcal{G}$ stand in what follows for a set of graphs recognizable in polynomial time. For any graph $G = (V,E)$ from $\mathcal{G}$ let $s := s_G$ denote an *edge function* $s : E \to \mathbb{Z}_0^+$ that is polynomial time computable. In this setting $\mathrm{div}_{s(e)}$ will denote a (polynomial time computable) transformation that maps $G \in \mathcal{G}$ to a graph G' obtained from G by $s(e)$-subdivision of each edge $e \in E$. Then Theorem 3.1 says that for any edge function s the operator $\mathrm{div}_{2s(e)}$ is a shift for the MAX-IS problem (respectively, for MIN-VC) restricted to $\mathcal{G}$ with difference $\sum_e s_G(e)$ (resp., $-\sum_e s_G(e)$). Similarly, Theorem 3.2 says that if P is any of problems MIN-DS, MIN-EDS, MAX-INDUCED-MATCH, or MAX-SIS, then for any edge function s, $\mathrm{div}_{3s(e)}$ is a shift for P restricted to $\mathcal{G}$ with difference $\sigma_P \sum_e s(e)$. The problem MIN-IDS is not self-reducible in this way but MIN-DS reduces to it (assuming $s_G(e) > 0$ for each edge $e \in E$ and $G = (V,E) \in \mathcal{G}$), namely $\mathrm{div}_{3s(e)}$ is a shift for ($\text{MIN-DS}\big|_{\mathcal{G}}$, MIN-IDS) with difference $-\sum_e s_G(e)$. Some graph problems are known to be related to those studied in Theorem 3.2. Translated to our terminology these relations can be stated as follows:

LEMMA 3.3. *The identity map is a shift for the following pairs of graph problems:*

(i) for (MIN-IDS, MAX-MINL-VC) *with difference* $|V|$,

(ii) for (MIN-EDS, MIN-MAXL-MATCH) *with difference* 0,

(iii) *for* (MIN-MAXL-MATCH, MAX-TOTAL-MATCH) *with difference* $|V|$*, and assuming that* $\mathcal{G}$ *is a set of graphs without isolated vertices,*

(iv) *for* $(\text{MIN-DS}|_{\mathcal{G}}, \text{MAX-MINL-EC}|_{\mathcal{G}})$ *with difference* $|V|$*,*

(v) *for* $(\text{MAX-MINL-EC}|_{\mathcal{G}}, \text{MAX-SF}|_{\mathcal{G}})$ *with difference* 0.

Obviously, the identity is a shift for (P, Q) with difference φ if and only if it is a shift for (Q, P) with difference $-\varphi$. Hence combining Theorem 3.2 with Lemma 3.3 we can obtain the following results: if P is one of the problems MIN-MAXL-MATCH, MAX-TOTAL-MATCH, MAX-MINL-EC, or MAX-SF, then for any edge function s $\text{div}_{3s(e)}$ is a shift for the problem P restricted to $\mathcal{G}$ with difference $-\sum_e s_G(e)$, $2\sum_e s_G(e)$, $2\sum_e s_G(e)$, and $2\sum_e s_G(e)$, respectively. Moreover, $\text{div}_{3s(e)}$ is a shift with difference $|V_G| + 2\sum_e s_G(e)$ for any of pairs $(\text{MIN-DS}|_{\mathcal{G}}, \text{MAX-MINL-EC})$, $(\text{MIN-DS}|_{\mathcal{G}}, \text{MAX-SF})$, and $(\text{MIN-EDS}|_{\mathcal{G}}, \text{MAX-TOTAL-MATCH})$. The problem MAX-MINL-VC (similarly as MIN-IDS) is not self-reducible in this way but MIN-DS reduces to it (assuming $s_G(e) > 0$ for each edge $e \in E$ and $G = (V, E) \in \mathcal{G}$), namely $\text{div}_{3s(e)}$ is a shift for $(\text{MIN-DS}|_{\mathcal{G}}, \text{MAX-MINL-VC})$ with difference $|V_G| + 2\sum_e s_G(e)$.

