Nonlinear Approximation Schemes in Relativistic Cosmology

Hedda Alice Gressel

Institute of Cosmology and Gravitation
University of Portsmouth

This thesis is submitted in partial fulfilment of
the requirements for the award of the degree of
Doctor of Philosophy
of the University of Portsmouth.

April 2019

"Higher mathematics! I have never learned higher mathematics. But there wasn't a day in my life when I didn't think, today I would have needed higher mathematics."[1]

Heine Exner

[1] Translated from German to English by the author.

Declaration

Whilst registered as a candidate for the above degree, I have not been registered for any other research award. The results and conclusions embodied in this thesis are the work of the named candidate and have not been submitted for any other academic award. (wordcount: 44288)

The work in chapter 3 was carried out in collaboration with Marco Bruni in Hedda Alice Gressel and Marco Bruni. "fNL gNL mixing in the matter density field at higher orders", Gressel H. A., Bruni M., JCAP, 1806(06):016, 2018.

The work in chapter 4 was carried out in collaboration with Camille Bonvin, Marco Bruni, and David Bacon in "Full-sky weak lensing: a nonlinear post-Friedmann treatment", Gressel H.A., Bonvin C., Bruni M., Bacon D., submitted. arXiv: 1902.00059

Hedda Alice Gressel
April 2019

Acknowledgements

My very sincere thanks go to my first supervisor Marco Bruni for his support, guidance, patience, encouragement, and for always taking time for me throughout the course of my PhD.

I would like express my deep gratitude to my second supervisor David Bacon for his support, advice, and very constructive suggestions.

Furthermore, I would like offer my special thanks to my third supervisor David Wands for valuable and constructive feedback and support.

I am particularly grateful for the support and encouragement given by my collaborator Camille Bonvin and for always thorougly answering all my questions.

I wish to acknowledge the scientific network CANTATA supported by COST (European Cooperation in Science and Technology) for its support to visit my collaborator.

I would like to very much thank all my friends and family for their support and friendship. Special thanks goes out to Matthew and Anna, without whom these years would have been certainly less joyful.

I am indebted to Bread Addiction for feeding me with sandwiches and coffee. A further thank you goes to Southsea Coffee for never failing to provide the best coffee in town.

A rather unusual thank goes to parkrun as I found my love for running there and it helped me to stay focused.

If ever someone felt the need to thank the Universe for the existence of chocolate, then it is me and the time to thank is now. Huge amounts of this deliciousness helped me through long working days.

Abstract

In this thesis I present two main projects, based on the application of different nonlinear approximation schemes in relativistic cosmology. The first project comprises effects on the matter distribution on very large scales sourced by the nonlinearity of General Relativity, while the second project examines a weak lensing analysis considering all scales.

In the first project, the results of which were published in [55], we used two different approximation schemes, the gradient expansion and standard perturbation theory, to examine the effects the nonlinear nature of General Relativity (GR) has on the density field allowing primordial non-Gaussianity in the inital conditions. The gradient expansion restricts the computations to very large scales, where the gradients of the metric perturbations are significantly smaller than their time derivatives. In this work, we neglect contributions of the order $\mathscr{O}\left(\nabla^4\right)$ in the gradient expansion. We show that at order $\mathscr{O}(3)$ and $\mathscr{O}(4)$ in standard perturbation theory the intrinsic nonlinearity of GR produces a mixing of the primordial non-Gaussianity in the density contrast. The main result of this project is that at higher orders due to the nonliearity of GR a mixing of f_{NL}, g_{NL}, and h_{NL} occurs in the density field.

In the second project, of which the results will be published in the paper [54], we computed the convergence and the shear up to higher orders using the post-Friedmann approximation scheme. This approximation scheme is especially beneficial for the subject of weak lensing due to its validity on all scales. In weak lensing, we integrate along the line of sight and thereby couple large scales to small scales. The post-Friedmann approximation is a post-Newtonian-type approximation scheme in a cosmological setting that combines both the fully nonlinear Newtonian dynamics on small scales and the relativistic perturbations on large scales. It comprises scalar, vector, and tensor perturbations, whereas the lowest order of the vector perturbation is sourced by Newtonian quantities, yet its effects are purely relativistic. The vector potential does not influence matter dynamics but affects the photon geodesic and therefore the weak lensing analysis. We present the convergence and shear projected on a spherical screen space, which allows us to go beyond the thin-lens or small-angle approximation, in

terms of the redshift z up to the order $\mathscr{O}\left(\frac{1}{c^4}\right)$. The main reason for the investigation was to have a formalism for the weak lensing analysis that includes the effect of the gravimagnetic potential on nonlinear scales. It was shown in [33] that the magnitude of this vector potential is small but not negligible on nonlinear scales. Hence, the main result of this project is the computation of the convergence and shear up to higher orders using a formalism that includes the gravimagnetic potential and is valid on all scales including nonlinear, small scales. In particular, we show the contribution of the vector potential to the convergence and shear.

Contents

List of Figures

Chapter 1

Introduction

The aim of Cosmology is to understand the evolution of our universe. The scientific theory contemplates the study of the structure formation on large scales of the observable regions of the universe and connects it to the local physical dynamics.

Observations from the very large scales or very early times to the smallest scales like the solar system indicate that the universe evolved from a homogeneous and isotropic matter distribution to structures such as galaxies clusters and voids. In order to study evolution of the structure formation, we need to establish scientific models and mathematical tools to describe them. When we look at the largest scales, on which the universe is homogeneous and isotropic, the observations are matched best by the Friedmann-Lemaitre-Robertson-Walker model (FLRW) (see section 1.3.1) with general relativistic dynamics (see section 1.1). Going to smaller scales, fluctuations arise in the matter distributions, which are believed to cause the structure formation we observe today via gravitational collapse. It is standard to describe these fluctuations with the use of perturbation theory (see section 2.1). If however, we look a small enough scales, the standard perturbative description breaks down and it is standard to use fully non-linear Newtonian dynamics e.g. N-body simulations. Yet there are approximation schemes that aim to unite the perturbative approaches on large scales with the Newtonian descriptions on small scales. One of these approximation schemes is the post-Friedmann approximation scheme [80], which will be summarised in this thesis in section 2.4.

In this thesis, I will explore the use of different approximation schemes in cosmology, applicable on different scales. We will focus on relativistic and higher order effects and will present the works on $f_{\rm NL} - g_{\rm NL}$ mixing at higher orders in the density contrast [55] (see chapter 3) and on the weak lensing analysis with a post-Friedmann approximation scheme [54] (see chapter 4).

The outline of the thesis is the following: in this chapter, chapter 2, we give a general overview of General Relativity (GR) and the topics of cosmology that will be dealt with later in the thesis. In chapter 3, we introduce different relativistic approximation schemes and perturbation theory. Furthermore, we discuss the theory of gauge transformations and specific gauges. Subsequently, I introduce the three different approximation schemes used in the latter chapters, namely standard perturbation theory, the gradient expansion, and the post-Friedmann approximation scheme. Chapter 3 and 4 are dedicated to the two main projects of my PhD study. The first of the two projects investigates relativistic effects on very large scales. Using the gradient expansion and standard perturbation theory, we can show that on very large scales, the intrinsic non-linearity of General Relativity affects the Gaussianity of matter density distribution by generating a mixing of $f_{\rm NL}$ and $g_{\rm NL}$ at higher orders. In chapter 4, I present the current work based on [54], where we provide a full-sky, all scales weak-lensing analysis valid in the fully non-linear regime of structure formation using the post-Friedmann approach. In the conclusion I will summarise the work of this thesis and will discuss the results and future directions.

1.1 General Relativity

General Relativity (GR) is a theory of gravitation, in which gravity is described by geometric properties of the space-time. Based on the universality of free fall, i.e. all bodies fall precisely in the same way in a gravitational field, and the equivalence principle[1], GR connects the magnitude of the gravitational field to the curvature of the space-time. The Einstein field equations (EFE) relate the curvature to the source of gravitation and the geodesic equation dictates how bodies fall. Subsequently, a self-consistent interpretation of the geometric properties of gravitational fields concludes that the space-time *is* the gravitational field.

1.1.1 Differential geometry

The theory of GR portrays gravitation as a geometrical property. In order to formulate gravity in terms of geometrical concepts, in this subsection I will introduce aspects of differential geometry and derive the field and geodesic equations of GR in the following subsection. I will follow the work of [111, 97, 91, 90].

[1]or in a gravitational field, there exists a local inertial system (LIS) in every neighbourhood of an event in space-time. In this LIS, the laws of special relativity hold and there exists a choice of local coordinates such that the gravitational field is canceled out.

Via the equivalence principle we can deduce that we can locally establish the theory of special relativity (SR). In SR it is assumed that the theory is true globally. In GR, on the other hand, this assumption cannot be made. Therefore, we introduce the concept of manifolds, which is a collection of subsets that are homeomorphic to $\mathbb{R}^4$ that are smoothly connected. On each open set, we can impose a coordinate system, on which we can establish a LIS. I will now introduce the quantities of differential geometry that we need for the formulation of GR in the next subsection.

Definition 1.1.1. A n-dimensional, real *manifold* M is a topological space [2] that is a collection of subsets $\{\mathscr{O}_\alpha\}$ of which each is homeomorphic to $\mathbb{R}^n$.

Definition 1.1.2. A *coordinate system* or *chart* is a map $\psi_\alpha : \mathscr{O}_\alpha \to U_\alpha$ with U_α is an open subset of $\mathbb{R}^4$.

Next we need to define the notion of differentiability of the charts on the manifold. Therefore, we introduce a differential structure:

Definition 1.1.3. A *differential* manifold M is a n-dimensional manifold with a globally defined differential structure: let $\mathscr{A} = \{U_\alpha, \psi_\alpha\} : M = \bigcup_\alpha U_\alpha, \psi_\alpha : U_\alpha \to \mathscr{O}_\alpha \subset \mathbb{R}^n$ be an atlas $\mathscr{A}$ of charts ψ. If $U_\alpha \cap U_\beta \neq \emptyset$, then $\psi_\alpha \circ \psi_\beta^{-1} : \psi_\alpha \left(U_\alpha \cap U_\beta\right) \to \psi_\beta \left(U_\alpha \cap U_\beta\right)$ is a differentiable map (*diffeomorphism*).

Definition 1.1.4. Let $\mathscr{F}$ denote the collection of $\mathscr{C}^\infty$ functions from M into $\mathbb{R}$. A *tangent vector* v at point $p \in M$ is defined as a map $v : \mathscr{F} \to \mathbb{R}$ which is linear and obeys the Leibnitz rule.

The tangent vectors v in a point p span the *tangent vector space* T_p with dim $T_P = n$.

Definition 1.1.5. Let $\psi : \mathscr{O} \to U \subset \mathbb{R}^n$ be a chart with $p \in \mathscr{O}$ and let $f \in \mathscr{F}$. The basis $\left\{X_\beta\right\}$ of T_P defined as $X_\beta : \mathscr{F} \to \mathbb{R}$ with $X_\beta(f) = \frac{\partial}{\partial x^\beta}\left(f \circ \psi^{-1}\right)\big|_{\psi(p)}$ is called *coordinate basis*.

The coordinate basis is dependent on the choice of the chart ψ. It can be expressed in terms of the partial derivative in all possible directions $\left\{\partial/\partial x^i|_P\right\}$. To change coordinates, and therefore chart, it follows that $\frac{\partial}{\partial x^\nu} = \frac{\partial \hat{x}^\mu}{\partial x^\nu}\frac{\partial}{\partial \hat{x}^\mu}$.

Definition 1.1.6. A *Cotangent space* in P is the dual space T_P* of the tangent space T_P. It's basis is the dual coordinate basis: $\{dx^\mu|_P\} : dx^\mu \frac{\partial}{\partial x^\nu} = \delta^\mu_\nu$.

[2] second countable Hausdorff space

If we want to define a (covariant) derivative for tensor fields, we need to establish an preferred isomophism (*parallel transport*) between the tangent space of any nearby two points. Only then the difference of a tensor evaluated at two different points can be invariantly formulated. An *affine manifold* is a (differentiable) manifold equipped with a connection L that connects nearby tangent spaces and thereby provides the topological requirement to define a covariant derivative of tensor fields:

Definition 1.1.7. The *covariant derivative* ∇ of a tensor field T on an affine manifold M is a map $\nabla : \text{type}(p,q) \to \text{type}(p,q+1)$[3] satisfying the following three properties: linearity, $\nabla f = df$ for a scalar function f, and the Leibnitz rule.

When we translate Definition 1.1.7 into index notation and apply it to a tensor field T of type(p,q) we obtain

$$\nabla_\zeta T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q} = T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q,\zeta} + L^{\alpha_1}{}_{\kappa\zeta} T^{\kappa\alpha_2\ldots\alpha_p}{}_{\beta_1\ldots\beta_q} + \cdots - L^{\kappa}{}_{\beta_1\zeta} T^{\alpha_1\ldots\alpha_p}{}_{\kappa\beta_2\ldots\beta_q} - \cdots \tag{1.1}$$

A covariant derivative along a curve which is parametrised by the affine parameter λ is denoted by the *absolute derivative* and is defined as

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu. \tag{1.2}$$

If the absolute derivative of an arbitrary tensor field vanishes, it is referred to as *parallel transport*[4]. By virtue of the definition of the parallel transport and its path dependence, an expression for the curvature can be established. Let v^μ be a vector field and p a point on an affine manifold M. We create a small closed loop on which we parallel transport the vector v^μ. v^μ_p refers to the initial vector and v^μ_∇ to the transported vector. If the two vectors v^μ_p and v^μ_∇ do not align at point p, the "defect" in the alignment corresponds to the curvature of the manifold:

$$\left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right) v^\mu = R^\mu{}_{\nu\alpha\beta} v^\nu \tag{1.3}$$

with the Riemann tensor $R^\nu{}_{\mu\alpha\beta}$

$$R^\nu{}_{\mu\alpha\beta} = L^\nu{}_{\mu\beta,\alpha} - L^\nu{}_{\mu\alpha,\beta} + L^\nu{}_{\kappa\alpha} L^\kappa{}_{\mu\beta} - L^\nu{}_{\kappa\beta} L^\kappa{}_{\mu\alpha}. \tag{1.4}$$

Furthermore, one defines a metric on the manifold that is parallel transported.

[3] a tensor field in P of type(p,q) is a multilinear map $\underbrace{T_P* \times T_P* \times \cdots \times T_P*}_{q} \times \underbrace{T_P \times \ldots T_P \times T_P}_{p} \to \mathbb{R}$

[4] which denotes the preferred isomorphism mentioned above

In order to identify the symmetry group of a tensor field, we introduce the Lie-derivative: a tensor field T at point $P \in \mathscr{M}$ is carried along to the point $\bar{P} \in \mathscr{M}$. The difference between the tensor field T carried along the flow of a vector field v from point P to $\bar{P}$ and the original tensor field T at $\bar{P}$ with $P \to \bar{P}$ denoting a infinitesimal transformation is regarded as the Lie derivative.

Definition 1.1.8. Let $\mathscr{M}$ be a manifold and let ϕ_t be a one-parameter group of diffeomorphism and v the vector field generated by ϕ_t. ϕ_t^* denotes the *pullback*[5] of ϕ_t, then

$$\mathscr{L}_v T^{a_1 \dots a_k}{}_{b_1 \dots b_l} \equiv \lim_{t \to 0} \left(\frac{\phi_{-t}^* T^{a_1 \dots a_k}{}_{b_1 \dots b_l} - T^{a_1 \dots a_k}{}_{b_1 \dots b_l}}{t} \right). \tag{1.5}$$

defines the *Lie-derivative*.

Definition 1.1.9. A *pseudo-Riemannian manifold (M,g)* is an n-dimensional manifold with a symmetric, covariant tensor field $g_{\alpha\beta}$ (*metric*), which does not need to be positive definit. In any tangent space T_P, the metric defines a nondegenerate biliniear form g (*inner product*).

The inner product defines an isomorphism $T_P \to T_P^* : v \mapsto g(\,.\,, v) \equiv v^*$, which in index notation reads $v_\mu = g_{\mu\nu} v^\nu$.

Definition 1.1.10. A *Lorentzian manifold* is a pseudo-Riemannian manifold with the signature of the metric being $diag(-1, 1, \dots, 1)$.

The fundamental theorem of Riemannian geometry states that there is a unique torsion-free metric connection Γ with $\nabla_\alpha g_{\beta\gamma} = 0$. Γ is referred to as the *Christoffel symbol* (or *Levi-Civita connection*) and takes the form

$$\Gamma^\mu_{\nu\gamma} \equiv \frac{1}{2} g^{\mu\kappa} \left(g_{\kappa\nu,\gamma} + g_{\kappa\gamma,\nu} - g_{\nu\gamma,\kappa} \right). \tag{1.6}$$

In this subsection, we introduced the quantities of differential geometry that we will need to construct the theory of GR. In order to formulate a field theory, field equations and an equation of motion, which determines how a test particle moves if subject to the field, are needed. In GR, these equations are the Einstein field equations and the geodesic equation. We will derive both in the next subsection.

[5]The pullback of a map is defined as follows: let $\phi : \mathscr{M} \to \mathscr{N}$ with $\mathscr{M}$, and $\mathscr{N}$ being manifolds and let $f : \mathscr{N} \to \mathbb{R}$. The vector field $v \in V_P$ is the tangent field at point P. Then $\phi_t^* : V_P \to V_{\phi(P)}$ with $(\phi^* v)(f) = v(f \circ \phi)$.

1.1.2 Einstein field equations and geodesic equation

We describe the space-time with a four dimensional, real Lorentzian manifold (M,g) with a corresponding four-dimensional metric tensor $g_{\alpha\beta}$[6], with which we define the distance between two events with the line element $ds = \sqrt{g_{\alpha\beta}dx^{\alpha}dx^{\beta}}$. The geodesic equation describes the trajectory of a test particle within a gravitational field and can be derived using calculus of variation to extremise the length ds of curves $\mathscr{C}$ between two events. We choose the curves $\mathscr{C}$ to be parametrised by the affine parameter λ[7] and obtain

$$-\int_{\mathscr{C}} ds = -S[x^{\mu}(\lambda)] = -\int_{\lambda_1}^{\lambda_2} \mathscr{L}(x^{\mu},\dot{x}^{\mu})d\lambda = min \tag{1.7}$$

$$\Leftrightarrow \frac{\delta S}{\delta x^{\alpha}} = 0 \tag{1.8}$$

$$\Leftrightarrow \ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\gamma}\dot{x}^{\nu}\dot{x}^{\gamma} = 0 \tag{1.9}$$

with the tangent vector field $\frac{dx^{\alpha}}{d\lambda} = \dot{x}^{\alpha}$, the Lagrangian[8] Equally, the geodesic equation (1.9) can be defined as a curve whose tangent vector is parallel transported along itself:

$$\frac{D}{d\lambda}\dot{x}^{\nu} \equiv \dot{x}^{\mu}\nabla_{\mu}\dot{x}^{\nu} = \dot{x}^{\mu}\dot{x}^{\nu}_{;\mu} = 0. \tag{1.10}$$

Equations (1.9) or (1.10) can be written as $\frac{D^2x^{\mu}}{d\lambda^2} = 0$ as well, which reflects covariantly the fact that in GR a body in free fall is not subject to acceleration.

In order to derive the EFE, we need to introduce two quantities connected to the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$: if we take the trace of the Riemann tensor, we obtain the Ricci tensor $R_{\mu\nu}$

$$R_{\mu\nu} \equiv R^{\alpha}_{\mu\nu\alpha}. \tag{1.11}$$

The trace of the Ricci tensor yields the Ricci scalar R

$$R \equiv g^{\alpha\beta}R_{\alpha\beta}. \tag{1.12}$$

[6]We will use the metric signature (-,+,+,+). Greek indices run from 0 to 3, whereas Latin indices run from 1 to 3. We denote the Minkowski metric as $\eta_{\alpha\beta} = diag(-1,1,1,1)$

[7]We choose $\lambda \propto s$. It follows that $\frac{d\mathscr{L}}{d\lambda} = 0$.

[8]This is the Langrangian for timelike and spacelike geodesics, because the line element and therefore the Lagrangian mentioned above vanishes. Hence, we cannot chose $\lambda \propto s$ anymore because $s = 0$ for null geodesics. For null geodesics the Lagrangian $\mathscr{L}_{null} = \frac{1}{2}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ leads to the geodesic equation, if one performs the calculus of variation.

Futhermore, the Riemann tensor $R^{\nu}_{\ \mu\alpha\beta}$ satisfies the Bianchi identity

$$R^{\nu}_{\ \mu\alpha\beta;\kappa} + R^{\nu}_{\ \mu\kappa\alpha;\beta} + R^{\nu}_{\ \mu\beta\kappa;\alpha} = 0. \tag{1.13}$$

If we contract the Bianchi identity by δ^{ν}_{α} and $g^{\mu\beta}$, we obtain the divergence of the Einstein tensor $G_{\alpha\beta}$:

$$G^{\beta}_{\ \kappa;\beta} = 0 \quad \text{with} \;\; G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.14}$$

In the theory of special relativity, the conservation of energy and momentum is given by the property that the energy-momentum tensor $T_{\mu\nu}$ is divergence-free. The simplest covariant generalisation would be

$$T^{\mu\nu}_{\ \ ;\nu} = 0. \tag{1.15}$$

However, it can no longer be seen as a global conservation equation, because we cannot consider the gravitational field as a part of a closed system: gravitational tidal forces can influence local densities. If however, we look at a region small enough for the influence of tidal forces to be small or negligible, the energy and momentum of the fluid is approximately conserved[9].

On the basis of the principle of minimal gravitational coupling, the Einstein field equations (EFE) combine (1.14) and (1.15) to[10]

$$G_{\mu\nu} = -\kappa T_{\mu\nu} \tag{1.16}$$

with $\kappa = 8\pi \frac{G}{c^4}$. Equally, the EFE can be derived via the calculus of variation from the Einstein-Hilbert action

$$S = \frac{1}{2\kappa} \int R\sqrt{-g}\, d^4x. \tag{1.17}$$

The EFE (1.16) relate the curvature of the space-time on the l.h.s to the energy and momentum of the source on the r.h.s.. The Einstein tensor $G_{\mu\nu}$ on the l.h.s. of (1.16)

[9]Let u^{μ}_{o} be the four-velocity field of a family of observers. For (1.15) to satisfy a conservation equation, we need $\nabla_{(\mu}\, u_{o\,\nu)} = 0$ such that $\nabla_{\alpha}\left(T^{\alpha}_{\ \beta} u^{\beta}_{o}\right) = 0$. In curved space-time, there is in general not a vector field satisfying $u^{\mu}_{o} u_{o\mu} = -1$ and $\nabla_{(\mu}\, u_{o\,\nu)} = 0$. But on a small enough region, one can find a $\nabla_{(\mu}\, u_{o\,\nu)} \approx 0$ and therefore conclude that the energy and momentum of a fluid is approximately conserved as measured by the observers.

[10]Note that in the convention of the Riemann tensor (1.4) used in this work leads to a minus sign on the l.h.s. in the EFE (1.16).

depends on derivatives of the metric $g_{\mu\nu}$ up to second. Furthermore, it is highly nonlinear in the metric $g_{\mu\nu}$. Therefore, the EFE can be seen as a set of coupled, nonlinear, second order partial differential equations for the components of the metric $g_{\mu\nu}$.

Both sides of (1.16) comprise a symmetric "2-index" tensor. Therefore, the EFE are 10 independent equations, because the symmetric property of the tensors reduces the number of independent equations from 16 to 10. However, the Bianchi-identity (1.13) represents 4 further constraints on the Ricci tensor and reduces the number of independent equations to 6. The metric tensor on the other hand is a symmetric rank-2 tensor with 10 unkown functions. The remaining 4 degrees of freedom correspond to choice of coordinates.

1.2 Cosmology

The scientific theory of cosmology comprises the structure formation on large scales and its evolution. However, our observations are limited to the past light cone. Even if we can measure observables up to a high accuracy, we need to impose assumptions about the structure of the universe beyond the light cone in order to formulate a cosmological model. For example, we observe that on a spatial avarage over a big enough region the universe is isotropic. In combination with the Copernican principle that we are not privileged observers, this empirical observation leads to the cosmological principle, which states that the universe is homogeneous and isotropic on large enough scales.

1.2.1 The cosmological principle

The cosmological principle claims that on large enough scales the universe is spatially homogeneous and isotropic (in every point). Spatial homogeneity means that the spatial universe looks the same from any point of view at a given cosmological time: there exists a one-parameter family of spatial hypersurfaces Σ_t with $M = \bigcup_t \Sigma_t$, $\Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset$ for $t_1 \neq t_2$, such that there exists for every t and arbitrary points $P, \in \Sigma_t$ an isometry between P and Q. Spatial isotropy in every point is defined such that there exists a congruence of timelike lines with the following properties: let P be an arbitrary point and u_P its tangent vector of the line in the congruence crossing the point P and let s_1 and s_2 two unit vectors in the tangent space T_P and orthogonal to u_P. Then there exists a isometry, which maintains P and u_P, but transforms s_1 into s_2[11].

[11] Isotropy *in every point* implies homogeneity.

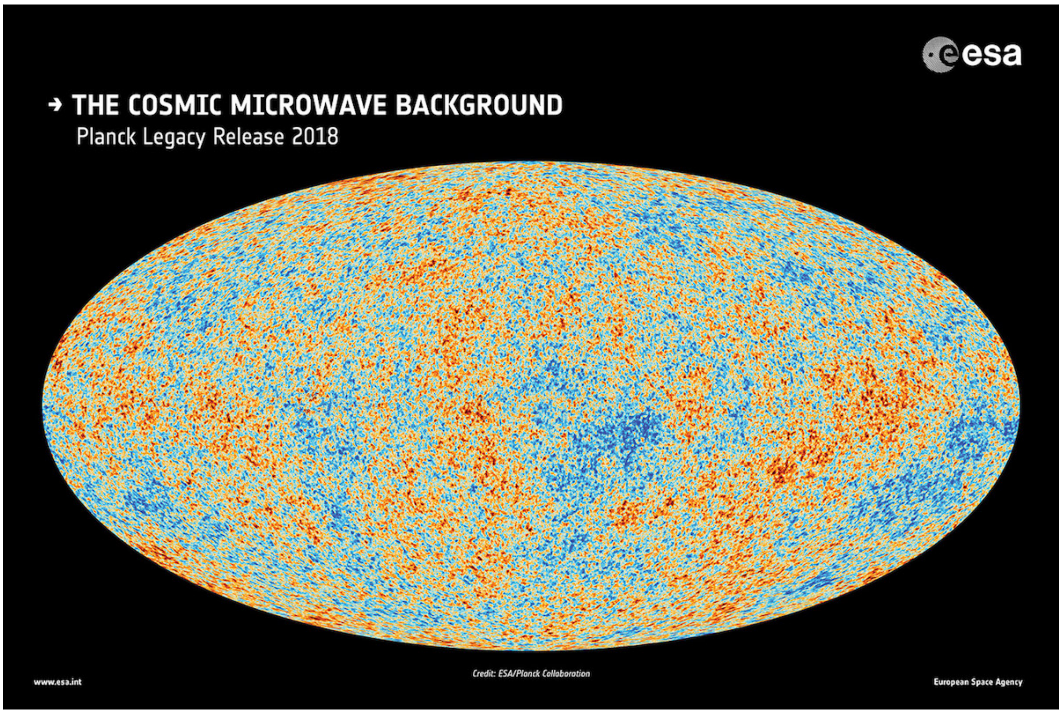

Figure 1.1 The anisotropies of the CMB as observed by ESA's Planck mission. (Copyright: ESA/Planck Collaboration)

1.2.2 Observational evidence

Our observations with telescopes, probes, and other instruments aim to detect electromagnetic radiation, neutrinos, and gravitational waves to learn about the distribution of matter in the universe. Electromagnetic radiation and gravitational waves travel at the speed of light, thus we can only observe the universe on the past light cone.

Hubble's observations of the redshift of spectral lines of distant galaxies and subsequently his formulation of the Hubble law showed that the universe is not as previously assumed static but expanding. Furthermore, the observation of the distribution of radio sources and the discovery of the *cosmic microwave background (CMB)* gave evidence to the assumption that the universe was evolving as it expands. The study of supernovae of type Ia at higher redshifts showed that the expansion rate is slower at higher redshifts than at lower redshifts. Hence, the universe is not only expanding but the expansion is accelerated.

The CMB - a electromagnetic radiation with a black body spectrum - is a remnant from the very early universe. It's discovery provides inter alia evidence for big bang cosmology models. The CMB has a remarkable high degree of isotropy with temperature fluctuations of order $|\Delta T/T| \lesssim 10^{-5}$. These temperature fluctuations indicate density fluctuations on very large scales.

There have been numerous successful test of GR in laboratories and within our solar system. When it comes to extrapolate the theory to scales beyond our solar system, we encounter discrepancies between the observations and the theory. For example, we can directly observe luminous baryonic matter but indirect measurements of the amount of matter (e.g. rotation curves of galaxies, gravitational lensing) suggests that the luminous baryonic matter is only a fraction of all the matter there is. In order to explain this bias in accordance with the theory of GR a new kind of matter, *dark matter*, which neither absorbs nor emits electromagnetic radiation, has been postulated. Another example of observations deviating from the theory is the accelerated expansion rate of the universe. As mentioned before, observations showed that the universe's expansion is accelerating. In order to agree with the predictions of GR, it is convenient to add a constant Λ to the source terms of the EFE (1.16). Λ is regarded as the *cosmological constant* or as the simplest form of *dark energy*.

1.3 Cosmological applications of GR

If we want to build a cosmological model based on GR, it must agree with the observations such as isotropy and homogeneity on the largest scales. The standard approach is a perturbative approximations with an isotropic and homogeneous background. In the EFE (1.16), the metric and its derivatives are connected to the energy-momentum tensor. So far we haven't discussed the components of the energy-momentum tensor and how to mathematically represent the matter. Our cosmological analysis is dependent on the scales we look at and the size of scale we average over. With the cosmological principle stating that the universe is homogeneous and isotropic at very large scales, we encounter a hierarchy of different scales with different descriptions from very large to very small ones. As for the description of matter, for very large scales to smaller scales, it is standard to use fluid description. This description holds as long as the matter dynamics can be describes as a congruence of fluid lines, thus with a 4-velocity u^μ. For the range of scales, in which the averaging scale is large enough such that fluctuations are smoothed out, the fluid approximation is valid. It breaks down once the averaged scale is small enough compared to the averaged density ρ such that the effect of individual particles on the density measurement cannot be smoothed by averaging. Mathematically, this means that the fluid cannot be described with one 4-velocity u^μ anymore and the fluid lines no longer form a congruence but cross each other.

1.3.1 FLRW models

The Friedmann-Lemaitre-Robertson-Walker models (FLRW) are models of constant spatial curvature that are exactly spatially homogeneous and isotropic. The metric of the FLRW models is

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right]. \tag{1.18}$$

$k = -1, 0, 1$ corresponds to negative, none, or positive curvature, respectively. A positive or negative value for k denotes a closed or open universe, respectively. Whereas $k = 0$ translates into a flat universe. The function $a(t)$ is solely dependent on the time t as any spatial dependency would breach the spatial isotropy and homogeneity. It increases or decreases the spatial line element with time. Hence, the coordinate r corresponds to a comoving distance within the increase or decrease of distances caused by $a(t)$. The time t is regarded as the cosmic time. It is convenient to introduce the *confromal time* τ with $dt = a(t)d\tau$. Consequently, the line element (1.18) changes to

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right]. \tag{1.19}$$

ΛCDM model:

The Λ cold dark matter (ΛCDM) model is regarded as the standard model of cosmology. It is a FLRW model with two main components: the cosmological constant Λ, which is the simplest form of dark energy, and a pressureless cold dark matter.

More in general, the EFE (1.16) with Λ are

$$G_{\alpha\beta} = -\kappa T_{\alpha\beta} + \Lambda g_{\alpha\beta}. \tag{1.20}$$

We assumes scales in which a perfect fluid description for the averaged matter is valid. The energy-momentum tensor for a perfect fluid takes the form

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2} \right) u^\alpha u^\beta + p g^{\alpha\beta} \tag{1.21}$$

with ρ being the mass-energy density and p the hydrostatic pressure. Due to the cosmological principle, both ρ and p are only dependent on the time t. If we substitute (1.18) and (1.21) into (1.20), we obtain

$$G^0_0 = 3\frac{\dot{a}^2 + k}{a^2} = \kappa\rho + \Lambda \tag{1.22}$$

$$G^i_{\ i} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} = \kappa p - \Lambda \tag{1.23}$$

We introduce the Hubble expansion scalar $H \equiv \dot{a}/a$ and (2.100) becomes the *first Friedmann equation*

$$3H^2 + 3\frac{k}{a^2} = \kappa\rho + \Lambda. \tag{1.24}$$

Both (2.100) and (1.23) combined yield the *second Friedmann equation*

$$\dot{H} + H^2 = -\frac{\kappa}{6}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{1.25}$$

The energy conservation equation (1.15) reduced to

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{1.26}$$

The first and second Friedmann equations (1.24) and (1.25), and the energy conversation equation (1.26) are not independent: taking the derivative of (1.24) and substituting (1.25) yields after some rearranging (1.26). Equivalently, (1.24) is a first integral of (1.25) with (1.26).

Thus, we have only two independent equations for three unknowns, which are the scale factor $a(t)$, the energy density $\rho(t)$, and the pressure $p(t)$. Therefore, another equation is needed to solve the system of differential equations: the *equation of state* relates the pressure to the energy density and in cosmology the simplest used is

$$P = w\rho. \tag{1.27}$$

E.g. for radiation, we obtain $w = 1/3$ and for preassureless fluid $w = 0$.

If we use the conformal time τ instead of the cosmic time t, the Friedmann equations (1.24) and (1.25) and the energy conservation equation (1.26) become

$$\mathscr{H}^2 = \frac{\kappa}{3}\rho a^2 - k + \frac{\Lambda}{3}a^2, \tag{1.28}$$

$$\mathscr{H}' = -\frac{\kappa}{6}a^2(\rho + 3P) + \frac{\Lambda}{3}a^2, \tag{1.29}$$

$$\rho' + 3\mathscr{H}(\rho + P) = 0 \tag{1.30}$$

with the prime denoting the derivative w.r.t. the conformal time τ and $\mathscr{H} = a'/a$.

1.3.2 Kinematics and dynamics in cosmology

On scales where the fluid description is valid, we can identify a 4-velocity u^μ corresponding to the matter flow. This indicates a preferred velocity and thus a preferred rest frame. Let us consider a family of time-like geodesics corresponding to fundamental observers[12]. Each world line has the tangent vector u^μ. We can (locally) decompose the space-time into a 3-dimensional local hyperspace H_t that is orthogonal to u^μ and into u^μ. The projection tensors along or perpendicular to the tangent vector u^μ read

$$U^\mu_\nu = -u^\mu u_\nu \text{ and } h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu. \tag{1.31}$$

We are interested in how the flow changes. Therefore, we split the covariant derivative of u^μ into

$$\nabla_\mu u_\nu = -u_\mu \dot{u}_\nu + \frac{1}{3}\Theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}. \tag{1.32}$$

$\Theta = \nabla_\mu u^\mu$ represents the increase or decrease in size and is regarded as *expansion*. $\sigma_{\mu\nu}$ is the traceless symmetric part of the $\nabla_\mu u_\nu$ and represents the *shear*. The antisymmetric part $\omega_{\mu\nu}$ is called *vorticity* and denotes the rotation of the fluid.

The *Raychaudhuri equation* denotes the evolution equation for the expansion Θ:

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 + \dot{u}^\mu \dot{u}_\mu + 2\left(\omega^2 - \sigma^2\right) + \bar{\nabla}_\mu \dot{u}^\mu - R_{\mu\nu} u^\mu u^\nu, \tag{1.33}$$

where $\omega = \sqrt{\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}}$ and $\sigma = \sqrt{\frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}}$ are the amplitudes of the vorticity and shear, respectively. The covariant derivative $\bar{\nabla}_\mu$ denotes the derivative projected on the hypersurface Σ_t.

If the vorticity ω vanishes, then $h_{\mu\nu}$ is effectively the metric of the 3-dimensional hypersurface Σ_t orthogonal to u^μ, i.e. all local subspaces H_t merge to form Σ_t.

1.3.3 Inhomogeneous universe

The homogeneous and isotropic FLRW-models give a very good description of the averaged universe on very large scales. If we look at smaller scales, however, structures such as galaxies, galaxy clusters, voids, ... provide small over- and under-densities and therefore sligthly break the homogeneity and isotropy. As long as the deviation in the density is small, it is an apt description of the over- and under-densities to use a perturbative approach. We split quantities into their FLRW background value and a

[12] comoving with the fluid flow

perturbation,

$$Q(t,x^{\mu}) = \bar{Q}(t) + \delta Q\left(t,x^{i}\right). \tag{1.34}$$

The bar denotes the background contribution. Because the background is spatially homogeneous and isotropic, the quantity $\bar{Q}$ can only be dependent on the time t. The perturbation δQ however can be dependent t and x^i. Mathematically, in equation (1.34) the background quantity $\bar{Q}(t)$ and the physical quantity $Q(t,x^{\mu})$ are quantities from different manifolds. Hence, we need to define a map between the manifolds. There are different maps from different points on the background manifold to the same point on the physical manifold. By choosing a specific map, we choose a gauge. If we change map, we are changing the gauge. In section 2.1 we provide a detailed discussion about perturbation theory and gauges.

1.3.4 The redshift z

Hubble's observations on the distance of other galaxies and eventually the formulation of Hubble's law are considered the beginning of modern cosmology. Hubble's observations displayed that the wavelength of the radiation submitted by a galaxy was lengthened and therefore appeared "reddened". This lead to the conclusion that the emitted radiation is experiencing a Doppler shift and as a consequence that the observed objects are moving away from the observer. The redshift z denotes the ratio of the emitted and observed wavelength of a light ray:

$$z \equiv \frac{\lambda_O - \lambda_e}{\lambda_e} \quad \Leftrightarrow \quad 1+z = \frac{\lambda_O}{\lambda_e}, \tag{1.35}$$

where λ_O refers to the wavelenth measured at the observer O and λ_e to the wavelength emitted at the source [46].

For FLRW models, the relation (1.35) simplifies to

$$1+z = \frac{a(t_O)}{a(t_e)} \tag{1.36}$$

with t_O and t_e being the time of observation and the time of emission, respectively. Equation (1.36) relates the physical time t to the redshift z. The physical time t is not a measurable quantity, but the redshift z is. In chapter 4, we therefore find it convenient to perform a change of the x^0 coordinate and display quantities in terms of the redshift z instead of the conformal time τ.

1.4 Weak gravitational lensing

Gravitational lensing is a general-relativistic effect that distorts images of galaxies. It follows from the equivalence principle that light is bent by gravitational masses, which consequently distorts the galaxy images along the line of sight. While strong gravitational lensing leads to visible distortions such as arc-like images of galaxies, weak gravitational lensing causes distortions small enough that they can only be detected statistically. This effect is regarded as the cosmic shear [20, 61, 82] and has been first detected in the early 2000 [7, 62, 108, 116]. In this thesis, I will focus on weak gravitational lensing. In chapter 4, we investigate relativistic effects in weak lensing using the post-Friedmann approach, which is a post-Newtonian like approximation scheme. Chapter 4 is based on the paper in preparation [54]. The statistical analysis of weak lensing reveals distortions alignements referring to the matter distribution in the universe. With a detailed map of the distribution of matter we may be able to to constrain modified gravity or the equation of state of dark energy. Furthermore, weak lensing promises to be a valuable tool to test GR on large scales.

As mentioned before this effect can only be measured statistically. The distortion is small enough that it is impossible to detect the lensing effect on a single object without the knowledge of the unlensed image. But if we take a sample of multiple galaxies into account, we are able to detect a deviation from the mean value of shape, orientation, and magnitude on the galaxy images. The different types of alterations of the galaxy images are categorised as convergence κ, shear γ, and rotation ω. The convergence κ refers to the increase or decrease in size leaving the shape of the galaxy unchanged. The shear γ alters the shape of the image while the rotation ω rotates the images without changing the size or shape. It is convenient to introduce the reduced shear g which refers to a change in the shape of the image without changing the size. In the mathematical analysis, the convergence κ, the shear γ, and the rotation ω are the components of the irreducible decomposition of the Jacobi map $\mathscr{D}^a_{\ b}$, which is a 2×2 matrix. A detailed derivation of $\mathscr{D}^a_{\ b}$, where we start with the geodesic deviation equation and eventually derive the Sachs equation involving $\mathscr{D}^a_{\ b}$, can be found in chapter 4 in section 4.2.

Let us assume two neighbouring geodesics $x^\mu(\lambda)$ and $y^\mu(\lambda)$ with $x^\mu(\lambda) = y^\mu(\lambda) + \xi^\mu$. The Jacobi map $\mathscr{D}^a_{\ b}$ connects the vectorial angle between two neighbouring geodesics at the observer θ^a_O to the deviation vector ξ^a, which has been projected onto the spatial 2D surface orthogonal to the spatial tangent vector of the geodesic:

$$\xi^a_n = \mathscr{D}^a_{\ b}\theta^b_O. \tag{1.37}$$

See also (4.5) in chapter 4 section 4.2. We decompose $\mathscr{D}^a_b$ in the following way:

$$\mathscr{D}_{ab} = \frac{1}{2}\mathscr{D}^c_c\delta_{ab} + \left(\mathscr{D}_{(ab)} - \frac{1}{2}\mathscr{D}^c_c\delta_{ab}\right) + \mathscr{D}_{[ab]} \tag{1.38}$$

$$= (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix} + \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \tag{1.39}$$

with the round and box brackets denoting the symmetrisation and antisymmetrisation of $\mathscr{D}_{ab}$, respectively. The first, second, and third term on the r.h.s. of equation (1.38) refers to the convergence κ, the shear γ, and the rotation ω, respectively, in (1.39). In chapter 4 we compute the convergence κ, the shear γ, and the rotation ω using the post-Friedmann approximation scheme.

Chapter 2

Relativistic approximations schemes in cosmology

The homogeneous and isotropic FLRW models describe the universe very well on large enough scales. If, however, we go to smaller scales, spatial inhomoegneities such as galaxy clusters and voids arise. Due to the complexity of the EFE, there are few exact solution for a spatially inhomogeneous universe and these are in any case not realistic. Therefore, for scales where the density fluctuations are small but not negligible, it is convenient to use a perturbative approach to describe the complex distribution of matter and energy, and find approximate solutions for the EFE. The concept of the perturbative approach is to consider a homogeneous background model on which we introduce perturbations to describe an inhomogeneous universe.

2.1 Perturbation theory

In this section, I will review perturbation theory and gauge transformations [76, 29, 64, 8, 75].

2.1.1 Background and Inhomogeneities

We split space-time into a background, which is a FRLW metric, and perturbations. The FLRW-metric reads

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) = a^2(\tau)\left(-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \quad (2.1)$$

where τ denotes the conformal time. It is a homogeneous and isotropic solution of the Einstein-field-equations incorporating the expansion of the Universe described by the scale factor $a(t)$ and the expansion rate $H = \frac{\dot{a}}{a}$. It is assumed that it approximately describes our observable Universe on large enough scales. By introducing inhomogeneities we may get an understanding of the formation and evolution on large scales. The factor k can take on the values $k = \pm 1$, 0 and indicates the spatial curvature. $k = -1$, $k = 0$, and $k = +1$ denote a hyperbolic, flat, and spherical spacetime, respectively. In this work, we assume $k = 0$.

We will denote quantities that refer to the background with a bar. Due to the spatial homogeneity and isotropy of the background, quantities corresponding to the background are solely depending on the cosmic time. The inhomogeneities are small deviations from the FLRW-metric with which physical quantities are described. For example, the perturbed density reads $\rho(\tau, x^i) = \bar{\rho}(\tau) + \delta\rho(\tau, x^i)$ with $\bar{\rho}$ denoting the background density and $\delta\rho$ being the perturbation. It is convenient to introduce the density contrast δ, which is a combination of the density perturbation and the background density: $\rho(\tau, x^i) = \bar{\rho}(\tau)\left(1 + \delta(\tau, x^i)\right)$.

We can split any tensorial quantity into a background and a perturbation:

$$\mathbf{T}\left(\tau, x^i\right) = \bar{\mathbf{T}}(\tau) + \delta\mathbf{T}\left(\tau, x^i\right), \tag{2.2}$$

where $\bar{\mathbf{T}}$ and $\delta\mathbf{T}$ denote the part corresponding to the background and inhomogeneities, respectively.
The perturbation can be expanded as a power series [76]

$$\delta\mathbf{T} = \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \delta\mathbf{T}_n. \tag{2.3}$$

Furthermore, we perform a standard (3+1) split, in which the space-time is decomposed into spatial hypersurfaces of constant time orthogonal to a timelike vector. Thereby, we are able to split any quantity into scalar and vector, or scalar, vector, and tensor components. The perturbed metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ reads

$$ds^2 = a^2(\tau)\left[-(1+2\phi)d\tau^2 + B_i dx^i d\tau + (\delta_{ij} + 2C_{ij})dx^i dx^j\right] \tag{2.4}$$

with

$$B_i = B_{,i} - S_i \quad \text{with} \quad \text{and} \quad S^i_{\ ,i} = 0, \tag{2.5}$$

$$\text{and} \quad C_{ij} = -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2}h_{ij}, \tag{2.6}$$

$$\text{with} \quad h^{ij}_{\ \ ,i} = 0, \quad \text{and} \quad h^i_i = 0. \tag{2.7}$$

ϕ, B, ψ, and E are scalar metric perturbations, S_i, F_i vector perturbations and h_{ij} are tensor perturbations. As mentioned before, the metric tensor comprises 10 independent functions, which are distributed in the following way: the tensor perturbation h_{ij} is a symmetric, transverse, and traceless tensor and has therefore 2 independent components[1]. The vector potentials F_i and S_i are both divergence-free and have therefore 4 independent components in total[2]. Then, there are the four remaining scalar potentials ϕ, B, ψ, and E. Thus, in total we have 10 independent functions, which matches the amount of components of the metric $g_{\mu\nu}$.