REMARK 3.3. *One can also easily prove that* div_1 *is a shift (in both directions) between* MIN-DS *and* MAX-INDUCED-MATCH*, resp.* MIN-EDS *and* MAX-SIS*, with difference* $|V_G|$*, resp.* $-|V_G|$*, for vice versa shifts (see* [24], [27]*). Moreover, it is also a shift from* MAX-INDUCED-MATCH *to* MIN-IDS *with difference* $-|V_G|$*. Using composition of* div_1 *with* $\text{div}_{3s(e)}$ *above we can obtain that, in general, a subdivision operation of the form* $\text{div}_{1+3s(e)}$ *shares these properties with* div_1 *(with an appropriate change of difference of the shift after composition). We omit the details, as that kind of subdivisions will not be used in what follows.*

One can observe the following

LEMMA 3.4. *Let* f *be a shift with difference* φ *for a pair* (P, Q) *of* NPO *problems. Assume further that there is a positive constant* C *such that for every* $x \in I_P$ *it holds* $|\varphi(x)| \leq C \cdot \text{OPT}_P(x)$*. Then* f *is an L-reduction from* P *to* Q *with* $\alpha = \sigma_Q \sigma_P + C$ *and* $\beta = 1$*. This is in particular true (with* $C := \frac{K}{c}$*) if for a positive constants* $c, K \in (0, \infty)$ *and for every* $x \in I_P$ *(i)* $\text{OPT}_P(x) \geq c|x|$*, (ii)* $|\varphi(x)| \leq K|x|$.

For a fixed $k \geq 0$ and $B \geq 3$ differences of our uniform shift reductions div_{3k} among problems restricted to $\mathcal{G}_B$ clearly satisfy (ii) from Lemma 3.4 due to bounds $|\varphi(G)| \leq |V| + 2k|E| \leq (1 + kB)|V|$. As we have already mentioned, (i) holds (in graphs $\mathcal{G}_B$) for any of problems we reduce from, namely MIN-DS and MIN-EDS. As all studied problems are clearly in APX when restricted to $\mathcal{G}_B$, the following theorem follows

THEOREM 3.6. *For any fixed integer* $k \geq 0$*, each of problems* MAX-MINL-VC*,* MAX-MINL-EC*,* MIN-MAXL-MATCH*,* MAX-TOTAL-MATCH*, and* MAX-SF *when restricted to* $3k$*-subdivisions of graphs of maximum degree* 3*, is* APX*-complete.*

4 Approximation Hardness Results in d-box Intersection Graphs

Our Theorem 2.1 shows that any graph obtained by at least 3-subdivision of each edge from another (arbitrary) graph is a d-box intersection graph for any fixed $d \geq 3$. This immediately shows that many optimization problems in intersection graphs are as hard to approximate as in general graphs. It is rather obvious to see for such problems as MINIMUM STEINER TREE or MINIMUM TRAVELING SALESMAN for which replacing edges by pairwise internally disjoint paths (and splitting edge weights properly) cannot make the problem easier to approximate.

On the other hand, many problems are known to be easier to approximate in d-box graphs than in general ones, and no approximation hardness results are known for them in such graphs. We provide first inapproximability results for some of them, namely, APX-hardness and hence non-existence of PTAS (unless P = NP). Moreover, all our hardness results apply even to the setting when the representation is given, not merely its intersection graph. This makes our hardness results stronger, as the reconstruction problem is known to be NP-hard. Rather straightforward application of Theorems 2.1, 3.3, 3.4, and 3.6 yields in the following

THEOREM 4.1. *Let* $d \geq 3$ *be a fixed integer. Each of the problems* MAX-IS*,* MIN-VC*,* MIN-DS*,* MIN-EDS*,* MAX-INDUCED-MATCH*,* MAX-SIS*,* MIN-IDS*,* MAX-MINL-VC*,* MAX-MINL-EC*,* MIN-MAXL-MATCH*,* MAX-TOTAL-MATCH*, and* MAX-SF *restricted to (intersection graphs of) sets of axis-parallel* d*-dimensional boxes, is* APX*-hard and does not admit* PTAS *(unless* P = NP*). These hardness results apply also to instances whose the intersection graph is simultaneously of maximum degree at most* 3*, of girth at least* k *(for any prescribed constant* k*), and, except* MAX-IS *and* MIN-VC*, bipartite as well.*

These results could be stated as explicit NP-hard gap type results and provide explicit lower bounds on their approximability. We will demonstrate it on the main problem of our interest, Maximum Independent Set.

Theorem 4.2. *Let $d \geq 3$ be a fixed integer and $\mathcal{R}$ be a set of d-boxes such that the intersection graph $G_R = (V_R, E_R)$ of $\mathcal{R}$ is the 4-subdivision graph of a 4-regular graph. Then the following decision problem is* NP*-hard: to decide whether the maximum number of pairwise disjoint d-boxes of $\mathcal{R}$ is less than $0.4966261808|V_R|$ or greater than $0.4977507872|V_R|$ (under promise that one of these two cases occurs). Consequently, it is* NP*-hard to approximate* Max-IS *problem in d-boxes within $1 + \frac{1}{442}$.*

Remark 4.1. *The same approximation hardness results as we obtained in Theorems 4.1 and 4.2 for (intersection graphs of) sets of d-boxes easily follow for (intersection graphs of) sets of axis-parallel lines, for any fixed $d \geq 3$, with slightly worse explicit lower bounds as in Theorem 4.2.*

4.1 Rectangle Intersection Graphs. Topological obstructions prevent to obtain similar inapproximability results in 2-dimensional case. However, we can use some of ideas above to give a generic approach to NP-hardness results for the same kind of graph optimization problems in rectangle intersection graphs, or even in unit square graphs (alternatively, in unit disk graphs).

First, we can prove easily planar variant of our structural Theorem 2.1. If G is a planar graph and G' is a graph obtained from G by at least 1-subdivision of every edge of G, then G' is a rectangle intersection graph. Moreover, its rectangle representation can be constructed in linear time. It easily follows from known algorithms on weak visibility representation of planar graphs [38]. Such algorithm draws a planar graph $G = (V, E)$ so that each vertex of G is represented by a horizontal segment, each edge is represented by a vertical segment, and incidence between vertices and edges translates to intersections of horizontal and vertical segments. Replacing segments properly by thin axis-parallel rectangles we obtain a set of rectangles whose intersection graph is exactly $\mathrm{div}_1(G)$. To obtain more general subdivisions of G, just replace the "vertical rectangle" corresponding to an edge $e \in E$ by more rectangles, creating a path of required length. As problems studied in this paper are known to be NP-hard in planar graphs (even in planar graphs of maximum degree 3), NP-hardness of the restriction of any of them to rectangle intersection graphs easily follows from Theorems 3.1, 3.2, and Lemma 3.3. Even more, we can prove NP-hardness result for any of studied problems restricted to unit square graphs (alternatively, unit disk graphs). Some of these results were well studied and are known for unit disk graphs.

Theorem 4.3. *Each of the problems* Max-IS, Min-VC, Min-DS, Min-EDS, Max-Induced-Match, Max-SIS, Min-IDS, Max-MinI-VC, Max-MinI-EC, Min-MaxI-Match, Max-Total-Match, *and* Max-SF *is* NP*-hard when restricted to (intersection graphs of) axis-parallel unit squares (alternatively, unit disks) in the plane. These hardness results apply also to instances whose intersection graph is simultaneously of maximum degree 3, of girth at least k (for any prescribed constant k), and, except* Max-IS *and* Min-VC, *bipartite as well.*

Remark 4.2. *The described method can be applied to more general geometric graphs, e.g., to contact graphs of unit discs, for which we can obtain the same* NP*-hardness results. Recall that contact graphs are a special kind of intersection graphs of geometrical objects in which we do not allow the objects to cross but only to touch one another.*

5 Maximum Weighted Projection-Independent Set of d-boxes

In this section we discuss another (weighted) Max-IS problem that is related to geometric graphs generated by sets of d-boxes. The problem was introduced by Bafna et al. as the Independent Subset of Rectangles (shortly, IR) problem. The research was motivated by a fundamental problem in computational molecular biology (see [4] for more details).