In the Introduction we discussed a (local) decomposition of space-time into 3 dimensional subspace H_t orthogonal to a time-like vector field u^μ. Here we will use the (3+1) split, where we define a unit time-like vector field, which is orthogonal to the constant τ-slices:

$$n_\mu \propto \frac{\partial \tau}{\partial x^\mu}. \tag{2.8}$$

For the FLRW background, n^μ coincides with the 4-velocity of matter, usually denotes as u^α.
Taking the covariant derivative of n_μ yields

$$n_{\mu:\nu} = \frac{1}{3}\hat{\theta}\hat{P}_{\mu\nu} + \hat{\sigma}_{\mu\nu} - \hat{a}_\mu n_\nu, \tag{2.9}$$

where n_μ has been decomposed into the expansion rate $\hat{\theta} = n^\mu{}_{:\mu}$, the projection tensor $\hat{P}_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, the shear $\hat{\sigma}_{\mu\nu} = \frac{1}{2}\hat{P}_\mu{}^\alpha \hat{P}_\nu{}^\beta \left(n_{\alpha:\beta} + n_{\beta:\alpha}\right) - \frac{1}{3}\hat{\theta}\hat{P}_{\mu\nu}$, and the acceleration $\hat{a}_\mu = n_{\mu:\nu}n^\nu$. By construction, n^α is hypersurface orthogonal and so is vorticity free. In the background, we have $\hat{\theta} = 3H$. Note that the hatted quantities are defined via the vector field n_μ. It is also possible to define these quantities, e.g. a expansion rate θ, using the matter four-velocity u^μ as was discussed in the Introduction in equation (1.32). In the case of a vanishing vorticity, the projection tensor becomes the 3 dimensional metric of the slice. Then, the extrinsic curvature is given by the Lie derivative of the projection tensor $\hat{P}_{\mu\nu}$ along the vector field n^μ:

$$K_{\mu\nu} = \frac{1}{2}\mathscr{L}_n \hat{P}_{\mu\nu} = \hat{P}_\nu{}^\lambda n_{\mu;\lambda} = \frac{1}{3}\hat{\theta}\hat{P}_{\mu\nu} + \hat{\sigma}_{\mu\nu}. \tag{2.10}$$

2.1.2 Gauge Transformations

In the perturbative approach, we assume that our physical, inhomogeneous universe can be split into a fictitious background space-time and perturbations. Thus, in order

[1] h_{ij}: 9 components - 3 (symmetric) - 3 (transverse) - 1 (trace-less)=2
[2] vector potentials F_i and S_i: 2 $\times$ 3 components - (1+1)(divergence-free)=4

to define perturbations, we need to choose a correspondence between the background and the physical universe. In terms of differential geometry, we have two manifolds, the idealised background and the physical space-time, which are connected by the chosen correspondence. A point on the physical space-time manifold can be connected to different points on the background manifold via different correspondences. Each correspondence refers to a gauge. If we want to change from one correspondence to another, we perform a gauge transformation.

"A gauge transformation induces a coordinate transformation in the physical space-time, but it also changes the point in the background space-time corresponding to a given point in the physical space-time" ([8], p. 1882)

Let us assume that $\bar{\mathscr{M}}$ denotes a 4-manifold representing the FLRW-background and $\mathscr{M}_\varepsilon$ 4-manifolds representing ε perturbed space-times. The manifold $\mathscr{M}_\varepsilon$ is associated with the physical, inhomogeneous space-time.

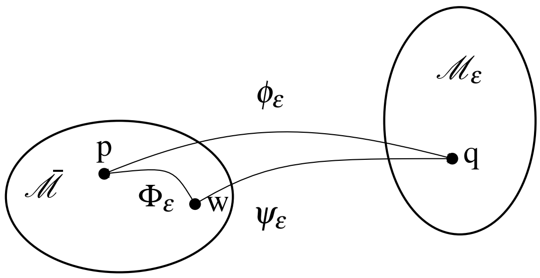

Figure 2.1 The gauge transformation $\Phi_\varepsilon = \phi_{-\varepsilon} \circ \psi_\varepsilon$.

Let the points $p \in \bar{\mathscr{M}}$ and $q \in \mathscr{M}_\varepsilon$. We choose a vector field X, which generates a one-parameter group of diffeomorphisms $\phi_\varepsilon : \mathbb{R} \times \bar{\mathscr{M}} \to \mathscr{M}_\varepsilon$. Thus, the correspondence ϕ_ε links the point p with q. Furthermore, we assign to q and p the same coordinates x^μ. If, however, we choose a different vector field Y (generating a diffeomorphism $\psi_\varepsilon : \bar{\mathscr{M}} \to \mathscr{M}_\varepsilon$), the point q will be identified with another point $w \in \bar{\mathscr{M}}$. The gauge transformation Φ_ε denotes the change from X to Y ($\Phi_\varepsilon = \phi_{-\varepsilon} \circ \psi_\varepsilon$), and thereby the transformation from p to w, leaving q unchanged.

There are two different approaches to gauge transformations. The procedure mentioned above is regarded as the active approach. The passive approach compares tensorial quantities in two different coordinate systems by evaluating these quantities at the same point in the background, e.g. w.

The gauge transformation of any tensorial quantity T is given by

$$\tilde{T} = \Phi_{*\varepsilon} T = (\phi_{-\varepsilon} \circ \psi_\varepsilon)_* T = \psi_{*\varepsilon} \circ \phi_{*-\varepsilon} T = e^{\varepsilon \mathscr{L}_{\psi_Y}} e^{-\varepsilon \mathscr{L}_{\phi_X}} T, \tag{2.11}$$

which can be expressed using the Baker-Hausdorff formula (X and Y do, in general, not commute) as $\Phi_{*\varepsilon} T = \exp\left(\sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \mathscr{L}_{\xi_n}\right) T$ with $\xi_1 = Y - X$, $\xi_2 = [X, Y], \ldots$ and with vector fields ξ_n^μ being functions of X and Y. Hence, we can express the gauge transforation (2.11) for an arbitrary tensorial quantity T as

$$\tilde{T}(w) = \exp\left(\sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!}\mathscr{L}_{\xi_n}\right) T(p) \tag{2.12}$$

with $\exp\left(\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\mathscr{L}_{\xi_n}\right) = 1+\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\mathscr{L}_{\xi_n}+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varepsilon^n}{n!}\frac{\varepsilon^m}{m!}\mathscr{L}_{\xi_n}\mathscr{L}_{\xi_m}+\dots$.

Equation (2.12) is the (active) gauge transformation with ξ_n generators. We specify the vector field as $\xi^\mu = \left(\alpha, \beta^{,i}+\gamma^i\right)$. Thus, the choice of α determines the time slicing, and β and γ^i the threading. We relate two coordinate systems x^μ and $\tilde{x}^\mu$ under an infinitesimal transformation via (2.12) and obtain:[3]

$$\begin{aligned}\tilde{x}^\mu(w) =& e^{\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\xi^{(n)\nu}\frac{\partial}{\partial x^\nu}|_p} x^\mu(p) &(2.13)\\ =& x^\mu(p)+\varepsilon\xi^{(1)\mu}(p)+\frac{1}{2}\varepsilon^2\left[\xi^{(2)\mu}(p)+\xi^{(1)\mu}_{,\nu}(x^\lambda(p))\xi^{(1)\nu}(x^\lambda(p))\right]+\dots &(2.14)\end{aligned}$$

In the passive approach, we choose $\tilde{x}^\mu(w) = x^\mu(p)$. Via (2.14), we obtain

$$\begin{aligned}\tilde{x}^\mu(w) =& x^\mu(p) &(2.15)\\ =& x^\mu(w) - \underbrace{\varepsilon\xi^{(1)\mu}(x^\nu(p))}_{\varepsilon\left(\xi^{(1)\mu}(x^\nu(w))-\varepsilon\xi^{(1)\nu}(x^\lambda p)\right)} - \frac{1}{2}\varepsilon^2\left[\xi^{(2)\mu}(p)+\xi^{(1)\mu}_{,\nu}(x^\lambda(p))\xi^{(1)\nu}(x^\lambda(p))\right]+\dots &(2.16)\\ =& x^\mu(w)-\varepsilon\xi^{(1)\mu}(w)-\varepsilon^2\frac{1}{2}\left[\xi^{(2)\mu}(w)-\xi^{(1)\mu}_{,\nu}(x^\lambda(w))\xi^{(1)\nu}(x^\lambda(w))\right]+\dots &(2.17)\end{aligned}$$

evaluated on the same point w. In practice this results in the gauge transformation formula for the perturbation of tensors.

For example the density $\rho(x^\mu)$ is a scalar and therefore invariant under coordinate transformations, i.e. $\tilde{\rho}(\tilde{x}^\mu) = \rho(x^\mu)$. However, it is not invariant under a gauge transformation. At first order, we have $\tilde{x}^\mu = x^\mu - \xi^{(1)\mu}$ (see (2.17)) and, thus,

$$\begin{aligned}\tilde{\rho}(\tilde{x}^\mu) =& \rho(x^\mu) &(2.18)\\ =& \tilde{\rho}(x^\mu - \xi^\mu) &(2.19)\\ =& \tilde{\rho}(x^\mu) - \tilde{\rho}_{,\lambda}\xi^\lambda &(2.20)\end{aligned}$$

[3] Note that the Lie-derivative on a scalar quantity is $\mathscr{L}_{\xi_n} = \xi_n^\nu\frac{\partial}{\partial x^\nu}$ and that $\xi_n^\mu = \xi^{(n)\mu}$

$$=\tilde{\bar{\rho}} + \delta\tilde{\rho}^{(1)} - \bar{\rho}'\alpha^{(1)} \tag{2.21}$$

$$=\bar{\rho} + \delta\rho^{(1)} \tag{2.22}$$

$$\Rightarrow \delta\tilde{\rho}^{(1)} = \delta\rho^{(1)} + \bar{\rho}'\alpha^{(1)} \tag{2.23}$$

with $\tilde{\bar{\rho}}(x^\mu) = \bar{\rho}(x^\mu)$.

There are several ways to deal with the coordinate dependence of perturbations regarding gauge transformations. One of which is to define gauge-invariant perturbations as combinations of gauge-dependent components, e.g. the Bardeen potentials [8]. Another possibility, referred to as *gauge fixing*, is to use the freedom to choose the functions α, β, and γ^i such that for example two scalar perturbations and one vector perturbation of the metric perturbations equal zero. A third possibility is to directly use the covariant perturbations, i.e. tensors that vanish in the background and are therefore gauge invariant (see Stewart-Walker Lemma in [99]).

2.1.3 Gauges

Throughout this work, we will mostly use the synchronous-comoving gauge in chapter 3 and the Poisson gauge in chapter 4. In this subsection, we will introduce both gauges up to first order. First, we impose the gauge condition and then we choose α, β, and γ^i accordingly.

Poisson Gauge

In this gauge, we wish to eliminate the shear term σ in eq. (2.9), thus we have to choose α so that the shear of the normal n^μ in this gauge $\tilde{\sigma}$ equals zero. The second condition is to choose β_1 such that the coordinate system is orthogonal ($\tilde{B}_1 = 0$, and, hence, $\tilde{E}_1 = 0$). [4]
Hence, the scalar metric perturbations transform from a arbitrary gauge to the longitudinal gauge (quantities with tilde and subscript l) as follows

$$\tilde{\phi}_{1l} = \phi_1 + \mathscr{H}(B_1 - E_1') + (B_1 - E_1')', \tag{2.24}$$

$$\tilde{\psi}_{1l} = \psi_1 - \mathscr{H}(B_1 - E_1') \tag{2.25}$$

[4] The gauge transformation behaviour of σ, E, and B is $\tilde{\sigma}_1 = \sigma_1 + \alpha_1$, $\tilde{E}_1 = E_1 + \beta_1$, and $\tilde{B}_1 = B_1 - \alpha_1 + \beta_1'$, respectively. Further, $\sigma_1 = E_1' - B_1$. We choose $\alpha_1 = -\sigma_1 = -E_1' + B_1$ and $\beta_1 = -E_1$.

Note that (2.24) and (2.25) coincide with the gauge invariant Bardeen potentials. The other scalar perturbations read

$$\delta\tilde{\rho}_{1l} = \delta\rho + \rho_0'\left(B_1 - E_1'\right) \tag{2.26}$$

$$\tilde{v}_{1l} = v_1 + E_1' \tag{2.27}$$

Synchronous Gauges

The synchronous gauge requires $\tilde{\phi} = \tilde{B}_i = 0$.
Thus, for first order we obtain

$$\alpha_{1s} = -\frac{1}{a}\left(\int a\phi_1 d\tau - C_{1\alpha}\left(x^a\right)\right), \tag{2.28}$$

$$\beta_{1s} = \int\left(\alpha_{1s} - B_1\right) d\tau + C_{1\beta}\left(x^a\right), \tag{2.29}$$

$$\gamma_{1s}^i = \int S_1^i d\tau + C_1^{i\gamma^i}\left(x^a\right). \tag{2.30}$$

The scalar perturbations become

$$\tilde{\psi}_{1s} = \psi + 1 + \frac{\mathscr{H}}{a}\left(\int a\phi + 1 d\tau - C(x^i)\right) \tag{2.31}$$

$$\tilde{\sigma}_{1s} = \sigma_1 + \alpha_1 - B_1 \tag{2.32}$$

$$\delta\tilde{\rho}_{1s} = \delta\rho_1 - \frac{\rho_0'}{a}\left(\int a\phi_1 d\tau - C_1(x^i)\right) \tag{2.33}$$

$$\tilde{v}_1 = v_{1s} + B_1 - \alpha_1 \tag{2.34}$$

Through the integration constants, we obtain an additional gauge freedom that needs to be fixed by assuming initial conditions, e.g. that the initial velocity of cold dark matter equals zero.

2.2 Standard Perturbation Theory

In this section we will follow the work of [18, 46, 85, 76, 64]. Previously, we have argued that we are looking at scales at which the fluid description is an appropriate approximation. The scales we look at with SPT are well within fluid description validity. In SPT, we assume that the perturbations are very small, e.g. $\delta \ll \bar{\rho}$. These small deviations from the homogeneous and isotropic background determine the growth of structure through gravitational collapse and eventually lead to the inhomogeneous structure we observe on smaller scales. In SPT, we perturb the following quantities: the metric potentials ϕ, ψ, B, E, S_i, F_i, h_{ij}, the velocity u_μ, the density ρ, and the pressure P. In this thesis, we will only consider pressureless fluid such as dust and we will therefore set $P = 0$ from now on.

2.2.1 The metric

The general line element using conformal time τ with the metric perturbations mentioned above reads

$$ds^2 = a^2(\tau)\left\{-(1+2\phi)\,d\tau^2 + B_i d\tau dx^i + \left[(1-2\psi)\,\delta_{ij} + 2E_{,ij} + 2F_{(i,j)} + h_{ij}\right] dx^i dx^j\right\}. \tag{2.35}$$

Within the regime of validity of SPT, perturbations are assumed to be very small. We split each quantity into different orders, whereby each order is sourced by the previous one. So is for example the second order of a quantity A sourced by the square of its first order contributions $(A^{(1)})^2$ and other quantities. It follows that $A^{(1)} > A^{(2)}$ and $\sum_{n=1}^{\infty} A^{(n)} = \delta A$.

The metric perturbations up to second order read [76]

$$\delta g_{00} = -2a^2\phi = -2a^2\left(\phi^{(1)} + \frac{1}{2}\phi^{(2)} + \dots\right) \tag{2.36}$$

$$\delta g_{0i} = a^2\left(B_{,i} + S_i\right) = a^2\left(B_{,i}^{(1)} + \frac{1}{2}B_{,i}^{(2)} + S_i^{(1)} + \frac{1}{2}S_i^{(2)} + \dots\right) \tag{2.37}$$

$$\delta g_{ij} = a^2\left(-\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2}h_{ij}\right) \tag{2.38}$$

$$= a^2\left[-\left(\psi^{(1)} + \frac{1}{2}\psi^{(2)}\right)\delta_{ij} + E_{,ij}^{(1)} + \frac{1}{2}E_{,ij}^{(2)} + F_{(i,j)}^{(1)} + \frac{1}{2}F_{(i,j)}^{(2)} + \frac{1}{2}h_{ij}^{(1)} + \frac{1}{4}h_{ij}^{(2)} + \dots\right] \tag{2.39}$$

Via $g_{\alpha\beta}g^{\beta\gamma} = \delta^\alpha_\gamma$ we can compute the contravariant metric tensor up to second order [76]:

$$\delta g^{00} = -a^{-2}\left[1 - 2\phi^{(1)} - \phi^{(2)} + 4\phi^{(1)2} - B_k^{(1)}B^{(1)k}\right], \tag{2.40}$$

$$\delta g^{0i} = a^{-2}\left[B^{(1)i} + \frac{1}{2}B^{(2)i} - 2\phi^{(1)}B^{(1)i} - 2B_k^{(1)}C^{(1)ki}\right], \tag{2.41}$$

$$\begin{aligned}\delta g^{ij} = a^{-2}\Big[&\delta^{ij}\left(1 + 2\psi^{(1)} - \psi^{(2)} + 4\psi^{(1)2}\right) - 2E^{(1),ij} - E^{(2),ij} + 4E^{(1),ik}E^{(1),j}_{,k} + \\ &- F^{(1)(i,j)} - F^{(2)(i,j)} + 4F^{(1)(i,k)}F^{(1)}_{(k,m)}\delta^{mj} - h^{(1)ij} - \frac{1}{2}h^{(2)ij} + h^{(1)ik}h^{(1)j}_{k} + \\ &- 8\psi^{(1)}E^{(1),ij} - 8\psi^{(1)}F^{(1)(i,j)} - 4\psi^{(1)}h^{(1)ij} + 8E^{(1),ik}F^{(1)}_{(k,m)}\delta^{mj} + 4E^{(1),ik}h^j_k +\end{aligned}$$

$$+4F^{(1)(i,k)}h^{j}_{k} - B^{(1)i}B^{(1)j}\Big] . \tag{2.42}$$

The 4-veloctiy of matter u^μ is a perturbed quantity and via the constraint $u^\mu u_\mu = -1$ we can compute both co- and contravariant components [76]:

$$u^0 = \frac{1}{a}\left[1 - \phi^{(1)} - \frac{1}{2}\phi^{(2)} + \frac{3}{2}\phi^{(1)2} + \frac{1}{2}v^{(1)}_k v^{(1)k} + v^{(1)}_k B^{(1)k}\right] \tag{2.43}$$

$$u^i = \frac{1}{a}\left(v^{(1)i} + \frac{1}{2}v^{(2)i}\right) \tag{2.44}$$

$$u_0 = -a\left[1 + \phi^{(1)} + \frac{1}{2}\phi^{(2)} - \frac{1}{2}\phi^{(1)2} + \frac{1}{2}v^{(1)}_k v^{(1)k}\right] \tag{2.45}$$

$$u_i = a\Big[v^{(1)}_i + B^{(1)}_i + \frac{1}{2}\left(v^{(2)}_i + B^{(2)}_i\right) - \phi^{(1)}B^{(1)}_i - 2\psi^{(1)}_{ik}v^{(1)k} + 2E^{(1)}_{,ik}v^{(1)k} + \tag{2.46}$$

$$+2F^{(1)}_{(i,k)}v^{(1)k} + h^{(1)}_{ik}v^{(1)k}\Big] . \tag{2.47}$$

Next we turn to the energy-momentum tensor $T_{\mu\nu}$ of a single fluid[5] which is defined as

$$T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + P g_{\mu\nu} + \pi_{\mu\nu}, \tag{2.48}$$

where $\pi_{\mu\nu}$ is the anisotropic stess tensor. In the next chapters, we will only consider pressureless, irrotational fluids. Therefore, we will set in this section the pressure P and the anisotropic stress tensor $\pi_{\mu\nu}$ to zero. For a full analysis see [76, 64]. It is standard to introduce the density contrast δ with $\rho = \bar{\rho}\,(1+\delta)$. Here we will use for the derivation of the EFE in SPT with

$$\rho = \bar{\rho} + \rho^{(1)} + \frac{1}{2}\rho^{(2)}. \tag{2.49}$$

Via $T^\mu_\nu u^\nu = -\rho u^\mu$ we can compute the perturbations of the EM tensor $T_{\mu\nu}$:

$$T^0_0 = -\bar{\rho} - \rho^{(1)} - \frac{1}{2}\rho^{(2)} - \bar{\rho}v^{(1)}_k\left(v^{(1)k} + B^{(1)k}\right) \tag{2.50}$$

$$T^0_i = \bar{\rho}\left(v^{(1)}_i + B^{(1)}_i\right) + \frac{1}{2}\bar{\rho}\Big[v^{(2)}_i + B^{(2)}_i + 4v^{(1)}_i\left(-\psi^{(1)}\delta_{ik} + E^{(1)}_{,ik} + F^{(1)}_{(i,k)} + \frac{1}{2}h^{(1)}_{ik}\right) +$$

$$-2\phi^{(1)}\left(v^{(1)}_i + 2B^{(1)}_i\right)\Big] + \rho^{(1)}\left(v^{(1)}_i + B^{(1)}_i\right), \tag{2.51}$$

$$T^i_j = \bar{\rho}v^{(1)i}\left(v^{(1)}_j + B^{(1)}_j\right). \tag{2.52}$$

[5] Note that we previously considers the EMT of a perfect fluid and thus $\pi_{\mu\nu} = 0$

2.2.2 The Einstein field equations

With the metric perturbations (2.36) - (2.42) and the perturbed energy-momentum tensor (2.50) - (2.52) we can formulate the EFE. Here we present the field equations in Poisson gauge[6] only up to first order (only scalar perturbations)[76]:

$$G^0_{\ 0} + \Lambda = -\kappa T^0_{\ 0} : \tag{2.53}$$

$$H^2 + \frac{2}{3}\frac{\nabla^2 \psi^{(1)}}{a^2} + 2H\dot{\psi}^{(1)} - 2H^2\phi^{(1)} = -\frac{1}{3}\kappa\left(\bar{\rho} + \rho^{(1)}\right) + \frac{1}{3}\Lambda, \tag{2.54}$$

$$G^j_{\ i} + \Lambda = -\kappa T^j_{\ i} : \tag{2.55}$$

$$\frac{1}{a^2}\left(\phi^{(1)} - \psi^{(1)}\right)^{,j}_{\ ,i} + \delta^j_i \left[3H^2 + 2\dot{H} - \frac{1}{a^2}\nabla^2\left(\phi^{(1)} - \psi^{(1)}\right) + \right.$$
$$\left. -2H\left(\dot{\phi}^{(1)} + 3H\dot{\psi}^{(1)}\right) - \left(4\dot{H} + 2H^2\right)\phi^{(1)} + 2\ddot{\psi}^{(1)}\right] = 0. \tag{2.56}$$

2.3 Gradient expansion

The gradient expansion is an approximation in which we expand in terms of spatial gradients. It is valid on scales where the spatial gradient is small compared to the time derivative [35, 93, 41, 88, 27].

We choose synchronous comoving coordinates and denote the metric as

$$ds^2 = -dt^2 + \gamma_{ij}(t, x^i)dx^i dx^j. \tag{2.57}$$

In a spatially homogeneous and isotropic space-time, γ_{ij} would be a function of the time t and $a^2 = \gamma^{1/3}$ with $\gamma = det(\gamma_{ij})$ and $\dot{\gamma}_{ij} \sim H\gamma_{ij}$. Let us assume that at each point we can define a *local* scale factor $\hat{a}$ for which we define a *local* pseudo-Hubble time $\hat{H}^{-1}$ with $\hat{a}^2 \equiv \hat{\gamma}^{1/3}$ and $\hat{H} \equiv \dot{\hat{a}}/\hat{a}$. Then, the time derivative of the spatial metric yields $\dot{\hat{\gamma}}_{ij} \sim \hat{H}\gamma_{ij}$, where the pseudo-Hubble time denotes the characteristic time period in which the metric evolves at each point. The spatial derivative of the spatial metric $\hat{\gamma}_{ij}$ denotes the comoving length $\hat{L}$ at which the metric varies with

$$\hat{\gamma}_{ij,k} \sim \hat{L}^{-1}\hat{\gamma}_{ij}. \tag{2.58}$$

[6]We want to compare the result to the post-Friedmann approach in the latter subsection. In this work, we only consider the post-Friedmann approximation in Poisson gauge

We now assume that there exists a reference frame, in which all spatial gradients are very small compared to the time derivative:

$$\frac{1}{\hat{a}}\hat{\gamma}_{ij,k} \ll \dot{\hat{\gamma}}_{ij}. \tag{2.59}$$

It follows that[7]

$$\hat{a}\hat{L} \gg \hat{H}^{-1}. \tag{2.60}$$

Thus, the scales at which the spatial metric varies is much bigger than the (pseudo-)Hubble radius. This approximation is regarded as the *long wavelength approximation*. Solving the EFE while neglecting the spatial derivatives offers a possibility to compute the dynamics on scales beyond the Hubble radius. The gradient expansion uses the long wavelength approximation as the first order in its expansion. The EFE are solved iteratively and the order of the expansion is determined by the order of spatial gradients while the condition (2.59) holds.

2.3.1 The Einstein field equations with the gradient expansion

In order to solve the EFE we need to determine the EMT. As mentioned before, the gradient expansion is well within the scales where the fluid description is valid. Furthermore, we assume a pressureless fluid such as dust. Then, the EMT becomes

$$T_{\mu\nu} = \rho u_\mu u_\nu. \tag{2.61}$$

The EFE using (2.57) and (2.61) yield [35]:

$$R^{i}_{\gamma_{\mathbf{ij}}\,j} + \frac{1}{2\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}K^{i}_{\ j}\right) = \frac{2\dot{K} + K^{l}_{\ k}K^{k}_{\ l}}{-4 - 8u^{k}u_{k}}\left[2u^{i}u_{j} + \delta^{i}_{\ j}\right] \tag{2.62}$$

$$\kappa\rho = \frac{2\dot{K} + K^{l}_{\ k}K^{k}_{\ l}}{-2 - 4u^{k}u_{k}} \tag{2.63}$$

$$\kappa\rho u_i = -\frac{1}{2\sqrt{1 + u^{k}u_{k}}}\left(K^{j}_{i;j} - K_{,i}\right) \tag{2.64}$$

[7] Note that the comoving Hubble radius $1/(Ha)$ varies with time and therefore this assumtion is not valid at all times.

Choosing a synchronous comoving frame, the extrinsic curvature tensor K_{ij} coincides with the time derivative of the spatial metrix γ_{ij}[8]. The scalar K denotes the trace of the extrinsic curvature $K = K^i_{\ i}$.

At first order, we neglect the spatial Ricci tensor $R^i_{\gamma_{ij}\,j}$ because its components are of second order in the gradients. Equation (2.64) shows that the velocity u_i is at least a first order quantity in the gradient expansion and we can therefore neglect all terms involving the product $u^i u_j$. The traceless part of equation (2.62) becomes

$$\frac{1}{2\sqrt{\gamma}}\frac{\partial}{\partial t}\left[\sqrt{\gamma}\left(K^i_{\ j} - \frac{1}{3}K\delta^i_{\ j}\right)\right] = 0. \tag{2.65}$$

We integrate equation (2.65) and obtain

$$K^i_{\ j} = \frac{1}{3}K\delta^i_{\ j} + \frac{1}{\sqrt{\gamma}}S^i_{\ j} \tag{2.66}$$

with $S^i_{\ j}$ being a time-independent, traceless tensor, which denotes a local anisotropy. Following [35], we define a function A with $A = \gamma^{1/3}$. The scalar K becomes $K = 3\dot{A}/A$ and we can formulate an evolution equation for A by taking the trace of (2.62) at first order:

$$\delta^j_i \frac{1}{2}A^{-3/2}\frac{\partial}{\partial t}\left(A^{3/2}K^i_{\ j}\right) = -\left[6\frac{\partial}{\partial t}\frac{\dot{A}}{A} + 3\left(\frac{\dot{A}}{A}\right)^2 + A^{-3}S^2\right]\frac{1}{4} \tag{2.67}$$

$$3\left(\frac{\dot{A}}{A}\right)^2 + 2\frac{\partial}{\partial t}\frac{\dot{A}}{A} = -6\frac{\partial}{\partial t}\frac{\dot{A}}{A} - 3\left(\frac{\dot{A}}{A}\right)^2 - A^{-3}S^2 \tag{2.68}$$

$$0 = 8\frac{\partial}{\partial t}\frac{\dot{A}}{A} + 6\left(\frac{\dot{A}}{A}\right)^2 + A^{-3}S^2 \tag{2.69}$$

with $S^i_{\ j}S^j_{\ i} = S^2$.

At sufficiently late times or equally sufficiently large scales, the local anisotropy is assumed to be negligible or very small. Thus, in equation (2.66) the first term on the r.h.s. will always dominate over the second term and we will therefore assume $S^i_{\ j} \approx 0$. With this assumption equation (2.66) yields

$$K^i_{\ j} = \frac{1}{3}K\delta^i_{\ j} = \frac{\dot{A}}{A}\delta^i_{\ j} \tag{2.70}$$

$$\Rightarrow \quad \gamma_{ij} = Aw_{ij}, \tag{2.71}$$

[8] The extrinsic curvature $K_{\mu\nu}$ is defined via the Lie derivative along the unit normal vector n^μ, which coincides with the 4-velocity u^μ in the synchronous comoving gauge: $K_{\mu\nu} \equiv \frac{1}{2}\mathscr{L}_n h_{\mu\nu}$ with $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$.

where w_{ij} is a time-independent.

For the first order in the gradient expansion, we can solve (2.69) and obtain:

$$A(t)^{(1)} = t^{4/3}. \tag{2.72}$$

At this order, the energy density and the peculiar velocity yield

$$\rho^{(1)} = \frac{4}{3t^2}, \quad \text{and} \quad u_i^{(1)} = 0. \tag{2.73}$$

The next step is to solve the EFE (2.62), (2.63), and (2.64) iteratively with

$$\gamma_{ij} = \gamma_{ij}^{(1)} + \gamma_{ij}^{(3)} + \dots. \tag{2.74}$$

The solution of any order n can be written as

$$\gamma_{ij} = t^{4/3}\left[w_{ij} + \sum_{p=1}^{n} t^{-p\frac{2}{3}} C_{ij}^{(p)}\right] \tag{2.75}$$

with $C_{ij}^{(p)}$ being a spatial tensors of order $\mathscr{O}\left(\nabla^{2p}\right)$.

In summary, the gradient expansion is an approximation scheme with a space-time as background (or zeroth order), where the spatial gradients are zero. This is e.g. a homogenous and isotropic FLRW space–time. The order of non-zero spatial gradients refers to the magnitude of anisotropy in a space-time.

2.4 Post-Friedmann approximation scheme

The post-Friedmann (PF) formalism is a post-Minkowskian (weak field) type approximation scheme in a cosmological setting [80]. The aim of the formalism is to unite perturbative schemes of all cosmological scales, from small scales, where the dynamics are well approximated by Newtonian approximations, to the largest scales, at which relativistic effects are taken into account. We expand in inverse powers of the speed of light c and assume a ΛCDM background and a fluid description for matter. When linearised, this formalism recovers the linear general-relativistic perturbation theory and can therefore be used to describe structure-formation on very large scales. At leading order, however, the post-Friedmann formalism yields nonlinear Newtonian physics. In the Newtonian regime, when derived consistently from the Einstein field equations, one recovers a metric vector potential additionally to the Newtonian scalar potentials. It is sourced by Newtonian quantities and has been computed from N-body simulations in

[103]. The PF formalism differs from post-Newtonian approximation schemes [86] in two ways: firstly, we choose a ΛCDM background instead of a Minkowski background, and secondly, the only the peculiar velocities are assumed to be small. The latter assumption ensures that the approach is valid on larger scales. Other cosmological applications of a post-Minkowskian type approximations are e.g. [2, 94, 53].

In this thesis, we look at effects derived from the vector potential such as gravimagnetic effects. Examples in other contexts are the geodesic precession and the Thirring-Lense effect (frame-dragging). The latter effect has been measured in the Solar System by Gravity Probe B [48]. The main incentives in developing this formalism are applications such as the calculation of the vector potential and the extension of Newtonian approximations to include relativistic effects.

2.4.1 The metric

The metric of the PF formalism in the Poisson gauge reads [80]

$$g_{00} = -e^{-\frac{2U_N}{c^2} - \frac{4U_P}{c^4}} + \mathscr{O}\left(\frac{1}{c^6}\right), \tag{2.76}$$

$$= -\left[1 - \frac{2U_N}{c^2} + \frac{1}{c^4}\left(2U_N^2 - 4U_P\right)\right] + \mathscr{O}\left(\frac{1}{c^6}\right) \tag{2.77}$$

$$g_{0i} = -\frac{a}{c^3}B_i^N - \frac{a}{c^5}B_i^P + \mathscr{O}\left(\frac{1}{c^7}\right), \tag{2.78}$$

$$g_{ij} = e^{\frac{2V_N}{c^2} + \frac{4V_P}{c^4}}\delta_{ij} + \frac{1}{c^4}h_{ij} + \mathscr{O}\left(\frac{1}{c^6}\right) \tag{2.79}$$

$$= a^2\left[\left(1 + \frac{2V_N}{c^2} + \frac{1}{c^4}\left(2V_N^2 + 4V_P\right)\right)\delta_{ij} + \frac{1}{c^4}h_{ij}\right] + \mathscr{O}\left(\frac{1}{c^6}\right). \tag{2.80}$$

The subscripts N and P of the metric potentials refer to Newtonian and post-Friedmann contributions, respectively. In Poisson gauge, the vector fields B_i^N and B_i^P are divergenceless and the tensor field h_{ij} is transverse and trace-free. If we compare the metric (2.77) - (2.80) to the metric in SPT (2.36) - (2.39) we see that the first two orders of SPT correspond to the orders up to $\mathscr{O}\left(\frac{1}{c^5}\right)$ in the PF representation. In particular, the PF orders $\mathscr{O}\left(\frac{1}{c^2}\right)$ and $\mathscr{O}\left(\frac{1}{c^3}\right)$ to some extend correspond to the first order in SPT, while the order $\mathscr{O}\left(\frac{1}{c^4}\right)$ and $\mathscr{O}\left(\frac{1}{c^5}\right)$ to the second order in SPT. Furthermore, we see that the tensor perturbation h_{ij} occurs only at higher orders in the PF approximation, while in SPT the tensor modes contribute to all orders. The contravariant expression for the metric (2.77) - (2.80) can be derived using $g_{\mu\nu}g^{\nu\kappa} = \delta^{\kappa}_{\mu}$:

$$g_{0\nu}g^{\nu 0} = g_{00}g^{00} + g_{0i}g^{i0} = 1 \tag{2.81}$$

$$= -e^{-\frac{2U_N}{c^2}-\frac{4U_P}{c^4}} g^{00} + \left[-\frac{a}{c^3}B^N_i - \frac{a}{c^5}B^P_i\right] g^{0i} = 1 \tag{2.82}$$

$$\Rightarrow \quad g^{00} = -e^{\frac{2U_N}{c^2}+\frac{4U_P}{c^4}} + \cdots + \frac{1}{c^6}B^N_i B^{Ni} + \cdots = -e^{\frac{2U_N}{c^2}+\frac{4U_P}{c^4}} + \mathcal{O}\left(\frac{1}{c^5}\right) \tag{2.83}$$

$$g^{0i} = \frac{a}{c^3}B^N_i + \frac{a}{c^5}B^P_i + \mathcal{O}\left(\frac{1}{c^6}\right) \quad \text{and} \tag{2.84}$$

$$g_{i\mu}g^{\mu j} = g_{i0}g^{0j} + g_{ik}g^{kj} = \delta^j_i \tag{2.85}$$

$$\frac{1}{c^6}B^N_i B^{Nj} + \cdots + \left(e^{\frac{2V_N}{c^2}+\frac{4V_P}{c^4}}\delta_{ij} + \frac{1}{c^4}h_{ij}\right) g^{jk} = \delta^j_i \tag{2.86}$$

$$\Rightarrow \quad g^{ij} = e^{-\frac{2V_N}{c^2}-\frac{4V_P}{c^4}}\delta^{ij} - \frac{1}{c^4}h^{ij} + \mathcal{O}\left(\frac{1}{c^5}\right) \tag{2.87}$$

Note that in comparison with the contravariant metric using SPT (2.40) - (2.42), we see that there are no vector contributions in the contravariant metric up to the order considered in this work. Because the vector field B^N_i is of order $\mathcal{O}\left(\frac{1}{c^3}\right)$ and we only consider perturbations up to order $\mathcal{O}\left(\frac{1}{c^4}\right)$, any squared term of the vector potential will be neglected. In SPT on the other hand, the vector and tensor potential have first order contributions and therefore any squared term will contribute to the second order.

Validity on All Scales

The PF formalism is an approximation scheme that is valid on both small and large scales. It differs from traditional post-Newtonian (PN) approximations in the following way: the PN formalism is derived from the post-Minkowski approximation with the assumption that velocities are small $v/c \ll 1$ [115, 87]. The PF approximation has a FLRW background instead of a Minkowskian and only *peculiar* velocities are assumed to be small $v_p/c \ll 1$. The latter assumption does not restrict the validity of the approximation to small scales: let us assume that x^i are comoving, spatial coordinates, then the physical coordinate of a fluid element is $r^i = ax^i$. The time derivative of r^i yields $\dot{r}^i = Hr^i + v^i$, which is the sum the Hubble flow and the deviation from it, i.e. the peculiar velocity. If we assume that $|\dot{r}^i| \ll c$, our approach would only be valid on scales much smaller than the Hubble horizon. However, if we only assume that $v^i_p \ll c$, we are not restricted to specific scales.

2.4.2 Einstein field equations

In order to formulate the EFE, we introduce and compute the 4-velocity u^μ and the EM tensor $T^{\mu\nu}$ [80]. Note that in the previous chapters we have set $c = 1$. In the PF approach, we expand in terms of the inverse of the speed of light c and therefore will not set c equals to one. Furthermore, every derivative w.r.t. the physical time t or the conformal time τ will add a factor of $1/c$:

$$\frac{d}{dx^0} = \frac{d}{cd\tau} \text{ with } x^0 = c\tau. \tag{2.88}$$

The 4-velocity u^μ is defined as

$$u^\mu \equiv \frac{dx^\mu}{cd\tau}. \tag{2.89}$$

We can express the spatial part of (2.89) in terms of u^0 via

$$u^i = \frac{dx^i}{cdt}\frac{dt}{d\tau} = \frac{v^i}{ca}u^0 \tag{2.90}$$

with $v^i \equiv a\frac{dx^i}{dt}$ being the physical peculiar velocity. With $g_{\mu\nu}u^\mu u^\nu = -1$ we obtain the remaining co- and contravariant components of the 4-velocity as follows [80]:

$$u^0 = 1 + \frac{1}{c^2}\left(U_N + \frac{1}{2}v^2\right) + \frac{1}{c^4}\left(\frac{1}{2} + 2U_P + v^2V_N + \frac{3}{2}v^2U_N + \frac{3}{8}v^4 - B^N_i v^i\right) \tag{2.91}$$

$$u_i = \frac{av^i}{c} + \frac{a}{c^3}\left(-B^N_i + v_iU_N + 2v_iV_N + \frac{1}{2}v_iv^2\right) \tag{2.92}$$

$$u_0 = -1 + \frac{1}{c^2}\left(U_N - \frac{1}{2}v^2\right) + \frac{1}{c^4}\left(2U_P - \frac{1}{2}U_N^2 - \frac{1}{2}v^2U_N - v^2V_N - \frac{3}{8}v^4\right). \tag{2.93}$$

Analogously to subsection 2.2 we compute the components of the EM tensor assuming a pressureless, irrotational fluid with $T^\mu_\nu = c^2\rho u^\mu u_\nu$ [80]:

$$T^0_0 = -c^2\rho - \rho v^2 - \frac{1}{c^2}\rho\left(4W_Nv^2 - B^N_i v^i + v^4\right), \tag{2.94}$$

$$T^0_i = c\rho a v_i + \frac{1}{c}\rho a\left[v_i\left(v^2 + 4W_N\right) - B^N_i\right], \tag{2.95}$$

$$T^i_0 = -c\frac{1}{a}\rho v^i - \frac{1}{ca}\rho v^2 v^i, \tag{2.96}$$

$$T^i_j = \rho v^i v_j + \frac{1}{c^2}\rho\left[v^i v_j\left(v^2 + 4W_N\right) - v^i B^N_j\right], \tag{2.97}$$

$$T^\mu_\mu = -\rho c^2 \tag{2.98}$$

with $W_N \equiv \frac{1}{2}\left(U_N + V_N\right)$.

With the expressions (2.94) - (2.98) for the EM tensor and the metric (2.77) - (2.80) we can formulate the EFE up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ [80]:

$$G^0_{\ 0} + \Lambda = -\kappa T^0_{\ 0} : \tag{2.99}$$

$$\frac{1}{c^2}\left(3H^2 - 2\frac{\nabla^2 V_N}{a^2}\right) + \frac{1}{c^4}\left(6H\dot{V}_N + 6H^2 U_N - \frac{4}{a^2}\nabla^2 V_P + \frac{2}{a^2}\nabla^2 V_N^2 - \frac{5}{a^2}V_N^{,i} V_{N,i}\right)$$
$$= -\frac{1}{c^2}\kappa\rho - \frac{1}{c^4}\kappa\rho v^2 + \Lambda, \tag{2.100}$$

$$G^j_{\ i} + \Lambda = -\kappa T^j_{\ i} : \tag{2.101}$$

$$\frac{1}{c^2}\left\{\frac{1}{a^2}\left(V_N - U_N\right)^{,j}_{,i} + \delta^j_i\left[3H^2 + 2\dot{H} - \frac{1}{a^2}\nabla^2\left(V_N - U_N\right)\right]\right\} + \frac{1}{c^4}\left\{-\frac{1}{a}H\left(B^{N,j}_i + B^{N\,j}_{\ \ ,i}\right) + \right.$$
$$-\frac{1}{2a}\left(\dot{B}^{N,j}_i + \dot{B}^{N\,j}_{\ \ ,i}\right) - \frac{2}{a^2}U_{P,i}^{\ ,j} + \frac{2}{a^2}V_{P,i}^{\ ,j} + \frac{1}{a^2}U_{N,i}U_N^{,j} - \frac{1}{a^2}V_{N,i}V_N^{,j} + \frac{2}{a^2}U_{N(,i}V_N^{\ ,j)} +$$
$$-\frac{2}{a^2}V_N\left(V_N - U_N\right)_{,i}^{\ ,j} + \delta_i^{\ j}\left[2H\dot{U}_N + \left(4\dot{H} + 2H^2\right)U_N + 6H\dot{V}_N + 2\ddot{V}_N + \right.$$
$$\left.\left. -\frac{2}{a^2}\nabla^2 U_P - \frac{2}{a^2}\nabla^2 V_P - \frac{1}{a^2}U_{N,k}U_N^{\ ,k} + \frac{2}{a^2}V_N\nabla^2\left(V_N - U_N\right)\right] + \frac{1}{2a^2}\nabla^2 h_i^{\ j}\right\}$$
$$= \Lambda\delta_i^{\ j} + \frac{1}{c^4}\kappa\rho v_i v^j, \tag{2.102}$$

$$G^0_{\ i} + \Lambda = -\kappa T^0_{\ i} : \tag{2.103}$$

$$\frac{1}{c^3}\left(-\frac{1}{2a}\nabla^2 B^N_i + 2HU_{N,i} + 2\dot{V}_{N,i}\right) = -\frac{1}{c^3}\kappa\rho a v_i. \tag{2.104}$$

If we compare (2.100) with the 00-component of the EFE (2.54) using SPT, one can see that terms of first order SPT can be found in order $\mathscr{O}\left(\frac{1}{c^2}\right)$ and $\mathscr{O}\left(\frac{1}{c^4}\right)$ in the PF approximation. This follows from the additional factor $1/c$ that stems from time derivatives, which affects the order in the PF approximation.In the PF approximation the peculiar velocity and the density contrast are not perturbative quantities compared to SPT. Yet, via (2.90) the peculiar velocity is often accompanied by a factor $1/c$.

Newtonian regime: at leading order, the Einstein Field Equations yield the standard equations of Newtonian cosmology [80]. One obtains the Poisson equation as well as constraint equations demanding $V_N = U_N$ from the leading order of equation (2.102). However, the EFE involving G^0_i also has leading order contributions, which determine the frame-dragging potential B^N_i. It follows that B^N_i is sourced by the purely Newtonian quantities $\bar{\rho}v_i$.

Relativistic regime: we define "resummed variables" $\phi = -\left(U_N + \frac{2}{c^2}U_P\right)$ and $\psi = -\left(V_N + \frac{2}{c^2}V_P\right)$. When one linearise the EFE substituting the resummed variables, we recover the first order of standard relativistic perturbation theory. For more details, see [80].

The validity on all scales is especially beneficial for the analysis of weak gravitational lensing (WL): we integrate along the line of sight from large to small scales. If e.g. two galaxies are far apart but almost aligned w.r.t. the line of sight, we can compute the correlation for the convergence and shear with the PF formalism.

Chapter 3

$f_{\mathrm{NL}} - g_{\mathrm{NL}}$ mixing in the matter density field at higher orders

3.1 Introduction

The aim of this chapter is to present my work based on the published paper [55] on non-Gaussianity at higher orders. Non-Gaussianity of primordial fluctuations, a residue from the inflationary era, is a powerful probe of the dynamics of the very early universe. The bispectrum of the cosmic microwave background radiation (CMB) provides the statistical measure for non-Gaussianity and insights into the conditions in the inflationary universe [10, 112, 66, 34]. Recently, high precision measurements with the Planck satellite were able to further constraint the value of the local type non-Gaussianity f_{NL} [3]. In upcoming galaxy surveys, primordial non- Gaussianity will be probed thanks to its scale-dependence on large scales [38], where however it is important to consider relativistic effects [24].

However, even with Gaussian primordial density fluctuations, the intrinsic nonlinearity of General Relativity produces non-Gaussian contributions in the matter density field [27, 78, 18, 13, 11]. In particular [26, 109] show how this effective non-Gaussianity and primordial non-Gaussianity add to the evolution of the density field up to second order. There have been recent discussions on the topic whether and how this effective non-Gaussianity contributes to the galaxy bias [12, 37, 40, 42]; however, in this chapter we restrict our attention to the underlying matter density field.

We use the gradient expansion approximation scheme, also known as long-wavelength approximation [70, 104, 71, 93, 41, 32, 88, 27], to investigate non-Gaussian contribution in the density field at very large scales, up to fourth order in standard perturbation theory, in the context of standard ΛCDM cosmology. Thus, we focus on scales large enough to neglect spatial gradients in comparison to the time derivatives. We discuss the contributions derived solely from the nonlinear nature of General Relativity as well as primordial non-Gaussianity up to fourth order. To describe collisionless matter, CDM, we consider a pressureless irrotational dust flow in synchronous-comoving gauge.

The outline of the chapter is as follows: in section 3.2, we summarise the essential exact equations that are needed in the following sections to study the nonlinear evolution of the density contrast. In later sections, we perturb and expand these equations, i.e. the exact continuity equation for the density contrast, the exact Raychaudhuri equation for the expansion scalar, and the exact energy constraint that links density contrast and expansion scalar to the spatial curvature. In general these equations are then nonlinearly coupled to the equations for the shear of the matter flow and for the Weyl tensor [47], but in the approximation used in this thesis these three equations are all is needed, at any perturbative order and at large scales, as we are going to show.

Section 3.3 is dedicated to the gradient expansion. We omit any quantities of order higher than $\mathscr{O}(\nabla^2)$. By splitting scales into long (superhorizon) and short, we can safely approximate the local evolution as that of a separate (homogeneous) universe with its own background density and curvature. This is commonly regarded as the separate universe conjecture. We find that within this approximation, the metric reduces to a conformally flat metric with an effective scale factor constructed from the scale factor a and the metric perturbation ζ. Within the approximation of the gradient expansion, the quantities still contain all orders from standard perturbation theory (SPT).

However, in section 3.4 we contrast the results of section 3.3 with SPT and study the first and second order of the evolution equations. Thereby, we find that the first order equations of SPT coincide with the equations obtained in the approximation of the gradient expansion.

In section 3.5, we express the density contrast in terms of a series expansion and compute all orders up to order $\mathscr{O}(4)$ in SPT. In addition, we add non-Gaussian contributions up to fourth order in the initial conditions to examine the evolution of non-Gaussian contributions reflecting on possible inflationary scenarios.

In the section 3.6 and 3.7 we relate the first order curvature perturbation $\zeta^{(1)}$ to the Poisson gauge metric potentials in order to subsequently compare our results to Newtonian dynamics.

3.2 Evolution equation for the density contrast δ

In this section we will provide the basis for deriving the evolution equations for the density contrast using the Einstein field equations, the deformation tensor, and the continuity equation for the density contrast.

A general cosmological line element can be written as

$$ds^2 = a^2(\eta)\left[-(1+2\phi)\,d\eta^2 + 2\omega_i d\eta dx^i + \gamma_{ij} dx^i dx^j\right], \tag{3.1}$$

where η is the conformal time, $a(\eta)$ the scale factor and γ_{ij} is the conformal spatial metric.
From now on, we will use the synchronous-comoving gauge, so that $\phi = \omega_i = 0$ [67] (see subsections 3.6 and 3.7 for relations to other gauges).

We consider a pressureless, irrotational fluid and comoving observers with four-velocity $u_\mu = (-a, 0, 0, 0)$. Thus, the four-velocity u^μ of the fluid and of the observers coincides with the normal n^μ of constant time hypersurfaces. Using u^μ we can covariantly define kinematical quantities, following the covariant fluid approach [45, 47, 46]; the projection tensor h^μ_ν coincides with the spatial metric $h^\mu_\nu \equiv g^\mu_\nu + u^\mu u_\nu$ in the constant time hypersurfaces.

The deformation tensor of the fluid is defined as

$$\vartheta^\mu_\nu \equiv a u^\mu_{;\nu} - \mathscr{H} h^\mu_\nu, \tag{3.2}$$

where $\mathscr{H} = a'/a$ is the conformal Hubble scalar, the prime indicates conformal time derivative, and the isotropic background expansion $3\mathscr{H}$ has been subtracted[1]. Then, the trace $\vartheta = \vartheta^\mu_\mu$ of the deformation tensor denotes the inhomogeneous volume expansion and the traceless part represents the matter shear tensor.

Due to our synchronous-comoving gauge choice, with $u^\mu = n^\nu$, the deformation tensor is purely spatial and coincides with the negative of the extrinsic curvature K^i_j of the conformal spatial metric γ_{ij}, which can be expressed as follows [111]:

$$\vartheta^i_j = -K^i_j \equiv \frac{1}{2}\gamma^{ik}\gamma'_{kj}. \tag{3.3}$$

[1] In the Introduction in eq. (1.32) we introduced the deformation tensor without subtracting the background.

The matter density field is characterised by a background part $\bar{\rho}$ and a density contrast δ, as in equation (3.2) for the deformation tensor,

$$\rho(\mathbf{x},\eta) = \bar{\rho}(\eta) + \delta\rho(\mathbf{x},\eta) = \bar{\rho}(\eta)\,(1+\delta(\mathbf{x},\eta))\,. \tag{3.4}$$

Using (3.4), the energy conservation equation $u_\alpha T^{\alpha\beta}_{\ \ ;\beta} = 0$ gives the continuity equation for the density contrast:

$$\delta' + (1+\delta)\,\vartheta = 0. \tag{3.5}$$

The evolution equation for the expansion, ϑ, is given by the Raychaudhuri equation (1.33), which reads in terms of ϑ:

$$\vartheta' + \mathscr{H}\vartheta + \vartheta^i_j\vartheta^j_i + 4\pi G a^2\bar{\rho}\,\delta = 0. \tag{3.6}$$

Furthermore, via (3.3) one obtains the energy constraint [111, 46]:

$$\vartheta^2 - \vartheta^i_j\vartheta^j_i + 4\mathscr{H}\vartheta + R = 16\pi G a^2\bar{\rho}\,\delta, \tag{3.7}$$

where R refers to the purely spatial Ricci scalar of the conformal spatial metric γ_{ij}. This is the 00 component of the Einstein field equations in the synchronous-comoving gauge.

Both in equation (3.6) and (3.7), the term $\vartheta^i_j\vartheta^j_i$ couples these equations to the evolution equations of the shear and the Weyl tensor [45, 47, 46]. However, in the approximation used in the following, we only need the equations above.

3.3 The gradient expansion

In order to compute the Ricci scalar in the gradient expansion, we first consider a more general spatial metric $g_{ij} = a^2 e^{2\zeta}\check{\gamma}_{ij} = a^2 e^{2\zeta}\left(\delta_{ij} + \check{\alpha}_{ij}\right)$ with $\check{\alpha}_{ij} = \partial_i\partial_j\check{\alpha}$ for scalar perturbations. It follows that any contribution from $\check{E}_{ij}$ from (2.6) to the spatial Ricci scalar is of order $\mathscr{O}\left(\nabla^4\right)$ and, therefore, it can be neglected in the gradient expansion. This is easily seen as follows. If two metric are related by a conformal transformation with conformal factor $e^{2\zeta}$, $\gamma_{ij} = e^{2\zeta}\check{\gamma}_{ij}$, then their Ricci scalars are related by [111]:

$$R = e^{-2\zeta}\left[-4\nabla^2\zeta - 2\,(\nabla\zeta)^2 + \check{R}\right], \tag{3.8}$$

where $\check{R} = \check{R}(\check{\gamma}_{ij})$. $\check{R}$ expressed in terms of the metric $\check{\gamma}_{ij}$ reads

$$\check{R} = \left(\check{\gamma}^{ij}\check{\gamma}^{kl} - \check{\gamma}^{ik}\check{\gamma}^{jl}\right)\check{\gamma}_{ij,kl} +$$

$$+\check{\gamma}_{ij,k}\check{\gamma}_{ab,c}\left(\frac{1}{2}\check{\gamma}^{ia}\check{\gamma}^{jc}\check{\gamma}^{kb}-\frac{3}{4}\check{\gamma}^{ia}\check{\gamma}^{jb}\check{\gamma}^{kc}+\check{\gamma}^{ia}\check{\gamma}^{jk}\check{\gamma}^{bc}+\frac{1}{4}\check{\gamma}^{ij}\check{\gamma}^{ab}\check{\gamma}^{kc}-\check{\gamma}^{ij}\check{\gamma}^{ac}\check{\gamma}^{kb}\right). \tag{3.9}$$

At order $\mathscr{O}\left(\nabla^2\right)$, we obtain $\check{R}=0$ given that $\check{\alpha}_{ij}$ is of order $\mathscr{O}\left(\nabla^2\right)$. The order $\mathscr{O}\left(\nabla^4\right)$ is the first order, at which we obtain non-zero contributions to the spatial Ricci scalar:

$$\check{R}\left(\nabla^4\right)=\left(\delta^{ij}\delta^{kl}-\delta^{ik}\delta^{jl}\right)\check{\alpha}_{ij,kl}+ \\ +\check{\alpha}_{ij,k}\check{\alpha}_{ab,c}\left(\frac{1}{2}\delta^{ia}\delta^{jc}\delta^{kb}-\frac{3}{4}\delta^{ia}\delta^{jb}\delta^{kc}+\delta^{ia}\delta^{jk}\delta^{bc}+\frac{1}{4}\delta^{ij}\delta^{ab}\delta^{kc}-\delta^{ij}\delta^{ac}\delta^{kb}\right). \tag{3.10}$$

In this work, we will only consider quantities up to order $\mathscr{O}\left(\nabla^2\right)$. Therefore, we choose the following representation of the spatial metric:

$$g_{ij}=a^2\gamma_{ij}=a^2e^{2\zeta}\delta_{ij}, \tag{3.11}$$

where ζ denotes the primordial curvature perturbation. This variable is customarily used to to deal with primordial non-Gaussianity from inflation [112] (see subsection 3.6 for a first-order gauge-invariant treatment). Furthermore, we only consider scalar perturbations.

In the standard model of cosmology, ΛCDM, it is assumed that inflation imposes the initial condition in the very early universe. For scalar perturbations, initial conditions are given by the primordial curvature perturbation ζ. This is convenient, because ζ remains constant after inflation ends and is almost scale-invariant [73]. By performing a gradient expansion up to second order, we focus on scales large enough that the spatial gradients are small compared to time derivatives and terms of order higher than $\mathscr{O}(\nabla^2)$ are negligible, where ∇ is the spatial gradient in comoving coordinates [70, 104, 71, 93, 41, 32, 88, 27]. In this approximation, one finds that

$$\delta\sim\vartheta\sim R\sim\nabla^2. \tag{3.12}$$

Consequently, the continuity equation (3.5) and the energy constraint (3.7) become

$$\delta'+\vartheta=\mathscr{O}(\nabla^4) \tag{3.13}$$

and

$$4\mathscr{H}\vartheta+R=16\pi Ga^2\bar{\rho}\delta+\mathscr{O}(\nabla^4). \tag{3.14}$$

Note that we have not perturbed any quantity in the conventional sense. Thus, from the point of view of the standard perturbative approach R, δ, and ϑ are nonlinear and contain all orders (at large scales).
We now combine (3.13) and (3.14) and, thereby, formulate an evolution equation for the density contrast:

$$4\mathscr{H}\delta' - R = 16\pi G a^2 \bar{\rho}\delta. \tag{3.15}$$

R remains constant, i.e. it is a conserved quantity in this large-scale approximation. This can easily be seen by taking the time derivative of (3.14) and combining the result with the Raychaudhuri equation (3.6), which reads

$$\vartheta' + \mathscr{H}\vartheta + 4\pi G a^2 \bar{\rho}\delta = 0. \tag{3.16}$$

within the gradient expansion up to $\mathscr{O}\left(\nabla^2\right)$.

The crucial feature of the above equations is that they take the same form as the first order equations in the standard perturbative approach (cf. section 3.4 and [26]). Therefore, the evolution equation (3.15) and its solution are formally equivalent to the first order evolution equation and solution, respectively. In the standard perturbation framework, the first order Ricci scalar[2], $R^{(1)}$, is conserved. At second order, $R^{(2)}$ comprises a time dependent and a conserved part. In the gradient expansion, we only need to take the conserved contribution of each order i of $R^{(i)}$ into account as the time-dependent part is of order $\mathscr{O}\left(\nabla^4\right)$.

At leading order on large scales, we can safely approximate the spatial metric as $\check{\gamma}_{ij} \simeq \delta_{ij}$, because non-flat contributions to $\check{\gamma}_{ij}$ are higher order in the gradient expansion as shown in (3.10). Hence, in this approximation the spatial metric (3.11) is conformally flat, with the conformal factor $a^2 e^{2\zeta}$; this conformal factor can be seen as an effective scale factor in the separate universe approach [68, 37, 113]. Given the conformal flatness of the spatial metric, the Ricci scalar is a nonlinear function solely of the curvature perturbation ζ and takes on the form [111, 26, 27]

$$R = e^{-2\zeta}\left[-4\nabla^2\zeta - 2\left(\nabla\zeta\right)^2\right]. \tag{3.17}$$

Performing a series expansion of the exponential, (3.17) yields

$$R = \sum_{n=0}^{\infty} \frac{(-2\zeta)^n}{n!}\left[-4\nabla^2\zeta - 2\left(\nabla\zeta\right)^2\right]$$

[2] In order to distinguish between the different approximation schemes, we refer with $R^{(i)}$ to the order i of R in the standard perturbative approach.

$$
\begin{aligned}
&= -4\nabla^2\zeta - \sum_{n=1}^{\infty}\frac{(-2\zeta)^n}{n!}4\nabla^2\zeta - \sum_{n=0}^{\infty}\frac{(-2\zeta)^n}{n!}2(\nabla\zeta)^2 \\
&= -4\nabla^2\zeta + \sum_{n=0}^{\infty}\frac{(-2)^{n+1}}{(n+1)!}\left[-4\zeta\nabla^2\zeta + (n+1)(\nabla\zeta)^2\right]\zeta^n. \qquad (3.18)
\end{aligned}
$$

As we shall see in section 3.5 this can be used to represent R up to any desired perturbative order in ζ.

3.4 Contrasting with the standard perturbative approach

So far we have used a gradient expansion, keeping the leading order, rather than applying the standard perturbative expansion to quantities representing inhomogeneities. We now clarify the relationship between the two approaches. [26, 27]

In the standard perturbative expansion, combining the first-order parts of (3.5) and (3.7), one obtains the following evolution equation for the density contrast:

$$
4\mathscr{H}\delta^{(1)\prime} + 6\mathscr{H}^2\Omega_m\delta^{(1)} - R^{(1)} = 0, \qquad (3.19)
$$

with

$$
R^{(1)\prime} = 0 \qquad (3.20)
$$

and $\Omega_m = 8\pi G a^2\bar{\rho}/3\mathscr{H}^2$. Equation (3.19) has exactly the same form of the evolution equation (3.15) obtained in the gradient expansion. Because $R^{(1)}$ is constant at first order, equation (3.19) is a first integral of the well known second-order homogeneous differential equation for $\delta^{(1)}$:

$$
\delta^{(1)\prime\prime} + \mathscr{H}\delta^{(1)\prime} - \frac{3}{2}\Omega_m\delta^{(1)} = 0. \qquad (3.21)
$$

The advantage of this fluid-flow approach to relativistic perturbations in the comoving-synchronous gauge is twofold. It is as close as possible to Newtonian perturbation theory (where equation (3.21) is exactly the same), with metric perturbations as secondary variables that can be expressed in terms of the density contrast and the curvature and expansion perturbations. Solving equation (3.19) directly shows that the well known decaying mode D_- and the growing mode D_+ of the solution of (3.21) correspond to the homogeneous solution of (3.19) and the particular solution sourced by the curvature perturbation $R^{(1)}$, respectively:

$$
D_- + \frac{3}{2}\mathscr{H}\Omega_m D_- = 0 \quad \text{and} \qquad (3.22)
$$

$$C(\mathbf{x})\left(\mathscr{H}D'_{+}+\frac{3}{2}\mathscr{H}^{2}\Omega_{m}D_{+}\right)-\frac{1}{4}R^{(1)}=0 \tag{3.23}$$

with

$$\delta^{(1)}(\eta,\mathbf{x})=C_{+}(\mathbf{x})D_{+}(\eta)+C_{-}(\mathbf{x})D_{-}(\eta). \tag{3.24}$$

Analogously to equation (3.19), we obtain the following for second order combining (3.5) and (3.7):

$$4\mathscr{H}\delta^{(2)\prime}+6\mathscr{H}^{2}\Omega_{m}\delta^{(2)}-R^{(2)}=2\vartheta^{(1)2}-2\vartheta^{(1)i}_{j}\vartheta^{(1)j}_{i}-8\mathscr{H}\delta^{(1)}\vartheta^{(1)}. \tag{3.25}$$

and [26]

$$R^{(2)\prime}=-4\vartheta^{(1)i}_{j}R^{(1)j}_{i}=2\left[\partial^{i}\partial_{j}\check{\alpha}^{(1)\prime}\partial^{j}\partial_{i}\zeta^{(1)}-\nabla^{2}\check{\alpha}^{(1)\prime}\nabla^{2}\zeta^{(1)}\right] \tag{3.26}$$

with $g_{ij}=a^{2}e^{2\zeta}\hat{\gamma}_{ij}=a^{2}e^{2\zeta}\left(\delta_{ij}+\check{\alpha}_{,ij}\right)$[3].

By nature of the perturbative expansion, these second-order equations are sourced by squared first order terms. Given the equivalence of the left-hand side of the systems of equations (3.19)-(3.20) and (3.25)-(3.26), it was shown in [26] that these equations are conveniently solved by splitting $\delta^{(2)}$ and $R^{(2)}$ into two parts:

$$\delta^{(2)}=\delta^{(2)}_{h}+\delta^{(2)}_{p},\quad R^{(2)}=R^{(2)}_{h}+R^{(2)}_{p}, \tag{3.27}$$

where $\delta^{(2)}_{h}$ and $R^{(2)}_{h}$ are the solutions of the homogeneous parts of (3.25)-(3.26) and $\delta^{(2)}_{p}$ and $R^{(2)}_{p}$ are the particular solutions sourced by the squared first-order terms. In particular, $R^{(2)}_{h}$ is time-independent as $R^{(1)}$ is.

Note that

$$\delta^{(1)}\sim\vartheta^{(1)}\sim R^{(1)}\sim\nabla^{2}, \tag{3.28}$$

and this holds true for $\delta^{(2)}_{h}$, $\vartheta^{(2)}_{h}$, and $R^{(2)}_{h}$, while it is clear from (3.25)-(3.26) that

$$\delta^{(2)}_{p}\sim\vartheta^{(2)}_{p}\sim R^{(2)}_{p}\sim\nabla^{4}. \tag{3.29}$$

Iterating the procedure at higher orders, it follows that at any order i,

$$\delta^{(i)}_{h}\sim\vartheta^{(i)}_{h}\sim R^{(i)}_{h}\sim\nabla^{2}. \tag{3.30}$$

[3]Note that at order $\mathscr{O}\left(\nabla^{2}\right)$ in the gradient expansion $\hat{\gamma}_{ij}\approx\delta_{ij}$ and therefore $\check{\alpha}_{,ij}$ is negligible.

Therefore, comparing with the equations in the previous section, it should be clear that the leading $\sim \nabla^2$ order in the gradient expansion is equivalent to the homogeneous solution of standard perturbation theory at all orders.

In particular, we can now assume that the Ricci scalar at higher orders can be split into a time-dependent part $R_p^{(i)}$ and a time-independent part $R_h^{(i)}$. Thus, the gradient expansion offers a unique possibility to compute the homogeneous solution of the evolution equation of δ at higher orders.

3.5 Third and fourth order

3.5.1 Growing mode solution in the large scale limit

Because the evolution equation in the gradient expansion (3.15) formally coincides with the evolution equation for the density contrast (3.19) at first order, the solution is formally the same. Thus, we solve for the density contrast δ using the same ansatz (3.24) as we used for the first-order solution. For the growing part of the density contrast sourced by the curvature perturbation we have

$$\delta = D_+(\eta)C(\mathbf{x}). \tag{3.31}$$

Furthermore, the decaying mode D_- is negligible in the matter dominated era. As long as this is well represented by the Einstein-de Sitter model, the growing mode is proportional to the scale factor $a(\eta)$ (see e.g. [18]).

Within the regime of the gradient expansion, the function $C(\mathbf{x})$ is related to the Ricci scalar by [27, 26]

$$C(\mathbf{x}) = \frac{R}{10\mathscr{H}_{\mathrm{IN}}^2 D_{+\mathrm{IN}}}, \tag{3.32}$$

where the subscript “IN” refers to the evaluation at the time η_{IN} early in the matter-dominated era.

3.5.2 First, second, third and fourth order solution

The calculations in this subsection are my responsibility.

Following the scheme outlined of Section 3.4, we now compute the homogeneous solution for the second, third, and fourth order density contrast, adding primordial non-Gaussianity to our initial conditions.

We expand ζ in terms of a Gaussian random field $\zeta^{(1)}$ [79, 72]:

$$\zeta = \zeta^{(1)} + \frac{3}{5} f_{\rm NL}\left(\zeta^{(1)2} - \langle\zeta^{(1)2}\rangle\right) + \frac{9}{25} g_{\rm NL}\zeta^{(1)3} + \frac{27}{125} h_{\rm NL}\left(\zeta^{(1)4} - \langle\zeta^{(1)4}\rangle\right) + \ldots. \tag{3.33}$$

where $f_{\rm NL}$, $g_{\rm NL}$, and $h_{\rm NL}$ denote the non-Gaussian deviations at different orders.

Now we substitute (3.33) into the series expansion for the spatial Ricci scalar (3.18). For $n=0$ (3.18) yields the second order expansion of R [26]. For $n=2$, we obtain the spatial Ricci scalar R up to the fourth perturbative order:

$$\begin{aligned} R \simeq & -4\nabla^2\zeta + (-2)\left[(\nabla\zeta)^2 - 4\zeta\nabla^2\zeta\right] + \frac{4}{2}\left[2(\nabla\zeta)^2 - 4\zeta\nabla^2\zeta\right]\zeta - \\ & -\frac{4}{3}\left[3\zeta^2(\nabla\zeta)^2 - 4\zeta^3\nabla^2\zeta\right] + \ldots \end{aligned} \tag{3.34}$$

$$\begin{aligned} = & -4\nabla^2\zeta^{(1)} + \left(\nabla\zeta^{(1)}\right)^2\left[-2 - \frac{24}{5}f_{\rm NL}\right] + \zeta^{(1)}\nabla^2\zeta^{(1)}\left[-\frac{24}{5}f_{\rm NL} + 8\right] + \\ & + \zeta^{(1)}\left(\nabla\zeta^{(1)}\right)^2\left[\frac{216}{25}g_{\rm NL} + \frac{24}{5}f_{\rm NL} + 4\right] + \zeta^{(1)2}\nabla^2\zeta^{(1)}\left[-\frac{108}{25}g_{\rm NL} - 8 + \frac{48}{5}f_{\rm NL}\right] + \\ & + \zeta^{(1)2}\left(\nabla\zeta^{(1)}\right)^2\left[-\frac{1296}{125}h_{\rm NL} + \frac{324}{125}g_{\rm NL} + \frac{72}{25}f_{\rm NL}^2 + \frac{12}{5}f_{\rm NL} - 4\right] + \\ & + \zeta^{(1)3}\nabla^2\zeta^{(1)}\left[-\frac{432}{125}h_{\rm NL} + \frac{288}{25}g_{\rm NL} + \frac{144}{25}f_{\rm NL}^2 - \frac{96}{5}f_{\rm NL} + \frac{16}{3}\right] + \ldots \end{aligned} \tag{3.35}$$

We expand δ up to fourth order

$$\delta = \delta^{(1)} + \frac{1}{2}\delta^{(2)} + \frac{1}{6}\delta^{(3)} + \frac{1}{24}\delta^{(4)} + \ldots \tag{3.36}$$

and substituting (3.35) into (3.31) yields for each order

$$\delta_h^{(1)} = D_+(\eta)\frac{1}{10\mathscr{H}_{\rm IN}^2 D_{+\rm IN}}\left(-4\nabla^2\zeta^{(1)}\right) \tag{3.37}$$

$$\begin{aligned} \frac{1}{2}\delta_h^{(2)} = & D_+(\eta)\frac{1}{10\mathscr{H}_{\rm IN}^2 D_{+\rm IN}} \\ & \frac{24}{5}\left[-\left(\nabla\zeta^{(1)}\right)^2\left(\frac{5}{12} + f_{\rm NL}\right) + \zeta^{(1)}\nabla^2\zeta^{(1)}\left(\frac{5}{3} - f_{\rm NL}\right)\right] \end{aligned} \tag{3.38}$$

$$\begin{aligned} \frac{1}{6}\delta_h^{(3)} = & D_+(\eta)\frac{1}{10\mathscr{H}_{\rm IN}^2 D_{+\rm IN}} \\ & \frac{108}{25}\left[\zeta^{(1)}\left(\nabla\zeta^{(1)}\right)^2 2\left(g_{\rm NL} + \frac{5}{9}f_{\rm NL} + \frac{25}{54}\right) + \right. \\ & \left. + \zeta^{(1)2}\nabla^2\zeta^{(1)}\left(-g_{\rm NL} - \frac{50}{27} + \frac{20}{9}f_{\rm NL}\right)\right] \end{aligned} \tag{3.39}$$

$$
\begin{aligned}
\frac{1}{24}\delta_h^{(4)} =&D_+(\eta)\frac{1}{10\mathscr{H}_{\rm IN}^2 D_{+\rm IN}} \\
&\frac{432}{125}\left[\zeta^{(1)3}\nabla^2\zeta^{(1)}\left(-h_{\rm NL}+\frac{10}{3}g_{\rm NL}+\frac{5}{3}f_{\rm NL}^2-\frac{50}{9}f_{\rm NL}+\frac{125}{81}\right)+\right.\\
&\left.+\zeta^{(1)2}\left(\nabla\zeta^{(1)}\right)^2 3\left(-h_{\rm NL}+\frac{1}{4}g_{\rm NL}+\frac{5}{18}f_{\rm NL}^2+\frac{25}{108}f_{\rm NL}-\frac{125}{324}\right)\right], \quad (3.40)
\end{aligned}
$$

where in a general ΛCDM model D_+ can be expressed as $D_+ = \frac{5}{2}\frac{\mathscr{H}_{\rm IN}^2 D_{+\rm IN}}{\mathscr{H}^2\left(f_1(\Omega_m)+\frac{3}{2}\Omega_m\right)}$ and f is the standard grow factor:

$$
f = \frac{D_+'}{\mathscr{H}D_+}. \quad (3.41)
$$

Eq. (3.38) is exactly the same solution for the homogeneous part of δ as in [26]. The third and fourth order homogeneous solution (3.39) are new results.

Following pioneering work on second-order perturbations in the nineties [28, 78, 98, 31], other second-order solutions have been provided by [105] and [109], cf. also [58], of which the homogeneous part is in accordance with the solution presented here. Solutions up to third order have been derived in [118] using a different gauge, cf. also [59, 100].

Furthermore, we are interested in the peaks of the density contrast, thus, we may focus on terms involving $\nabla^2\zeta$ as $\nabla\zeta$ vanishes for extremal values. At second order, the amplitude is decreased by $f_{\rm NL}^{\rm GR} = -\frac{5}{3}$ (cf. [27]), in third order by $g_{\rm NL}^{\rm GR} = \frac{50}{27} - \frac{20}{9}f_{\rm NL}$, and in fourth order by $h_{\rm NL}^{\rm GR} = -\frac{10}{9}g_{\rm NL} - \frac{5}{9}f_{\rm NL}^2 + \frac{50}{27}f_{\rm NL} - \frac{125}{81}$.

It is remarkable that at third order $f_{\rm NL}$ contributes to the non-Gaussianity of the density field and at fourth order, additional contributions appear involving $f_{\rm NL}^2$ and $g_{\rm NL}$. The reason becomes quite obvious, when we look at the series expansion of the spatial Ricci scalar R (3.18). The third order terms comprise combinations of third and zeroth order, or first and second order. The second order terms contain the non-Gaussianity $f_{\rm NL}$ and consequently the combination of first and second order contributes an $f_{\rm NL}$ term to the third order result. The fourth order term, on the other hand, comprises combinations of first and third order, second order squared, or two first order and one second order terms, which results in the mixed non-Gaussian terms.

3.6 The curvature perturbation ζ and the scalar potential ψ: relating to the Poisson gauge

We now relate the first-order curvature perturbation $\zeta^{(1)}$ to first order gauge-invariant (GI) Bardeen potentials Φ and Ψ [8] and to the Poisson gauge metric variables ϕ_{P} and ψ_{P}.

The Ricci scalar $R^{(3)}$ of the comoving slicing, i.e. the slicing orthogonal to the irrotational fluid flow with four-velocity u^{α}, is a gauge invariant perturbation once we assume a flat FLRW background. We then have (see equation (107) in [25])

$$R^{(3)} = -a^{-2} 4\nabla^2 \left(\Psi + \mathscr{H} V_S\right) \tag{3.42}$$

with $V_S = v + \chi'$ being the GI velocity perturbation. Again, using a covariant approximation for a perfect fluid with equation of state parameter w, one can derive (see equation (127) in [25]):

$$0 = -a \left[3\mathscr{H}^2 \frac{1}{a^2 (1+w)} V_S - 2a^{-2} \left(\Psi' - \mathscr{H}\Phi\right)\right] \tag{3.43}$$

Combining, (3.42) and (3.43) and using that $\Psi = -\Phi$ yields

$$a^2 \delta^{(3)} R = -4\nabla^2 \left(\Psi + \mathscr{H} \frac{2}{3\mathscr{H}^2 (1+w)} \left(\Psi' - \mathscr{H}\Phi\right)\right) \tag{3.44}$$

$$= -4\nabla^2 \left[-\Phi - \frac{2}{3\mathscr{H} (1+w)} \left(\Phi' + \mathscr{H}\Phi\right)\right] \tag{3.45}$$

$$= -4\nabla^2 \left[-\Phi - \frac{2}{3 (1+w)} \left(\mathscr{H}^{-1}\Phi' + \Phi\right)\right] \tag{3.46}$$

We compare (3.46) with the first order part of (3.17):

$$-4\nabla^2 \zeta^{(1)} = -4\nabla^2 \left[-\Phi - \frac{2}{3 (1+w)} \left(\mathscr{H}^{-1}\Phi' + \Phi\right)\right] \tag{3.47}$$

$$\zeta^{(1)} = -\Phi - \frac{2}{3 (1+w)} \left(\mathscr{H}^{-1}\Phi' + \Phi\right) \tag{3.48}$$

which coincides with the definition of ζ_{BST} in [77].

In an Einstein-de Sitter universe, we have $\Phi' = 0$ and $w = 0$. We then obtain for (3.48)

$$\zeta^{(1)} = -\frac{5}{3}\Phi. \tag{3.49}$$

In Poisson gauge, the Bardeen potentials Φ and Ψ are expressed in terms of the scalar potentials $\phi_{\mathrm{P}}^{(1)}$ and $\psi_{\mathrm{P}}^{(1)}$ as follows:

$$\Phi = \phi_{\mathrm{P}}^{(1)} \quad \text{and} \quad \Psi = -\psi_{\mathrm{P}}^{(1)}. \tag{3.50}$$

For the scalar potentials, the line element reads

$$ds^2 = a^2 \left[-\left(1 + 2\phi_{\mathrm{P}}^{(1)}\right) d\eta^2 + \left(1 - 2\psi_{\mathrm{P}}^{(1)}\right) \delta_{ij} dx^i dx^j \right]. \tag{3.51}$$

Therefore, equation (3.49) becomes

$$\zeta^{(1)} = -\frac{5}{3} \psi_{\mathrm{P}}^{(1)}. \tag{3.52}$$

In the main body of the chapter, we have used the synchronous-comoving gauge. A general metric in the synchronous-comoving gauge reads

$$ds^2 = a^2 \left\{ -d\eta^2 + \left[(1 - 2\psi_{\mathrm{S}}) \delta_{ij} + \chi_{\mathrm{Sij}} \right] dx^i dx^j \right\} \tag{3.53}$$

with $\chi_{ij} = \left(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\chi$.

The following calculations of this subsection are my work.

At first-order, the metric (3.53) is related to the metric (3.11), which we used in this paper, via

$$e^{2\zeta^{(1)}} \check{\gamma}_{ij} = \left(1 - 2\psi_{\mathrm{S}}^{(1)}\right) \delta_{ij} + \chi_{\mathrm{Sij}}^{(1)} \tag{3.54}$$

$$\left(1 + 2\zeta^{(1)}\right) \delta_{ij} + e^{2\zeta^{(1)}} \check{\alpha}_{,ij}^{(1)} = \left[1 - 2\left(\psi_{\mathrm{S}}^{(1)} + \frac{1}{6}\nabla^2 \chi_{\mathrm{S}}^{(1)} \right) \right] \delta_{ij} + \chi_{\mathrm{S,ij}}^{(1)} \tag{3.55}$$

$$2\zeta^{(1)} \delta_{ij} + \check{\alpha}_{,ij}^{(1)} = -2\left(\psi_{\mathrm{S}}^{(1)} + \frac{1}{6}\nabla^2 \chi_{\mathrm{S}}^{(1)} \right) \delta_{ij} + \chi_{\mathrm{S,ij}}^{(1)} \tag{3.56}$$

$$\Rightarrow \quad \zeta^{(1)} = -\left(\psi_{\mathrm{S}}^{(1)} + \frac{1}{6}\nabla^2 \chi_{\mathrm{S}}^{(1)} \right) = -\mathscr{R}_c \tag{3.57}$$

with $\mathscr{R}_c$ being the comoving curvature perturbation and $\check{\alpha}^{(1)} = \chi_{\mathrm{S}}^{(1)}$ at first order. Using the gradient expansion approximation, equation (3.57) becomes

$$\zeta^{(1)} = -\psi_{\mathrm{S}}^{(1)}. \tag{3.58}$$

To confirm the relation between $\psi_{\mathrm{P}}^{(1)}$ and $\psi_{\mathrm{S}}^{(1)}$ via $\zeta^{(1)}$ from equation (3.52) and (3.57), we perform a gauge transformation of $\psi^{(1)}$ from Poisson gauge to synchronous-comoving gauge:

$$\phi_{\rm P}^{(1)} = \alpha'_{\rm PS} + \mathscr{H}\alpha_{\rm PS}, \tag{3.59}$$

$$\psi_{\rm P}^{(1)} = \psi_{\rm S}^{(1)} - \frac{1}{3}\nabla^2\beta_{\rm PS} - \mathscr{H}\alpha_{\rm PS} \tag{3.60}$$

$$\text{with} \quad \alpha_{\rm PS} = \beta'_{\rm PS} = -\frac{1}{2}\chi_{\rm S}^{(1)\prime} \tag{3.61}$$

In [26], the first-order scalar potential χ is expressed in terms of the density contrast $\delta^{(1)}$. For an Einstein-de Sitter universe, we obtain the following relation:

$$\chi_{\rm S}^{(1)} = -2\nabla^{-2}\delta_{\rm S}^{(1)} = -\frac{\eta^2}{5}\mathscr{R}_c \tag{3.62}$$

with $\mathscr{H} = \frac{2}{\eta}$. Substituting equation (3.62) into equation (3.60) yields

$$\psi_{\rm P}^{(1)} = \psi_{\rm S}^{(1)} - \frac{1}{3}\nabla^2\beta_{\rm PS} - \mathscr{H}\alpha_{\rm PS} \tag{3.63}$$

$$= \psi_{\rm S}^{(1)} + \frac{1}{6}\nabla^2\chi_{\rm S}^{(1)} + \frac{2}{\eta}\frac{1}{2}\left(-\frac{2\eta}{5}\mathscr{R}_c\right) \tag{3.64}$$

$$= \psi_{\rm S}^{(1)} + \frac{1}{6}\nabla^2\chi_{\rm S}^{(1)} - \frac{2}{5}\mathscr{R}_c \tag{3.65}$$

$$= \psi_{\rm S}^{(1)} + \frac{1}{6}\nabla^2\chi_{\rm S}^{(1)} - \frac{2}{5}\left(\psi_{\rm S}^{(1)} + \frac{1}{6}\nabla^2\chi_{\rm S}^{(1)}\right) \tag{3.66}$$

$$= \frac{3}{5}\left(\psi_{\rm S}^{(1)} + \frac{1}{6}\nabla^2\chi_{\rm S}^{(1)}\right) \tag{3.67}$$

Within the approximation of the gradient expansion, equation (3.67) simplifies to

$$\psi_{\rm P}^{(1)} = \frac{3}{5}\psi_{\rm S}^{(1)}, \tag{3.68}$$

which is in accordance with equations (3.58) and (3.52).

3.7 Long and short wavelength split

Equations (3.69) - (3.71) in this section are my responsiblity.

In the ΛCDM model, we assume that galaxies evolve in virialised dark matter halos. The halos collapse once the matter density field reaches a critical value. This matter density is determined by the spatial amplitude, $C(\mathbf{x})$, in particular by the nonlinear, spatial Ricci scalar R, which comprises of spatial derivatives of ζ. While we don't address here issues related to the halo density, we derive formulas for the matter density field, performing a peak-background split, where we decompose ζ into a longer-wavelength modes ζ_l and shorter-wavelength modes ζ_s using $\zeta^{(1)} = \zeta_s + \zeta_l$ [27]. The short wavelength mode represents modes attributed to local peak formation, whereas the long wavelength modes are assumed to be absorbed into the background. We already did a gradient expansion and by that we are limiting our analysis to large scale wavelengths, $\lambda > \lambda_{\text{min}}$. In the peak-background split, the gradient of the shorter wavelength modes $\left(\lambda_{\text{min}} < \lambda_s < \lambda_{\text{split}}\right)$ still remains small and the gradient of the long wavelength modes $(\lambda_l > \lambda_{\text{split}})$ is small enough to be neglected.

Hence, the series expansion of the Ricci scalar (3.35) up to fourth order simplifies to

$$\begin{aligned} R \simeq & -4\nabla^2\zeta_s - \frac{24}{5}\left(\nabla\zeta_s\right)^2\left(f_{\text{NL}} + \frac{5}{12}\right) - \\ & - \left(\zeta_s+\zeta_l\right)\nabla^2\zeta_s\frac{24}{5}\left(f_{\text{NL}} - \frac{5}{3}\right) - \\ & - \left(\zeta_s+\zeta_l\right)\left(\nabla\zeta_s\right)^2\frac{216}{25}\left(g_{\text{NL}} - \frac{5}{9}f_{\text{NL}} - \frac{25}{54}\right) - \\ & - \left(\zeta_s+\zeta_l\right)^2\nabla^2\zeta_s\frac{108}{25}\left(g_{\text{NL}} - \frac{20}{9}f_{\text{NL}} + \frac{50}{27}\right) - \\ & - \left(\zeta_s+\zeta_l\right)^2\left(\nabla\zeta_s\right)^2\frac{1296}{125}\left(h_{\text{NL}} - \frac{1}{4}g_{\text{NL}} - \frac{5}{18}f_{\text{NL}}^2 - \frac{25}{108}f_{\text{NL}} - \frac{125}{324}\right) - \\ & - \left(\zeta_s+\zeta_l\right)^3\nabla^2\zeta_s\frac{432}{125}\left(h_{\text{NL}} - \frac{10}{3}g_{\text{NL}} - \frac{5}{3}f_{\text{NL}}^2 - \frac{50}{9}f_{\text{NL}} + \frac{125}{81}\right) \end{aligned} \tag{3.69}$$

Substituting this result (3.69) into the expression for the density contrast (3.31), where we use (3.32) for the spatial function $C(\mathbf{x})$, gives

$$\begin{aligned} \delta = & \frac{1}{\left(f_1\left(\Omega_m\right) + \frac{3}{2}\Omega_m\right)\mathscr{H}^2}\left[-\nabla^2\zeta_s - \frac{6}{5}\left(\nabla\zeta_s\right)^2\left(f_{\text{NL}} + \frac{5}{12}\right) - \right. \\ & - \left(\zeta_s+\zeta_l\right)\nabla^2\zeta_s\frac{6}{5}\left(f_{\text{NL}} - \frac{5}{3}\right) - \\ & - \left(\zeta_s+\zeta_l\right)\left(\nabla\zeta_s\right)^2\frac{54}{25}\left(g_{\text{NL}} - \frac{5}{9}f_{\text{NL}} - \frac{25}{54}\right) - \\ & - \left(\zeta_s+\zeta_l\right)^2\nabla^2\zeta_s\frac{27}{25}\left(g_{\text{NL}} - \frac{20}{9}f_{\text{NL}} + \frac{50}{27}\right) - \\ & - \left(\zeta_s+\zeta_l\right)^2\left(\nabla\zeta_s\right)^2\frac{54}{125}\left(h_{\text{NL}} - \frac{1}{4}g_{\text{NL}} - \frac{5}{18}f_{\text{NL}}^2 - \frac{25}{108}f_{\text{NL}} - \frac{125}{324}\right) - \\ & \left. - \left(\zeta_s+\zeta_l\right)^3\nabla^2\zeta_s\frac{18}{125}\left(h_{\text{NL}} - \frac{10}{3}g_{\text{NL}} - \frac{5}{3}f_{\text{NL}}^2 - \frac{50}{9}f_{\text{NL}} + \frac{125}{81}\right) + \dots\right] \end{aligned} \tag{3.70}$$

$$= \delta^{(1)} + \frac{1}{2}\delta^{(2)} + \frac{1}{6}\delta^{(3)} + \frac{1}{24}\delta^{(4)} + \dots \tag{3.71}$$

From the above, one can read off the different contributions to matter density field at different orders.

If, however, $f_{\mathrm{NL}} = g_{\mathrm{NL}} = h_{\mathrm{NL}} = 0$, the long-wavelength contribution ζ_l in (3.70) is derived from the series expansion of the exponential $e^{-2\zeta}$ in (3.17) [27]:

$$R = e^{-2\zeta}\left[-4\nabla^2\zeta - 2\left(\nabla\zeta\right)^2\right] \tag{3.72}$$

$$= e^{-2\zeta^{(1)}}\left[-4\nabla^2\zeta^{(1)} - 2\left(\nabla\zeta^{(1)}\right)^2\right] \tag{3.73}$$

$$= e^{-2(\zeta_l+\zeta_s)}\left[-4\nabla^2\zeta_s - 2\left(\nabla\zeta_s\right)^2\right] + \mathscr{O}\left(\nabla\zeta_l\right) \tag{3.74}$$

$$\approx e^{-2\zeta_l}R_s \tag{3.75}$$

with $R_s = e^{-2\zeta_s}\left[-4\nabla^2\zeta_s - 2\left(\nabla\zeta_s\right)^2\right]$. In this case, the forefactor $e^{-2\zeta_l}$ can be absorbed into the background scale factor

$$a \to a_l = e^{\zeta_l}a. \tag{3.76}$$

Note that we perform the split into long and short wavelengths after expanding ζ in terms of $\zeta^{(1)}$ using eq. (3.33). As a consequence, the absorption of the long mode curvature perturbation ζ_l into the scale factor in eq. (3.76) is only valid for $f_{\mathrm{NL}} = g_{\mathrm{NL}} = h_{\mathrm{NL}} = 0$.

3.8 Relation between Newtonian and relativistic non-Gaussianities in the matter-dominated era

The calculations in this section have been performed by me.