Recall that a graph is an *interval graph* if its vertices can be assigned to intervals on the real line so that vertices are adjacent iff the corresponding intervals intersect. Such assignment is called the *interval representation* of the interval graph. In case of the *proper interval representation*, no interval *properly* contains another one. A graph is a *proper interval graph* if it admits a proper interval representation. Define two d-boxes to be *projection-independent* if for each axis their projections on this axis are disjoint.

The IR problem was formulated as follows: given a set $\mathcal{R}$ of positively weighted axis-parallel d-boxes such that for each axis, the projection of a d-box on this axis does not contain another one. The goal of IR is to find a maximum weighted subset $\mathcal{R}^* \subseteq \mathcal{R}$ of d-boxes that are pairwise projection-independent. Equivalently, it is Max-w-IS in the corresponding project-intersection graph $G_{\mathcal{R}}$, where vertex set of $G_{\mathcal{R}}$ is $\mathcal{R}$ and two d-boxes of $\mathcal{R}$ are adjacent by an edge in $G_{\mathcal{R}}$ iff their projections on (at least) one axis intersect. Graphs

$G_{\mathcal{R}}$ (for arbitrary set $\mathcal{R}$) are termed as d-union interval graphs, as it can be expressed as the union of d interval graphs ([6]). Hence the IR problem yields to MAX-w-IS in proper d-union graphs. The MAX-w-IS problem in proper d-union graphs was studied in more details for $d = 2$. Bafna et al. [4] showed that in this case the corresponding graphs are 5-claw free and using known result of Berman for MAX-w-IS in t-claw free graphs ([8]), the current best approximation for MAX-w-IS in proper 2-union graphs has a factor of $\frac{5}{2}$.

In [4] it is noted that the projection-intersection graph $G_{\mathcal{R}}$ for a set $\mathcal{R}$ satisfying assumptions of the IR problem is $(2^d + 1)$-claw free for $d \geq 3$. Best known algorithms for MAX-w-IS in t-claw free graphs provide approximation factor for this problem that is exponential in d (as repeated as well in [6]). Also in [10] (and [9]) it is repeated the question of an existence of efficient algorithms for d-dimensional version with approximation factor increasing less drastically (e.g., linearly in d). However, it is rather easy to prove that proper d-union graphs are in fact $(2d + 1)$-claw free, so the answer is straightforward.

LEMMA 5.1. *Proper d-union graphs are $(2d + 1)$-claw free.*

Proof. Let $G = (V, E)$ be a proper d-union graph. Hence there are d proper interval graphs $G_i = (V, E_i)$, $i = 1, 2, \ldots, d$ such that $E = \cup_{i=1}^{d} E_i$. Assume that G contains induced $(2d+1)$-claw, and let L be the set of its $(2d + 1)$ edges. Clearly, $(2d + 1) = |L| \leq \sum_{i=1}^{d} |L \cap E_i|$, hence there is an index i_0 for which $|L \cap E_{i_0}| > 2$. Consequently, G_{i_0} contains induced 3-claw. However, this contradicts with the fact that G_{i_0} is a proper interval graph, because an interval graph is a proper interval graph if and only if it is 3-claw free [36]. This contradiction shows that G is $(2d + 1)$-claw free.□

Using results of [8] we immediately get the following theorem.

THEOREM 5.1. *For any fixed $d \geq 2$ the* MAX-w-IS *problem in proper d-union graphs can be approximated within $(d + \frac{1}{2})$ in polynomial time.*

The best known algorithms for t-claw free graphs are rather impractical, hence Berman et al. [9], [10] showed a simple $O(n \log n)$-time algorithm with approximation factor 3 for 2-dimensional version. They observe that the algorithm can be extended to yield a $(2^d - 1)$-approximation for d-dimensional version. But the straightforward extension of their algorithm to d-dimensional version gives $(2d - 1)$-approximation. Recall that for more general d-interval graphs Bar-Yehuda et al. [6] achieved $2d$-approximation algorithm.

THEOREM 5.2. *For any fixed $d \geq 2$ the* MAX-w-IS *problem in proper d-union graphs (with their representation given) can be approximated within $2d - 1$ in nearly linear time.*

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