We now want to relate our relativistic results, obtained with the gradient expansion, with the local-type primordial non-Gaussianity described in a Newtonian fashion, generalising the results in [27]. We now focus on the matter-dominated era, assuming therefore $f = \Omega_m = 1$ and $\bar{\rho} = \frac{3}{\kappa}\mathscr{H}^2$. First, we use the standard expansion for the Newtonian potential [65]:

$$\phi_{\mathrm{N}} = \underbrace{\phi_1}_{\phi^{(1)}} + \underbrace{f_{\mathrm{NL}}^{\mathrm{N}}\left(\phi_1^2 - \langle\phi_1^2\rangle\right)}_{\frac{1}{2}\phi^{(2)}} + \underbrace{g_{\mathrm{NL}}^{\mathrm{N}}\phi_1^3}_{\frac{1}{6}\phi^{(3)}} + \underbrace{h_{\mathrm{NL}}^{\mathrm{N}}\left(\phi_1^4 - \langle\phi_1^4\rangle\right)}_{\frac{1}{24}\phi^{(4)}} + \dots \tag{3.77}$$

Note that $f_{\mathrm{NL}}^{\mathrm{N}}$, g_{NL}^{N}, and $h_{\mathrm{NL}}^{\mathrm{N}}$ do not refer to primordial non-Gaussianity such as f_{NL}, g_{NL}, and h_{NL}, respectively, but to non-Gaussianity in the Newtonian picture at some initial time in the matter dominated era. We now want to split the Newtonian potential into long and short wavelength modes $\phi_1 = \phi_s + \phi_l$, and substitute them into the Poisson equation. To this end, first consider its gauge-invariant first-order version in terms of the Bardeen potential Φ and the gauge-invariant density perturbation δ_{GI} [8]:

$$\nabla^2 \Phi = \frac{\kappa}{2} \bar{\rho}\, \delta_{\mathrm{GI}}, \tag{3.78}$$

where $\kappa = 8\pi G$. Given that Φ reduces to $\phi_{\mathrm{P}}^{(1)}$ in Poisson gauge and δ_{GI} reduces to $\delta_{\mathrm{S}}^{(1)}$ in synchronous-comoving gauge, we get[4] [8, 25, 26]:

$$\nabla^2 \phi_{\mathrm{N}} = -\nabla^2 \phi_{\mathrm{P}}^{(1)} = -\frac{\kappa}{2} \bar{\rho}\, \delta_{\mathrm{S}}^{(1)}, \tag{3.79}$$

Where the Newtonian potential ϕ_{N} can been clearly identified with the Poisson gauge metric perturbation ϕ_{P} when a post-Newtonian-like expansion is used [81, 89]. As discussed in section 3.4, the equations in the gradient expansion at leading order formally coincide with those of first-order perturbation theory, while including the homogeneous contributions at all orders. Therefore, we can assume that in this approximation the Poisson equation (3.79) relates ϕ and δ at all orders. It follows that

$$\begin{aligned}
\nabla^2 \phi_{\mathrm{N}}^{(1)} &= \nabla^2 \phi_1 = -\frac{\kappa}{2} a^2 \bar{\rho}\, \delta^{(1)}, && (3.80)\\
\frac{1}{2} \nabla^2 \phi_{\mathrm{N}}^{(2)} &= \nabla^2 f_{\mathrm{NL}}^{\mathrm{N}} \left(\phi_1^2 - \langle \phi_1^2 \rangle \right) = -\frac{\kappa}{2} a^2 \bar{\rho} \frac{1}{2} \delta^{(2)}, && (3.81)\\
\frac{1}{6} \nabla^2 \phi_{\mathrm{N}}^{(3)} &= g_{\mathrm{NL}}^{\mathrm{N}} \nabla^2 \phi_1^3 = -\frac{\kappa}{2} a^2 \bar{\rho} \frac{1}{6} \delta^{(3)}, \text{ and} && (3.82)\\
\frac{1}{24} \nabla^2 \phi_{\mathrm{N}}^{(4)} &= h_{\mathrm{NL}}^{N} \nabla^2 \left(\phi_1^4 - \langle \phi_1^4 \rangle \right) = -\frac{\kappa}{2} a^2 \bar{\rho} \frac{1}{24} \delta^{(4)}. && (3.83)
\end{aligned}$$

and subsequently, we omit the gradients of the long wavelength terms.

Second order:

Using the second-order part of equation (3.70) in equation (3.81) yields

$$f_{\mathrm{NL}}^{\mathrm{N}} \nabla^2 \left(\phi_1^2 - \langle \phi_1^2 \rangle \right) = \frac{\kappa}{2} \bar{\rho} \frac{6}{5} \frac{(\nabla \zeta_s)^2 \left(\frac{5}{12} + f_{\mathrm{NL}} \right) + (\zeta_s + \zeta_l) \nabla^2 \zeta_s \left(f_{\mathrm{NL}} - \frac{5}{3} \right)}{\mathscr{H}^2 \frac{5}{2}} \tag{3.84}$$

$$2 f_{\mathrm{NL}}^{\mathrm{N}} \left((\nabla \phi_s)^2 + (\phi_s + \phi_l) \nabla^2 \phi_s \right) = \frac{18}{25} a^{-2} \left[(\nabla \zeta_s)^2 \left(\frac{5}{12} + f_{\mathrm{NL}} \right) + (\zeta_s + \zeta_l) \nabla^2 \zeta_s \left(f_{\mathrm{NL}} - \frac{5}{3} \right) \right] \tag{3.85}$$

[4]See [112] for a discussion of different sign conventions.

In the matter-dominated era, the first-order scalar potential ϕ_1 is linearly related to the first order curvature perturbation ζ in equation (3.49), which reads for the longer and shorter wavelength modes [27]:

$$\phi_{1\,i} = \frac{3}{5}\zeta_i \quad \text{with the index } \; i = l, s. \tag{3.86}$$

In order to discuss our result and compare it to the Newtonian dynamics, we focus on the peaks of the metric perturbations and, therefore, omit the terms involving $(\nabla\zeta_s)^2$ (or $(\nabla\phi_s)^2$).

$$\frac{18}{25} f_{\mathrm{NL}}^{\mathrm{N}} (\zeta_s + \zeta_l) \nabla^2 \zeta_s = \frac{18}{25} (\zeta_s + \zeta_l) \nabla^2 \zeta_s \left(f_{\mathrm{NL}} - \frac{5}{3} \right) \tag{3.87}$$

$$f_{\mathrm{NL}}^{\mathrm{N}} = \left(f_{\mathrm{NL}} - \frac{5}{3} \right) \tag{3.88}$$

Equation (3.88) shows that the non-Gaussianity $f_{\mathrm{NL}}^{\mathrm{N}}$ derived in the Newtonian picture consists of the primordial non-Gaussianity f_{NL} and an additional term, which has its origin in the nonlinearity of General Relativity. Even if there is no primordial non-Gaussianity ($f_{\mathrm{NL}} = 0$), there remains an effective non-Gaussianity of magnitude $f_{\mathrm{NL}}^{\mathrm{N}} = -\frac{5}{3}$. (See also [27, 26])

Third order:

Analogously to (3.84), we combine the third-order part of equation (3.70) with (3.82) neglecting any terms involving $(\nabla\zeta_s)^2$:

$$3 g_{\mathrm{NL}}^{\mathrm{N}} (\phi_s + \phi_l)^2 \nabla^2 \phi_s = -\frac{\kappa}{2}\bar{\rho} \frac{1}{\frac{5}{2}\mathscr{H}^2} \left[-\frac{27}{25} (\zeta_s + \zeta_l)^2 \nabla^2 \zeta_s \left(g_{\mathrm{NL}} - \frac{20}{9} f_{\mathrm{NL}} + \frac{50}{27} \right) \right] \tag{3.89}$$

and use the relationship (3.86):

$$3 \frac{27}{125} g_{\mathrm{NL}}^{\mathrm{N}} (\zeta_s + \zeta_l)^2 \nabla^2 \zeta_s - \frac{3}{5}\frac{27}{25} (\zeta_s + \zeta_l)^2 \nabla^2 \zeta_s \left(g_{\mathrm{NL}} - \frac{20}{9} f_{\mathrm{NL}} + \frac{50}{27} \right) \tag{3.90}$$

$$g_{\mathrm{NL}}^{\mathrm{N}} = \left(g_{\mathrm{NL}} - \frac{20}{9} f_{\mathrm{NL}} + \frac{50}{27} \right), \tag{3.91}$$

which, analogously to the second order approach, is what we aimed to show. We see that we obtain the same non-Gaussian contribution as in (3.39).

Fourth order:

The recursive process above can be extended to arbitrarily large orders. As an example, here we use the fourth-order part of (3.70), substituting it into (3.83) neglecting any terms involving $(\nabla\zeta_s)^2$:

$$\begin{aligned}
&4h^{\mathrm{N}}_{\mathrm{NL}}\left(\phi_s+\phi_l\right)^3\nabla^2\phi_s = \\
&\quad = \frac{\kappa}{2}\bar{\rho}\frac{108}{125}\frac{\left(\zeta_s+\zeta_l\right)^3\nabla^2\zeta_s}{\frac{5}{2}\mathscr{H}^2}\left(h_{\mathrm{NL}}-\frac{10}{3}g_{\mathrm{NL}}-\frac{5}{3}f^2_{\mathrm{NL}}-\frac{50}{9}f_{\mathrm{NL}}+\frac{125}{81}\right)
\end{aligned} \tag{3.92}$$

Again we make us of (3.86)

$$\begin{aligned}
4\left(\frac{3}{5}\right)^4 h^{\mathrm{N}}_{\mathrm{NL}}\left(\zeta_s+\zeta_l\right)^3\nabla^2\zeta_s =& \frac{3}{5}\frac{108}{125}\left(\zeta_s+\zeta_l\right)^3\nabla^2\zeta_s\left(h_{\mathrm{NL}}-\frac{10}{3}g_{\mathrm{NL}}+\right. \\
&\left.-\frac{5}{3}f^2_{\mathrm{NL}}-\frac{50}{9}f_{\mathrm{NL}}+\frac{125}{81}\right) \qquad (3.93) \\
h^{\mathrm{N}}_{\mathrm{NL}} =& h_{\mathrm{NL}}-\frac{10}{3}g_{\mathrm{NL}}-\frac{5}{3}f^2_{\mathrm{NL}}-\frac{50}{9}f_{\mathrm{NL}}+\frac{125}{81} \qquad (3.94)
\end{aligned}$$

As in second and third order, we aimed to show that in comparison with the Newtonian gravitational dynamics, we obtain an effective non-Gaussian contribution even with Gaussian primordial initial conditions. $(f_{\mathrm{NL}}=g_{\mathrm{NL}}=h_{\mathrm{NL}}=0)$

3.9 Conclusions

In this chapter we have investigated non-Gaussian contributions to the density field at very large scales, extending the results of [27] and [26] up to fourth order in standard perturbation theory, within the regime and at the leading order of the gradient expansion (aka long wavelength approximation). At second order, our result agrees with the result of [26], [105], and [109] (cf. also [58] for other second order results). At third and fourth order, our solutions are new (other third order results, using a different gauge, were derived by [118], cf. also [59, 100]) and have been published in [55].

By performing a gradient expansion, we only consider very large scales of the order of the Hubble radius, at which the spatial gradients are negligible with respect to the time derivatives. We consider spatial gradient up to $\mathscr{O}\left(\nabla^2\right)$, thus including expressions linear in the density contrast, δ, the inhomogeneous expansion, ϑ, etc.. In this regime, the evolution equation for the density contrast takes the same form as the first order equation using standard perturbation theory. In particular, the density contrast δ, as well as the expansion ϑ are of $\mathscr{O}(\nabla^2)$, thus any squared term or combination of the two quantities is negligible. At these scales the spatial Ricci scalar remains constant. Using the synchronous-comoving gauge, it is a valid approximation to assume a conformal flat spatial metric, i. e. to neglect anisotropic metric perturbations on these scales. As a consequence, the spatial Ricci scalar can be written as a series expansion.

The evolution equation for the density contrast in the gradient expansion is effectively equivalent to the first order standard perturbation theory evolution equation. Therefore, the same ansatz for δ, in which the density contrast is split into a time and a space dependent part, can be used. The spatial amplitude $C(x^i)$ is proportional to the spatial Ricci scalar, and thereby determined by its nonlinearity.

Our solution is the homogeneous solution of [26], [105], and [109]. In the gradient expansion we neglect the terms that source the particular solution.

We show how non-Gaussianity contributes to third and fourth order of SPT within the regime of the gradient expansion. At third order, we obtain terms of order $\mathcal{O}(3)$, $\mathcal{O}(1)\,\mathcal{O}(2)$, and $\mathcal{O}(1)\,\mathcal{O}(1)\mathcal{O}(1)$ in the density contrast. Naturally, the combinations $\mathcal{O}(3)$ and $\mathcal{O}(1)\,\mathcal{O}(2)$ will involve terms with $g_{\rm NL}$ and $f_{\rm NL}$, respectively. Hence, at this order, both $f_{\rm NL}$ and $g_{\rm NL}$ contribute to the density contrast. At fourth order, we obtain terms containing $h_{\rm NL}$, $g_{\rm NL}$, $f_{\rm NL}^2$, and $f_{\rm NL}$ in the density contrast.

We should keep in mind that for terms of order $\mathcal{O}(3)$ or higher, our homogeneous solution is not the only contribution in the full solution containing $f_{\rm NL}$ terms and other non-Gaussianities. At third order, the particular solution is sourced by terms containing the second order density contrast, which involves $f_{\rm NL}$. At fourth order, the source terms for the particular solution will contain the second and third order density contrast and, thus, terms containing both $f_{\rm NL}$ and $g_{\rm NL}$.

Futhermore, we perform a peak-bachground split by decomposing $\zeta^{(1)}$ into longer-wavelenth modes ζ_l and shorter-wavelenth modes ζ_s following the work of [27] and aim to compare our relativistic result to local-type non-Gaussianity using Newtonian dynamics in the matter dominated era. We show how the mixing of the non-Gaussian parameters $f_{\rm NL}$, $g_{\rm NL}$, and $h_{\rm NL}$ translates into a Newtonian treatment.

In summary, we compute the nonlinear, relativistic contributions in the density field at higher orders at very large scales. In addition we impose initial conditions involving primordial non-Gaussianity up to fourth order. In this context, we see that the nonlinear nature of GR generates both effective non-Gaussian terms and a mixing of the primordial non-Gaussian parameters $f_{\rm NL}$, $g_{\rm NL}$, and $h_{\rm NL}$ at higher orders.

Our results should be relevant in the discussion of higher-order contributions to observables [31, 117], e.g. for higher-order statistics such as the bispectrum, cf. [118, 100]. In addition, they may help in setting initial conditions - and extract relativistic effects - from simulations of the growth of large scale structure in cosmology, both Newtonian, cf. [33, 103] (see also [110]) and [49] (and references therein), and in full numerical relativity [50, 15, 74, 14, 51], cf. also [2, 39, 44]. In turn, fully general relativistic simulations will help to establish the range of validity of higher-order standard perturbation theory and of the long-wavelength approximation we used in this paper, as well as other nonlinear relativistic approximations such as the post-Friedmann scheme [81, 89, 33], see also [102, 103, 101], and other approximations [30, 57, 30]. We leave all of this for future work.

Chapter 4

Full-sky and full scale weak lensing analysis with the Post-Friedmann approximation

This chapter provides the basis for the publication [54].

4.1 Motivation

Until now, cosmic shear surveys have covered only parts of the sky, see for example DES [1] with a coverage of around 5000 square degrees. But future surveys such as Euclid and LSST will deliver high precision data of large sky areas. With this vast amount of high precision data, weak lensing is becoming more and more a promising tool to map the universe.

Yet, the WL analysis is challenging; while we use different approximation schemes for large and small scales, by its own nature weak lensing necessarily involves different scales, because by integrating along the light path, large and small scales couple. E.g. two galaxies are far apart but aligned along the line of sight. The two galaxies act as consecutive lenses and thereby couple large and small scales. Furthermore, the majority of the high precision data from these surveys will be data from smaller, nonlinear scales. It is standard to use Newtonian dynamics for the structure formation on small scales, but with e.g. Euclid aiming at 1% accuracy, is the Newtonian treatment still sufficient or do relativistic effects come into play? In this chapter, we will use a different approximation scheme, the post-Friedmann (PF) formalism [80], which is a post-Newtonian-type approximation scheme in a cosmological setting that combines both the Newtonian treatment on small, nonlinear scales and the relativistic analysis on large scales. Therefore, it seems to be the ideal approximation for a thorough weak lensing analysis. Furthermore, the PF formalism provides an apt framework for n-body simulations with relativistic corrections [102, 33, 103].

With the coming full-sky surveys in mind, we need to go beyond the small-angle approximation, which restricts to angular scales much smaller than the angular diameter distance at the source. On these angular scales relativistic effects as well as fluctuations of the gravitational potentials along the line of sight are neglected. The dominant contributions come from derivatives of the gravitational potentials transverse to the line of sight [19]. In order to perform a full-sky weak lensing analysis, we follow the work of [17, 23], in which the distortions of the galaxy images are projected onto a spherical screen space and the shear and convergence are expressed in terms of spherical spin operators.

The outline of this chapter is the following: in section 4.2, we derive the magnification matrix. Starting from the geodesic deviation equation, we derive the Jacobi mapping up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ with the PF formalism, thereby including both leading order $\frac{1}{c^3}$ gravimagnetic contributions and the first relativistic corrections to the scalar contributions at order $\mathscr{O}\left(\frac{1}{c^4}\right)$. Furthermore, we present the Jacobi mapping in terms of the redshift z instead of the affine parameter χ and add redshift perturbations up to the required order. In section 4.3, we change to spherical coordinates and spin operators. Furthermore, we split the Jacobi mapping into two functions according to their rotational symmetry s. Thereby, we map the distortions of the galaxy images onto a sphere instead of a plane. This circumvents the thin-lens approximation and offers the possibility of a full-sky analysis. Furthermore, we extract the reduced shear and convergence from the spin-weighted functions we derived from the Jacobi mapping. In the next section 4.4, we show that if we take the limit for large scales within the PF framework, we retain the first-order convergence and reduced shear in standard perturbation theory (SPT).

4.2 Derivation of the Magnification Matrix

In weak lensing we study the distortion of light through gravitational masses. We are interested how a light bundle changes, when it is propagated [95, 111, 92, 96, 69, 107]. We consider two neighbouring geodesics $x^\mu(\lambda)$ and $y^\mu(\lambda) = x^\mu(\lambda) + \xi^\mu(\lambda)$, which start at $\lambda = 0$ at the observer O. The relative acceleration between the two geodesics is expressed by the geodesic deviation equation

$$\frac{D^2\xi^\mu}{D\lambda^2} = R^\mu_{\nu\alpha\beta}\xi^\beta k^\nu k^\alpha, \tag{4.1}$$

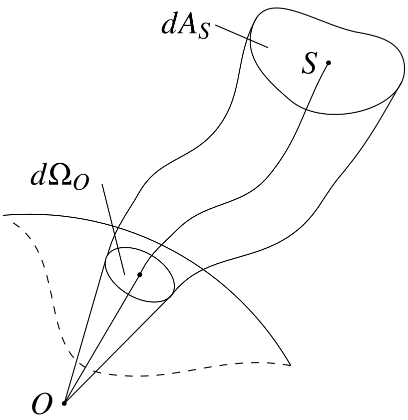

Figure 4.1 The surface dA_S is related to the solid angle $d\Omega_O$ at the observer O.

where $k^\mu = \frac{dx^\mu}{d\lambda}$ denotes the tangent vector to the congruence of light rays. Let us assume that a light beam is emitted at the source S and measured at the observer O. Furthermore, v^μ_O denotes the 4-velocity of the observer in O. We define an orthonormal spacelike basis with n^μ_a with $a = 1, 2$, which is orthogonal to k^μ and to u^μ_O. It is standard to refer to the two dimensional space spanned by n^μ_a as the *screen space*. Thus, we can create a basis with $\{n^\mu_1, n^\mu_2, k^\mu, u^\mu_O\}$, which is parallel transported along the geodesic:

$$\frac{Dn^\mu_a}{D\lambda} = 0 \quad \text{and} \quad \frac{Du^\mu_O}{D\lambda} = 0. \tag{4.2}$$

The deviation vector expressed in this basis reads

$$\xi^\mu = \xi^a_{\mathbf{n}} n^\mu_a + \xi_{\mathbf{k}} k^\mu + \xi_{\mathbf{u}} u^\mu_O, \tag{4.3}$$

with $\xi^a(0) = 0$, $\xi_k(0) = 0$, and $\xi_v(0) = 0$. If we substitute (4.3) into (4.1) and express the result in terms of the basis $\{n^\mu_1, n^\mu_2, k^\mu, v^\mu_O\}$, it follows from $\xi^\mu k_\mu = 0$ that $\xi_{\mathbf{u}} = 0$ and that the relative acceleration has only contributions on the spacelike basis n^μ_a [17, 92]:

$$\frac{D^2\xi^c}{D\lambda^2} = \mathscr{R}^c{}_a\xi^a, \text{ with } \mathscr{R}^c{}_a = R^\mu{}_{\nu\alpha\beta}k^\nu k^\alpha n^c_\mu n^\beta_a. \tag{4.4}$$

Let θ^b_O be the vectorial angle between the two neighbouring geodesics at the observer O. We can assume that $|\theta|$ is small enough such that ξ^a can be linearised in terms of θ^b_O:

$$\xi^a_n = \mathscr{D}^a_b\theta^b_O \tag{4.5}$$

with $\theta_O^b \equiv \frac{d\xi^b}{d\lambda}\Big|_{\lambda=0}$. The matrix $\mathscr{D}_{ab}$ denotes the linear Jacobi mapping, which relates the angle θ_O^b between the two neighbouring geodesics at the observer O to the distance ξ^c between the geodesics at the source S.

We substitute (4.5) into the geodesic deviation equation (4.4) projected onto the screen space n_a^μ and obtain [17, 96, 69, 107, 56]

$$\frac{d^2}{d\lambda^2}\mathscr{D}_{ab} = \mathscr{R}_{ac}\mathscr{D}^c{}_b \tag{4.6}$$

with $\mathscr{D}_{ab}(0) = 0$ and $\frac{d\mathscr{D}_{ab}}{d\lambda}\Big|_{\lambda=0} = \delta_{ab}$. The affine parameter λ is a perturbative quantity. In order to take these perturbations into account, we rewrite the evolution equation (4.6) in terms of the unperturbed parameter χ, defined as $\chi \equiv c\,(\eta_O - \eta)$, where η denotes conformal time. Note that we choose our time coordinate as $x^0 = c\,(\eta_O - \eta) = \chi$ (see section 4.2.1 for a more detailed discussion). As a consequence $d\chi/d\lambda = dx^0/d\lambda = k^0$. The total derivative with respect to λ transforms then into

$$\frac{d}{d\lambda} = \frac{d\chi}{d\lambda}\frac{d}{d\chi} = k^0\frac{d}{d\chi}\,. \tag{4.7}$$

Substituting (4.7) into (4.6) we obtain

$$\frac{d^2}{d\chi^2}\mathscr{D}_{ab} + \frac{1}{k^0}\frac{dk^0}{d\chi}\frac{d}{d\chi}\mathscr{D}_{ab} = \frac{1}{(k^0)^2}\mathscr{R}_{ac}\mathscr{D}_{cb}\,. \tag{4.8}$$

To solve eq. (4.8), we need to calculate k^0 and $\mathscr{R}_{ab}$ and solve the equation order by order in powers of $1/c$. The calculation can be simplified by using the fact that null geodesics are not affected by conformal transformations. As a consequence, the calculation can be performed without the Friedmann expansion, i.e. for the metric ds^2 defined through

$$d\tilde{s}^2 = a^2 ds^2\,, \tag{4.9}$$

where $d\tilde{s}^2$ is the line element associated to the metric (2.77)-(2.80). The effect of the expansion can then simply be taken into account at the end by rescaling the mapping by the conformal factor [17]

$$\tilde{\mathscr{D}}_{ab}(\chi_S) = a(\chi_S)\mathscr{D}_{ab}(\chi_S)\,, \tag{4.10}$$

where $\mathscr{D}_{ab}$ denotes the Jacobi mapping for the metric ds^2, and $\tilde{\mathscr{D}}_{ab}$ is the expression for the metric $d\tilde{s}^2$.

The matrix $\tilde{\mathscr{D}}_{ab}(\chi_S)$ represents the Jacobi mapping for sources situated at constant conformal time χ_S. However, observationally we select sources at constant redshift z_S. Since the observed redshift is itself affected by perturbations, $z_S = \bar{z}_S + \delta z_S$, where $1+\bar{z}_S = 1/a_S$, this will modify the expression of the Jacobi mapping [1]. In particular, we can write

$$\begin{aligned}\tilde{\mathscr{D}}_{ab}(\chi_S) &= \tilde{\mathscr{D}}_{ab}(\chi_S(\bar{z}_S)) = \tilde{\mathscr{D}}_{ab}(\bar{z}_S) = \tilde{\mathscr{D}}_{ab}(z_S - \delta z_S) \\ &= \tilde{\mathscr{D}}_{ab}(z_S) - \frac{d}{dz_S}\tilde{\mathscr{D}}_{ab}(z_S)\,\delta z_S + \frac{1}{2}\frac{d^2}{dz_S^2}\tilde{\mathscr{D}}_{ab}(z_S)\,\delta z_S^2 \\ &- \frac{1}{3!}\frac{d^3}{dz_S^3}\tilde{\mathscr{D}}_{ab}(z_S)\,\delta z_S^3 + \frac{1}{4!}\frac{d^4}{dz_S^4}\tilde{\mathscr{D}}_{ab}(z_S)\,\delta z_S^4 + \mathscr{O}\left(\frac{1}{c^5}\right),\end{aligned} \tag{4.11}$$

where in the second and third lines the matrix $\tilde{\mathscr{D}}_{ab}(z_S)$ and its derivatives [2] are formally given by eqs. (4.8) and (4.10) where χ_S can now be interpreted as $\chi(z_S)$ and $1+\bar{z}_S$ can be replaced by $1+z_S$.

The Jacobi mapping $\tilde{\mathscr{D}}_{ab}(z_S)$ is usually decomposed into a convergence κ, a shear $\gamma = \gamma_1 + i\gamma_2$ and a rotation ω

$$\tilde{\mathscr{D}}_{ab} = \frac{\chi(z_S)}{1+z_S}\begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2-\omega \\ -\gamma_2+\omega & 1-\kappa+\gamma_1 \end{pmatrix}. \tag{4.12}$$

The prefactor $\chi(z_S)/(1+z_S)$ represents the magnification of images due to the background expansion of the Universe, for sources situated at the observed redshift z_S. The convergence κ denotes the magnification or demagnification of images due to perturbations. The shear γ is the trace-free, symmetric part of $\tilde{\mathscr{D}}_{ab}$ and refers to the change in the shape. The rotation ω, the antisymmetric part of $\tilde{\mathscr{D}}_{ab}$, represents a rotation without any change in the shape. Note that what we observe when we measure the ellipticity of galaxies is not directly the shear γ but rather the reduced shear g which is the ratio of the anisotropic and isotropic deformations [9, 17, 63]

$$g \equiv \frac{\gamma}{1-\kappa}. \tag{4.13}$$

The rotation ω does in principle contribute to ellipticity orientation (see [17]). However, we will see that the rotation is of order $\mathscr{O}\left(\frac{1}{c^4}\right)$ and contributes consequently to the

[1] Note that we normalise the scale factor to 1 today: $a_O = 1$.

[2] The derivatives in eq. (4.11) are formally given by $d^n\tilde{D}_{ab}(\bar{z}_S)/d\bar{z}_S^n\Big|_{\bar{z}_S=z_S}$

ellipticity at the order $\mathscr{O}\left(\frac{1}{c^6}\right)$. Therefore our ellipticity measurement is dominated by the reduced shear.

Following [17], the convergence, shear and rotation can be expressed in terms of the spin-0 and spin-2 components of the Jacobi mapping $\tilde{\mathscr{D}}_{ab}$

$$ {}_0\tilde{\mathscr{D}} \equiv \tilde{\mathscr{D}}_{11} + \tilde{\mathscr{D}}_{22} + i\left(\tilde{\mathscr{D}}_{12} - \tilde{\mathscr{D}}_{21}\right) , \tag{4.14} $$

$$ {}_2\tilde{\mathscr{D}} \equiv \tilde{\mathscr{D}}_{11} - \tilde{\mathscr{D}}_{22} + i\left(\tilde{\mathscr{D}}_{12} + \tilde{\mathscr{D}}_{21}\right) . \tag{4.15} $$

The spin-0 field contains the contribution from the magnification and the rotation, whereas the spin-2 field is related to the shear distortion. Comparing eqs. (4.14) and (4.15) with (4.12) we obtain

$$ \kappa = 1 - \frac{1+z_S}{2\chi_S}\mathrm{Re}\left[{}_0\tilde{\mathscr{D}}\right] , \qquad \omega = -\frac{1+z_S}{2\chi_S}\mathrm{Im}\left[{}_0\tilde{\mathscr{D}}\right] , \tag{4.16} $$

$$ \gamma = -\frac{1+z_S}{2\chi_S}\,{}_2\tilde{\mathscr{D}} , \tag{4.17} $$

where Re and Im denote the real and imaginary parts of the spin-0 component. The reduced shear then becomes

$$ g = -\frac{{}_2\tilde{\mathscr{D}}}{\mathrm{Re}\left[{}_0\tilde{\mathscr{D}}\right]} . \tag{4.18} $$

The advantage of expressing the shear in terms of the spin-2 component of the magnification matrix is that this allows us to expand it onto spin-weighted spherical harmonics. We can then uniquely decompose it into an E-component (or scalar gradient) and a B-component (or curl) [83]. Contrary to the γ_1 and γ_2 components, the E and B components are invariant under a rotation of the coordinate system around the line of sight. As a consequence, this decomposition is particularly well adapted to a full-sky survey where the line of sight direction varies from patch to patch of the sky. In approximations in which we expand in powers of $1/c$, the time-derivative of a quantity changes its order because an additional factor $1/c$ is introduced. This reflects the assumption that the motion of matter is regarded as slow. In our case, however, all time-derivatives originate from the parallel transport along the geodesic, which is the covariant derivative w.r.t λ[3]. Let $A(\lambda)$ be a scalar function, then

$$ \frac{dA}{d\lambda} = A_{,0}k^0 + A_{,i}k^i = k^0\frac{dA}{d\chi} = k^0\left(A_{,0} + \frac{k^i}{k^0}A_{,i}\right) , \tag{4.19} $$

[3]We will see in the next section that the contracted Riemann tensor $\mathscr{R}^a_b$, and therefore the Jacobi mapping $\mathscr{D}^a_b$, only contain time-derivatives that are derived from $\frac{d}{d\lambda}$.

$$k^0 = \frac{d\chi}{d\lambda} = -c\frac{d\eta}{d\lambda} \quad \text{and} \quad A_{,0} = -\frac{1}{c}\dot{A}, \;\Rightarrow\; A_{,0}k^0 = \dot{A}\frac{d\eta}{d\lambda} \tag{4.20}$$

In the following calculations, all time-derivatives appears always alongside a factor k^0. Therefore they will not change the order in this approximation scheme.

4.2.1 The Jacobi mapping $\mathscr{D}_{ab}$

In this section, we compute the Jacobi mapping for the orders $\mathscr{O}\left(\frac{1}{c^2}\right)$, $\mathscr{O}\left(\frac{1}{c^3}\right)$, and $\mathscr{O}\left(\frac{1}{c^4}\right)$ using the post-Friedmann formalism, which was introduced in section 2.4. First we compute in subsections 4.2.1 various terms in the evolution equation (4.8) up to the required order. Then, in subsection 4.2.1 we use the our previous results to solve for the Jacobi mapping $\mathscr{D}_{ab}$ at different orders.

We introduced the conformal time η so that the line element ds^2 of the metric (2.77)-(2.80) changes to

$$d\tilde{s}^2 = a^2 ds^2. \tag{4.21}$$

Then the Jacobi mapping $\mathscr{D}_{ab}$ conformally transforms as

$$\tilde{\mathscr{D}}_{ab} = a\mathscr{D}_{ab} \tag{4.22}$$

(see Appendix A of [17]). This highlights the fact that a conformal transformation does not affect angles but only distances. In this subsection, we will compute the Jacobi mapping for the metric $d\tilde{s}^2$, whereas in the next subsection 4.2.2, we will use equation (4.22) to reintroduce the scale factor a.

Before we solve the evolution equation (4.8) for each order, we need expressions for both k^0 and $\mathscr{R}_{ab}$ up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$, and n_a^μ and k^i up to order $\mathscr{O}\left(\frac{1}{c^2}\right)$.

The wave-vector k^μ and the screen-space basis n_a^μ

All following calculations of chapter 4 are my responsibility.

The two quantities k^μ and n_a^μ do not change under parallel transport along the null geodesic and we can solve the transport equations up to the required order. First, we want to solve the geodesic equation

$$\frac{Dk^\mu}{D\lambda} = 0 \tag{4.23}$$

with $k^\mu = \frac{dx^\mu}{d\lambda}$. Analogously to the previous section, we substitute the derivative w.r.t. λ with a derivative w.r.t. $\chi = c(\eta_0 - \eta)$. Then the geodesic equation (4.23) reads

$$k^0 \frac{dk^\mu}{d\chi} = -\Gamma^\mu_{\nu\alpha} k^\nu k^\alpha. \tag{4.24}$$

The solution for (4.24) up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ yields

$$\begin{aligned} k^0 =& \bar{k}^0 \exp\left(2U_N \frac{1}{c^2} - 2\int_0^\chi W_{N,0} d\chi' \frac{1}{c^2} + 4U_P \frac{1}{c^4} - 4\int_0^\chi d\chi' W_{P,0} \frac{1}{c^4}\right) - \\ & - \frac{1}{c^3}\int_0^\chi B_{Ni,j}\bar{k}^j \frac{1}{\bar{k}^0}\bar{k}^i d\chi' - \frac{1}{c^4}\frac{1}{2\bar{k}^0} h_{ij}\bar{k}^i\bar{k}^j + \frac{1}{c^4}\frac{1}{2\bar{k}^0}\int_0^\chi d\chi' h_{ij,m}\bar{k}^i\bar{k}^j\bar{k}^m + \\ & - \frac{4}{c^4} U_{N,i}\bar{k}^i \int_0^\chi d\chi' \left[W_N - (\chi - \chi')\left(\bar{k}^0 W_N^{;i} - W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\bar{k}^0}\right)\right] + \\ & + \frac{4}{c^4}\int_0^\chi d\chi' U_{N,i}\bar{k}^i \left[W_N - \int_0^{\chi'} d\chi'' \left(\bar{k}^0 W_N^{;i} - W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\bar{k}^0}\right)\right] + \\ & + \frac{4}{c^4}\int_0^\chi d\chi' W_{N,0i}\bar{k}^i \int_0^{\chi'} d\chi'' \left[W_N - (\chi' - \chi'')\left(\bar{k}^0 W_N^{;i} - W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\bar{k}^0}\right)\right] \end{aligned} \tag{4.25}$$

with the Weyl potential $W_A = \frac{1}{2}(U_A + V_A)$[4] with $A = N, P$. The subscript N refers to Newtonian contributions, while the potentials with subscript P are considered to be relativistic contributions. At order $\mathscr{O}\left(\frac{1}{c^2}\right)$ we obtain the leading order Newtonian contributions. At order $\mathscr{O}\left(\frac{1}{c^3}\right)$, a vector field perturbation B_N^i is introduced, which is sourced by Newtonian quantities but represents a relativistic contribution.

In order to solve k^i up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ we look at the spatial part of (4.23)

$$k^0 \frac{dk^i}{d\chi} = -\Gamma^i{}_{\nu\alpha} k^\nu k^\alpha \tag{4.26}$$

and find the solution

$$\begin{aligned} k^i =& \bar{k}^i\left(1 - 2V_N\frac{1}{c^2}\right) + 2\frac{1}{c^2}\int_0^\chi W_N^{;i} d\chi' + \frac{1}{c^3}B_N^i - \frac{1}{c^3}\int_0^\chi B_{Nm}^{;i}\bar{k}^m d\chi' + \\ & - \frac{1}{c^4}\bar{k}^i\left(4V_P - 2V_N^2\right) - 4\frac{1}{c^4}V_N\int_0^\chi W_N^{;i} d\chi' + 4\frac{1}{c^4}\int_0^\chi d\chi' W_N^{;i} W_N + \\ & + 2\frac{1}{c^4}\int_0^\chi d\chi' \left(2W_{P,} - W_N^2\right)_{,j}\delta^{ij} + \delta^{ij} 4\frac{1}{c^4}\int_0^\chi d\chi' \left(W_{N,j}\int_0^{\chi'} W_{N,n}\bar{k}^n d\chi''\right). \end{aligned} \tag{4.27}$$

[4] At order $\mathscr{O}\left(\frac{1}{c^2}\right)$, the Einstein field equations yield that $V_N = U_N$ and therefore $W_N = V_N = U_N$. We decided to keep U_N and V_N throughout the calculations. Potentially, they could differ, e.g. in a modified gravity theory.

The expressions for k^0 and k^i in (4.52) and (4.27) are derived as follows:

k^0 up to $\mathscr{O}\left(\frac{1}{c^4}\right)$:

For simplicity, we set $A = 2U_N\frac{1}{c^2} + 4U_P\frac{1}{c^4}$, $C = 2V_N\frac{1}{c^2} + 4V_P\frac{1}{c^4}$. Then the PF metric yields

$$g_{00} = -e^{-A}, \quad g_{0i} = -B_i\frac{1}{c^3}, \quad \text{and} \quad g_{ij} = e^C\delta_{ij} + h_{ij}\frac{1}{c^4}. \tag{4.28}$$

The geodesic equation (4.24) yields

$$k^0\frac{dk^0}{d\chi} = -\frac{1}{2}g^{0\gamma}\left(2g_{\gamma\nu,\alpha} - g_{\alpha\nu,\gamma}\right)k^\nu k^\alpha \tag{4.29}$$

$$= -\frac{1}{2}g^{00}\left(2g_{0\nu,\alpha}k^\nu k^\alpha - g_{\alpha\nu,0}k^\nu k^\alpha\right) \tag{4.30}$$

$$= -\frac{1}{2}g^{00}\left(2\frac{dg_{00}}{d\chi}k^0k^0 + 2\frac{dg_{0i}}{d\chi}k^0k^i - g_{00,0}k^0k^0 - 2g_{0i,0}k^0k^i - g_{ij,0}k^ik^j\right) \tag{4.31}$$

$$\frac{dk^0}{d\chi} = -g^{00}\left(\frac{dg_{00}}{d\chi}k^0 + \frac{dg_{0i}}{d\chi}k^i - \frac{1}{2}g_{00,0}k^0 - g_{0i,0}k^i - \frac{1}{2k^0}g_{ij,0}k^ik^j\right) \tag{4.32}$$

$$\begin{aligned}
&= -\left(-\frac{dA}{d\chi}\right)k^0 + \left(-\frac{dB_i}{d\chi}\right)k^i\frac{1}{c^3} + \\
&\quad + \frac{g_{00}}{2k^0}\left[(-A_{,0})\,g_{00}k^0k^0 - 2B_{i,0}k^0k^i\frac{1}{c^3} + \left(C_{,0}e^C\delta_{ij} + h_{ij,0}\frac{1}{c^4}\right)k^ik^j\right] \\
&= \frac{dA}{d\chi}k^0 - \frac{dB_i}{d\chi}k^i\frac{1}{c^3} - \frac{1}{2}A_{,0}k^0 - g^{00}B_{i,0}k^i\frac{1}{c^3} + \\
&\quad + \frac{g^{00}}{2}C_{,0}\left(-g_{00}k^0 + 2B_ik^i\frac{1}{c^3} - h_{ij}k^ik^j\frac{1}{k^0}\right) + \frac{g^{00}}{2k^0}h_{ij,0}k^ik^j \\
&= \frac{dA}{d\chi}k^0 - \frac{dB_i}{d\chi}k^i\frac{1}{c^3} - \frac{1}{2}A_{,0}k^0 - g^{00}B_{i,0}k^i\frac{1}{c^3} + \frac{1}{2}C_{,0}\left(-k^0\right) + \frac{g^{00}}{2k^0}h_{ij,0}k^ik^j \\
&= \frac{dA}{d\chi}k^0 - \frac{dB_i}{d\chi}k^i\frac{1}{c^3} - \frac{1}{2}\left(A+C\right)_{,0}k^0 - g^{00}\left(\frac{dB_i}{d\chi} - \frac{1}{c^3}B_{i,j}k^j\frac{1}{k^0}\right)k^i + \frac{g^{00}}{2k^0}h_{ij,0}k^ik^j \\
&= \frac{dA}{d\chi}k^0 - \frac{1}{2}\left(A+C\right)_{,0}k^0 - \frac{1}{c^3}B_{i,j}\bar{k}^j\frac{1}{\bar{k}^0}\bar{k}^i - \frac{1}{2\bar{k}^0}h_{ij,0}\bar{k}^i\bar{k}^j
\end{aligned} \tag{4.33}$$

Now we will extract the different orders from the solution (4.33). We will start with the order $\mathscr{O}\left(\frac{1}{c^2}\right)$:

$$\frac{dk^{0(2)}}{d\chi} = \frac{dA^{(2)}}{d\chi}\bar{k}^0 - \frac{1}{2}\left(A^{(2)} + C^{(2)}\right)_{,0}\bar{k}^0 \tag{4.34}$$

$$\Rightarrow \quad k^{0(2)} = \bar{k}^0 \frac{1}{c^2}\left[A^{(2)} - \frac{1}{2}\int_0^{\chi}\left(A^{(2)} + C^{(2)}\right)_{,0} d\chi\right] \tag{4.35}$$

$$= \bar{k}^0 \frac{1}{c^2}\left(2U_N - \int_0^{\chi} W_{N,0} d\chi'\right) \tag{4.36}$$

$$= \bar{k}^0 \frac{1}{c^2}\left(2U_N - 2W_N + \int_0^{\chi} W_{N,m}\bar{k}^m d\chi'\right). \tag{4.37}$$

For $\mathcal{O}\left(\frac{1}{c^3}\right)$ we have

$$k^0 \frac{dk^0}{d\chi} = -\frac{1}{2} g^{00}\left(2g_{0\nu,\alpha}k^\nu k^\alpha - g_{\alpha\nu,0}k^\nu k^\alpha\right) \tag{4.38}$$

$$= -\frac{1}{2} g^{00}\left(2g_{0i,\alpha}\bar{k}^i k^\alpha - 2g_{0i,0}\bar{k}^0\bar{k}^i\right) \tag{4.39}$$

$$= \frac{dg_{0i}}{d\chi}\bar{k}^i - g_{0i,0}\bar{k}^i \tag{4.40}$$

$$= -\frac{1}{c^3}\frac{dB_{Ni}}{d\chi}\bar{k}^i + B_{N,0}\bar{k}^i \tag{4.41}$$

$$\Rightarrow \quad k^{0(3)} = -B_{Ni}\bar{k}^i \frac{1}{c^3} + \frac{1}{c^3}\int_0^{\chi} B_{Ni,0}\bar{k}^i d\chi \tag{4.42}$$

$$= -\frac{1}{c^3}\int_0^{\chi} B_{Ni,j}\bar{k}^i\bar{k}^j d\chi \tag{4.43}$$

To solve for k^0 up to order $\mathcal{O}\left(\frac{1}{c^4}\right)$ we need to go beyond the so-called Born approximation, and integrate eq. (4.24) along the perturbed geodesic. We have

$$k^0(\chi) = \bar{k}^0 + \int_0^{\chi} d\chi' G(\chi'), \tag{4.44}$$

where we have defined

$$G = -\Gamma^0_{\nu\alpha}\frac{k^\nu k^\alpha}{k^0}. \tag{4.45}$$

At order $\mathcal{O}\left(\frac{1}{c^4}\right)$ on the other hand, we need to integrate along the perturbed trajectory $x^\mu(\chi) = \bar{x}^\mu(\chi) + \delta x^\mu(\chi)$. We have

$$G\big(x^\mu(\chi')\big) = G\big(\bar{x}^\mu(\chi') + \delta x^\mu(\chi')\big)$$

$$= G\big(\bar{x}^\mu(\chi')\big) + \delta x^\mu(\chi')\partial_\mu G\big(\bar{x}^\mu(\chi')\big) + \mathcal{O}\left(\frac{1}{c^6}\right) \tag{4.46}$$

$$\rightarrow \quad k^0(\chi) = \bar{k}^0 + \int_0^{\chi} d\chi' G\big(\bar{x}^\mu(\chi')\big) + \int_0^{\chi} d\chi' \delta x^\mu(\chi')\partial_\mu G\big(\bar{x}^\mu(\chi')\big) + \mathcal{O}\left(\frac{1}{c^6}\right)$$

$$= \bar{k}^0 + k^0_{\mathrm{G}} + k^0_{\delta\mathrm{G}} + \mathcal{O}\left(\frac{1}{c^6}\right) \tag{4.47}$$

Since G is at least of order $1/c^2$, it is enough to consider only the first term of the Taylor expansion in eq. (4.46). We need to calculate $\delta x^\mu(\chi)$ at order $1/c^2$. We have

$$\frac{dx^\mu}{d\chi} = \frac{dx^\mu}{d\lambda}\frac{d\lambda}{d\chi} = \frac{k^\mu}{k^0}\,. \tag{4.48}$$

Using eq. (4.27) and (4.37) we obtain

$$\frac{d}{d\chi}x^i_{\text{pert}} = \frac{k^i}{k^0} = \bar{k}^i - \frac{2}{c^2}W_N\bar{k}^i + 2\frac{1}{c^2}\int_0^\chi \left(W_N^{;i} - W_{N,j}\bar{k}^i\bar{k}^j\right)d\chi' + \mathscr{O}\left(\frac{1}{c^3}\right) \tag{4.49}$$

$$\delta x^i = -\frac{2}{c^2}\int_0^\chi d\chi' W_N \frac{\bar{k}^i}{\bar{k}^0} + \frac{2}{c^2}\int_0^\chi d\chi' \int_0^{\chi'} d\chi'' \left(W_N^{;i} - W_{N,j}\frac{\bar{k}^i\bar{k}^j}{(\bar{k}^0)^2}\right)\,, \tag{4.50}$$

$$\frac{d}{d\chi}x^0_{\text{pert}} = 1 \quad \delta x^0 = 0\,. \tag{4.51}$$

Inserting this into eqs. (4.46) and (4.44) we obtain for $k^0_{\delta\mathrm{G}}$

$$\begin{aligned} k^0_{\delta\mathrm{G}} =& -\frac{4}{c^4}U_{N,i}\bar{k}^i\int_0^\chi d\chi'\left[W_N - (\chi-\chi')\left(\bar{k}^0 W_N^{;i} - W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\bar{k}^0}\right)\right] + \\ &+\frac{4}{c^4}\int_0^\chi d\chi' U_{N,i}\bar{k}^i\left[W_N - \int_0^{\chi'} d\chi''\left(\bar{k}^0 W_N^{;i} - W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\bar{k}^0}\right)\right] + \\ &+\frac{4}{c^4}\int_0^\chi d\chi' W_{N,0i}\bar{k}^i\int_0^{\chi'} d\chi''\left[W_N - (\chi'-\chi'')\left(\bar{k}^0 W_N^{;i} - W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\bar{k}^0}\right)\right]\,. \end{aligned} \tag{4.52}$$

To solve for k^0_{G}, we look at the geodesic equation (4.24):

$$\begin{aligned} \frac{dk^0}{d\chi} =& \frac{dA^{(2)}}{d\chi}k^{0(2)} + \frac{dA^{(4)}}{d\chi}\bar{k}^0 - \frac{1}{2}\left(A^{(2)}+C^{(2)}\right)_{,0}k^{0(2)} + \\ &-\frac{1}{2}\left(A^{(4)}+C^{(4)}\right)_{,0}\bar{k}^0 - \frac{1}{2\bar{k}^0}h_{ij,0}\bar{k}^i\bar{k}^j\frac{1}{c^4} \\ =& \frac{dA^{(2)}}{d\chi}\bar{k}^0\left[A^{(2)} - \frac{1}{2}\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi\right] + \frac{dA^{(4)}}{d\chi}\bar{k}^0 - \\ &-\frac{1}{2}\left(A^{(2)}+C^{(2)}\right)_{,0}\bar{k}^0\left[A^{(2)} - \frac{1}{2}\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi\right] + \\ &-\frac{1}{2}\left(A^{(4)}+C^{(4)}\right)_{,0}\bar{k}^0 - \frac{1}{2\bar{k}^0}h_{ij,0}\bar{k}^i\bar{k}^j\frac{1}{c^4} \\ =& \bar{k}^0\frac{1}{2}\frac{d\left(A^{(2)}\right)^2}{d\chi}\bar{k}^0 - \bar{k}^0\bar{k}^0\left[\frac{d}{d\chi}\left(A^{(2)}\frac{1}{2}\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi\right) + \right. \\ &\left. -A^{(2)}\frac{1}{2}\left(A^{(2)}+C^{(2)}\right)_{,0}\right] + \end{aligned}$$

$$+\frac{dA^{(4)}}{d\chi}\bar{k}^0 - \frac{1}{2}\left(A^{(2)}+C^{(2)}\right)_{,0}\bar{k}^0 A^{(2)} + \frac{1}{8}\frac{d}{d\chi}\left[\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi\right]^2 +$$
$$-\frac{1}{2}\left(A^{(4)}+C^{(4)}\right)_{,0}\bar{k}^0 - \frac{1}{2\bar{k}^0}h_{ij,0}\bar{k}^i\bar{k}^j\frac{1}{c^4}$$
$$=\bar{k}^0\frac{1}{2}\frac{d\left(A^{(2)}\right)^2}{d\chi}\bar{k}^0 - \bar{k}^0\bar{k}^0\frac{d}{d\chi}\left(A^{(2)}\frac{1}{2}\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi\right) + \frac{dA^{(4)}}{d\chi}\bar{k}^0 -$$
$$+\frac{1}{8}\frac{d}{d\chi}\left[\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi\right]^2 - \frac{1}{2}\left(A^{(4)}+C^{(4)}\right)_{,0}\bar{k}^0 - \frac{1}{2\bar{k}^0}h_{ij,0}\bar{k}^i\bar{k}^j\frac{1}{c^4}$$
$$\Rightarrow \quad k_{\mathrm{G}}^{0(4)} = A^{(4)}\bar{k}^0 + \bar{k}^0\frac{1}{2}\left(A^{(2)}\right)^2\bar{k}^0 - \bar{k}^0\bar{k}^0\left(A^{(2)}\frac{1}{2}\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi\right) +$$
$$+\frac{1}{2}\left[\frac{1}{2}\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi\right]^2 - \frac{1}{2}\int d\chi\left(A^{(4)}+C^{(4)}\right)_{,0}\bar{k}^0 +$$
$$-\frac{1}{2\bar{k}^0}\int d\chi h_{ij,0}\bar{k}^i\bar{k}^j\frac{1}{c^4} \tag{4.53}$$

If we compare (4.53) with the following exponential ansatz

$$e^{a+b+c} = 1+a+b+c+\frac{1}{2}\left(a^2+b^2+c^2+2ab+2bc+2ac\right)+\dots \tag{4.54}$$
$$a \equiv A^{(2)}, \quad b \equiv -\frac{1}{2}\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi, \quad c \equiv A^{(4)} - \frac{1}{2}\int d\chi\left(A^{(4)}+C^{(4)}\right)_{,0}\bar{k}^0 \tag{4.55}$$

$$\rightarrow \quad e^{a+b+c} = 1+a+b+c+\frac{1}{2}\left(a^2+b^2+2ab+\right)+\dots \tag{4.56}$$
$$=1+A^{(2)} - \frac{1}{2}\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi + A^{(4)} +$$
$$-\frac{1}{2}\int d\chi\left(A^{(4)}+C^{(4)}\right)_{,0}\bar{k}^0 + \frac{1}{2}\left(A^{(2)}\right)^2 +$$
$$+\frac{1}{8}\left[\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi\right]^2 - A^{(2)}\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi \tag{4.57}$$

we see that we can rewrite k_{G}^0 as

$$k_{\mathrm{G}}^0 = \bar{k}^0\exp\left[A^{(2)} - \frac{1}{2}\int\left(A^{(2)}+C^{(2)}\right)_{,0}d\chi + A^{(4)} - \frac{1}{2}\int d\chi\left(A^{(4)}+C^{(4)}\right)_{,0}\right] -$$
$$-\int B_{i,j}\bar{k}^j\frac{1}{\bar{k}^0}\bar{k}^i d\chi - \frac{1}{2\bar{k}^0}\int d\chi h_{ij,0}\bar{k}^i\bar{k}^j - \bar{k}^0 \tag{4.58}$$

and in terms of the metric potentials:

$$k_{\mathrm{G}}^0 = \bar{k}^0\exp\left[2U_N\frac{1}{c^2} - \int_0^{\chi}\left(U_N+V_N\right)_{,0}d\chi'\frac{1}{c^2} + 4U_P\frac{1}{c^4} - \int_0^{\chi}d\chi' 2\left(U_P+V_P\right)_{,0}\frac{1}{c^4}\right] -$$

$$-\frac{1}{c^3}\int_0^{\chi} B_{Ni,j}\bar{k}^j\frac{1}{\bar{k}^0}\bar{k}^i d\chi' - \frac{1}{c^4}\frac{1}{2\bar{k}^0}\int_0^{\chi} d\chi' h_{ij,0}\bar{k}^i\bar{k}^j - \bar{k}^0 \tag{4.59}$$

The next step is to solve equation (4.26) up to $\mathscr{O}\left(\frac{1}{c^3}\right)$, which reads

$$k^0\frac{dk^i}{d\chi} = -\Gamma^i{}_{\nu\alpha}k^\nu k^\alpha \tag{4.60}$$

for k^i. For order $\mathscr{O}\left(\frac{1}{c^2}\right)$ we obtain

$$\begin{aligned}
\frac{dk^i}{d\chi} =& -\frac{1}{2k^0}\delta^{ij}\left(2g_{j\nu,\alpha} - g_{\nu\alpha,j}\right)k^\nu k^\alpha && (4.61)\\
=& -\frac{1}{2k^0}\delta^{ik}\left(2\frac{dg_{kj}}{d\chi}k^jk^0 - g_{00,k}k^0k^0 - g_{jp,k}k^jk^p\right) && (4.62)\\
=& -2\frac{1}{c^2}\frac{dV_N}{d\chi}k^i + \frac{1}{c^2k^0}\left(U_N^{;i}k^0k^0 + V_N^{;i}\delta_{jp}k^jk^p\right) && (4.63)\\
=& -2\frac{dV_N}{d\chi}k^i + \left(U_N + V_N\right)^{,i} && (4.64)\\
\Leftrightarrow \quad k^i =& \bar{k}^i - 2V_N\bar{k}^i + 2\int_0^{\chi} W_N^{;i}d\chi'. && (4.65)
\end{aligned}$$

and for order $\mathscr{O}\left(\frac{1}{c^3}\right)$

$$\begin{aligned}
\frac{dk^i}{d\chi} =& -\frac{1}{2\bar{k}^0}\delta^{ij}\left(2g_{j\nu,\alpha} - g_{\nu\alpha,j}\right)k^\nu k^\alpha && (4.66)\\
=& -\frac{1}{2}\delta^{ij}\left(2\frac{dg_{j0}}{d\chi} - 2g_{0m,j}\bar{k}^m\right) && (4.67)\\
=& \frac{dB_N^i}{d\chi} - B_{Nm}^{;i}\bar{k}^m && (4.68)\\
k^i =& B_N^i - \int B_{Nm}^{;i}\bar{k}^m d\chi. && (4.69)
\end{aligned}$$

As the next step we want to find an expression for the screen space n_a^μ by solving (4.2) up to order $\mathscr{O}\left(\frac{1}{c^2}\right)$:

$$\begin{aligned}
\frac{Dn_a^\mu}{D\lambda} =& 0 \quad \text{with} && (4.70)\\
n_a^0 =& \frac{1}{c^2}\int_0^{\chi} U_{N,i}\bar{n}_a^i d\chi' \quad \text{and} && (4.71)\\
n_a^i =& \left(1 - \frac{V_N}{c^2}\right)\bar{n}_a^i - \frac{\bar{k}^i}{\bar{k}^0}\int_0^{\chi}\frac{V_{N,j}}{c^2}\bar{n}_a^j d\chi'. && (4.72)
\end{aligned}$$

The derivation of (4.71) and (4.72) is the following: as well as the phase vector k^μ, the screen space unit vectors n_1^μ and n_2^μ are parallel transported along the line of sight (4.70). In order to solve the evolution equation for the Jacobi mapping $\mathscr{D}$, we will only need an expression for n_s^μ up to order $\mathscr{O}\left(\frac{1}{c^2}\right)$. Thus, we obtain

$$k^0 \frac{dn_a^\mu}{d\chi} = -\frac{1}{2} g^{\mu\beta} \left(g_{\beta\nu,\alpha} + g_{\beta\alpha,\nu} - g_{\nu\alpha,\beta}\right) k^\nu n_a^\alpha \tag{4.73}$$

$$= -\frac{1}{2} g^{\mu\beta} \left(g_{\beta\nu,\alpha} - g_{\nu\alpha,\beta}\right) k^\nu n_a^\alpha - \frac{1}{2} g^{\mu\beta} \frac{g_{\beta\alpha}}{d\chi} n_a^\alpha \tag{4.74}$$

For n_a^0 we obtain

$$k^0 \frac{dn_a^0}{d\chi} = -\frac{1}{2} g^{00} \left(g_{00,\alpha} k^0 n_a^\alpha - g_{\nu\alpha,0} k^\nu n_a^\alpha\right) - \frac{1}{2} g^{00} \frac{dg_{00}}{d\chi} n_a^0 \tag{4.75}$$

$$\frac{dn_a^0}{d\chi} = -\frac{1}{2} \bar{g}^{00} \left(g_{00,\alpha} \bar{n}_a^\alpha - g_{\nu\alpha,0} \bar{k}^\nu \bar{n}_a^\alpha\right) - \frac{1}{2} \bar{g}^{00} \frac{dg_{00}}{d\chi} \bar{n}_a^0 \tag{4.76}$$

$$\frac{dn_a^0}{d\chi} = \frac{1}{2} \left(g_{00,i} \bar{n}_a^i - g_{ji,0} \bar{k}^j \bar{n}_a^i\right) \tag{4.77}$$

$$\frac{dn_a^0}{d\chi} = U_{N,i} \bar{n}_a^i - V_{N,0} \delta_{ji} \bar{k}^j \bar{n}_a^i \tag{4.78}$$

$$\frac{dn_a^0}{d\chi} = U_{N,i} \bar{n}_a^i \tag{4.79}$$

$$\Leftrightarrow \quad n_a^0 = \int_0^\chi U_{N,i} \bar{n}_a^i d\chi' + const = \int_0^\chi U_{N,i} \bar{n}_a^i d\chi' \tag{4.80}$$

and for n_a^i

$$k^0 \frac{dn_a^i}{d\chi} = -\frac{1}{2} g^{ij} \left(g_{j\nu,\alpha} k^\nu n_a^\alpha - g_{\nu\alpha,j} k^\nu n_a^\alpha\right) - \frac{1}{2} g^{ij} \frac{dg_{jk}}{d\chi} n_a^k \tag{4.81}$$

$$\frac{dn_a^i}{d\chi} = -\frac{1}{2} \bar{g}^{ij} \left(g_{j\nu,\alpha} \bar{k}^\nu \bar{n}_a^\alpha - g_{\nu\alpha,j} \bar{k}^\nu \bar{n}_a^\alpha\right) - \frac{1}{2} \bar{g}^{ij} \frac{dg_{jk}}{d\chi} \bar{n}_a^k \tag{4.82}$$

$$\frac{dn_a^i}{d\chi} = -\frac{1}{2} \delta^{ij} \left(g_{jl,\alpha} \bar{k}^l \bar{n}_a^\alpha - g_{pl,j} \bar{k}^p \bar{n}_a^l\right) - \frac{dV_N}{d\chi} \bar{n}_a^i \tag{4.83}$$

$$\frac{dn_a^i}{d\chi} = -V_{N,\alpha} \bar{k}^i \bar{n}_a^\alpha + \delta^{ij} \delta_{pl} V_{N,j} \bar{k}^p \bar{n}_a^l - \frac{dV_N}{d\chi} \bar{n}_a^i \tag{4.84}$$

$$\frac{dn_a^i}{d\chi} = -V_{N,j} \bar{k}^i \bar{n}_a^j - \frac{dV_N}{d\chi} \bar{n}_a^i \tag{4.85}$$

$$\Leftrightarrow \quad n_a^i = -V_N \bar{n}_a^i - \int_0^\chi V_{N,j} \bar{n}_a^j \bar{k}^i d\chi' + const \tag{4.86}$$

$$\Rightarrow \quad n_a^i = \left(1 - \frac{V_N}{c^2}\right) \bar{n}_a^i - \bar{k}^i \int_0^\chi \frac{V_{N,j}}{c^2} \bar{n}_a^j d\chi'. \tag{4.87}$$

In this subsection, we have computed an expression for k^0, k^i, n_a^0, and n_a^i in (4.52), (4.27), (4.71), and (4.72), respectively, up to the desired order.

The next quantity we have to compute in order to solve for the Jacobi mapping $\mathscr{D}_{ab}$ in (4.8), is the contracted Riemann tensor $\mathscr{R}_{ab}$ (4.4) up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$. In (4.4), the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$ is contracted either by n_a^i or k^α. Thus, every derivative within the Riemann tensor will either be a spatial derivative (contracted by n_a^i) or parallel transported along the line of sight (contracted by k^α). Therefore, every time-derivative in $\mathscr{R}^a_b$ will be accompanied by a factor k^0 and, consequently, will not change the order of any metric potential in $\mathscr{R}^a_b$. Furthermore, up to $\mathscr{O}\left(\frac{1}{c^4}\right)$, any quantity of $\mathscr{O}\left(\frac{1}{c^3}\right)$ will not mix with quantities of $\mathscr{O}\left(\frac{1}{c^2}\right)$ and $\mathscr{O}\left(\frac{1}{c^4}\right)$. Thus, $\mathscr{R}^{(3)a}{}_b$[5] can be computed separately.

We start with the contracted Riemann tensor at order $\mathscr{O}\left(\frac{1}{c^3}\right)$, which is solely composed of contributions of the vector potential B_i^N:

$$\mathscr{R}^{(3)a}{}_c = n_i^a n_c^j R^i{}_{\nu\alpha j} k^\nu k^\alpha = \bar{n}^{ai}\bar{n}_c^j \frac{1}{c^3}\left(\frac{dB^N_{(i,j)}}{d\chi} - \bar{k}^k B^N_{k,ij}\right). \tag{4.88}$$

In order to compute the orders $\mathscr{O}\left(\frac{1}{c^2}\right)$ and $\mathscr{O}\left(\frac{1}{c^4}\right)$, we perform a pseudo-conformal transformation

$$\begin{aligned} ds^2 =& -e^{-\frac{2}{c^2}U_N - \frac{4}{c^4}U_P}dt^2 + \left(e^{\frac{2}{c^2}V_N + \frac{4}{c^4}V_P}\delta_{ij} + \frac{1}{c^4}h_{ij}\right)dx^i dx^j && (4.89)\\ =& e^{\frac{2}{c^2}V_N + \frac{4}{c^4}V_P}\left[-e^{-\frac{4}{c^2}W_N - \frac{8}{c^4}W_P}dt^2 + \left(\delta_{ij} + \frac{1}{c^4}h_{ij}e^{-\frac{2}{c^2}V_N - \frac{4}{c^4}V_P}\right)dx^i dx^j\right] && (4.90)\\ =& e^{\frac{2}{c^2}V_N + \frac{4}{c^4}V_P}\left[-e^{-\frac{4}{c^2}W_N - \frac{8}{c^4}W_P}dt^2 + \left(\delta_{ij} + \frac{1}{c^4}h_{ij}\right)dx^i dx^j\right] + \mathscr{O}\left(\frac{1}{c^6}\right) && (4.91) \end{aligned}$$

and use the expression for the conformally transformed Riemann tensor: [111]

$$\begin{aligned} R_{\alpha\beta\gamma\delta} =& \hat{R}_{\alpha\beta\gamma\delta} - 2g_{\alpha[\gamma}\nabla_{\delta]}\nabla_\beta \ln\Omega + 2g_{\beta[\gamma}\nabla_{\delta]}\nabla_\alpha \ln\Omega - 2\left(\nabla_{[\gamma}\ln\Omega\right)g_{\delta]\alpha}\nabla_\beta\ln\Omega + \\ &+ 2\left(\nabla_{[\gamma}\ln\Omega\right)g_{\delta]\beta}\nabla_\alpha\ln\Omega + 2g_{\beta[\gamma}g_{\delta]\alpha}g^{\varepsilon\zeta}\left(\nabla_\varepsilon\ln\Omega\right)\nabla_\zeta\ln\Omega \end{aligned} \tag{4.92}$$

with $g_{\alpha\beta} = \Omega^2\hat{g}_{\alpha\beta}$. Substituting the metric (4.91) into (4.92) with $\Omega = e^{\frac{1}{c^2}V_N + \frac{2}{c^4}V_P}$ we obtain

$$\mathscr{R}_{ab} = R_{\alpha\beta\delta\gamma}k^\alpha k^\delta n_a^\gamma n_b^\beta = e^{-\frac{4}{c^2}W_N - \frac{8}{c^4}W_P}\left[-k^i k^j n_a^0 n_b^0 + k^j k^0 n_a^0 n_b^i + k^j k^0 n_a^i n_b^0 - \left(k^0\right)^2 n_a^i n_b^j\right]$$

[5] We refer to the different orders as $X^{(n)}$, where n denotes the inverse power of the speed of light c.

$$\left[\left(\frac{2}{c^2}W_{N,i}+\frac{4}{c^4}W_{P,i}\right)\left(\frac{2}{c^2}W_{N,j}+\frac{4}{c^4}W_{P,j}\right)-\frac{2}{c^2}W_{N,ij}-\frac{4}{c^4}W_{P,ij}\right]+$$
$$+\delta_{ab}\left[k^\alpha k^\beta\left(\frac{1}{c^2}V_{N,\delta\beta}+\frac{2}{c^4}V_{P,\delta\beta}\right)-k^\alpha k^\beta\left(\frac{1}{c^2}V_{N,\delta}+\frac{2}{c^4}V_{P,\delta}\right)\left(\frac{1}{c^2}V_{N,\beta}+\frac{2}{c^4}V_{P,\beta}\right)\right]+$$
$$+\frac{1}{2c^4}n_a^i n_b^j\left[\frac{d^2}{d\chi^2}h_{ij}-\frac{d}{d\chi}\left(h_{jp,i}+h_{ip,j}\right)\bar{k}^p+h_{mp,ij}k^p k^m\right] \tag{4.93}$$

The contracted Riemann tensor at order $\mathscr{O}\left(\frac{1}{c^2}\right)$ yields

$$\begin{aligned}\mathscr{R}_{ab}^{(2)} &= \left(\bar{k}^0\right)^2\bar{n}_a^i\bar{n}_b^j\frac{2}{c^2}W_{N,ij}+\delta_{ab}\bar{k}^\alpha\bar{k}^\beta\frac{1}{c^2}V_{N,\delta\beta}\\ &=\bar{n}_a^i\bar{n}_b^j\frac{2}{c^2}W_{N,ij}+\delta_{ab}\frac{1}{c^2}\frac{d^2V_N}{d\chi^2}\end{aligned} \tag{4.94}$$

and at order $\mathscr{O}\left(\frac{1}{c^4}\right)$

$$\begin{aligned}\mathscr{R}_{ab}^{(4)} =&\delta_{ab}\frac{1}{c^4}\left(\bar{k}^0\right)^2\left[2\frac{d^2}{d\chi^2}V_P-\frac{d}{d\chi}V_N\frac{d}{d\chi}V_N+2\left(\frac{dU_N}{d\chi}-W_{N,0}\right)\frac{dV_N}{d\chi}\right]+\\ &+\bar{n}_a^i\bar{n}_b^j\frac{1}{c^4}\left(\bar{k}^0\right)^2\left[4W_{P,ij}-4W_{N,ij}^2+4W_{N,i}W_{N,j}-4W_{N,ij}V_N+\right.\\ &\left.-4W_{N,ij}\frac{1}{c^4}\bar{k}^0\left(\bar{k}^i\bar{n}_a^m\bar{n}_b^j+\bar{k}^j\bar{n}_b^m\bar{n}_a^i\right)\int_0^\chi W_{N,m}d\chi'\right]+\\ &+\frac{1}{2c^4}\bar{n}_a^i\bar{n}_b^j\left(\bar{k}^0\right)^2\left[\frac{d^2}{d\chi^2}h_{ij}-\frac{d}{d\chi}\left(h_{jp,i}+h_{ip,j}\right)\bar{k}^p+h_{mp,ij}k^p k^m\right]\end{aligned} \tag{4.95}$$

From now on, we will set $\bar{k}^0=1$ keeping in mind that $\bar{k}^0$ is of order $\mathscr{O}(c)$.

The Jacobi mapping $\mathscr{D}_{ab}(\chi)$ up to $\mathscr{O}\left(\frac{1}{c^4}\right)$

Previously we derived k^0, k^i, n_a^0, and n_a^i in (4.52), (4.27), (4.71), and (4.72), respectively, as well as $\mathscr{R}_{ab}$ up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ in (4.94), (4.88), and (4.95). In this subsection, we will use these results and substitute them into the evolution equation (4.8). We will now solve (4.8) for the order $\mathscr{O}\left(\frac{1}{c^2}\right)$, $\mathscr{O}\left(\frac{1}{c^3}\right)$, and $\mathscr{O}\left(\frac{1}{c^4}\right)$:

Order $\mathscr{O}\left(\frac{1}{c^2}\right)$

At this order, the evolution equation for $\mathscr{D}_{ab}^{(2)}$ (4.8) reduces to

$$\frac{d^2}{d\chi^2}\mathscr{D}_{ab}^{(2)}=-\frac{dk^{(2)0}}{d\chi}\delta_{ab}+\mathscr{R}_{ab}^{(2)}\chi, \tag{4.96}$$

with $\bar{\mathscr{D}}_{ab} = \chi \delta_{ab}$. We integrate (4.96) two times

$$\mathscr{D}_{ab} = \int_0^{\chi_S} d\chi \left(2 - k^0\right) \delta_{ab} + \int_0^{\chi_S} d\chi \left(\chi_S - \chi\right) \chi \mathscr{R}_{ab}. \tag{4.97}$$

and substitute (4.94) and (4.52) up to order $\mathscr{O}\left(\frac{1}{c^2}\right)$ into (4.97):

$$\begin{aligned}\mathscr{D}_{ab}\left(\chi_S\right) =& \chi_S \delta_{ab} \left(1 + V_N \frac{1}{c^2}\right) - 2\frac{1}{c^2} \int_0^{\chi_S} d\chi \left[W_N + (\chi_S - \chi) W_{N,i} \bar{k}^i\right] \delta_{ab} + \\ &+ 2\frac{1}{c^2} \int_0^{\chi_S} d\chi \left(\chi_S - \chi\right) \chi \bar{n}_a^i \bar{n}_b^j W_{N,ij}.\end{aligned} \tag{4.98}$$

The solution (4.98) formally coincides with the first order solution for $\mathscr{D}_{ab}$ in SPT [17, 23, 21]. This follows from the fact that the metric of order $\mathscr{O}\left(\frac{1}{c^2}\right)$ in the PF formalism is mathematically identical to the metric first order SPT. We derived the evolution equation of the Jacobi mapping from the geodesic deviation equation, which is a mathematical identity. Thus, as long as we don't substitute field equations into the evolution equation (4.8), the solution for the Jacobi mapping will mathematically coincide.

In (4.98), the Jacobi mapping $\mathscr{D}_{ab}$ involves the Weyl potential W_N as well as the scalar potential V_N. Because we are using a conformal metric with the Weyl potential in the conformal factor and because null geodesics are invariant under conformal transformations, one would expect that $\mathscr{D}_{ab}$ only comprises the Weyl potential W_N. But as noted in [17], the term involving V_N is derived from the parallel transport of the basis n_a^μ, which is not conformally invariant.

Order $\mathscr{O}\left(\frac{1}{c^3}\right)$

For order $\mathscr{O}\left(\frac{1}{c^3}\right)$, (4.8) yields

$$\frac{d^2}{d\chi^2} \mathscr{D}_{ab}^{(3)} + \frac{1}{\bar{k}^0} \frac{dk^{(3)0}}{d\chi} \frac{d}{d\chi} \bar{\mathscr{D}}_{ab} = \frac{1}{(\bar{k}^0)^2} \mathscr{R}_a^{(3)c} \bar{\mathscr{D}}_{cb}, \tag{4.99}$$

Analogously to the previous order, we integrate along the background geodesic and substitute (4.52) and (4.88) into (4.99):

$$\begin{aligned}\mathscr{D}_{ab}^{(3)}\left(\chi_S\right) =& - \int_0^{\chi_S} d\chi k^{(3)0} \delta_{ab} + \int_0^{\chi_S} d\chi \left(\chi_S - \chi\right) \chi \mathscr{R}_{ab}^{(3)} \\ =& \frac{1}{c^3} \int_0^{\chi_S} d\chi \left(\chi_S - \chi\right) B_{Ni,j} \bar{k}^j \bar{k}^i \delta_{ab} +\end{aligned} \tag{4.100}$$

$$+\frac{1}{c^3}\int_0^{\chi_S} d\chi\,(\chi_S-\chi)\,\chi\bar{n}_a^i\bar{n}_b^j\left[\frac{dB^N_{(i,j)}}{d\chi}-\bar{k}^m B^N_{m,ij}\right] \tag{4.101}$$

The second part of (4.101) coincides with the vector part of [17], in which the shear has been computed up to second order in SPT. The first part of (4.101) will contribute to the convergence.

Order $\mathscr{O}\left(\frac{1}{c^4}\right)$

The Jacobi mapping $\mathscr{D}^{(4)}_{ab}$ is the solution of the following differential equation

$$\frac{d^2}{d\chi^2}\mathscr{D}^{(4)}_{ab}=-\left(\frac{1}{k^0}\frac{dk^0}{d\chi}\right)^{(4)}\delta_{ab}-\left(\frac{1}{k^0}\frac{dk^0}{d\chi}\right)^{(2)}\frac{d}{d\chi}\mathscr{D}^{(2)}_{ab}+ \\ +\left(\frac{1}{(k^0)^2}\mathscr{R}^c_a\right)^{(4)}\chi\delta_{cb}+\left(\frac{1}{(k^0)^2}\mathscr{R}^c_a\right)^{(2)}\mathscr{D}^{(2)}_{cb} \tag{4.102}$$

In this section, we will derive the expressions for each term and subsequently combine them. For simplicity , we define a function $A(\chi)$ as

$$A(\chi)\equiv 2(U_N-W_N)\frac{1}{c^2}+\int d\chi 2W_{N,l}k^l d\chi\frac{1}{c^2}+4(U_P-W_P)\frac{1}{c^4}+\int d\chi 4W_{P,l}k^l\frac{1}{c^4} \tag{4.103}$$

so that k^0 reduces to

$$k^0=\bar{k}^0e^{A(\chi)}-\frac{1}{c^3}\int B_{Ni,j}\bar{k}^j\frac{1}{\bar{k}^0}\bar{k}^i d\chi-\frac{1}{c^4}\frac{1}{2\bar{k}^0}h_{ij}\bar{k}^i\bar{k}^j+\frac{1}{c^4}\frac{1}{2(\bar{k}^0)^2}\int d\chi h_{ij,l}\bar{k}^i\bar{k}^j\bar{k}^l \tag{4.104}$$

Using the expression (4.104), the first term in (4.102) yields

$$-\left(\frac{1}{k^0}\frac{d}{d\chi}k^0\right)^{(4)}=-e^{-A(\chi)}\left[e^{A(\chi)}\frac{dA(\chi)}{d\chi}-\frac{1}{c^4}\frac{1}{2\bar{k}^0}\frac{dh_{ij}}{d\chi}\bar{k}^i\bar{k}^j+\frac{1}{c^4}\frac{1}{2(\bar{k}^0)^2}h_{ij,l}\bar{k}^i\bar{k}^j\bar{k}^l\right] \\ =-4\frac{d}{d\chi}(U_P-W_P)\frac{1}{c^4}-4W_{P,i}\bar{k}^i\frac{1}{c^4}+\frac{1}{c^4}\frac{1}{2\bar{k}^0}\frac{dh_{ij}}{d\chi}\bar{k}^i\bar{k}^j+ \\ -\frac{1}{c^4}\frac{1}{2(\bar{k}^0)^2}h_{ij,l}\bar{k}^i\bar{k}^j\bar{k}^l. \tag{4.105}$$

The second term in (4.102) becomes

$$
\begin{aligned}
-\left(\frac{1}{k^0}\frac{dk^0}{d\chi}\right)^{(2)}\frac{d}{d\chi}\mathscr{D}^{(2)}_{ab} =& -\frac{1}{\bar{k}^0}\frac{d}{d\chi}k^{0(2)}\left(-k^{0(2)}\delta_{ab}+\int d\chi'\mathscr{R}_{ab}\chi'\right)\\
=&\frac{1}{2\bar{k}^0}\frac{d}{d\chi}\left(k^{0(2)}\right)^2\delta_{ab}-\frac{1}{\bar{k}^0}\frac{d}{d\chi}k^{0(2)}\int d\chi'\mathscr{R}_{ab}\chi'\\
=&\frac{1}{2}\frac{d}{d\chi}\left(2U_N-2W_N+2\int_0^\chi W_{N,l}\bar{k}^l\right)^2\delta_{ab}-\\
&-4\frac{d}{d\chi}\left[\left(U_N-W_N\right)\int\bar{n}^i_a\bar{n}^j_b\chi W_{N,ij}d\chi'\right]+\\
&+4\left(U_N-W_N\right)\bar{n}^i_a\bar{n}^j_b\chi W_{N,ij}+\\
&-2\frac{d}{d\chi}\left(U_N-W_N\right)\left(-V_N+\chi\frac{d}{d\chi}V_N\right)\delta_{ab}\\
&-2W_{N,l}\bar{k}^l\left[\int 2\bar{n}^i_a\bar{n}^j_b\chi W_{N,ij}d\chi'+\left(-V_N+\chi\frac{d}{d\chi}V_N\right)\delta_{ab}\right]
\end{aligned}
\tag{4.106}
$$

with

$$
\int d\chi'\mathscr{R}_{ab}\chi' = \int 2\bar{n}^i_a\bar{n}^j_b\chi W_{N,ij}d\chi'+\left(-V_N+\chi\frac{d}{d\chi}V_N\right)\delta_{ab}\quad\text{and}\tag{4.107}
$$

$$
-V_N+\chi\frac{d}{d\chi}V_N=\chi^2\frac{d}{d\chi}\left(\frac{1}{\chi}V_N\right)=\frac{d}{d\chi}\left(\chi V_N\right)-2V_N.\tag{4.108}
$$

The third term in (4.102) is given by

$$
\begin{aligned}
\left(\frac{1}{k^0k^0}\mathscr{R}_{ac}\right)^{(4)}\chi\delta^c_b =\chi\Bigg\{&\delta_{ab}\frac{1}{c^4}\left(2\frac{d^2}{d\chi^2}V_P-\frac{d}{d\chi}V_N\frac{d}{d\chi}V_N+\right.\\
&\left.+2\frac{d}{d\chi}\left(U_N-W_N\right)\frac{dV_N}{d\chi}+2W_{N,l}\bar{k}^l\frac{dV_N}{d\chi}\right)+\\
&+\bar{n}^i_a\bar{n}^j_b\frac{1}{c^4}\left[4W_{P,ij}-4W^2_{N,ij}+4W_{N,i}W_{N,j}-4W_{N,ij}V_N-\right.\\
&\left.-4\bar{k}^lW_{N,lj}\int_0^\chi W_{N,i}d\chi-4\bar{k}^lW_{N,il}\int_0^\chi W_{N,j}d\chi\right]+\\
&+\frac{1}{2}\left[\frac{d^2}{d\chi^2}h_{ij}-\frac{d}{d\chi}\left(h_{jp,i}+h_{ip,j}\right)\bar{k}^p+h_{mp,ij}k^pk^m\right]\Bigg\}
\end{aligned}
\tag{4.109}
$$

The last term in (4.102) yields

$$
\left(\frac{1}{k^0k^0}\mathscr{R}_{ac}\right)^{(2)}\mathscr{D}^{(2)c}_{\ \ b}=\left(\frac{d^2}{d\chi^2}V_N\delta_{ac}+2\bar{n}^q_a\bar{n}^p_cW_{N,qp}\right)[\chi V_N\delta^c_b+
$$

$$
\begin{aligned}
&-\int d\chi'\left(2W_N+2\int W_{N,m}\bar{k}^m\right)\delta_b^c+\\
&+2\int_0^{\chi}\int_0^{\chi'}d\chi' d\chi''\chi''\bar{n}_c^q\bar{n}_b^p W_{N,qp}\Big]\\
=&\delta_{ab}\chi V_N\frac{d^2}{d\chi^2}V_N-\frac{d^2}{d\chi^2}V_N\delta_{ab}\int d\chi'\left(2W_N+2\int W_{N,m}\bar{k}^m\right)+\\
&+2\frac{d^2}{d\chi^2}V_N\int_0^{\chi}\int_0^{\chi'}d\chi' d\chi''\chi''\bar{n}_a^q\bar{n}_b^p W_{N,qp}+\\
&+2\bar{n}_a^q\bar{n}_b^p W_{N,qp}\chi V_N+\\
&-2\bar{n}_a^q\bar{n}_b^p W_{N,qp}\int_0^{\chi}d\chi'\left(2W_N+2\int_0^{\chi'}d\chi'' W_{N,m}\bar{k}^m\right)+\\
&+4\bar{n}_a^q\bar{n}_c^p\bar{n}^{rc}\bar{n}_b^s W_{N,qp}\int_0^{\chi}\int_0^{\chi'}d\chi' d\chi''\chi'' W_{N,rs}\\
=&\delta_{ab}\left[\frac{\chi}{2}\frac{d^2}{d\chi^2}V_N^2-\chi\left(\frac{dV_N}{d\chi}\right)^2-2\frac{d^2}{d\chi^2}\left(V_N\int_0^{\chi}d\chi' W_N\right)+\right.\\
&+2\frac{d}{d\chi}\left(V_N W_N\right)+2W_N\frac{d}{d\chi}V_N-2\frac{d^2}{d\chi^2}\left(V_N\int d\chi'\int d\chi'' W_{N,m}\bar{k}^m\right)\\
&\left.+4\frac{d}{d\chi}\left(V_N\int d\chi' W_{N,m}\bar{k}^m\right)-2V_N W_{N,m}\bar{k}^m\right]+\\
&+\bar{n}_a^i\bar{n}_b^j\left[2\frac{d^2}{d\chi^2}\left(V_N\int_0^{\chi}\int_0^{\chi'}d\chi' d\chi''\chi'' W_{N,ij}\right)+\right.\\
&-4\frac{d}{d\chi}\left(V_N\int d\chi'\chi' W_{N,ij}\right)+4V_N\chi W_{N,ij}+\\
&-4W_{N,ij}\int_0^{\chi}d\chi' W_N-4W_{N,ij}\int_0^{\chi}\int_0^{\chi'}W_{N,m}\bar{k}^m+\\
&\left.+4\bar{n}_c^s\bar{n}^{rc}W_{N,is}\int_0^{\chi}\int_0^{\chi'}d\chi' d\chi''\chi'' W_{N,rj}\right]
\end{aligned}
\tag{4.110}
$$

At order $\mathscr{O}\left(\frac{1}{c^4}\right)$, we have to go beyond the Born approximation and take perturbations of the null geodesic into account. Therefore, we rearrange the evolution equation (4.8) and define a function $\mathscr{S}(\chi)$, which will perturb later.

$$
\frac{d^2}{d\chi^2}\mathscr{D}_{ab}=\frac{1}{\chi^2}\mathscr{S}(\chi) \tag{4.111}
$$

$$
\text{with}\quad \mathscr{S}_{ab}\equiv\chi^2\left(-\frac{1}{k^0}\frac{dk^0}{d\chi}\frac{d}{d\chi}\mathscr{D}_{ab}+\frac{1}{(k^0)^2}\mathscr{R}_a{}^c\mathscr{D}_{cb}\right) \tag{4.112}
$$

Perturbation along the geodesic: analogously to $k^{(4)0}$ in equation (4.44) and (4.46) we can no longer integrate along the background null geodesic $\bar{x}^{\mu}$ but need to take perturbations of the geodesic up to order $\mathscr{O}\left(\frac{1}{c^2}\right)$ into account. We perturb the geodesic $x^\mu(\chi)$ and evaluate the function $\mathscr{S}$ at the geodesic position $x^\mu_{\text{pert}}(\chi) = x^\mu(\chi) + \delta x^\mu(\chi)$. In equations (4.50) and (4.51) we solved for δx^i and δx^0, respectively. Next, we expand $\mathscr{S}_{ab}\left(x^i_{\text{pert}}\right) = \mathscr{S}_{ab}(x^i) + \delta x^j \cdot \delta\left(\mathscr{S}_{ab}\right)_j\big|_x$. Considering the definition of the function $\mathscr{S}_{ab}$ in (4.112), we see that we obtain two additional terms

$$\left(\bar{\mathscr{D}}_{ac}\mathscr{R}^{(2)c}{}_{b}\right)_{,j}\delta x^j \quad \text{and} \quad -\left(\bar{\mathscr{D}}_{ab}\frac{dk^{0(2)}}{d\chi}\right)_{,j}\delta x^j \tag{4.113}$$

which are computed as follows:

$$\begin{aligned}
&\left(\bar{\mathscr{D}}_{ac}\mathscr{R}^{(2)c}{}_{b}\right)_{,j}\delta x^j = \\
=&\bar{n}^i_a\bar{n}^j_b\frac{1}{c^4}\left[-\chi 4W_{N,ijm}\int_0^\chi W_N d\chi'\bar{k}^m + \right.\\
&\left.+\chi 4W_{N,ijm}\int_0^\chi\int_0^{\chi'}\left(W_N^{;m} - W_{N,l}\bar{k}^m\bar{k}^l\right)d\chi''d\chi'\right] + \\
&+\delta_{ab}\frac{1}{c^4}\left\{-2\frac{d^2}{d\chi^2}\left(\chi V_{N,m}\int_0^\chi W_N d\chi'\bar{k}^m\right) + \right.\\
&+2\frac{d}{d\chi}\left(2V_{N,m}\int_0^\chi W_N d\chi'\bar{k}^m + \chi V_{N,m}\bar{k}^m W_N\right) + \\
&-2\bar{k}^m V_{N,m}W_N + 2\bar{k}^m\chi W_N\frac{d}{d\chi}V_{N,m} + \\
&+2\frac{d^2}{d\chi^2}\left[\chi V_{N,m}\int_0^\chi\int_0^{\chi'}\left(W_N^{;m} - W_{N,l}\bar{k}^m\bar{k}^l\right)d\chi''d\chi'\right] + \\
&-4\frac{d}{d\chi}\left[V_{N,m}\int_0^\chi\int_0^{\chi'}\left(W_N^{;i} - W_{N,j}\bar{k}^i\bar{k}^j\right)d\chi''d\chi' + \right.\\
&\left.+\chi V_{N,m}\int_0^\chi\left(W_N^{;m} - W_{N,l}\bar{k}^m\bar{k}^l\right)d\chi'\right] + \\
&\left.+4V_{N,m}\int_0^\chi d\chi\left(W_N^{;m} - W_{N,l}\bar{k}^m\bar{k}^l\right) + 2\chi V_{N,m}\left(W_N^{;m} - W_{N,l}\bar{k}^m\bar{k}^l\right)\right\}
\end{aligned}\tag{4.114}$$

and

$$\begin{aligned}
-\left(\frac{d}{d\chi}\bar{\mathscr{D}}_{ab}\frac{dk^{0(2)}}{d\chi}\right)_{,m}\delta x^m =&\delta_{ab}\frac{1}{c^4}\left[4\frac{d}{d\chi}\left(U_{N,m}\int_0^\chi W_N d\chi'\bar{k}^m\right) - 4U_{N,m}\bar{k}^m W_N + \right.\\
&-4\frac{d}{d\chi}\left(W_{N,m}\int_0^\chi W_N d\chi'\bar{k}^m\right) + 4W_{N,m}\bar{k}^m W_N + \\
&+4W_{N,nm}\bar{k}^n\bar{k}^m\int_0^\chi W_N d\chi' +
\end{aligned}$$

$$
\begin{aligned}
&-4\frac{d}{d\chi}\left(U_{N,m}\int_0^{\chi}\int_0^{\chi'}\left(W_N^{;m}-W_{N,j}\bar{k}^m\bar{k}^j\right)d\chi''d\chi'\right)+\\
&+4U_{N,m}\int_0^{\chi}\left(W_N^{;m}-W_{N,j}\bar{k}^m\bar{k}^j\right)d\chi'+\\
&+4\frac{d}{d\chi}\left(W_{N,m}\int_0^{\chi}\int_0^{\chi'}\left(W_N^{;m}-W_{N,j}\bar{k}^m\bar{k}^j\right)d\chi''d\chi'\right)+\\
&-4W_{N,m}\int_0^{\chi}\left(W_N^{;m}-W_{N,j}\bar{k}^m\bar{k}^j\right)d\chi'+\\
&-4W_{N,nm}\bar{k}^n\int_0^{\chi}\int_0^{\chi'}\left(W_N^{;m}-W_{N,j}\bar{k}^m\bar{k}^j\right)d\chi''d\chi'\Bigg].
\end{aligned}
\tag{4.115}
$$

We substitute (4.105) - (4.110) into (4.102) and add the terms (4.114) and (4.115). Furthermore, we perform the integrals to obtain an expression for $\mathscr{D}_{ab}^{(4)}$. The Jacobi map can be divided into an isotropic part, which involves δ_{ab} and an anisotropic part involving $n_a^i n_b^j$. Only the latter contributes to the shear, whereas both parts contribute to the convergence. In this section we will group the terms of $\mathscr{D}_{ab}$ according to the potentials or their couplings and will discuss each category separately. We split $\mathscr{D}_{ab}$ into

$$
\mathscr{D}_{ab}^{(4)}=\mathscr{D}_{ab}^{(P)}+\mathscr{D}_{ab}^{(VV)}+\mathscr{D}_{ab}^{(WW)}+\mathscr{D}_{ab}^{(UW)}+\mathscr{D}_{ab}^{(VW)}+\mathscr{D}_{ab}^{(h)},
\tag{4.116}
$$

where the subscripts refer to the couplings of the potentials. The contribution $\mathscr{D}_{ab}^{(P)}$ is a purely relativistic contribution generated by the relativistic potentials U_P, V_P, and W_P:

$$
\begin{aligned}
\mathscr{D}_{ab}^{(P)}=&\frac{2}{c^4}\chi_S V_{PS}\delta_{ab}-\frac{4}{c^4}\int_0^{\chi_S}d\chi\left[W_P+(\chi_S-\chi)W_{P,i}\frac{\bar{k}^i}{\bar{k}^0}\right]\delta_{ab}\\
&+\frac{4}{c^4}\int_0^{\chi_S}d\chi\,(\chi_S-\chi)\,\chi\bar{n}_a^i\bar{n}_b^j W_{P,ij}\,.
\end{aligned}
\tag{4.117}
$$

$\mathscr{D}_{ab}^{(P)}$ take the same form as the Jacobi mapping $\mathscr{D}_{ab}^{(2)}$ in (4.98), which is due to the form of the metric (2.77) - (2.80). The $\mathscr{D}_{ab}^{(VV)}$ contribution contains all the terms quadratic in V_N:

$$
\mathscr{D}_{ab}^{(VV)}=\frac{1}{c^4}\left[\frac{\chi_S}{2}V_{NS}^2-2\int_0^{\chi_S}d\chi(\chi_S-\chi)\chi\left(\frac{dV_N}{d\chi}\right)^2\right]\delta_{ab}\,.
\tag{4.118}
$$

This terms contributes only to the convergence, since they are proportional to δ_{ab}. The $\mathscr{D}_{ab}^{(WW)}$ contribution contains all the terms quadratic in W_N. It can be split into a part proportional to δ_{ab} and a part proportional to $n_a n_b$, which contributes also to the shear and rotation:

$$
\begin{aligned}
\mathscr{D}_{ab}^{(WW)} =& \delta_{ab}\frac{1}{c^4}\left\{\int_0^{\chi_S} d\chi \left[4W_{N,m}\int_0^{\chi} d\chi' W_N \frac{\bar{k}^m}{\bar{k}^0} + 2W_N^2 + 2\left(\int_0^{\chi} d\chi' W_{N,l}\frac{\bar{k}^l}{\bar{k}^0}\right)^2 + \right.\right. \\
& -4W_{N,m}\int_0^{\chi} d\chi'\left(\chi-\chi'\right)\left(W_N^{;m} - W_{N,j}\frac{\bar{k}^m\bar{k}^j}{\left(\bar{k}^0\right)^2}\right) + 4W_N\int_0^{\chi} d\chi' W_{N,l}\frac{\bar{k}^l}{\bar{k}^0}\Bigg] + \\
& +4\int_0^{\chi_S} d\chi(\chi_S-\chi)\left[W_{N,m}\left(\int_0^{\chi} d\chi'\left(W_N^{;m} - W_{N,j}\frac{\bar{k}^m\bar{k}^j}{\left(\bar{k}^0\right)^2}\right) - W_N\frac{\bar{k}^m}{\bar{k}^0}\right) + \right. \\
& -W_{N,nm}\frac{\bar{k}^n}{\bar{k}^0}\int_0^{\chi} d\chi'\left(\left(\chi-\chi'\right)\left(W_N^{;m} - W_{N,j}\frac{\bar{k}^m\bar{k}^j}{\left(\bar{k}^0\right)^2}\right) - W_N\frac{\bar{k}^m}{\bar{k}^0}\right) + \\
& \left.\left. + \frac{1}{\bar{k}^0}W_{N,0i}\int_0^{\chi} d\chi'\left(\left(\chi-\chi'\right)\left(W_N^{;i} - W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2}\right) - \bar{k}^i W_N\right)\right]\right\} + \\
& +\bar{n}_a^i\bar{n}_b^j\frac{1}{c^4}\left\{-\int_0^{\chi_S} d\chi\left(4W_N\int_0^{\chi} d\chi'\chi' W_{N,ij}\right) + \right. \\
& +\int_0^{\chi_S} d\chi(\chi_S-\chi)\left[-2\chi W_{N,ij}^2 - 4W_{N,l}\frac{\bar{k}^l}{\bar{k}^0}\int_0^{\chi} d\chi'\chi' W_{N,ij} + \right. \\
& -4\chi\frac{\bar{k}^l}{\bar{k}^0}W_{N,lj}\int_0^{\chi} d\chi' W_{N,i} - 4\chi\frac{\bar{k}^l}{\bar{k}^0}W_{N,il}\int_0^{\chi} d\chi' W_{N,j} + \\
& -4W_{N,ij}\int_0^{\chi} d\chi'\left(W_N + \left(\chi-\chi'\right)W_{N,m}\frac{\bar{k}^m}{\bar{k}^0}\right) + \\
& +4\bar{n}_c^s\bar{n}^{rc}W_{N,is}\int_0^{\chi} d\chi'\left(\chi-\chi'\right)\chi' W_{N,rj} + \\
& \left.\left. +\chi 4W_{N,ijm}\int_0^{\chi} d\chi'\left(\left(\chi-\chi'\right)\left(W_N^{;m} - W_{N,l}\frac{\bar{k}^m\bar{k}^l}{\left(\bar{k}^0\right)^2}\right) - W_N\frac{\bar{k}^m}{\bar{k}^0}\right)\right]\right\}.
\end{aligned}
\tag{4.119}
$$

with

$$
\begin{aligned}
2\left(U_N - W_N + \int_0^{\chi} d\chi W_{N,l}\bar{k}^l\right)^2 - V_N^2 + 4V_N W_N + 4V_N\int_0^{\chi} d\chi W_{N,l}\bar{k}^l = \\
2W_N^2 + V_N^2 + 2\left(\int_0^{\chi} d\chi W_{N,l}\bar{k}^l\right)^2 + 4W_N\int_0^{\chi} d\chi W_{N,l}\bar{k}^l
\end{aligned}
\tag{4.120}
$$

and

$$
\frac{d}{d\chi}V_N^2 - 2\frac{d}{d\chi}\left(U_N - W_N\right)\left(\chi\frac{d}{d\chi}V_N - V_N\right) - 2W_{N,l}\bar{k}^l\left(\chi\frac{d}{d\chi}V_N - V_N\right) - 2V_N\frac{d}{d\chi}W_N +
$$

$$-2V_N W_{N,m}\bar{k}^m + 2\chi \frac{d}{d\chi}\left(U_N - W_N\right)\frac{dV_N}{d\chi} + 2\chi W_{N,l}\bar{k}^l \frac{dV_N}{d\chi} = 0 \tag{4.121}$$

The remaining tensor contributions read

$$\begin{aligned}\mathscr{D}_{\mathbf{h_{ij}}ab} =& \delta_{ab}\left[\frac{1}{2\bar{k}^0}\int_0^{\chi_S} d\chi h_{ij}\bar{k}^i\bar{k}^j - \frac{1}{2(\bar{k}^0)^2}\int_0^{\chi_S} d\chi(\chi_S-\chi)h_{ij,l}\bar{k}^i\bar{k}^j\bar{k}^l\right] + \\ & \bar{n}_a^i\bar{n}_b^j\left[\frac{1}{2}\chi_S h_{ij} + \int_0^{\chi_S} d\chi\left[-h_{ij} - \frac{\chi}{2}\left(h_{jp,i}+h_{ip,j}\right)\bar{k}^p\right] + \right. \\ & \left. + \int_0^{\chi_S} d\chi(\chi_S-\chi)\left[\frac{\chi}{2}\left(h_{jp,i}+h_{ip,j}\right)\bar{k}^p + \frac{\chi}{2}h_{mp,ij}k^p k^m\right]\right]\end{aligned} \tag{4.122}$$

4.2.2 Redshift Perturbations

In the previous section 4.2.1 we have calculated the Jacobi mapping $\mathscr{D}_{ab}$ in an non-expanding universe, as a function of the coordinate χ_S. We now use eqs. (4.10) and (4.11) to calculate the Jacobi mapping $\tilde{\mathscr{D}}_{ab}$ in an expanding universe, as a function of the observed redshift z_S. Let us start by calculating the redshift perturbations up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$.

The redshift of a photon emitted at S and measured at O is given by

$$1+z_S = \frac{\left(\tilde{g}_{\mu\nu}\tilde{k}^\mu\tilde{u}^\nu\right)_S}{\left(\tilde{g}_{\mu\nu}\tilde{k}^\mu\tilde{u}^\nu\right)_O} = \frac{1}{a_S}\frac{\left(g_{\mu\nu}k^\mu u^\nu\right)_S}{\left(g_{\mu\nu}k^\mu u^\nu\right)_O} = \frac{1}{a_S}(1+\delta f)\,, \tag{4.123}$$

where we have defined

$$\delta f \equiv \frac{\left(g_{\mu\nu}k^\mu u^\nu\right)_S}{\left(g_{\mu\nu}k^\mu u^\nu\right)_O} - 1\,, \tag{4.124}$$

and we have used that under conformal transformation the photon wave vector transforms as $\tilde{k}^\mu = k^\mu/a^2$ and the four-velocity as $\tilde{u}^\mu = u^\mu/a$ (see e.g. [23] for more detail). Note that we normalise the scale factor to $a_O = 1$. Using that $1+\bar{z}_S = 1/a_S$ we can write

$$1+z_S = (1+\bar{z}_S)(1+\delta f) = (1+z_S-\delta z_S)(1+\delta f)\,, \tag{4.125}$$

leading to

$$\delta z_S = (1+z_S)\frac{\delta f}{1+\delta f}\,. \tag{4.126}$$

The perturbation δf depends on the metric potentials and the peculiar velocity at both the source and observer positions. However, as argued at the beginning of section 4.2.1 the metric perturbations at the observer do not contribute to the observed shear, convergence and rotation. The observer velocity would generate a global dipole variation in the convergence, which is degenerated with the vector contribution at the observer. This dipole can be subtracted from the observables, and we therefore do not consider it here.

Using eq. (4.124), the perturbation δf is calculated as a function of χ_S. We need to express it in terms of the observed redshift z_S. We obtain

$$\begin{aligned}\delta z_S =&(1+z_S)\delta F(\chi_S) = (1+z_S)\delta F(z_S - \delta z_S) \qquad (4.127)\\ =&(1+z_S)\left[\delta F(z_S) - \frac{d}{dz_S}\delta F(z_S)\delta z_S + \frac{1}{2}\frac{d^2}{dz_S^2}\delta F(z_S)\delta z_S^2 - \frac{1}{3!}\frac{d^3}{dz_S^3}\delta F(z_S)\delta z_S^3\right] + \mathscr{O}\left(\frac{1}{c^5}\right),\end{aligned}$$

where we have defined

$$\delta F = \frac{\delta f}{1+\delta f} \simeq \delta f - \delta f^2 + \delta f^3 - \delta f^4 + \dots . \qquad (4.128)$$

The perturbation δf depends on the four velocity u^μ. In the PF formalism it is given by [80]

$$\begin{aligned}u^i =&\frac{v^i}{c}u^0 \qquad (4.129)\\ u^0 =&1 + \frac{1}{c^2}\left(U_N + \frac{1}{2}v^2\right) + \frac{1}{c^4}\left(\frac{1}{2}U_N^2 + 2U_P + v^2 V_N + \frac{3}{2}v^2 U_N + \frac{3}{8}v^4 - B_{Ni}v^i\right), \qquad (4.130)\end{aligned}$$

where $v^2 = \sum_{ij}\delta_{ij}v_i v_j$. Inserting this into (4.124) and neglecting terms at the observer we obtain

$$\begin{aligned}\delta f(\chi_S) = &\left(g_{00}k^0u^0 + g_{0i}k^0u^i + g_{0i}k^iu^0 + g_{ij}k^iu^j\right)(\chi_S) - 1\\ =&e^{-2U_N\frac{1}{c^2} - 4U_P\frac{1}{c^4}}\frac{k^0}{\bar{k}^0}u^0 + B_{Ni}\frac{1}{c^3}\frac{k^0}{\bar{k}^0}\frac{1}{c}v^iu^0 + B_{Ni}\frac{1}{c^3}\frac{k^i}{\bar{k}^0}u^0\\ &- e^{2V_N\frac{1}{c^2} + 4V_P\frac{1}{c^4}}\delta_{ij}\frac{k^i}{\bar{k}^0}\frac{1}{c}v^iu^0 - 1, \qquad (4.131)\end{aligned}$$

where all terms are evaluated at the source position χ_S.

The derivation of δF, δf and δz_S

In this section we will compute the different orders of δz_S, δF, as well as δf using

$$\delta z_S =(1+z_S)\left[\delta F(z_S) - \frac{d}{dz_S}\delta F(z_S)\delta z_S + \frac{1}{2}\frac{d^2}{dz_S^2}\delta F(z_S)\delta z_S^2 - \frac{1}{3!}\frac{d^3}{dz_S^3}\delta F(z_S)\delta z_S^3\right] + \qquad (4.132)$$

$$+\mathscr{O}\left(\frac{1}{c^5}\right) \tag{4.133}$$

and

$$\delta F = \frac{\delta f}{1+\delta f} = \delta f - \delta f^2 + \delta f^3 - \delta f^4 + \dots, \tag{4.134}$$

and

$$\delta f \equiv \frac{g_{\mu\nu} k^\mu u^\nu|_S}{g_{\mu\nu} k^\mu u^\nu|_O} - 1. \tag{4.135}$$

δf up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ reads

$$\delta f^{(1)} = -\frac{1}{c} v_{S\|} \tag{4.136}$$

$$\delta f^{(2)} = \frac{1}{c^2}\left(U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2} v_S^2\right) \tag{4.137}$$

$$\delta f^{(3)} = \frac{1}{c^3}\left[\int_0^{\chi_S} d\chi B_{Ni,0}\bar{k}^i - v_{S\|}\left(V_{NS} + \frac{1}{2} v_S^2\right) - v_S^i \delta_{ij} \int_0^{\chi_S} d\chi W_N^{;i}\right] \tag{4.138}$$

$$\begin{aligned}
\delta f^{(4)} = & \frac{1}{c^4}\left\{2U_{PS} - 4\int_0^{\chi_S} d\chi W_{P,0} - \frac{3}{2} U_{NS}^2 + 2\left(\int_0^{\chi_S} d\chi W_{N,0}\right) - \frac{1}{2}\int_0^{\chi_S} d\chi h_{ij,0} \frac{\bar{k}^i \bar{k}^j}{\left(\bar{k}^0\right)^2} + \right.\\
& + v_S^2\left(V_{NS} + \frac{1}{2} U_{NS} + \frac{3}{8}\right) - B_{NSi} v^i + v_S^i \int_0^{\chi} d\chi B_{Nm}{}^{,i} \frac{\bar{k}^m}{\bar{k}^0} + \\
& - \frac{4}{c^4}\left(U_{NS,i} \frac{\bar{k}^i}{\bar{k}^0} \int_0^{\chi} d\chi W_N\right) + \frac{4}{c^4}\int_0^{\chi_S} d\chi U_{N,i} \frac{\bar{k}^i}{\bar{k}^0} W_N + \\
& + \frac{4}{c^4}\int_0^{\chi_S} d\chi \left(W_{N,0i} \frac{\bar{k}^i}{\bar{k}^0} \int_0^{\chi} d\chi' W_N\right) + \\
& + \frac{4}{c^4} U_{NS,i} \int_0^{\chi_S}\int_0^{\chi} d\chi d\chi' \left(W_N^{;i} - W_{N,j} \frac{\bar{k}^i \bar{k}^j}{\left(\bar{k}^0\right)^2}\right) + \\
& - \frac{4}{c^4}\int_0^{\chi_S} d\chi \left[U_{N,i} \int_0^{\chi} d\chi' \left(W_N^{;i} - W_{N,j} \frac{\bar{k}^i \bar{k}^j}{\left(\bar{k}^0\right)^2}\right)\right] + \\
& \left. - \frac{4}{c^4}\int_0^{\chi_S} d\chi \left[W_{N,0i} \int_0^{\chi}\int_0^{\chi''} d\chi d\chi' \left(W_N^{;i} - W_{N,j} \frac{\bar{k}^i \bar{k}^j}{\left(\bar{k}^0\right)^2}\right)\right]\right\}
\end{aligned} \tag{4.139}$$

For δF up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$, we obtain

$$\delta F^{(1)} = \delta f^{(1)} = -\frac{1}{c} v_{S\|} \tag{4.140}$$

$$\delta F^{(2)} = \delta f^{(2)} - \delta f^{(1)2}$$

$$=\frac{1}{c^2}\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2-v_{S\|}^2\right) \tag{4.141}$$

$$\delta F^{(3)}=\delta f^{(3)}-2\delta f^{(2)}\delta f^{(1)}+\delta f^{(1)3} \tag{4.142}$$

$$=\frac{1}{c^3}\left[\int_0^{\chi_S}d\chi B_{Ni,0}\bar{k}^i-v_{S\|}\left(V_{NS}+\frac{1}{2}v_S^2\right)-v_S^i\delta_{ij}\int_0^{\chi_S}d\chi W_N^{,i}\right.$$
$$\left.+2\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2\right)v_{S\|}-v_{S\|}^3\right] \tag{4.143}$$

$$\delta F^{(4)}=\delta f^{(4)}-\delta f^{(2)2}+3\delta f^{(2)}\delta f^{(1)2}-\delta^{(1)4} \tag{4.144}$$

$$=\frac{1}{c^4}\left\{2U_{PS}-4\int_0^{\chi_S}d\chi W_{P,0}-\frac{3}{2}U_{NS}^2+2\left(\int_0^{\chi_S}d\chi W_{N,0}\right)+\right.$$
$$-\frac{1}{2}\int_0^{\chi_S}d\chi h_{ij,0}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2}+v_S^2\left(V_{NS}+\frac{1}{2}U_{NS}+\frac{3}{8}\right)-B_{NSi}v^i+$$
$$+v_S^i\int_0^{\chi}d\chi B_{Nm}^{,i}\frac{\bar{k}^m}{\bar{k}^0}-\frac{4}{c^4}\left(U_{NS,i}\frac{\bar{k}^i}{\bar{k}^0}\int_0^{\chi_S}d\chi W_N\right)+$$
$$+\frac{4}{c^4}\int_0^{\chi_S}d\chi U_{N,i}\frac{\bar{k}^i}{\bar{k}^0}W_N+\frac{4}{c^4}\int_0^{\chi_S}d\chi\left(W_{N,0i}\frac{\bar{k}^i}{\bar{k}^0}\int_0^{\chi}d\chi' W_N\right)+$$
$$+\frac{4}{c^4}U_{NS,i}\int_0^{\chi_S}\int_0^{\chi'}d\chi d\chi'\left(W_N^{,i}-W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2}\right)+$$
$$-\frac{4}{c^4}\int_0^{\chi_S}d\chi\left[U_{N,i}\int_0^{\chi}d\chi'\left(W_N^{,i}-W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2}\right)\right]+$$
$$-\frac{4}{c^4}\int_0^{\chi_S}d\chi\left[W_{N,0i}\int_0^{\chi}\int_0^{\chi'}d\chi' d\chi''\left(W_N^{,i}-W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2}\right)\right]+$$
$$-\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2\right)^2+$$
$$\left.+3\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2\right)v_{S\|}^2-v_{S\|}^4\right\}. \tag{4.145}$$

The derivatives w.r.t. z_S in (4.132) read

$$-\frac{d}{dz_S}\delta F(z_S)^{(1)}=-\frac{d\chi_S}{dz_S}\frac{d}{d\chi_S}\delta F(z_S)^{(1)}=\frac{a_S^2}{a_S'}v_{S\|}'\frac{1}{c}=-a_S\frac{c}{\mathscr{H}_S c}v_{S\|}' \tag{4.146}$$

$$-\frac{d}{dz_S}\delta F(z_S)^{(2)}=-\frac{d\chi_S}{dz_S}\frac{d}{\chi_S}\delta F(z_S)^{(2)}=\frac{a_S^2}{a_S'}\frac{d}{\chi_S}\delta F(z_S)^{(2)}$$
$$=-\frac{a_S c}{\mathscr{H}_S}\frac{1}{c^2}\left(\frac{dU_{NS}}{d\chi_S}-2W_{NS,0}+v_Sv_S'-2v_{S\|}v_{S\|}'\right) \tag{4.147}$$

$$-\frac{d}{dz_S}\delta F(z_S)^{(3)}=-\frac{a_S c}{\mathscr{H}_S}\frac{1}{c^3}\left[B_{NSi,0}\bar{k}^i-v_{S\|}'\left(V_{NS}+\frac{1}{2}v_S^2\right)+\right.$$

$$
\begin{aligned}
& - v_{S\|}\left(\frac{d}{d\chi_S}V_{NS} + v_S v_S' v_{S\|}'\right) - v_S^{i\prime}\delta_{ij}\int_0^{\chi_S} d\chi W_N^{;i} \\
& - v_S^i \delta_{ij} W_{NS}^{;i} + 2\left(\frac{d}{d\chi_S}U_{NS} - 2W_{NS,0} + v_S v_S'\right) v_{S\|} + \\
& + 2\left(U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2}v_S^2\right) v_{S\|}' - 3v_{S\|}^2 v_{S\|}'\Big] \qquad (4.148)
\end{aligned}
$$

$$
\frac{d^2}{dz_S^2}\delta F^{(1)} = \frac{d\chi_S}{dz_S}\frac{d}{d\chi_S}\left(a_S\frac{c}{c\mathscr{H}_S}v_{S\|}'\right) = \frac{a_S^2 c}{c\mathscr{H}_S}\left[v_{S\|}'\left(1 + \frac{\mathscr{H}_S' c}{\mathscr{H}_S^2}\right) - \frac{c}{\mathscr{H}_S}v_{S\|}''\right] \tag{4.149}
$$

$$
\begin{aligned}
\frac{d^2}{dz_S^2}\delta F^{(2)} = & -\frac{1}{c^2}\frac{a_S^2 c}{\mathscr{H}_S}\left\{\left(1 + \frac{\mathscr{H}_S' c}{\mathscr{H}_S^2}\right)\left[\frac{dU_{NS}}{d\chi_S} - 2W_{NS,0} + v_S v_S' - 2v_{S\|}v_{S\|}'\right] + \right. \\
& \left. - \frac{c}{\mathscr{H}_S}\left[\frac{d^2U_{NS}}{d\chi_S^2} - 2\frac{dW_{NS,0}}{d\chi_S} + v_S'^2 + v_S v_S'' - 2v_{S\|}'^2 - 2v_{S\|}v_{S\|}''\right]\right\}
\end{aligned} \tag{4.150}
$$

$$
\begin{aligned}
\frac{d^3}{dz_S^3}\delta F^{(1)} = & \frac{a_S^3 c}{c\mathscr{H}_S}\left\{\left(2 + \frac{\mathscr{H}_S' c}{\mathscr{H}_S^2}\right)\left[v_{S\|}'\left(1 + \frac{\mathscr{H}_S' c}{\mathscr{H}_S^2}\right) - \frac{c}{\mathscr{H}_S}v_{S\|}''\right] + \right. \qquad (4.151) \\
& \left. - \frac{c}{\mathscr{H}_S}\left[v_{S\|}''\left(1 + \frac{\mathscr{H}_S' c}{\mathscr{H}_S^2}\right) + v_{S\|}'\left(\frac{\mathscr{H}_S'' c}{\mathscr{H}_S^2} + \frac{2\mathscr{H}_S'^2}{\mathscr{H}_S^2}\right) - \frac{v_{S\|}''' c}{\mathscr{H}_S} + \frac{\mathscr{H}_S' v_{S\|}'' c}{\mathscr{H}_S^2}\right]\right\}
\end{aligned} \tag{4.152}
$$

Note that the prime denotes the derivative w.r.t. the parameter χ. However, we define $\mathscr{H} \equiv \dot{a}(\eta)/a(\eta)$ using the derivative w.r.t. the conformal time η. Since $\frac{d}{d\chi} = -\frac{1}{c}\frac{d}{d\eta}$, every $\mathscr{H}$ comes with a factor $\left(-\frac{1}{c}\right)$. This factor c does not change the order of the expression because it only appears due to the convention we choose for $\mathscr{H}$.

Using (4.136) - (4.152), we obtain for δz_S up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ the following:

$$
\delta z_S^{(1)} = (1+z_S)\delta F^{(1)} = -(1+z_S)\frac{1}{c}v_{S\|} \tag{4.153}
$$

$$
\begin{aligned}
\delta z_S^{(2)} = & (1+z_S)\left(\delta F^{(2)} - \frac{d}{dz_S}\delta F^{(1)}\delta z^{(1)}\right) \\
= & (1+z_S)\frac{1}{c^2}\left[U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2}v_S^2 - v_{S\|}^2 + \frac{c}{\mathscr{H}_S}v_{S\|}' v_{S\|}\right]
\end{aligned} \tag{4.154}
$$

$$
\delta z_S^{(3)} = (1+z_S)\left(\delta F^{(3)} - \frac{d}{dz_S}\delta F^{(1)}\delta z^{(2)} - \frac{d}{dz_S}\delta F^{(2)}\delta z^{(1)} + \frac{1}{2}\frac{d^2}{dz_S^2}\delta F^{(1)}\delta z^{(1)2}\right) \tag{4.155}
$$

$$
\begin{aligned}
= & (1+z_S)\frac{1}{c^3}\left\{\int_0^{\chi_S} d\chi B_{Ni,0}\bar{k}^i - v_{S\|}\left(V_{NS} + \frac{1}{2}v_S^2\right) - v_S^i\delta_{ij}\int_0^{\chi_S} d\chi W_N^{;i}\right. \\
& + 2\left(U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2}v_S^2\right) v_{S\|} - v_{S\|}^3 +
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{c}{\mathscr{H}_S}v'_{S\|}\left[U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2-v_{S\|}^2+\frac{1}{\mathscr{H}_S}v'_{S\|}v_{S\|}\right]+\\
&+\frac{c}{\mathscr{H}_S}\left[\frac{dU_{NS}}{d\chi}-2W_{NS,0}+v_Sv'_S-2v_{S\|}v'_{S\|}\right]v_{S\|}+\\
&+\frac{c}{2\mathscr{H}_S}\left[v'_{S\|}\left(1+\frac{\mathscr{H}'_Sc}{\mathscr{H}_S^2}\right)-\frac{c}{\mathscr{H}_S}v''_{S\|}\right]v_{S\|}^2\Bigg\}
\end{aligned}
\tag{4.156}
$$

$$
\delta z_S^{(4)}=(1+z_S)\left[\delta F^{(4)}-\frac{d}{dz_S}\delta F^{(1)}\delta z^{(3)}-\frac{d}{dz_S}\delta F^{(2)}\delta z^{(2)}-\frac{d}{dz_S}\delta F^{(3)}\delta z^{(1)}+\right.
\tag{4.157}
$$

$$
\begin{aligned}
&\left.+\frac{1}{2}\frac{d^2}{dz_S^2}\delta F^{(2)}\delta z^{(1)2}+\frac{d^2}{dz_S^2}\delta F^{(1)}\delta z^{(1)}\delta z^{(2)}-\frac{1}{6}\frac{d^3}{dz_S^3}\delta F^{(1)}\delta z^{(1)3}\right]\\
&=(1+z_S)\frac{1}{c^4}\Bigg\{2U_{PS}-4\int_0^{\chi_S}d\chi W_{P,0}-\frac{5}{2}U_{NS}^2-2\left(\int_0^{\chi_S}d\chi W_{N,0}\right)^2+\\
&+4U_{NS}\int_0^{\chi_S}d\chi W_{NS,0}+4U_{NS,i}\int_0^{\chi_S}d\chi\,(\chi_S-\chi)\left(W_N^{,i}-W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2}\right)+\\
&+3\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2\right)v_{S\|}^2-v_{S\|}^4+\\
&-\frac{1}{2}\int_0^{\chi_S}d\chi h_{ij,0}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2}+v_S^2\left(V_{NS}-\frac{1}{2}U_{NS}+\frac{1}{8}v_S^2-2\int_0^{\chi_S}d\chi W_{N,0}\right)+\\
&-B_{NSi}v_S^i+v_S^i\int_0^{\chi_S}d\chi B_{Nm}^{\ ,i}\frac{\bar{k^m}}{\bar{k}^0}-4U_{NS,i}\frac{\bar{k}^i}{\bar{k}^0}\int_0^{\chi_S}d\chi''W_N+\\
&+\int_0^{\chi_S}d\chi\left[4U_{N,i}\frac{\bar{k}^i}{\bar{k}^0}W_N+4\left(W_{N,0i}\frac{\bar{k}^i}{\bar{k}^0}\int_0^{\chi}d\chi'W_N\right)+\right.\\
&-4U_{N,i}\int_0^{\chi}d\chi'\left(W_N^{,i}-W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2}\right)+\\
&\left.-4W_{N,0i}\int_0^{\chi}d\chi'\,(\chi-\chi')\left(W_N^{,i}-W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2}\right)\right]+\\
&-\frac{c}{\mathscr{H}_S}v'_{S\|}\Bigg\{\int_0^{\chi_S}d\chi B_{Ni,0}\bar{k}^i-v_{S\|}\left(V_{NS}+\frac{1}{2}v_S^2\right)-v_S^i\delta_{ij}\int_0^{\chi_S}d\chi W_N^{,i}\\
&+2\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2\right)v_{S\|}-v_{S\|}^3+\\
&+\frac{c}{\mathscr{H}_S}v'_{S\|}\left[U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2-v_{S\|}^2+\frac{1}{\mathscr{H}_S}v'_{S\|}v_{S\|}\right]+\\
&-\frac{c}{\mathscr{H}_S}\left[\frac{dU_{NS}}{d\chi}-2W_{NS,0}+v_sv'_S-2v_{S\|}v'_{S\|}\right]v_{S\|}+\\
&+\frac{c}{2\mathscr{H}_S}\left[v'_{S\|}\left(1+\frac{\mathscr{H}'_Sc}{\mathscr{H}_S^2}\right)-\frac{c}{\mathscr{H}_S}v''_{S\|}\right]v_{S\|}^2\Bigg\}+\frac{c}{\mathscr{H}_S}\left[\frac{dU_{NS}}{d\chi}-2W_{NS,0}+v_sv'_S\right.
\end{aligned}
$$

$$-2v_{S\|}v'_{S\|}\Big]\left[U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2-v_{S\|}^2-\frac{c}{\mathscr{H}_S}v'_{S\|}v_{S\|}\right]+$$
$$+\frac{c}{\mathscr{H}_S}\left[B_{NSi,0}\bar{k}^i-v'_{S\|}\left(V_{NS}+\frac{1}{2}v_S^2\right)-v_{S\|}\left(\frac{d}{d\chi}V_{NS}+v_Sv'_Sv'_{S\|}\right)+\right.$$
$$-v_S^{i\prime}\delta_{ij}\int_0^{\chi_S}d\chi W_N^{,i}-v_S^i\delta_{ij}W_{NS}^{,i}+2\left(\frac{d}{d\chi}U_{NS}-2W_{NS,0}+v_Sv'_S\right)v_{S\|}+$$
$$+2\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2\right)v'_{S\|}-3v_{S\|}^2v'_{S\|}\Big]v_{S\|}+$$
$$+\frac{1}{2}\frac{v_{S\|}^2c}{\mathscr{H}_S}\left\{\left(1+\frac{\mathscr{H}'_Sc}{\mathscr{H}_S^2}\right)\left[\frac{dU_{NS}}{d\chi_S}-2W_{NS,0}+\left(v_Sv'_S-2v_{S\|}v'_{S\|}\right)\right]+\right.$$
$$\left.-\frac{c}{\mathscr{H}_S}\left[\frac{d^2U_{NS}}{d\chi_S^2}-2\frac{dW_{NS,0}}{d\chi_S}+\left(v_S'^2+v_Sv''_S-2v_{S\|}'^2-2v_{S\|}v''_{S\|}\right)\right]\right\}+$$
$$-\frac{v_{S\|}c}{\mathscr{H}_S}\left[v'_{S\|}\left(1+\frac{\mathscr{H}'_Sc}{\mathscr{H}_S^2}\right)-\frac{c}{\mathscr{H}_S}v''_{S\|}\right]\left[U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2+\right.$$
$$\left.-v_{S\|}^2+\frac{c}{\mathscr{H}_S}v'_{S\|}v_{S\|}\right]+\frac{v_{S\|}^3}{6}\frac{c}{\mathscr{H}_S}\left\{\left(2+\frac{\mathscr{H}'_Sc}{\mathscr{H}_S^2}\right)\left[v'_{S\|}\left(1+\frac{\mathscr{H}'_Sc}{\mathscr{H}_S^2}\right)-\frac{c}{\mathscr{H}_S}v''_{S\|}\right]+\right.$$
$$\left.-\frac{c}{\mathscr{H}_S}\left[v''_{S\|}\left(1+\frac{\mathscr{H}'_Sc}{\mathscr{H}_S^2}\right)+v'_{S\|}\left(\frac{\mathscr{H}''_Sc}{\mathscr{H}_S^2}+\frac{2\mathscr{H}_S'^2}{\mathscr{H}_S^2}\right)-\frac{v'''_{S\|}c}{\mathscr{H}_S}+\frac{\mathscr{H}'_Sv''_{S\|}c}{\mathscr{H}_S^2}\right]\right\}. \tag{4.158}$$

From eq. (4.11) we see that to calculate $\tilde{D}_{ab}$ up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ we need the first derivative of $\tilde{D}_{ab}$ up to order $\mathscr{O}\left(\frac{1}{c^3}\right)$, the second derivative up to order $\mathscr{O}\left(\frac{1}{c^2}\right)$ and the third and fourth derivatives for the background $\tilde{D}_{ab}$ only.

$$\frac{d}{dz_S}\tilde{\mathscr{D}}_{ab}(z_S)=\frac{\chi_S}{(1+z_S)^2}\left(\frac{c}{\mathscr{H}_S\chi_S}-1\right)\delta_{ab} \tag{4.159}$$
$$+\frac{1}{(1+z_S)^2}\left[-(1+z_S)\,\delta\tilde{\mathscr{D}}_{ab}+\frac{c}{\mathscr{H}_S}\frac{d}{d\chi_S}((1+z_S)\delta\tilde{\mathscr{D}}_{ab})\right]$$
$$\frac{d^2}{dz_S^2}\tilde{\mathscr{D}}_{ab}(z_S)=\frac{\chi_S}{(z_S+1)^3}\left(2-\frac{\mathscr{H}'_Sc^2}{\chi_S\mathscr{H}_S^3}-\frac{3c}{\mathscr{H}_S\chi_S}\right)\delta_{ab}+ \tag{4.160}$$
$$+\frac{1}{(z_S+1)^3}\left[2(1+z_S)\delta\tilde{\mathscr{D}}_{ab}-\frac{d}{d\chi_S}\left((1+z_S)\delta\tilde{\mathscr{D}}_{ab}\right)\frac{c}{\mathscr{H}_S}\left(3+\frac{\mathscr{H}'c}{\mathscr{H}^2}\right)+\right.$$
$$\left.+\frac{c^2}{\mathscr{H}_S^2}\frac{d^2}{d\chi_S^2}\left((1+z_S)\delta\tilde{\mathscr{D}}_{ab}\right)\right] \tag{4.161}$$
$$\frac{d^3}{dz_S^3}\tilde{\tilde{\mathscr{D}}}_{ab}(z_S)=\delta_{ab}\frac{\chi_S}{(z_S+1)^4}\left(\frac{11c}{\chi_S\mathscr{H}_S}-6-\frac{\mathscr{H}''_Sc^3}{\chi_S\mathscr{H}_S^4}+\frac{3c^3\mathscr{H}'^2}{\chi_S\mathscr{H}_S^5}+\frac{6c^2\mathscr{H}'_S}{\chi_S\mathscr{H}_S^3}\right) \tag{4.162}$$

$$\frac{d^4}{dz_S^4}\bar{\tilde{\mathscr{D}}}_{ab}(z_S) = \delta_{ab}\frac{\chi_S}{(z_S+1)^5}\left(24 - \frac{\mathscr{H}_S'''c^4}{\chi_S\mathscr{H}_S^5} + \frac{10c^3\mathscr{H}_S''}{\chi_S\mathscr{H}^4} - \frac{15c^4\mathscr{H}_S'^3}{\chi_S\mathscr{H}_S^7} - \frac{30c^3\mathscr{H}_S'^2}{\chi_S\mathscr{H}_S^5} + \right.$$
$$\left. - \frac{35c^2\mathscr{H}_S'}{\chi_S\mathscr{H}_S^3} + \frac{10c^4\mathscr{H}_S'\mathscr{H}_S''}{\chi_S\mathscr{H}^6} - \frac{50c}{\chi_S\mathscr{H}}\right). \tag{4.163}$$

The expression for the Jacobi mapping $\tilde{\mathscr{D}}_{ab}(z_S)$ in terms of the redshift z_S in (4.11) reads for the orders $\mathscr{O}\left(\frac{1}{c}\right)$ - $\mathscr{O}\left(\frac{1}{c^4}\right)$

$$\tilde{\mathscr{D}}_{ab}^{(1)}(\chi_S) = \tilde{\mathscr{D}}_{ab}^{(1)}(\bar{z}_S) = -\frac{d}{d\bar{z}_S}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(1)} \tag{4.164}$$

$$\tilde{\mathscr{D}}_{ab}^{(2)}(\chi_S) = \tilde{\mathscr{D}}_{ab}^{(2)}(z_S) - \frac{d}{d\bar{z}_S}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(2)} + \frac{1}{2}\frac{d^2}{d\bar{z}_S^2}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(1)2}, \tag{4.165}$$

$$\tilde{\mathscr{D}}_{ab}^{(3)}(\bar{z}_S) = \tilde{\mathscr{D}}_{ab}^{(3)}(z_S) - \frac{d}{d\bar{z}_S}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(3)} - \frac{d}{d\bar{z}_S}\tilde{\mathscr{D}}_{ab}^{(2)}(z_S)\,\delta z_S^{(1)} +$$
$$+ \frac{d^2}{d\bar{z}_S^2}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(1)}\delta z_S^{(2)} - \frac{1}{6}\frac{d^3}{d\bar{z}_S^3}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(1)3}, \text{ and} \tag{4.166}$$

$$\tilde{\mathscr{D}}_{ab}^{(4)}(\bar{z}_S) = \tilde{\mathscr{D}}_{ab}^{(4)}(z_S) - \frac{d}{d\bar{z}_S}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(4)} - \frac{d}{d\bar{z}_S}\tilde{\mathscr{D}}_{ab}^{(2)}(z_S)\,\delta z_S^{(2)} +$$
$$- \frac{d}{d\bar{z}_S}\tilde{\mathscr{D}}_{ab}^{(3)}(z_S)\,\delta z S^{(1)} + \frac{1}{2}\frac{d^2}{d\bar{z}_S^2}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\left(\delta z_S^{(2)}\right)^2 +$$
$$+ \frac{1}{2}\frac{d^2}{d\bar{z}_S^2}\tilde{\mathscr{D}}_{ab}^{(2)}(z_S)\left(\delta z_S^{(1)}\right)^2 + \frac{d^2}{d\bar{z}_S^2}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(1)}\delta z^{(3)} +$$
$$- \frac{1}{2}\frac{d^3}{d\bar{z}_S^3}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(2)}\left(\delta z_S^{(1)}\right)^2 + \frac{1}{4!}\frac{d^4}{d\bar{z}_S^4}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\left(\delta z_S^{(1)}\right)^4. \tag{4.167}$$

Due to the length of the full expression of $\tilde{\mathscr{D}}_{ab}(z_S)$ in (4.167) we list the different terms in this section of the appendix:

$$-\frac{d}{d\bar{z}_S}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(4)} = \frac{\chi_S}{(1+z_S)}\left(1 - \frac{c}{\mathscr{H}_S\chi_S}\right)\delta_{ab}\frac{1}{c^4}\left\{2U_{PS} - 4\int_0^{\chi_S} d\chi W_{P,0} - \frac{5}{2}U_{NS}^2 + \right.$$
$$- 2\left(\int_0^{\chi_S} d\chi W_{N,0}\right)^2 + 4U_{NS}\int_0^{\chi_S} d\chi W_{NS,0} +$$
$$+ 4U_{NS,i}\int_0^{\chi_S} d\chi\,(\chi_S - \chi)\left(W_N^{;i} - W_{N,j}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2}\right) +$$
$$+ 3\left(U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2}v_S^2\right)v_{S\parallel}^2 - v_{S\parallel}^4 +$$
$$- \frac{1}{2}\int_0^{\chi_S} d\chi h_{ij,0}\frac{\bar{k}^i\bar{k}^j}{\left(\bar{k}^0\right)^2} + v_S^2\left(V_{NS} - \frac{1}{2}U_{NS} + \frac{1}{8}v_S^2 - 2\int_0^{\chi_S} d\chi W_{N,0}\right) +$$

$$
\begin{aligned}
&- B_{NSi} v_S^i + v_S^i \int_0^{\chi_S} d\chi B_{Nm}{}^{,i} \frac{\bar{k}^m}{\bar{k}^0} - 4U_{NS,i} \frac{\bar{k}^i}{\bar{k}^0} \int_0^{\chi_S} d\chi'' W_N + \\
&+ \int_0^{\chi_S} d\chi \left[4U_{N,i} \frac{\bar{k}^i}{\bar{k}^0} W_N + 4 \left(W_{N,0i} \frac{\bar{k}^i}{\bar{k}^0} \int_0^{\chi} d\chi' W_N \right) + \right. \\
&- 4U_{N,i} \int_0^{\chi} d\chi' \left(W_N^{;i} - W_{N,j} \frac{\bar{k}^i \bar{k}^j}{(\bar{k}^0)^2} \right) + \\
&\left. -4W_{N,0i} \int_0^{\chi} d\chi' \int_0^{\chi'} d\chi'' \left(W_N^{;i} - W_{N,j} \frac{\bar{k}^i \bar{k}^j}{(\bar{k}^0)^2} \right) \right] + \\
&- \frac{c}{\mathscr{H}_S} v'_{S\|} \left\{ \int_0^{\chi_S} d\chi B_{Ni,0} \bar{k}^i - v_{S\|} \left(V_{NS} + \frac{1}{2} v_S^2 \right) - v_S^i \delta_{ij} \int_0^{\chi_S} d\chi W_N^{;i} \right. \\
&+ 2 \left(U_{NS} - 2 \int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2} v_S^2 \right) v_{S\|} - v_{S\|}^3 + \\
&+ \frac{c}{\mathscr{H}_S} v'_{S\|} \left[U_{NS} - 2 \int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2} v_S^2 - v_{S\|}^2 + \frac{1}{\mathscr{H}_S} v'_{S\|} v_{S\|} \right] + \\
&- \frac{c}{\mathscr{H}_S} \left[\frac{dU_{NS}}{d\chi} - 2W_{NS,0} + v_S v'_S - 2v_{S\|} v'_{S\|} \right] v_{S\|} + \\
&\left. + \frac{c}{2\mathscr{H}_S} \left[v'_{S\|} \left(1 + \frac{\mathscr{H}'_S c}{\mathscr{H}_S^2} \right) - \frac{c}{\mathscr{H}_S} v''_{S\|} \right] v_{S\|}^2 \right\} + \frac{c}{\mathscr{H}_S} \left[\frac{dU_{NS}}{d\chi} - 2W_{NS,0} + v_S v'_S \right. \\
&\left. -2v_{S\|} v'_{S\|} \right] \left[U_{NS} - 2 \int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2} v_S^2 - v_{S\|}^2 - \frac{c}{\mathscr{H}_S} v'_{S\|} v_{S\|} \right] + \\
&+ \frac{c}{\mathscr{H}_S} \left[B_{NSi,0} \bar{k}^i - v'_{S\|} \left(V_{NS} + \frac{1}{2} v_S^2 \right) + \right. \\
&- v_{S\|} \left(\frac{d}{d\chi} V_{NS} + v_S v'_S \right) - v_S^{i\prime} \delta_{ij} \int_0^{\chi_S} d\chi W_N^{;i} \\
&- v_S^i \delta_{ij} W_{NS}^{;i} + 2 \left(\frac{d}{d\chi} U_{NS} - 2W_{NS,0} + v_S v'_S \right) v_{S\|} + \\
&\left. +2 \left(U_{NS} - 2 \int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2} v_S^2 \right) v'_{S\|} - 3v_{S\|}^2 v'_{S\|} \right] v_{S\|} + \\
&+ \frac{1}{2} \frac{v_{S\|}^2 c}{\mathscr{H}_S} \left\{ \left(1 + \frac{\mathscr{H}'_S c}{\mathscr{H}_S^2} \right) \left[\frac{dU_{NS}}{d\chi_S} - 2W_{NS,0} + \left(v_S v'_S - 2v_{S\|} v'_{S\|} \right) \right] + \right. \\
&\left. - \frac{c}{\mathscr{H}_S} \left[\frac{d^2 U_{NS}}{d\chi_S^2} - 2 \frac{dW_{NS,0}}{d\chi_S} + \left(v_S'^2 + v_S v''_S - 2v_{S\|}'^2 - 2v_{S\|} v''_{S\|} \right) \right] \right\} + \\
&- \frac{v_{S\|} c}{\mathscr{H}_S} \left[v'_{S\|} \left(1 + \frac{\mathscr{H}'_S c}{\mathscr{H}_S^2} \right) - \frac{c}{\mathscr{H}_S} v''_{S\|} \right] \left[U_{NS} - 2 \int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2} v_S^2 - v_{S\|}^2 + \frac{c}{\mathscr{H}_S} v'_{S\|} v_{S\|} \right] + \\
&+ \frac{v_{S\|}^3}{6} \frac{c}{\mathscr{H}_S} \left\{ \left(2 + \frac{\mathscr{H}'_S c}{\mathscr{H}_S^2} \right) \left[v'_{S\|} \left(1 + \frac{\mathscr{H}'_S c}{\mathscr{H}_S^2} \right) - \frac{c}{\mathscr{H}_S} v''_{S\|} \right] + \right.
\end{aligned}
$$

$$-\frac{c}{\mathcal{H}_S}\left[v''_{S\|}\left(1+\frac{\mathcal{H}'_S c}{\mathcal{H}_S^2}\right)+v'_{S\|}\left(\frac{\mathcal{H}''_S c}{\mathcal{H}_S^2}+\frac{2\mathcal{H}'^2_S}{\mathcal{H}_S^2}\right)-\frac{v'''_{S\|}c}{\mathcal{H}_S}+\frac{\mathcal{H}'_S v''_{S\|}c}{\mathcal{H}_S^2}\right]\Bigg\}, \tag{4.168}$$

$$-\frac{d}{d\bar{z}_S}\tilde{\mathscr{D}}^{(2)}_{ab}(z_S)\,\delta\tilde{z}^{(2)}_S=\delta_{ab}\frac{\chi_S}{1+z_S}\left[\left(V_{NS}-2\frac{1}{\chi_S}\int_0^{\chi_S}d\chi\left[W_N+(\chi_S-\chi)W_{N,i}\frac{\bar{k}^i}{\bar{k}^0}\right]\right)+\right.$$
$$-\frac{c}{\chi_S\mathcal{H}_S}\left(V_{NS}+\chi_S\frac{dV_{NS}}{d\chi_S}-2\left[W_{NS}+\int_0^{\chi_S}d\chi W_{N,i}\frac{\bar{k}^i}{\bar{k}^0}\right]\right)\Bigg]\left[U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\right.$$
$$\left.+\frac{1}{2}v_S^2-v_{S\|}^2+\frac{c}{\mathcal{H}_S}v'_{S\|}v_{S\|}\right]+$$
$$-\bar{n}^i_a\bar{n}^j_b\frac{\chi_S}{1+z_S}\left[\frac{1}{\chi_S}\left(2\int_0^{\chi_S}d\chi\,(\chi_S-\chi)\,\chi W_{N,ij}\right)\right.$$
$$\left.-\frac{c}{\chi_S\mathcal{H}_S}2\int_0^{\chi_S}d\chi\chi W_{N,ij}\right]\left[U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2-v_{S\|}^2+\frac{c}{\mathcal{H}_S}v'_{S\|}v_{S\|}\right], \tag{4.169}$$

$$-\frac{d}{d\bar{z}_S}\tilde{\mathscr{D}}^{(3)}_{ab}(z_S)\,\delta\tilde{z}^{(1)}_S=\delta_{ab}\frac{\chi_S}{1+z_S}\left[-\frac{v_{S\|}}{\chi_S}\int_0^{\chi_S}d\chi\,(\chi_S-\chi)\,B_{Ni,j}\frac{\bar{k}^j\bar{k}^i}{\left(\bar{k}^0\right)^2}\right.$$
$$\left.+\frac{cv_{S\|}}{\chi_S\mathcal{H}_S}\int_0^{\chi_S}d\chi B_{Ni,j}\frac{\bar{k}^j\bar{k}^i}{\left(\bar{k}^0\right)^2}\right]+$$
$$+\bar{n}^i_a\bar{n}^j_b\frac{\chi_S}{1+z_S}\left[-\frac{v_{S\|}}{\chi_S}\int_0^{\chi_S}d\chi\,(\chi_S-\chi)\,\chi\left(\frac{dB^N_{(i,j)}}{d\chi}-\frac{\bar{k}^m}{\bar{k}^0}B^N_{m,ij}\right)\right.$$
$$\left.+\frac{cv_{S\|}}{\chi_S\mathcal{H}_S}\int_0^{\chi_S}d\chi\chi\left(\frac{dB^N_{(i,j)}}{d\chi}-\frac{\bar{k}^m}{\bar{k}^0}B^N_{m,ij}\right)\right], \tag{4.170}$$

$$\frac{1}{2}\frac{d^2}{d\bar{z}_S^2}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\left(\delta\tilde{z}^{(2)}_S\right)^2=\delta_{ab}\frac{\chi_S}{z_S+1}\left(1-\frac{\mathcal{H}'_S c^2}{2\chi_S\mathcal{H}_S^3}-\frac{3c}{2\mathcal{H}_S\chi_S}\right)[U_{NS}+$$
$$-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2-v_{S\|}^2+\frac{c}{\mathcal{H}_S}v'_{S\|}v_{S\|}\Bigg]^2, \tag{4.171}$$

$$\frac{1}{2}\frac{d^2}{d\bar{z}_S^2}\tilde{\mathscr{D}}^{(2)}_{ab}(z_S)\left(\delta\tilde{z}^{(1)}_S\right)^2=\delta_{ab}\frac{\chi_S}{z_S+1}\left[V_{NS}-2\frac{1}{\chi_S}\int_0^{\chi_S}d\chi\left[W_{NS}+(\chi_S-\chi)W_{N,i}\frac{\bar{k}^i}{\bar{k}^0}\right]+\right.$$
$$-\left(V_{NS}+\chi_S\frac{d}{d\chi_S}V_{NS}-2\left[W_{NS}+\int_0^{\chi_S}d\chi W_{NS,i}\frac{\bar{k}^i}{\bar{k}^0}\right]\right)\frac{c}{2\mathcal{H}_S\chi_S}\left(3+\frac{\mathcal{H}'c}{\mathcal{H}^2}\right)+$$

$$
+\frac{c^2}{2\mathscr{H}_S^2\chi_S}\left(2\frac{d}{d\chi_S}V_{NS}+\chi_S\frac{d^2}{d\chi_S^2}V_{NS}-2\left[\frac{d}{d\chi_S}W_{NS}+W_{NS,i}\frac{\bar{k}^i}{\bar{k}^0}\right]\right)\Bigg]\,v_{S\|}^2
$$
$$
+\bar{n}_a^i\bar{n}_b^j\frac{\chi_S}{z_S+1}\left[2\frac{1}{\chi_S}\int_0^{\chi_S}d\chi\,(\chi_S-\chi)\,\chi W_{N,ij}-\int_0^{\chi_S}d\chi\chi W_{N,ij}\frac{c}{\mathscr{H}_S\chi_S}\left(3+\frac{\mathscr{H}'c}{2\mathscr{H}^2\chi_S}\right)+\right.
$$
$$
\left.+\frac{c^2}{\mathscr{H}_S^2}\chi_S W_{NS,ij}\right]v_{S\|}^2, \tag{4.172}
$$

$$
\frac{d^2}{d\bar{z}_S^2}\bar{\bar{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(1)}\delta z^{(3)}=-\frac{\chi_S}{z_S+1}\delta_{ab}v_{S\|}\left(2-\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-\frac{3c}{\mathscr{H}_S\chi_S}\right)\left\{\int_0^{\chi_S}d\chi B_{Ni,0}\bar{k}^i+\right.
$$
$$
-v_{S\|}\left(V_{NS}+\frac{1}{2}v_S^2\right)-v_S^i\delta_{ij}\int_0^{\chi_S}d\chi W_N^{;i}
$$
$$
+2\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2\right)v_{S\|}-v_{S\|}^3+
$$
$$
-\frac{c}{\mathscr{H}_S}v_{S\|}'\left[U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2-v_{S\|}^2+\frac{1}{\mathscr{H}_S}v_{S\|}'v_{S\|}\right]+
$$
$$
+\frac{c}{\mathscr{H}_S}\left[\frac{dU_{NS}}{d\chi}-2W_{NS,0}+v_Sv_S'-2v_{S\|}v_{S\|}'\right]v_{S\|}+
$$
$$
\left.+\frac{c}{2\mathscr{H}_S}\left[v_{S\|}'\left(1+\frac{\mathscr{H}_S'c}{\mathscr{H}_S^2}\right)-\frac{c}{\mathscr{H}_S}v_{S\|}''\right]v_{S\|}^2\right\}, \tag{4.173}
$$

$$
-\frac{1}{2}\frac{d^3}{d\bar{z}_S^3}\bar{\bar{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(2)}\left(\delta z_S^{(1)}\right)^2=\delta_{ab}\frac{\chi_S}{z_S+1}\left(-\frac{11c}{2\chi_S\mathscr{H}_S}+3+\frac{\mathscr{H}_S''c^3}{2\chi_S\mathscr{H}_S^4}-\frac{3\mathscr{H}'^2c^3}{2\chi_S\mathscr{H}_S^5}+\right.
$$
$$
\left.-\frac{3\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}\right)v_{S\|}^2\left[U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}+\frac{1}{2}v_S^2-v_{S\|}^2+\frac{c}{\mathscr{H}_S}v_{S\|}'v_{S\|}\right], \tag{4.174}
$$

$$
\frac{1}{4!}\frac{d^4}{d\bar{z}_S^4}\bar{\bar{\mathscr{D}}}_{ab}(z_S)\left(\delta z_S^{(1)}\right)^4=\frac{1}{24}\delta_{ab}\frac{\chi_S}{z_S+1}\left(24-\frac{\mathscr{H}_S'''c^4}{\chi_S\mathscr{H}_S^5}+\frac{10c^3\mathscr{H}_S''}{\chi_S\mathscr{H}^4}-\frac{15c^4\mathscr{H}_S'^3}{\chi_S\mathscr{H}_S^7}+\right.
$$
$$
\left.-\frac{30c^3\mathscr{H}_S'^2}{\chi_S\mathscr{H}_S^5}-\frac{35c^2\mathscr{H}_S'}{\chi_S\mathscr{H}_S^3}+\frac{10c^4\mathscr{H}_S'\mathscr{H}_S''}{\chi_S\mathscr{H}^6}-\frac{50c}{\chi_S\mathscr{H}}\right)v_{S\|}^4. \tag{4.175}
$$

The Jacobi mapping $\tilde{\mathscr{D}}(z_S)$

We can now express the Jacobi mapping as a function of z_S. Substituting the expressions for δz_S in (4.153)-(4.158) as well as the expressions of the derivatives (4.159)-(4.163) into eq. (4.11), we obtain $\tilde{\mathscr{D}}_{ab}(z_S)$ up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$.

We see that the redshift perturbations generate a new order $\mathscr{O}\left(\frac{1}{c}\right)$ in the expansion, proportional to the galaxy peculiar velocity $v_{S\|}$:

$$\tilde{\mathscr{D}}^{(1)}_{ab} = \left(\frac{c}{\mathscr{H}_S\chi_S} - 1\right)\frac{\chi_S}{1+z_S}\frac{v_{S\|}}{c}\delta_{ab}\,, \tag{4.176}$$

with $\mathscr{H} \equiv \frac{1}{a}\frac{da}{d\eta}$. Note that because we define $\mathscr{H}$ using the time derivative and not the derivative w.r.t. χ, we obtain an additional factor c: $da/d\eta = -c\,da/d\chi$. This factor does not influence the order of the expression. The contribution in eq. (4.176) has been called Doppler magnification, and it is the dominant contribution to the convergence at low redshift [21, 6, 22]. Note that in standard perturbation theory, since the peculiar velocity is a perturbative quantity it contributes to the Jacobi mapping at the same order as the gravitational potentials. In the PF framework however, the peculiar velocity is non-perturbative, but it is always weighted by a factor $1/c$. As such it is of lower order than the Newtonian gravitational potentials in the expansion $1/c$. This illustrates nicely the difference between the PF formalism and standard perturbation theory. The Doppler term is usually neglected in lensing analyses, first because at lowest order it does not contribute to the shear, and second because at high redshift, its contribution to the convergence is subdominant with respect to the Newtonian contribution of order $1/c^2$. From the PF formalism we see however that the velocity contribution will dominate in the regime where $v_{S\|}/c$ is larger than the lensing potential integrated along the photon trajectory, see eq. (4.98).

At order $\mathscr{O}\left(\frac{1}{c^2}\right)$ we obtain

$$\begin{aligned}\tilde{\mathscr{D}}^{(2)}_{ab} = \tilde{\mathscr{D}}^{(2)}_{ab}(z_S) + \frac{1}{c^2}\frac{\chi_S}{1+z_S}\Bigg\{&\left(1-\frac{c}{\mathscr{H}_S\chi_S}\right)\left(U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2}v_S^2 + \right.\\ &\left. + v_{S\|}^2 - \frac{c}{\mathscr{H}_S}v'_{S\|}v_{S\|}\right) + \left(1 - \frac{\mathscr{H}'_S c^2}{2\chi_S\mathscr{H}_S^3} - \frac{3c}{2\mathscr{H}_S\chi_S}\right)v_{S\|}^2\Bigg\}\delta_{ab}\,,\end{aligned} \tag{4.177}$$

where a prime denotes a derivative with respect to χ. In the first line, $\tilde{\mathscr{D}}^{(2)}_{ab}(z_S)$ is obtained from eq. (4.98) where the background coordinate χ_S can be replaced by its value at the observed redshift z_S. We see that at this order, the Jacobi mapping is not only affected by the radial component of the peculiar velocity $v_{S\|}$ but also by its transverse part through $v_S^2 = v_{S\|}^2 + v_{S\perp}^2$. Note that since the redshift corrections at this order are proportional to δ_{ab} they will only affect the convergence, and leave the shear and rotation unchanged.

At the order $\mathscr{O}\left(\frac{1}{c^3}\right)$ we obtain

$$\begin{aligned}
\tilde{\mathscr{D}}^{(3)}_{ab} =& \tilde{\mathscr{D}}^{(3)}_{ab}(z_S) + \frac{1}{c^3}\frac{\chi_S}{1+z_S}\delta_{ab}\left\{ v_{S\|}\left(2 - \frac{\mathscr{H}'_S c^2}{\chi_S \mathscr{H}_S^3} - \frac{3c}{\mathscr{H}_S \chi_S}\right)\left[U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0} + \right.\right. \\
&\left. + \frac{1}{2}v_S^2 - v_{S\|}^2 + \frac{c}{\mathscr{H}_S} v'_{S\|} v_{S\|}\right] + v_{S\|}^3\left(1 - \frac{11c}{6\chi_S \mathscr{H}_S} + \frac{\mathscr{H}''_S c^3}{6\chi_S \mathscr{H}_S^4} - \frac{\mathscr{H}'^2 c^3}{2\chi_S \mathscr{H}_S^5} - \frac{\mathscr{H}'_S c^2}{\chi_S \mathscr{H}_S^3}\right) + \\
&- \left(\frac{c}{\mathscr{H}_S \chi_S} - 1\right)\left[\int_0^{\chi_S} d\chi B_{Ni,0}\bar{k}^i - v_S^i \delta_{ij}\int_0^{\chi_S} d\chi W_N^{,i} - v_{S\|}^3 + \right. \\
&+ \left(2U_{NS} - 2V_{NS} - 4\int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2}v_S^2\right) v_{S\|} + \\
&- \frac{c}{\mathscr{H}_S} v'_{S\|}\left(U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2}v_S^2 - v_{S\|}^2 + \frac{1}{\mathscr{H}_S} v'_{S\|} v_{S\|}\right) + \\
&+ \frac{c}{\mathscr{H}_S}\left(\frac{dU_{NS}}{d\chi} - 2W_{NS,0} + v_S v'_S - 2v_{S\|} v'_{S\|}\right) v_{S\|} + \\
&+ \frac{c}{2\mathscr{H}_S}\left(v'_{S\|}\left(1 + \frac{\mathscr{H}'_S c}{\mathscr{H}_S^2}\right) - \frac{c}{\mathscr{H}_S} v''_{S\|}\right) v_{S\|}^2 + \\
&\left. + v_{S\|}\int_0^{\chi_S} d\chi\left(W_N + (\chi_S - \chi) W_{N,i}\frac{\bar{k}^i}{\bar{k}^0}\right)\right] + \\
&\left. + \frac{c v_{S\|}}{\mathscr{H}_S}\left(V_{NS} + \chi_S \frac{dV_{NS}}{d\chi_S} - 2W_{NS} - 2\int_0^{\chi_S} d\chi W_{N,i}\frac{\bar{k}^i}{\bar{k}^0}\right)\right\} \\
&+ \bar{n}_a^i \bar{n}_b^j \frac{1}{c^3}\frac{\chi_S}{1+z_S} v_{S\|}\left[-\frac{2}{\chi_S}\int_0^{\chi_S} d\chi\,(\chi_S - \chi)\,\chi W_{N,ij} + \right. \\
&\left. + \frac{c}{\chi_S \mathscr{H}_S} 2\int_0^{\chi_S} d\chi \chi W_{N,ij}\right]. \qquad (4.178)
\end{aligned}$$

Without redshift perturbations, only the vector potential B_N^i contributes to the Jacobi mapping at the order $\mathscr{O}\left(\frac{1}{c^3}\right)$, see eq. (4.101). However, since the peculiar velocity comes at order $\mathscr{O}\left(\frac{1}{c}\right)$, we obtain couplings between the velocity and the Newtonian potentials that also contribute at this order, as well as terms cubic in the velocity. Note that at this order the peculiar velocity modifies not only the convergence, but also the shear.

Finally, at the order $\mathscr{O}\left(\frac{1}{c^4}\right)$ we obtain

$$\begin{aligned}
\tilde{\mathscr{D}}^{(4)}_{ab} =& \tilde{\mathscr{D}}^{(4)}_{ab}(z_S) - \frac{d}{d\bar{z}_S}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(4)} - \frac{d}{d\bar{z}_S}\tilde{\mathscr{D}}^{(2)}_{ab}(z_S)\,\delta z_S^{(2)} + \\
&- \frac{d}{d\bar{z}_S}\tilde{\mathscr{D}}^{(3)}_{ab}(z_S)\,\delta z_S^{(1)} + \frac{1}{2}\frac{d^2}{d\bar{z}_S^2}\bar{\tilde{\mathscr{D}}}_{ab}(z_S)\left(\delta z_S^{(2)}\right)^2 +
\end{aligned}$$

$$+\frac{1}{2}\frac{d^2}{d\bar{z}_S^2}\tilde{\mathscr{D}}^{(2)}_{ab}(z_S)\left(\delta z_S^{(1)}\right)^2+\frac{d^2}{d\bar{z}_S^2}\bar{\bar{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(1)}\delta z^{(3)}$$
$$-\frac{1}{2}\frac{d^3}{d\bar{z}_S^3}\bar{\bar{\mathscr{D}}}_{ab}(z_S)\,\delta z_S^{(2)}\left(\delta z_S^{(1)}\right)^2+\frac{1}{4!}\frac{d^4}{d\bar{z}_S^4}\bar{\bar{\mathscr{D}}}_{ab}(z_S)\left(\delta z_S^{(1)}\right)^4, \qquad (4.179)$$

where we list the individual terms of (4.179) in the appendix in equation (4.168) - (4.175).

4.3 Extraction of Shear and Convergence

A convenient way to extract the shear and convergence is to introduce spin-fields. [17, 16, 52, 84]. In section 4.2 we already related spin-0 and spin-2 fields to the convergence and the shear via (4.14) and (4.15), respectively. In this section, we will introduce spin operators on a sphere. In weak lensing, the projection on a sphere using these operators has the advantage that we don't have to rely on the small-angle or thin-lens approximation.

4.3.1 Spin Operators on a Sphere

We introduce unit vectors e^i_+ and e^i_- which are defined as

$$e^i_\pm = e^i_\theta \pm i e^i_\phi, \qquad (4.180)$$

where e^i_θ and e^i_ϕ denote the angular unit vectors in spherical coordinates. The unit vectors e^i_+ and e^i_- span the screen space. To each point on the screen space, we can associate a spin-s field ${}_sX$, which transform under a rotation about the e^i_r-axis as ${}_sX \to e^{i\delta s}{}_sX$. Furthermore, we introduce the derivatives $\eth$ and $\bar{\eth}$, which increase and decrease the spin s by 1, respectively:

$$\eth {}_sX \equiv -\sin^s\theta\left(\partial_\theta+\frac{i}{\sin\theta}\partial_\phi\right)\sin^{-s}\theta\,{}_sX = -\left(\partial_\theta+\frac{i}{\sin\theta}\partial_\phi\right){}_sX + s\cot\theta\,{}_sX \qquad (4.181)$$

$$\bar{\eth} {}_sX \equiv -\sin^{-s}\theta\left(\partial_\theta-\frac{i}{\sin\theta}\partial_\phi\right)\sin^{s}\theta\,{}_sX = -\left(\partial_\theta-\frac{i}{\sin\theta}\partial_\phi\right){}_sX - s\cot\theta\,{}_sX \qquad (4.182)$$

The derivatives $\eth$ and $\bar{\eth}$ are effectively angular covariant derivatives on a sphere. If we apply both $\bar{\eth}$ and $\eth$ consecutively, the spin s remains unchanged and we obtain an expression corresponding to the angular Laplace operator $\Delta_{\phi\psi}$. The spin-s field ${}_sX$ can be associated with a symmetric and trace-free tensor $X_{a_1\dots a_s}$ for $s \geq 0$ in the following way

$$X^{a_1\dots a_s} \equiv 2^{-s}{}_sXe_-^{a_1}\dots e_-^{a_s} \tag{4.183}$$

$$\text{and inversely} \quad {}_sX \equiv e_+^{a_1}\dots e_+^{a_s}X_{a_1\dots a_s}. \tag{4.184}$$

For $s < 0$, we define $X^{a_1\dots a_{|s|}} \equiv 2^{-|s|}{}_sXe_-^{a_1}\dots e_-^{a_{|s|}}$ [17].

The PF metric includes vector and tensor potentials, which can be decomposed into spin fields:

$$B^i = B_r e_r^i + \frac{1}{2}{}_{-1}Be_+^i + \frac{1}{2}{}_1Be_-^i \tag{4.185}$$

$$h^{ij} = h_{rr}\left(e_r^ie_r^j - \frac{1}{2}e_+^{(i}e_-^{j)}\right) + {}_{-1}h_re_+^{(i}e_r^{j)} + {}_1h_re_-^{(i}e_r^{j)} + \frac{1}{4}{}_{-2}he_+^ie_+^j + \frac{1}{4}{}_2he_-^ie_-^j \tag{4.186}$$

The spin-0 and spin-2 fields ${}_0\mathscr{D}$ and ${}_2\mathscr{D}$:

We begin with the spin-2 field ${}_2\mathscr{D}$. Using (4.184), ${}_2\mathscr{D}$ reads

$${}_2\mathscr{D} = e_+^ae_+^b\mathscr{D}_{ab}. \tag{4.187}$$

For the following sections we choose to normalise $\bar{k}^\mu$ such that $\bar{k}^0 = \bar{k}^i\bar{k}^j\delta_{ij} = 1$, because the Jacobi mapping does not depend on the normalisation of $\bar{k}^\mu$, since it only depends on the ratio $\frac{\bar{k}^i}{\bar{k}^0}$.

We begin with the spin-2 field ${}_2\mathscr{D}$. Using (4.187), ${}_2\mathscr{D}$ reads up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ the following

$${}_2\tilde{\mathscr{D}}^{(2)}(z_S) = \frac{1}{z_S+1}2\frac{1}{c^2}\int_0^{\chi_S} d\chi\frac{\chi_S-\chi}{\chi}\eth^2W_N, \tag{4.188}$$

$$\begin{aligned}{}_2\tilde{\mathscr{D}}^{(3)}(z_S) =& -\frac{1}{z_S+1}\frac{1}{c^3}\left\{\int_0^{\chi_S} d\chi'\frac{\chi_S-\chi}{\chi}\left[\frac{d}{d\chi}\left(\chi\eth_1B\right) + \eth^2B_r\right] + \right.\\ &- 2v_{S\|}\int_0^{\chi_S} d\chi\frac{\chi_S-\chi}{\chi}\eth^2W_N + \\ &\left. + \frac{2c}{\mathscr{H}_S}v_{S\|}\int_0^{\chi_S} d\chi\frac{1}{\chi}\eth^2W_N\right\}, \quad \text{and}\end{aligned} \tag{4.189}$$

$$\begin{aligned}{}_2\tilde{\mathscr{D}}^{(4)}(z_S) =& \frac{1}{z_S+1}\frac{1}{c^4}\left\{\frac{1}{2}\chi_{S2}h(\chi_S) + \int_0^{\chi_S} d\chi\left[-4W_N\int_0^{\chi}\frac{1}{\chi'}\eth^2W_Nd\chi' + \right.\right.\\ &\left.-4\frac{1}{\chi}\eth^2\left(W_N\int_0^{\chi}W_Nd\chi'\right)\right] + \int_0^{\chi_S} d\chi(\chi_S-\chi)\left[2\frac{1}{\chi}\eth^2\left(2W_P+W_N^2\right) + \right.\end{aligned}$$

$$
\begin{aligned}
&+2V_{NS}\frac{1}{\chi}\eth^2 W_N + 2\frac{1}{\chi^2}\eth\left(\eth^2 W_N\int_0^{\chi} d\chi'\frac{\chi-\chi'}{\chi'}\bar{\eth} W_N\right)+\\
&+2\frac{1}{\chi^2}\eth\left(\eth\bar{\eth} W_N\int_0^{\chi} d\chi'\frac{\chi-\chi'}{\chi'}\eth W_N\right)+\\
&+\frac{4}{\chi^2}\eth W_N\int_0^{\chi} d\chi'\frac{\chi-\chi'}{\chi'}\eth W_N - 4\frac{1}{\chi^2}W_N\int_0^{\chi}\eth^2 W_N d\chi'+\\
&+4\frac{1}{\chi}\eth^2\left(W_{N,0}\int_0^{\chi} W_N d\chi'\right)+\frac{1}{2\chi}\eth^2 h_{rr}+\frac{\chi_S}{\chi}\eth_1 h_r+\\
&+\left[\frac{c}{\mathscr{H}_S}2\int_0^{\chi_S} d\chi\frac{1}{\chi}\eth^2 W_N - 2\int_0^{\chi_S} d\chi\frac{\chi_S-\chi}{\chi}\eth^2 W_N\right]\Big(U_{NS}+\\
&-2\int_0^{\chi_S} d\chi W_{N,0}+\frac{1}{2}v_S^2 - v_{S\|}^2+\frac{c}{\mathscr{H}_S}v'_{S\|}v_{S\|}\Big)+\\
&-v_{S\|}\int_0^{\chi_S} d\chi\frac{(\chi_S-\chi)}{\chi}\left[\frac{d}{d\chi}\left(\chi\eth_1 B^N\right)+\eth^2 B_r^N\right]+\\
&-\frac{c v_{S\|}}{\mathscr{H}_S}\int_0^{\chi_S} d\chi\frac{1}{\chi}\left[\frac{d}{d\chi}\left(\chi\eth_1 B^N\right)+\eth^2 B_r^N\right]+\\
&-v_{S\|}^2\int_0^{\chi_S} d\chi\frac{1}{\chi}\eth^2 W_N\frac{c}{\mathscr{H}_S}\left(3+\frac{\mathscr{H}'c}{2\mathscr{H}^2}\right)+\\
&+2v_{S\|}^2\int_0^{\chi_S} d\chi\frac{\chi_S-\chi}{\chi}\eth^2 W_N+\frac{v_{S\|}^2 c^2}{\mathscr{H}_S^2}\frac{1}{\chi_S}\eth^2 W_{NS}\Bigg\}
\end{aligned}
\tag{4.190}
$$

with $W_{N,r} = W_{N,i}\bar{k}^i$ and using (A.22)-(A.24) for

$$
\begin{aligned}
e_+^i e_+^j 4\bar{n}_c^s\bar{n}^{rc} W_{N,is}&\int_0^{\chi}\int_0^{\chi'} d\chi' d\chi''\chi'' W_{N,rj} =\\
=&e_+^i e_+^j 4\frac{1}{2}\left(e_+^s e_-^r + e_-^s e_+^r\right)W_{N,is}\int_0^{\chi}\int_0^{\chi'} d\chi' d\chi''\chi'' W_{N,rj}\\
=&2\frac{1}{\chi^2}\eth^2 W_N\int_0^{\chi}\int_0^{\chi'} d\chi' d\chi''\left(\frac{1}{\chi''}\bar{\eth}\eth W_N + 2W_{N,r}\right)+\\
&+2\left(\frac{1}{\chi^2}\bar{\eth}\eth W_N+\frac{2}{\chi}W_{N,r}\right)\int_0^{\chi}\int_0^{\chi'} d\chi' d\chi''\frac{1}{\chi''}\eth^2 W_N
\end{aligned}
\tag{4.191}
$$

and

$$
-e_+^i e_+^j \chi 4 W_{N,ijm}\int_0^{\chi} W_N d\chi'\bar{k}^m = \left(-\frac{4}{\chi}\eth^2 W_{N,r}+\frac{8}{\chi^2}\eth^2 W_N\right)\int_0^{\chi} W_N d\chi' \tag{4.192}
$$

and

$$
e_+^i e_+^j \chi 4 W_{N,ijm}\int_0^{\chi}\int_0^{\chi'}\left(W_N^{,m} - W_{N,l}\bar{k}^m\bar{k}^l\right)d\chi'' d\chi' =
$$

$$
\begin{aligned}
=& e^i_+ e^j_+ \chi 4 W_{N,ijm} \frac{1}{2}\left(e^m_+ e^n_- + e^m_- e^n_+\right) \int_0^{\chi}\int_0^{\chi'} W_{N,n} d\chi'' d\chi' \\
=& + \frac{2}{\chi^2} \eth^3 W_N \int_0^{\chi}\int_0^{\chi'} d\chi' d\chi'' \frac{1}{\chi''} \bar{\eth} W_N + \\
&+ \left(\frac{2}{\chi^2}\eth\bar{\eth}\eth W_N + \frac{8}{\chi}\eth W_{N,r} - \frac{4}{\chi^2}\eth W_N\right) \int_0^{\chi}\int_0^{\chi'} d\chi' d\chi'' \frac{1}{\chi''}\eth W_N. \qquad (4.193)
\end{aligned}
$$

The last six lines in (4.190) are contributions from the redshift perturbations. The spin-0 field ${}_0\tilde{\mathscr{D}}$ is obtained by $e^a_- e^b_+ \tilde{\mathscr{D}}_{ab}$. The real and imaginary part of ${}_0\tilde{\mathscr{D}}$ is related to the convergence and rotation, respectively. The combination of e^a_+ and e^b_- neither lowers nor raises the spin s and thus results in a spin-0 expression. We rearrange $e^a_- e^b_+ \tilde{\mathscr{D}}_{ab}$ to

$$
\begin{aligned}
e^a_- e^b_+ \tilde{\mathscr{D}}_{ab} =& \frac{1}{2}\left(e^a_+ e^b_- + e^a_- e^b_+\right)\tilde{\mathscr{D}}_{ij} + \frac{1}{2}\left(e^a_- e^b_+ - e^a_+ e^b_-\right)\tilde{\mathscr{D}}_{ab} \qquad (4.194)\\
=& \Re\left({}_0\tilde{\mathscr{D}}\right) + i\Im\left({}_0\tilde{\mathscr{D}}\right) \equiv {}_0\tilde{\mathscr{D}}_{\mathrm{R}} + {}_0\tilde{\mathscr{D}}_{\mathrm{I}} \qquad (4.195)
\end{aligned}
$$

and split the expression into its real and imaginary part. First, we compute the real part ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}$ and obtain for the background and up to order $\mathscr{O}\left(\frac{1}{c^3}\right)$:

$$
\begin{aligned}
{}_0\bar{\tilde{\mathscr{D}}}_{\mathrm{R}}(z_S) =& \frac{1}{z_S+1} 2\chi_S \qquad (4.196)\\
{}_0\tilde{\mathscr{D}}^{(1)}_{\mathrm{R}}(z_S) =& \frac{1}{z_S+1}\frac{1}{c} 2\chi_S \left(\frac{c}{\mathscr{H}_S\chi_S} - 1\right) v_{S\|} \qquad (4.197)\\
{}_0\tilde{\mathscr{D}}^{(2)}_{\mathrm{R}}(z_S) =& \frac{1}{z_S+1}\frac{1}{c^2}\left[\chi_S 2V_{NS} - 2\int_0^{\chi_S} d\chi \left(2W_N - (\chi_S-\chi)\frac{1}{\chi}\bar{\eth}\eth W_N\right) + \right.\\
&+ \left(1 - \frac{c}{\mathscr{H}_S\chi_S}\right) 2\chi_S \left(U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0} + \frac{1}{2}v_S^2\right) + \\
&\left. + \left(1 - \frac{3c}{2\mathscr{H}_S\chi_S} - \frac{c^2\mathscr{H}'_S}{2\mathscr{H}_S^3\chi_S}\right) 2\chi_S v^2_{Sk}\right] \quad \text{and} \qquad (4.198)\\
{}_0\tilde{\mathscr{D}}^{(3)}_{\mathrm{R}}(\bar{z}_S) =& \frac{1}{z_S+1}\frac{1}{c^3} 2\left\{-\frac{1}{4}\int_0^{\chi_S} d\chi \left(\bar{\eth}_1 B_N + \eth_{-1}B_N - 4B_{Nr}\right) + \right.\\
&- \int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{2\chi}\left(\frac{1}{2}\bar{\eth}_1 B_N + \frac{1}{2}\eth_{-1}B_N + \bar{\eth}\eth B_{Nr}\right) + \\
&- \left(\frac{c}{\mathscr{H}_S} - \chi_S\right)\int_0^{\chi_S} d\chi B_{Nr,0} \\
&+ v_{S\|}\left[\left(2\chi_S - \frac{\mathscr{H}'_S c^2}{\mathscr{H}_S^3} - \frac{3c}{\mathscr{H}_S}\right)\left(U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0}\right) + \right.\\
&- \int_0^{\chi_S} d\chi \left(\chi_S - \chi - \frac{c}{\mathscr{H}_s}\right)\frac{1}{\chi}\bar{\eth}\eth W_N +
\end{aligned}
$$

$$
\begin{aligned}
&-2\chi_S V_{NS}+2\chi_S U_{NS}-\chi_S W_{NS}+\int_0^{\chi_S}d\chi(\chi_S-\chi)W_{N,0}+\\
&-3\chi_S\int_0^{\chi_S}d\chi W_{N,0}+\frac{c\chi_S}{\mathscr{H}}\left(\frac{dU_{NS}}{d\chi}-2W_{NS,0}\right)+\\
&+\frac{c}{\mathscr{H}_S}\left[3V_{NS}-2U_{NS}-W_{NS}+\chi_S\frac{dV_{NS}}{d\chi_S}+\right.\\
&-\int_0^{\chi_S}d\chi\left(2W_N-(\chi_S-\chi)W_{N,0}\right)+3\int_0^{\chi_S}d\chi W_{N,0}+\\
&\left.\left.-\frac{c}{\mathscr{H}}\left(\frac{dU_{NS}}{d\chi}-2W_{NS,0}\right)\right]\right]+\\
&+v'_{S\parallel}\left(\frac{c}{\mathscr{H}_S}-\chi_S\right)\frac{c}{\mathscr{H}_S}\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}\right)+\\
&-\left(\frac{c}{\mathscr{H}_S}-\chi_S\right)\left(\frac{1}{2}{}_1v_S\int_0^{\chi_S}d\chi\frac{1}{\chi}\bar{\eth}W_N+\frac{1}{2}{}_{-1}v_S\int_0^{\chi_S}d\chi\frac{1}{\chi}\eth W_N\right)+\\
&+\left(3\chi_S-\frac{\mathscr{H}'_S c^2}{\mathscr{H}_S^3}-\frac{4c}{\mathscr{H}_S}\right)v_{S\parallel}\frac{1}{2}v_S^2+\\
&+\left(\frac{3}{2}\chi_S-\frac{3}{2}\frac{\mathscr{H}'_S c^2}{\mathscr{H}_S^3}-\frac{4c}{\mathscr{H}_S}+\frac{3}{2}\frac{c}{\mathscr{H}_S}+\frac{1}{2}\frac{\mathscr{H}'_S c}{\mathscr{H}_S^2}\right)\frac{c}{\mathscr{H}_S}v_{S\parallel}^2v'_{S\parallel}+\\
&+v_{S\parallel}^3\left(-2\chi_S+\frac{8c}{\mathscr{H}_S}+\frac{\mathscr{H}''_S c^3}{6\mathscr{H}_S^4}-\frac{\mathscr{H}'^2c^3}{2\mathscr{H}_S^5}\right)+\\
&+\left(\frac{c}{\mathscr{H}_S}-\chi_S\right)\frac{c}{\mathscr{H}}v'_{S\parallel}\left(\frac{1}{2}v_S^2+\frac{1}{\mathscr{H}_S}v'_{S\parallel}v_{S\parallel}\right)+\\
&\left.-\left(\frac{c}{\mathscr{H}_S}-\chi_S\right)\frac{c}{\mathscr{H}}v_{S\parallel}v_S v'_S+\left(\frac{c}{\mathscr{H}_S}-\chi_S\right)\frac{c^2}{2\mathscr{H}_S^2}v''_{S\parallel}v_{S\parallel}^2\right\}. \qquad (4.199)
\end{aligned}
$$

As at order $\mathscr{O}\left(\frac{1}{c^4}\right)$ the expression for ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(4)}$ is very long, we split it into nine parts:

$$
\begin{aligned}
{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(4)}=&{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(P)}+{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(VW)}+{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(UW)}+{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(WW)}+{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(h)}+\\
&+{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(\delta z)}+{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v)}+{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v^2)}+{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v^4)} \qquad (4.200)
\end{aligned}
$$

where the superscripts (UW), (VW), and (WW) refer to the respective couplings and the superscripts (P) to terms involving the quantities U_P, V_P, and W_P. The superscript (h) denotes terms with the tensor potential h_{ij}. The contributions of the redshift perturbations are split into two categories, which read ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(\delta z)}$ and ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v)}$. The superscripts (v), (v^2), and (v^4) refer to terms with the peculiar velocity v_S or its projection along the line of sight $v_{S\parallel}$, while the superscript (δz) denotes all terms stemming from redshift perturbations independent of the peculiar velocity v_S.

We begin with ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(P)}$. Note that due to the form of the metric (2.77) - (2.80), the contributions of the potentials V_P and W_P in ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(4)}$ will be of the same form as the potentials V_N and W_N in ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(2)}$ in (4.198) without redshift perturbations. The terms with U_P and W_P, which are derived from the redshift perturbations in (4.179), will appear in the part ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(\delta z)}$.

$$ {}_0\tilde{\mathscr{D}}_{\mathbf{P}} = \frac{1}{z_S+1}\frac{1}{c^4}\left[\chi_S 4V_P - \int_0^{\chi_S} d\chi \left(8W_P - 4\frac{\chi_S-\chi}{\chi}\bar{\eth}\eth W_P\right)\right]. \tag{4.201} $$

$$ \begin{aligned} {}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(UW)} =& \frac{1}{z_S+1}\frac{1}{c^4}\left\{\int_0^{\chi_S} d\chi 8\left[U_{NS}W_N - U_N W_N - U_{N,0}\int_0^{\chi} d\chi' W_N + \right.\right. \\ & \left. + (\chi_S-\chi)\left(U_{N,0}W_N - W_N\frac{d}{d\chi}U_N\right)\right] + \qquad (4.202) \\ & -4\int_0^{\chi_S} d\chi \frac{1}{\chi}\eth U_N \int_0^{\chi} d\chi' \left(\chi-\chi'\right)\frac{1}{\chi}\bar{\eth}W_N + \\ & -4\int_0^{\chi_S} d\chi \frac{1}{\chi}\bar{\eth} U_N \int_0^{\chi} d\chi' \left(\chi-\chi'\right)\frac{1}{\chi}\eth W_N + \\ & +4\int_0^{\chi_S} d\chi (\chi_S-\chi)\frac{1}{\chi}\eth U_N \int_0^{\chi} d\chi' \frac{1}{\chi}\bar{\eth} W_N + \\ & \left. +4\int_0^{\chi_S} d\chi (\chi_S-\chi)\frac{1}{\chi}\bar{\eth} U_N \int_0^{\chi} d\chi' \frac{1}{\chi}\eth W_N \right\}. \end{aligned} \tag{4.203} $$

For ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(VV)}$ and ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(VW)}$, we obtain

$$ {}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(VV)} = \frac{1}{c^4}\left[\frac{\chi_S}{2}V_{NS}^2 - 4\int_0^{\chi_S} d\chi (\chi_S-\chi)\chi'\left(\frac{d}{d\chi}V_N\right)^2\right] \tag{4.204} $$

and

$$ \begin{aligned} {}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(VW)} =& \frac{1}{z_S+1}2\frac{1}{c^4}\left\{\int_0^{\chi_S} d\chi \left[-2V_{NS}W_N - 2\chi W_N\frac{d}{d\chi}V_{NS} + 2\chi V_{NS,0}W_N + 2\chi W_N\frac{d}{d\chi}V_N + \right.\right. \\ & \left. -2W_N\chi V_{N,0} - 2\eth V_N \int_0^{\chi} d\chi' \frac{1}{\chi'}\bar{\eth}W_N - 2\bar{\eth}V_N\int_0^{\chi} d\chi' \frac{1}{\chi'}\eth W_N + V_{NS}W_N\right] \\ & + \int_0^{\chi_S} d\chi (\chi_S-\chi)\left[2W_N\frac{d}{d\chi}V_N - 2W_N V_{N,0} + 2\chi W_N\left(\frac{d^2}{d\chi^2}V_N - \frac{d}{d\chi}V_{N,0}\right)\right. \\ & + \eth V_N\frac{1}{\chi}\bar{\eth}W_N + \bar{\eth}V_N\frac{1}{\chi}\eth W_N + \eth V_{NS}\frac{1}{\chi}\bar{\eth}W_N + \bar{\eth}V_{NS}\frac{1}{\chi}\eth W_N + \\ & \left.\left. + V_{NS}\frac{1}{\chi}\bar{\eth}\eth W_N - V_{NS}W_{N,0}\right]\right\}. \end{aligned} \tag{4.205} $$

respectively. The next two terms contain the couplings of the lensing potentials $W_N - W_N$ as well as h_{ij}:

$$
\begin{aligned}
{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(WW)} = & \frac{1}{z_S+1}\frac{1}{c^4}\left\{\int_0^{\chi_S} d\chi \left[-8W_N^2 + 8W_{NS}W_N - 16W_{N,0}\int_0^{\chi} d\chi' W_N + \right.\right. \\
& + \frac{24}{\chi}\eth W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\bar{\eth} W_N + \frac{24}{\chi}\bar{\eth} W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\eth W_N + \qquad (4.206) \\
& - \frac{4}{\chi} W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\bar{\eth}\eth W_N + 4\eth\bar{\eth} W_N \int_0^{\chi} d\chi' W_N \frac{\chi-\chi'}{\chi\chi'} + \\
& \left. -4W_N \int_0^{\chi} \frac{1}{\chi'}\bar{\eth}\eth W_N d\chi' - 4\eth\bar{\eth}\left(W_N \int_0^{\chi}\frac{1}{\chi'}W_N d\chi\right)\right] \qquad (4.207) \\
& + \int_0^{\chi_S} d\chi (\chi_S - \chi)\left[16W_N W_{N,0} + 8W_{N,00}\int_0^{\chi} W_N d\chi' + \right. \\
& - \frac{24}{\chi}\eth W_N \int_0^{\chi} d\chi' \frac{1}{\chi'}\bar{\eth} W_N - \frac{24}{\chi}\bar{\eth} W_N \int_0^{\chi} d\chi' \frac{1}{\chi'}\eth W_N + \\
& - \frac{16}{\chi}\eth W_{N,0} \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\bar{\eth} W_N - \frac{16}{\chi}\bar{\eth} W_{N,0} \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\eth W_N + \\
& + \frac{4}{\chi^2}\eth W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\bar{\eth} W_N + \frac{4}{\chi^2}\bar{\eth} W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\eth W_N + \\
& + 4\bar{\eth}\eth\left(W_{N,0}\int_0^{\chi}\frac{1}{\chi'}W_N d\chi'\right) - 4\bar{\eth}\eth W_{N,0}\int_0^{\chi}\frac{1}{\chi'}W_N d\chi' + \\
& + \frac{1}{\chi}\eth W_N \int_0^{\chi}\frac{1}{\chi'}\bar{\eth} W_N d\chi + 4\frac{1}{\chi}\bar{\eth} W_N \int_0^{\chi}\frac{1}{\chi}\eth W_N d\chi + \\
& - 4\frac{1}{\chi^2}\bar{\eth}\eth W_N \int_0^{\chi} d\chi' W_N + 8\frac{1}{\chi^2}\bar{\eth}\eth W_N \int_0^{\chi} d\chi' \left(\chi-\chi'\right) W_{N,0} \\
& - 4\left(\frac{d}{d\chi}W_N\right)^2 + 8W_{N,0}\frac{d}{d\chi}W_N - 4W_{N,0}^2 + 2\bar{\eth}\eth W_N^2 \frac{1}{\chi} \\
& + \frac{1}{\chi^2}\eth^2 W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\bar{\eth}^2 W_N + \frac{1}{\chi^2}\bar{\eth}^2 W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\eth^2 W_N + \\
& + 2\frac{1}{\chi^2}\bar{\eth}\eth W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\bar{\eth}\eth W_N - 4\frac{1}{\chi^2}\bar{\eth}\eth W_N \int_0^{\chi} d\chi' W_N + \\
& - \frac{4}{\chi^2} W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\bar{\eth}\eth W_N + \frac{4}{\chi} W_N \int_0^{\chi} d\chi' \frac{1}{\chi}\bar{\eth}\eth W_N + \\
& + \frac{4}{\chi} W_{N,0} \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\bar{\eth}\eth W_N + \frac{4}{\chi}\eth\bar{\eth} W_{N,0}\int_0^{\chi} d\chi' W_N + \\
& + \frac{2}{\chi^2}\eth\bar{\eth}\eth W_N \int_0^{\chi}\frac{\chi-\chi'}{\chi'}\bar{\eth} W_N d\chi' + \qquad (4.208) \\
& \left.\left. + \frac{2}{\chi^2}\eth\bar{\eth}^2 W_N \int_0^{\chi}\frac{\chi-\chi'}{\chi'}\eth W_N d\chi'\right]\right\}
\end{aligned}
$$

and

$$
{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(h)} = \frac{1}{z_S+1}\frac{1}{c^4}\left\{-\frac{\chi_S}{2}h_{rr}(\chi_S)+\frac{1}{2}\int_0^{\chi_S}d\chi\frac{\chi_S}{\chi}\left(\eth_{-1}h_r+\bar{\eth}_1 h_r-h_{rr}\right)+\right.
$$
$$
\left.+\int_0^{\chi_S}d\chi(\chi_S-\chi)\frac{1}{2\chi}\left(\eth\bar{\eth}h_{rr}-\chi h_{rr,0}\right)\right. . \tag{4.209}
$$

The contributions to ${}_0\tilde{\mathscr{D}}^{(4)}$ from the redshift perturbations are divided into ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(\delta z)}$, ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v)}$, ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v^2)}$, and ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v^4)}$, where ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(\delta z)}$ denotes the perturbations independent of the peculiar velocity and ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v)}$, ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v^2)}$, and ${}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v^4)}$ refer to the terms involving the peculiar velocity at different powers:

$$
\begin{aligned}
{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(\delta z)} =& \frac{2\chi_S}{(1+z_S)}\frac{1}{c^4}\left\{\left[-\frac{1}{\chi_S}\int_0^{\chi_S}d\chi\left(2W_N-\left(\chi_S-\chi-\frac{c}{\mathscr{H}_s}\right)\frac{1}{\chi}\bar{\eth}\eth W_N\right)+\right.\right.\\
&+\left.\frac{c}{\chi_S\mathscr{H}_S}\left(-\chi_S\frac{dV_{NS}}{d\chi_S}+2W_{NS}\right)\right]\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}\right)+\\
&+\left(1-\frac{\mathscr{H}_S'c^2}{2\chi_S\mathscr{H}_S^3}-\frac{3c}{2\mathscr{H}_S\chi_S}\right)\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}\right)^2+\\
&+\left(1-\frac{c}{\mathscr{H}_S\chi_S}\right)\left\{2U_{PS}-4\int_0^{\chi_S}d\chi W_{P,0}-\frac{5}{2}U_{NS}^2-\frac{1}{2}\int_0^{\chi_S}d\chi h_{rr,0}+\right.\\
&-2\left(\int_0^{\chi_S}d\chi W_{N,0}\right)^2+4\left(U_{NS}\int_0^{\chi_S}d\chi W_{NS}\right)_{,0}+4W_{NS,0}\int_0^{\chi_S}d\chi W_N+\\
&+2\frac{1}{\chi_S}\bar{\eth}U_{NS}\int_0^{\chi_S}d\chi\,(\chi_S-\chi)\frac{1}{\chi}\eth W_N+2\frac{1}{\chi_S}\eth U_{NS}\int_0^{\chi_S}d\chi\,(\chi_S-\chi)\frac{1}{\chi}\bar{\eth}W_N+\\
&-4\frac{d}{\chi_S}U_{NS}\int_0^{\chi_S}d\chi''W_N+V_{NS}\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}\right)+\\
&+\frac{c}{\mathscr{H}_S}\left(\frac{dU_{NS}}{d\chi_S}U_{NS}-2\frac{dU_{NS}}{d\chi_S}\int_0^{\chi_S}d\chi W_{N,0}-2W_{NS,0}U_{NS}+2W_{NS,0}2\int_0^{\chi_S}d\chi W_{N,0}\right)\\
&+\int_0^{\chi_S}d\chi\left(4W_N\frac{d}{d\chi}U_N-4U_{N,0}W_N-4W_{N,0}W_N-4W_{N,00}\int_0^{\chi}d\chi'W_N+\right.\\
&-2\frac{1}{\chi}\eth U_N\int_0^{\chi}d\chi'\frac{1}{\chi'}\bar{\eth}W_N-2\frac{1}{\chi}\bar{\eth}U_N\int_0^{\chi}d\chi'\frac{1}{\chi'}\eth W_N+\\
&-2\frac{1}{\chi}\eth W_{N,0}\int_0^{\chi}d\chi\frac{\chi-\chi'}{\chi}\bar{\eth}W_N-2\frac{1}{\chi^2}\eth W_N\int_0^{\chi}d\chi\frac{\chi-\chi'}{\chi}\bar{\eth}W_N+\\
&\left.\left.\left.-2\frac{1}{\chi}\bar{\eth}W_{N,0}\int_0^{\chi}d\chi'\frac{\chi-\chi'}{\chi}\eth W_N-2\frac{1}{\chi^2}\bar{\eth}W_N\int_0^{\chi}d\chi'\frac{\chi-\chi'}{\chi}\eth W_N\right)\right\}\right\},
\end{aligned} \tag{4.210}
$$

$$
\begin{aligned}
{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v)} = & \frac{2\chi_S}{(1+z_S)}\frac{1}{c^4}\left\{\frac{v_{S\parallel}}{\chi_S}\left[\int_0^{\chi_S} d\chi \frac{\chi_S-\chi-\frac{c}{\mathscr{H}_S}}{\chi}\frac{1}{2}\left(\frac{d}{d\chi}\left(\chi\eth_1 B_N\right)+\eth^2 B_{Nr}-2\chi B_{Nr,0}\right)+\right.\right.\\
& +\frac{c}{\mathscr{H}_S}B_{NSr}-\int_0^{\chi_S} d\chi\left(B_{Nr}+\left(2\chi_S-\frac{\mathscr{H}_S' c^2}{\mathscr{H}_S^3}-\frac{3c}{\mathscr{H}_S}\right)B_{Nr,0}\right)\Bigg]+\\
& +\left(1-\frac{c}{\mathscr{H}_S\chi_S}\right)\left\{-{}_{-1}v_S\left[\frac{1}{2}{}_{-1}B_{NS}+\frac{1}{2}\int_0^{\chi_S} d\chi\frac{1}{\chi}\left(\bar{\eth}B_{Nr}+{}_{-1}B_N\right)+\right]+\right.\\
& -{}_{-1}v_S\left[\frac{1}{2}{}_1B_{NS}+\frac{1}{2}\int_0^{\chi_S} d\chi\frac{1}{\chi}\left(\eth B_{Nr}+{}_1B_N\right)\right]+\\
& \left.\left.+v_{S\parallel}v_{S\parallel}'\frac{c}{\mathscr{H}_S}\left[V_{NS}-\int_0^{\chi_S} d\chi W_{N,0}-\left(3+\frac{\mathscr{H}_S' c}{\mathscr{H}_S^2}\right)\left(U_{NS}-2\int_0^{\chi_S} d\chi W_{N,0}\right)\right]\right\}\right\},
\end{aligned}
\tag{4.211}
$$

$$
\begin{aligned}
{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v^2)} = & \frac{2\chi_S}{(1+z_S)}\frac{1}{c^4}\left\{\left(1-\frac{c}{\mathscr{H}_S\chi_S}\right)\left\{v_S^2\left(\frac{3}{2}V_{NS}-\frac{1}{2}U_{NS}+\right.\right.\right.\\
& -2\int_0^{\chi_S} d\chi W_{N,0}+\frac{1}{2}\frac{c}{\mathscr{H}_S}\frac{dU_{NS}}{d\chi}-\frac{c}{\mathscr{H}_S}W_{NS,0}\Bigg)+v_{S\parallel}^2\Bigg[3U_{NS}-V_{NS}+\\
& -6\int_0^{\chi_S} d\chi W_{N,0}-3\frac{c}{\mathscr{H}_S}\frac{dW_{NS}}{d\chi}+3\frac{c}{\mathscr{H}_S}W_{NS,0}+\\
& +\left(5\frac{c}{\mathscr{H}_S}+\frac{\mathscr{H}_S' c^2}{\mathscr{H}_S^3}-\frac{1}{2}\frac{c^2}{\mathscr{H}_S^2}\frac{d}{d\chi_S}\right)\left(\frac{1}{2}\frac{dU_{NS}}{d\chi_S}-W_{NS,0}\right)\Bigg]+\\
& +v_{S\parallel}v_{S\parallel}'\frac{c}{\mathscr{H}_S}\left[V_{NS}-\int_0^{\chi_S} d\chi W_{N,0}-\left(3+\frac{\mathscr{H}_S' c}{\mathscr{H}_S^2}\right)\left(U_{NS}-2\int_0^{\chi_S} d\chi W_{N,0}\right)\right]\\
& +v_{S\parallel}'^2\left(2\frac{c^2}{\mathscr{H}_S^2}\int_0^{\chi_S} d\chi W_{N,0}-\frac{c^2}{\mathscr{H}_S^2}U_{NS}\right)+v_S v_S'\frac{c}{\mathscr{H}_S}\left(U_{NS}-2\int_0^{\chi_S} d\chi W_{N,0}\right)+\\
& -\frac{c}{\mathscr{H}_S}v_{S\parallel}'\,{}_{-1}v_S\frac{1}{2}\int_0^{\chi_S} d\chi'\frac{1}{\chi'}\eth W_N-\frac{c}{\mathscr{H}_S}v_{S\parallel}'\,{}_1v_S\frac{1}{2}\int_0^{\chi_S} d\chi'\frac{1}{\chi'}\bar{\eth}W_N+\\
& +v_{S\parallel}\frac{c}{\mathscr{H}_S}{}_{-1}v_S'\frac{1}{2}\int_0^{\chi_S} d\chi'\frac{1}{\chi'}\eth W_N+v_{S\parallel}\frac{c}{\mathscr{H}_S}{}_1v_S'\frac{1}{2}\int_0^{\chi_S} d\chi'\frac{1}{\chi'}\bar{\eth}W_N+\\
& +v_{S\parallel}\,{}_{-1}v_S\frac{1}{2}\frac{1}{\chi_S}\eth W_{NS}+v_{S\parallel}\,{}_1v_S\frac{c}{2\mathscr{H}_S\chi_S}\bar{\eth}W_{NS}+\\
& \left.+\frac{v_{S\parallel}c^2}{\mathscr{H}_S^2}v_{S\parallel}''\left(U_{NS}-2\int_0^{\chi_S} d\chi W_{N,0}\right)\right\}+\\
& +\left[-\frac{1}{\chi_S}\int_0^{\chi_S} d\chi\left(2W_N-\left(\chi_S-\chi-\frac{c}{\mathscr{H}_s}\right)\frac{1}{\chi}\bar{\eth}\eth W_N\right)+\right.\\
& \left.+\frac{c}{\chi_S\mathscr{H}_S}\left(-\chi_S\frac{dV_{NS}}{d\chi_S}+2W_{NS}\right)\right]\left(\frac{1}{2}v_S^2-v_{S\parallel}^2+\frac{c}{\mathscr{H}_S}v_{S\parallel}'v_{S\parallel}\right)+
\end{aligned}
$$

$$
\begin{aligned}
&+\left(2-\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-\frac{3c}{\mathscr{H}_S\chi_S}\right)\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}\right)\left(\frac{1}{2}v_S^2+\frac{c}{\mathscr{H}_S}v_{S\|}'v_{S\|}\right)+\\
&-\left(2-\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-\frac{3c}{\mathscr{H}_S\chi_S}\right)\frac{c}{\mathscr{H}_S}\left[-v_{S\|}'v_{S\|}U_{NS}+2v_{S\|}'v_{S\|}\int_0^{\chi_S}d\chi W_{N,0}\right]+\\
&+\left\{4V_{NS}-2\frac{1}{\chi_S}\int_0^{\chi_S}d\chi\left[W_{NS}-(\chi_S-\chi)\frac{1}{2\chi}\bar{\eth}\eth W_N\right]+\right.\\
&-\left(V_{NS}+\chi_S\frac{d}{d\chi_S}V_{NS}+4W_{NS}+\int_0^{\chi_S}d\chi\frac{1}{\chi}\bar{\eth}\eth W_N\right)\frac{c}{2\mathscr{H}_S\chi_S}\left(3+\frac{\mathscr{H}'c}{\mathscr{H}^2}\right)+\\
&+\left(\frac{13c}{2\chi_S\mathscr{H}_S}+\frac{\mathscr{H}_S''c^3}{2\chi_S\mathscr{H}_S^4}-\frac{3\mathscr{H}'^2c^3}{2\chi_S\mathscr{H}_S^5}-2+\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}\right)U_{NS}+\\
&+\left(4-\frac{\mathscr{H}_S''c^3}{\chi_S\mathscr{H}_S^4}+\frac{3\mathscr{H}'^2c^3}{\chi_S\mathscr{H}_S^5}+\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-\frac{4c}{\mathscr{H}_S\chi_S}\right)\int_0^{\chi_S}d\chi W_{N,0}+\\
&\left.+\frac{c^2}{\mathscr{H}_S^2\chi_S}\left(\frac{d}{d\chi_S}V_{NS}+\frac{1}{2}\chi_S\frac{d^2}{d\chi_S^2}V_{NS}-\frac{d}{d\chi_S}W_{NS}-\frac{1}{2}\bar{\eth}\eth W_{NS}\right)\right\}v_{S\|}^2+\\
&-\left(2-\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-\frac{3c}{\mathscr{H}_S\chi_S}\right)v_{S\|}\left({}_{-1}v_S\frac{1}{2}\int_0^{\chi_S}d\chi'\frac{1}{\chi'}\eth W_N+\right.\\
&\left.\left.+{}_1v_S\frac{1}{2}\int_0^{\chi_S}d\chi'\frac{1}{\chi'}\bar{\eth}W_N\right)\right\},
\end{aligned}
\tag{4.212}
$$

and

$$
\begin{aligned}
{}_0\tilde{\mathscr{D}}_{\mathrm{R}}^{(v^4)}=&\frac{2\chi_S}{(1+z_S)}\frac{1}{c^4}\left\{\left(1-\frac{c}{\mathscr{H}_S\chi_S}\right)\left\{\frac{3}{2}v_{S\|}^2v_S^2-v_{S\|}^4+\frac{1}{8}v_S^4+\frac{1}{2}v_{S\|}\frac{c}{\mathscr{H}_S}v_{S\|}v_Sv_S'\right.\right.\\
&+\frac{\mathscr{H}_S'c}{\mathscr{H}_S^2}\frac{1}{2}\frac{v_{S\|}^2c}{\mathscr{H}_S}v_Sv_S'+\left(2+3\frac{\mathscr{H}_S'c}{\mathscr{H}_S^2}-\frac{\mathscr{H}_S'^2c^2}{\mathscr{H}_S^4}-\frac{\mathscr{H}_S''c^2}{\mathscr{H}_S^4}\right)\frac{v_{S\|}^3}{6}\frac{c}{\mathscr{H}_S}v_{S\|}'-\frac{1}{2}\frac{v_{S\|}^2c^2}{\mathscr{H}_S^2}v_S'^2+\\
&-\frac{v_{S\|}c}{\mathscr{H}_S}v_{S\|}'\left(3+\frac{\mathscr{H}_S'c}{\mathscr{H}_S^2}\right)\frac{1}{2}v_S^2+\frac{c}{\mathscr{H}_S}v_Sv_S'\frac{1}{2}v_S^2-\frac{1}{2}\frac{v_{S\|}^2c^2}{\mathscr{H}_S^2}v_Sv_S''+\\
&-\frac{c^2}{\mathscr{H}_S^2}v_{S\|}'^2\frac{1}{2}v_S^2-\frac{c^2}{\mathscr{H}_S^2}v_{S\|}'^3\frac{1}{\mathscr{H}_S}v_{S\|}-3\left(1+\frac{\mathscr{H}_S'c}{\mathscr{H}_S^2}\right)\frac{v_{S\|}^3}{6}\frac{c^2}{\mathscr{H}_S^2}v_{S\|}''+\\
&+\frac{v_{S\|}^3}{6}\frac{c^2}{\mathscr{H}_S^2}\frac{v_{S\|}'''c}{\mathscr{H}_S}+\frac{v_{S\|}^2c}{\mathscr{H}_S}v_{S\|}'^2\left(\frac{1}{2}-\frac{3}{2}\frac{\mathscr{H}_S'c}{\mathscr{H}_S^2}\right)\frac{c}{\mathscr{H}_S}+\\
&\left.+v_{S\|}\frac{c^2}{\mathscr{H}_S^2}v_{S\|}''\frac{1}{2}v_S^2+\frac{3c^2}{2\mathscr{H}_S^2}\frac{c}{\mathscr{H}_S}v_{S\|}^2v_{S\|}'v_{S\|}''++v_{S\|}^2\frac{c}{\mathscr{H}_S}v_{S\|}'^2\int_0^{\chi_S}d\chi W_{N,0}\right\}+\\
&+\left(1-\frac{\mathscr{H}_S'c^2}{2\chi_S\mathscr{H}_S^3}-\frac{3c}{2\mathscr{H}_S\chi_S}\right)\left[\frac{1}{4}v_S^4+\frac{c^2}{\mathscr{H}_S^2}v_{S\|}'^2v_{S\|}^2+\frac{c}{\mathscr{H}_S}v_{S\|}'v_{S\|}v_S^2\right]+\\
&-\left(-\frac{11c}{2\chi_S\mathscr{H}_S}+3+\frac{\mathscr{H}_S''c^3}{2\chi_S\mathscr{H}_S^4}-\frac{3\mathscr{H}'^2c^3}{2\chi_S\mathscr{H}_S^5}-\frac{3\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}\right)\frac{1}{2}v_S^2v_{S\|}^2+
\end{aligned}
$$

$$
\begin{aligned}
&-\left(2-\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-\frac{3c}{\mathscr{H}_S\chi_S}\right)\frac{c}{\mathscr{H}_S}\left[-v_{S\|}'v_{S\|}\frac{1}{2}v_S^2-v_{S\|}'^2v_{S\|}^2\frac{1}{\mathscr{H}_S}+\right.\\
&\left.+v_{S\|}^2\frac{dU_{NS}}{d\chi}-v_{S\|}^2 2W_{NS,0}+v_{S\|}^2v_Sv_S'+-v_{S\|}^3\frac{c}{\mathscr{H}_S}v_{S\|}''\right]+\\
&+\left(-\frac{5c}{2\chi_S\mathscr{H}_S}+1+\frac{\mathscr{H}_S''c^3}{2\chi_S\mathscr{H}_S^4}-\frac{\mathscr{H}'^2c^3}{2\chi_S\mathscr{H}_S^5}+\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-2\frac{\mathscr{H}_S'c}{\mathscr{H}_S^2}\right)\frac{c}{\mathscr{H}_S}v_{S\|}'v_{S\|}^3\\
&+\left(1+\frac{\mathscr{H}_S'c^2}{24\chi_S\mathscr{H}_S^3}-\frac{13c}{12\chi_S\mathscr{H}_S}-\frac{c^3\mathscr{H}_S''}{12\chi_S\mathscr{H}^4}+\frac{1\mathscr{H}'^2c^3}{4\chi_S\mathscr{H}_S^5}-\frac{\mathscr{H}_S'''c^4}{24\chi_S\mathscr{H}_S^5}+\right.\\
&\left.-\frac{15c^4\mathscr{H}_S'^3}{24\chi_S\mathscr{H}_S^7}+\frac{5c^4\mathscr{H}_S'\mathscr{H}_S''}{12\chi_S\mathscr{H}^6}\right)v_{S\|}^4\Bigg\}
\end{aligned}
\tag{4.213}
$$

Now we compute the imaginary part ${}_0\tilde{\mathscr{D}}_{\rm I}$ of ${}_0\tilde{\mathscr{D}}$. Note that ${}_0\tilde{\mathscr{D}}_{\rm I}$ comprises only terms off the diagonal of the Jacobi mapping $\tilde{\mathscr{D}}_{ab}$ and therefore only consists of terms involving $\bar{n}_a^i\bar{n}_b^j$. At both order $\mathscr{O}\left(\frac{1}{c^2}\right)$ and order $\mathscr{O}\left(\frac{1}{c^3}\right)$ the Jacobi mapping $\mathscr{D}_{ab}$ is symmetric in the indices a and b. Consequently, using (A.22) and (A.28) the rotation ω vanishes at these orders. At order $\mathscr{O}\left(\frac{1}{c^4}\right)$ the only term that contributes to ${}_0\tilde{\mathscr{D}}_{\rm I}^{(4)}$ stems from the product of the second order contracted Riemann with the second order Jacobi mapping $\mathscr{R}_a^{(2)b}\mathscr{D}_{bc}^{(2)}$. Note that all contributions from the redshift perturbations in $\tilde{\mathscr{D}}_{ab}^{(4)}$ in (4.179) are symmetric and consequently do not contribute to the rotation ω.

$$
\begin{aligned}
{}_0\tilde{\mathscr{D}}_{\rm I}^{(4)}=&\frac{1}{c^4}\frac{1}{z_S+1}\int_0^{\chi_S}d\chi(\chi_S-\chi)\frac{1}{2}\left(e_-^ae_+^b-e_+^ae_-^b\right)4\bar{n}_c^s\bar{n}^{cr}W_{N,as}\int_0^{\chi}\int_0^{\chi'}d\chi'd\chi''\chi''W_{N,rb}\\
=&\frac{1}{z_S+1}\int_0^{\chi_S}d\chi\frac{\chi_S-\chi}{\chi^2}\left(\bar{\eth}^2W_N\int_0^{\chi}d\chi'\frac{\chi-\chi'}{\chi'}\eth^2W_N+\right.\\
&\left.-\eth^2W_N\int_0^{\chi}d\chi'\frac{\chi-\chi'}{\chi'}\bar{\eth}^2W_N\right).
\end{aligned}
\tag{4.214}
$$

4.3.2 The reduced shear g

The reduced shear g is measured from the ellipticity of galaxies. It is given by

$$
g=\frac{\gamma}{1-\kappa}=-\frac{{}_2\tilde{\mathscr{D}}}{\mathrm{Re}\left[{}_0\tilde{\mathscr{D}}\right]}. \tag{4.215}
$$

At order $\mathscr{O}\left(\frac{1}{c^2}\right)$, the shear and reduced shear are equal, since $\bar{\kappa}=0$. We have

$$g^{(2)} = -\frac{{}_2\tilde{\mathscr{D}}^{(2)}}{{}_0\bar{\tilde{\mathscr{D}}}} = -\frac{1}{c^2}\int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi_S \chi} \eth^2 W_N . \tag{4.216}$$

This is the standard Newtonian expression for the shear, written in terms of derivatives on the sphere.

At order $\mathscr{O}\left(\frac{1}{c^3}\right)$ both ${}_2\tilde{\mathscr{D}}$ and ${}_0\tilde{\mathscr{D}}$ contribute to the reduced shear. Since the imaginary part of ${}_0\tilde{\mathscr{D}}$ is of order $\mathscr{O}\left(\frac{1}{c^4}\right)$ we can write

$$\begin{aligned} g^{(3)} =& -\frac{{}_2\tilde{\mathscr{D}}^{(2)} + {}_2\tilde{\mathscr{D}}^{(3)}}{{}_0\bar{\tilde{\mathscr{D}}} + {}_0\tilde{\mathscr{D}}^{(1)}} = -\frac{1}{{}_0\bar{\tilde{\mathscr{D}}}}\left(-\frac{{}_0\tilde{\mathscr{D}}^{(1)}}{{}_0\bar{\tilde{\mathscr{D}}}}{}_2\tilde{\mathscr{D}}^{(2)} + {}_2\tilde{\mathscr{D}}^{(3)}\right) \\ =& \frac{1}{c^3}\left\{\int_0^{\chi_S} d\chi' \frac{\chi_S - \chi}{2\chi_S \chi}\left[\frac{d}{d\chi}\left(\chi \eth_1 B\right) + \eth^2 B_r\right] + \right. \\ & \left. -\frac{c}{\mathscr{H}_S \chi_S^2} v_{S\|} \int_0^{\chi_S} d\chi \eth^2 W_N \right\} . \end{aligned} \tag{4.217}$$

Both (4.216) and (4.217) are complex as can be seen in (A.6) and (A.14). Because we perform a parallel transport along the line of sight, every time derivative is accompanied by a factor k^0, which is of the order $\mathscr{O}(c)$, see (4.20). Therefore, the time derivatives of the scalar potentials at order $\mathscr{O}\left(\frac{1}{c^2}\right)$ and the vector potentials at $\mathscr{O}\left(\frac{1}{c^3}\right)$ do not change to $\mathscr{O}\left(\frac{1}{c^3}\right)$ and $\mathscr{O}\left(\frac{1}{c^4}\right)$, respectively and at this order, the only contribution to the reduced shear comes from the vector potential B^i_N. Although the vector potential B^i_N does not influence the matter dynamics at $\mathscr{O}\left(\frac{1}{c^2}\right)$, it affects the photon geodesic and consequently contributes to both the convergence and shear. In the thin-lens or small-angle approximation, all derivatives along the geodesic are neglected. The contributions of the vector field are by definition beyond the Newtonian approximation and are regarded as relativstic effects. However, in the PF formalism the B^i_N field is sourced by Newtonian quantities, i.e. the product $(1+\delta)\,v^i$ with v^i being the peculiar velocity. In summary, we see that the dominant correction to the Newtonian expression (4.216) is due to two different effects: the vector potential B^i_N and the peculiar velocity of the galaxies $v_{S\|}$. The vector potential in the PF approximation has been computed from N-body simulations on non-linear scales [103, 33, 102]. It was found that the power spectrum of the vector field B^i_N/c^3 is of the order of 10^{-5} the power spectrum of the scalar potential W_N/c^2 over a range of scales and redshifts. Comparing eq. (4.217) with eq. (4.216) we see that part of the vector contribution enters in the reduced shear with exactly the same kernel as the Newtonian scalar part. As such we expect that the impact of the vector potential on the reduced shear will be of the order of $\sim \sqrt{10^{-5}} g^{(2)} \sim 3\times 10^{-3} g^{(2)} \sim 3\times 10^{-5}$, since $g^{(2)}$ is of order 10^{-2} [43].

The second contribution at order $\mathscr{O}\left(\frac{1}{c^3}\right)$ is due to the peculiar velocity of galaxies $v_{S\|}$, coupled to the standard Newtonian shear. This contribution is due to two effects. First, the reduced shear g is measured as a function of redshift, which is affected by the source peculiar velocity. To understand this effect, let us assume that we measure g for two different galaxies that are at the same redshift. One of the galaxies has no peculiar velocity, whereas the other has a velocity directed towards the observer. As a consequence, the second galaxy is physically situated at a larger distance than the first one. The impact of a given lens on the two galaxies will then be different, since the distance between the lens and the source is different. The second velocity contribution to g simply comes from the fact that the shear at second order $\gamma^{(2)}$ is divided by the convergence at first order $\kappa^{(1)}$ which is affected by peculiar velocity. This effect reflects the fact that peculiar velocities change the apparent size of galaxies, which has then an impact on the reduced shear. Note that in standard perturbation theory, this term appears at the next order, i.e. at the same order as the lens-lens coupling and Born correction, see eq. (4.218). However here since velocities are of order $\mathscr{O}\left(\frac{1}{c}\right)$, this coupling is already present at order $\mathscr{O}\left(\frac{1}{c^3}\right)$.

At order $\mathscr{O}\left(\frac{1}{c^4}\right)$, the reduced shear contains contributions from the shear up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ and from the convergence up to order $\mathscr{O}\left(\frac{1}{c^2}\right)$. We obtain

$$
\begin{aligned}
g^{(4)} =& -\frac{{}_2\tilde{\mathscr{D}}}{{}_0\tilde{\mathscr{D}}} = -\frac{{}_2\tilde{\mathscr{D}}^{(2)} + {}_2\tilde{\mathscr{D}}^{(3)} + {}_2\tilde{\mathscr{D}}^{(4)}}{{}_0\tilde{\mathscr{D}}^{(0)} + {}_0\tilde{\mathscr{D}}^{(1)} + {}_0\tilde{\mathscr{D}}^{(2)}} \\
=& \frac{1}{c^4}\Bigg\{ -\int_0^{\chi_S} d\chi \frac{\chi_S-\chi}{\chi_S\chi}\eth^2\left(2W_P + W_N^2\right) \qquad (4.218) \\
& -\int_0^{\chi_S} d\chi \frac{\chi_S-\chi}{\chi_S\chi^2}\eth\left[\eth^2 W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\bar{\eth} W_N + \eth\bar{\eth} W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\eth W_N\right] \\
& +\int_0^{\chi_S} d\chi \frac{\chi_S-\chi}{\chi_S\chi}\eth^2 W_N \cdot \left[\int_0^{\chi_S} d\chi' \frac{\chi_S-\chi'}{\chi'\chi_S}\bar{\eth}\eth W_N - \frac{2}{\chi_S}\int_0^{\chi_S} d\chi' W_N\right] \\
& +2\int_0^{\chi_S} d\chi \frac{\chi_S-\chi}{\chi_S\chi}\left[\frac{1}{\chi} W_N \int_0^{\chi} d\chi' \eth^2 W_N - \frac{1}{\chi}\eth W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\eth W_N\right. \\
& \left. -\eth^2\left(W_{N,0}\int_0^{\chi} W_N d\chi'\right)\right] + \frac{2}{\chi_S}\int_0^{\chi_S} d\chi \left[W_N \int_0^{\chi}\frac{d\chi'}{\chi'}\eth^2 W_N + \frac{1}{\chi}\eth^2\left(W_N\int_0^{\chi} d\chi' W_N\right)\right] \\
& -\frac{1}{4}{}_2h_S - \frac{1}{2}\int_0^{\chi_S} d\chi \left(\frac{\chi_S-\chi}{\chi_S\chi}\frac{1}{2}\eth^2 h_{rr} + \frac{1}{\chi}\eth_1 h_r\right) + \\
& +\left[2\int_0^{\chi_S} d\chi \frac{\chi_S-\chi}{\chi_S\chi}\eth^2 W_N + \frac{c}{\mathscr{H}_S\chi_S}\int_0^{\chi_S} d\chi \frac{1}{\chi_S}\eth^2 W_N\right]\left(U_{NS} - 2\int_0^{\chi_S} d\chi W_{N,0}\right)
\end{aligned}
$$

$$+\left(v_S^2+\frac{c}{\mathscr{H}_S}v_{S\|}v'_{S\|}\right)\int_0^{\chi_S}d\chi\frac{\chi_S-\chi}{\chi_S\chi}\eth^2W_N$$
$$+\frac{1}{2\mathscr{H}_S\chi_S^2}\int_0^{\chi_S}d\chi\eth^2W_N\left(\frac{\chi_S}{\chi}\frac{2c}{\mathscr{H}_S}v_{S\|}v'_{S\|}+v_S^2\right)$$
$$+v_{S\|}^2\left[-\frac{c^2}{\mathscr{H}_S^2}\frac{1}{2\chi_S^2}\eth^2W_{NS}-3\left(1-\frac{c}{2\mathscr{H}_S\chi_S}+\frac{c^2\mathscr{H}'_S}{2\mathscr{H}_S^3\chi_S}\right)\int_0^{\chi_S}d\chi\frac{\chi_S-\chi}{\chi_S\chi}\eth^2W_N\right.$$
$$\left.+\frac{1}{2\chi_S}\frac{c}{\mathscr{H}_S}\left(3+\frac{\mathscr{H}'_Sc}{2\mathscr{H}_S^2}-\frac{2c}{\mathscr{H}_S\chi_S}\right)\int_0^{\chi_S}d\chi\frac{1}{\chi}\eth^2W_N\right]$$
$$+v_{S\|}\left[\int_0^{\chi_S}d\chi\frac{(\chi_S-\chi)}{\chi_S\chi}\left(\frac{d}{d\chi}\left(\chi\eth_1B^N\right)+\eth^2B_r^N\right)\right.$$
$$\left.\left.+\frac{c}{2\mathscr{H}_S\chi_S}\int_0^{\chi_S}d\chi\frac{1}{\chi_S}\left(\frac{d}{d\chi}\left(\chi\eth_1B^N\right)+\eth^2B_r^N\right)\right]\right\}.$$

The first line is the standard shear contribution, where the Newtonian potential W_N has been replaced by the relativistic potential W_P and the square of W_N. This term encodes the fact that large-scale structures along the photon trajectory are not completely described by the Newtonian potential W_N, and that W_P and W_N^2 both give corrections to the potential felt by the photons. The second line contains the lens-lens coupling and correction to Born approximation [36]. These terms have four transverse derivatives of the potential, and they are therefore expected to dominate at small scales. The third line contains the product between the shear and the convergence at order $\mathscr{O}\left(\frac{1}{c^2}\right)$. The first term in this line also has four transverse derivatives and is therefore of the same order of magnitude as the lens-lens coupling and post-Born correction. Note that the boundary term in the convergence, proportional to V_{NS} (see eq. (4.221)) cancels with a similar term in ${}_2\tilde{\mathscr{D}}$ and does not contribute to the reduced shear. Lines 4 and 5 contain various couplings along the photon trajectory. These terms have been computed for the first time using standard perturbation theory up to second order in [17] and the expressions agree. Line 6 contains the contribution from the tensor modes, which also appear at second order in SPT. Finally, the last 6 lines contain the contributions due to redshift perturbations. In line 7, we have the redshift perturbations due to gravitational redshift and the integrated Sachs-Wolfe effect. In the following 3 lines, we have the Doppler contributions coupled with the scalar potential. Since the velocity is of order $1/c$, the reduced shear at order $\mathscr{O}\left(\frac{1}{c^4}\right)$ contains contribution from the second order Doppler, i.e. from both $v_{S\|}$ and $v_{S\perp}$ through the transverse Doppler effect. Finally, in the last two lines we have couplings between the first order Doppler contribution and the vector potential.

Note that in harmonic space, the angular derivatives $\eth$ and $\bar{\eth}$ are linked to an l factor. Therefore, within the small angle approximation, terms with higher order derivatives $\eth$ and $\bar{\eth}$ will be dominant. (4.218) is the equivalent of the solution in [17] but using the PF formalism in terms of the redshift z including redshift perturbations instead of using standard perturbation theory with affine parameter χ. In [17], the coupling terms in the reduced shear are discussed that occur once one goes beyond the thin-lens approximation. In this thesis I extend the work of [17] by computing the convergence in terms of spherical spin operators.

4.3.3 The convergence κ

We now calculate the convergence, i.e. the part of the Jacobi map which modifies only the size of the galaxy. As discussed in section 4.2 it is given by the real part of the spin-0 contribution

$$\kappa = 1 - \frac{1+z_S}{2\chi_S}\mathrm{Re}\left[{}_0\tilde{\mathscr{D}}\right]. \tag{4.219}$$

The convergence corresponds to the rotational symmetry of a spin-0 field. When using the spin operators $\eth$ and $\bar{\eth}$, the only possible combination of these operators is an equal order of spin raising operators $\eth$ and spin lowering operators $\bar{\eth}$. In the thin-lens approximation terms containing derivatives would dominate over derivatives along the line of sight.

At order $\mathscr{O}\left(\frac{1}{c}\right)$ the convergence becomes

$$\kappa^{(1)} = \left(1 - \frac{c}{\mathscr{H}_S\chi_S}\right)\frac{v_{S\|}}{c}. \tag{4.220}$$

This contribution, called Doppler magnification, has been derived in [21] for the first time and studied in detail in [6, 22]. Since it is directly sensitive to the galaxy peculiar velocity, it provides an alternative way of measuring velocities, independently from redshift-space distortions, and to test theories of modified gravity [5]. In the PF formalism, this term is the dominant contribution to the convergence. As shown in [6, 22] this is effectively the case at low redshift $z \leq 0.5$. At high redshift however, the order $\mathscr{O}\left(\frac{1}{c^2}\right)$ derived below dominates over the Doppler magnification, because the deviations generated by $\bar{\eth}\eth W_N$ accumulate along the photon trajectory, whereas the peculiar velocity decreases with redshift. Nevertheless, the Doppler term is still measurable in this regime due to its dipole around overdensities [22]. At order $\mathscr{O}\left(\frac{1}{c^2}\right)$

the convergence is given by

$$\begin{aligned}\kappa^{(2)} =& \frac{1}{c^2}\left[-\int_0^{\chi_S} d\chi \frac{\chi_S-\chi}{\chi\chi_S}\bar{\eth}\eth W_N - V_{NS} + \frac{2}{\chi_S}\int_0^{\chi_S} d\chi W_N \right.\\ &+\left(\frac{c}{\mathscr{H}_S\chi_S}-1\right)\left(U_{NS}-2\int_0^{\chi_S} d\chi W_{N,0}+\frac{1}{2}v_S^2-\frac{c}{\mathscr{H}_S}v'_{S\|}v_{S\|}\right)\\ &\left.-\left(2-\frac{5c}{2\mathscr{H}_S\chi_S}-\frac{c\mathscr{H}'_S}{2\mathscr{H}_S^3\chi_S}\right)v_{S\|}^2\right] \qquad (4.221)\end{aligned}$$

The first term in (4.221) is the standard Newtonian contribution to the convergence. Since it contains two transverse derivatives, it dominates over the other terms when one correlates galaxies at small separations. This term is the only one which changes the apparent size of galaxies through a real focusing of the light beam. The other two terms in the first line modify the length of the geodesic between the source and the observer, and consequently they change the apparent size of galaxies. The terms in the second and third line are due to the fact that we observe the size of galaxies as a function of redshift, which is a perturbed quantity. In particular, the first term in the second line is the contribution from gravitational redshift and the second one is the integrated Sachs-Wolfe contribution. The terms proportional to peculiar velocities in the second and third lines are second-order Doppler contributions. These contributions are sensitive not only to the radial part of the peculiar velocity $v_{S\|}$ but also to its transverse part since $v_S^2 = v_{S\|}^2 + v_{S\perp}^2$. Note that one contribution depends also on the time derivative of the peculiar velocity $v'_{S\|}$, which contributes to the redshift perturbation at second-order, see eq. (4.154).

The convergence at order $\mathscr{O}\left(\frac{1}{c^3}\right)$ is given by

$$\begin{aligned}\kappa^{(3)} =& \frac{1}{c^3}\left\{\int_0^{\chi_S} d\chi\left[\frac{\chi_S-\chi}{2\chi_S\chi}\bar{\eth}\eth B_{Nr}+\frac{1}{4\chi}\left(\bar{\eth}_1 B_N+\eth_{-1}B_N\right)-\frac{1}{\chi_S}B_{Nr}+\right.\right.\\ &\left.+\left(\frac{c}{\mathscr{H}_S\chi_S}-1\right)B_{Nr,0}\right]+v_{S\|}\left[\left(-4+\frac{\mathscr{H}'_S c^2}{\chi_S\mathscr{H}_S^3}+\frac{5c}{\mathscr{H}_S\chi_S}\right)U_{NS}\right.\\ &+\int_0^{\chi_S} d\chi\left(6-\frac{2\mathscr{H}'_S c^2}{\chi_S\mathscr{H}_S^3}-\frac{10c}{\mathscr{H}_S\chi_S}+\frac{\chi}{\chi_S}+\frac{c\chi}{\mathscr{H}_S\chi_S^2}\right)W_{N,0}+\\ &+\int_0^{\chi_S} d\chi\left(\frac{\chi_S-\chi}{\chi\chi_S}-\frac{c}{\chi\chi_S\mathscr{H}_S}\right)\bar{\eth}\eth W_N+2V_{NS}+W_{NS}+\\ &+\frac{c}{\mathscr{H}_S}\left(\frac{dU_{NS}}{d\chi}-2W_{NS,0}\right)+\frac{c}{\mathscr{H}_S\chi_S}\left[-3V_{NS}+W_{NS}+\right.\\ &\left.\left.-\chi_S\frac{dV_{NS}}{d\chi_S}+\int_0^{\chi_S} d\chi\frac{2}{\chi_S}W_N+\frac{c}{\mathscr{H}}\left(\frac{dU_{NS}}{d\chi}-2W_{NS,0}\right)\right]\right]+\end{aligned}$$

$$
\begin{aligned}
&+v'_{S\|}\left(\frac{c}{\mathscr{H}_S\chi_S}-1\right)\frac{c}{\mathscr{H}_S}\left(-U_{NS}+2\int_0^{\chi_S}d\chi W_{N,0}\right)+\\
&+\left(\frac{c}{\mathscr{H}_S\chi_S}-1\right)\left(\frac{1}{2}{}_1v_S\int_0^{\chi_S}d\chi\frac{1}{\chi}\bar{\eth}W_N+\frac{1}{2}{}_{-1}v_S\int_0^{\chi_S}d\chi\frac{1}{\chi}\eth W_N\right)+\\
&+\left(\frac{\mathscr{H}'_Sc^2}{\chi_S\mathscr{H}_S^3}+\frac{4c}{\mathscr{H}_S\chi_S}-3\right)v_{S\|}\frac{1}{2}v_S^2+\frac{1}{2}\frac{c}{\mathscr{H}_S}v_{S\|}^2v'_{S\|}\left(3\frac{\mathscr{H}'_Sc^2}{\chi_S\mathscr{H}_S^3}+\right.\\
&\left.+\frac{5c}{\mathscr{H}_S\chi_S}-\frac{\mathscr{H}'_Sc}{\mathscr{H}_S^2\chi_S}-3\right)-\left(\frac{c}{\mathscr{H}_S\chi_S}-1\right)\frac{c}{\mathscr{H}}v'_{S\|}\left(\frac{1}{2}v_S^2+\frac{1}{\mathscr{H}_S}v'_{S\|}v_{S\|}\right)+\\
&+\left(\frac{c}{\mathscr{H}_S\chi_S}-1\right)\frac{c}{\mathscr{H}}v_{S\|}v_Sv'_S-\left(\frac{c}{\mathscr{H}_S\chi_S}-1\right)\frac{c^2}{2\mathscr{H}_S^2}v''_{S\|}v_{S\|}^2+\\
&\left.+v_{S\|}^3\left(2-\frac{8c}{\mathscr{H}_S\chi_S}-\frac{\mathscr{H}''_Sc^3}{6\chi_S\mathscr{H}_S^4}+\frac{\mathscr{H}'^2c^3}{2\chi_S\mathscr{H}_S^5}\right)\right\}.
\end{aligned}
\tag{4.222}
$$

As for the reduced shear, at this order the convergence contains two types of contributions. First, contributions from the vector potential B_N^i. The dominant contribution at small scales is given by the first term, which contains two transverse derivatives, and is equivalent to the shear contribution in eq. (4.217). In addition, since the convergence is a spin-0 field it contains contributions from the spin-1 and -1 part of B_N^i, on which the transverse operators $\eth$ and $\bar{\eth}$ act once. The second type of contributions to the convergence are due to the coupling between the first order Doppler contribution and the convergence at second order. The spin-1 and -1 contributions ${}_1v_S$ and ${}_{-1}v_S$, respectively, stem from the decomposition of the peculiar velocity field v_S^i into $v_S^i = v_{S\|}^i + \frac{1}{2}{}_{-1}v_Se_+^i + \frac{1}{2}{}_1v_Se_-^i$ and occur when the vector field v_S^i is coupled with the derivative of the scalar potential W_N. Finally, the convergence contains also a pure Doppler contribution, proportional to the velocity cubed, in the last line of (4.222).

The expression for the convergence at order $\mathscr{O}\left(\frac{1}{c^4}\right)$ is grouped into various terms according to the potentials or their couplings plus various contributions from redshift perturbations:

$$
\kappa^{(4)} = \kappa^{(P)} + \kappa^{(UW)} + \kappa^{(VV)} + \kappa^{(VW)} + \kappa^{(WW)} + \kappa^{(h)} + \kappa^{(\delta z)} + \kappa^{(v)} + \kappa^{(v^2)} + \kappa^{(v^4)}.
\tag{4.223}
$$

The superscripts (UW), (VW), (VV), and (WW) refer to the couplings of the Newtonian potentials U_N, V_N, and W_N, whereas the superscript (P) denotes the contributions of the post-Friedmann potentials U_P, V_P, and W_P. The last four terms in eq. (4.223) with the superscripts (δz), (v), (v^2), and (v^4) refer to the terms that are introduced via the redshift perturbations in eq. (4.179). In particular (δz) regroups all redshift perturbations not due to peculiar velocity, whereas the other terms regroup the velocity terms at each relevant order.

The first contribution of eq. (4.223) reads

$$\kappa^{(P)} = \frac{2}{c^4}\left(\int_0^{\chi_S} d\chi \frac{2}{\chi_S} W_P - V_P - \int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi_S \chi} \bar{\eth}\eth W_P\right), \tag{4.224}$$

and is of purely relativistic origin. Note that $\kappa^{(P)}$ takes on the same form as $\kappa^{(2)}$ in eq. (4.221) with the relativistic potentials $2V_P$ and $2W_P$ replacing the Newtonian potentials V_N and W_N. The terms derived from the redshift perturbations in $\kappa^{(2)}$ have their relativistic analogue in $\kappa^{(\delta z)}$.

The next contributions $\kappa^{(UW)}$, $\kappa^{(VV)}$, $\kappa^{(VW)}$, and $\kappa^{(WW)}$ collect the coupling terms with the Newtonian potentials U_N, V_N, and W_N:

$$\begin{aligned}\kappa^{(UW)} =& \frac{1}{\chi_S}\frac{1}{c^4}\left\{\int_0^{\chi_S} d\chi \left[-4W_N\left(U_{NS} + U_N + (\chi_S - \chi)\frac{d}{d\chi}U_N\right) + \right.\right.\\ & \left. -4U_{N,0}\left(W_N - (\chi_S - \chi)\int_0^{\chi} d\chi' W_N\right)\right] + \\ & +2\int_0^{\chi_S} d\chi \left(\eth U_N \int_0^{\chi} d\chi' \frac{\chi - \chi'}{\chi\chi'}\bar{\eth} W_N + \bar{\eth} U_N \int_0^{\chi} d\chi' \frac{\chi - \chi'}{\chi\chi'}\eth W_N\right) + \\ & \left. -2\int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi}\left(\eth U_N \int_0^{\chi} d\chi' \frac{1}{\chi'}\bar{\eth} W_N + \bar{\eth} U_N \int_0^{\chi} d\chi' \frac{1}{\chi'}\eth W_N\right)\right\},\end{aligned} \tag{4.225}$$

$$\kappa^{(VV)} = \frac{1}{c^4}\left[2\int_0^{\chi_S} d\chi \frac{(\chi_S - \chi)\chi}{\chi_S}\left(\frac{d}{d\chi}V_N\right)^2 - \frac{1}{4}V_{NS}^2\right], \tag{4.226}$$

$$\begin{aligned}\kappa^{(VW)} =& \frac{1}{c^4}\left\{\int_0^{\chi_S} d\chi \frac{1}{\chi_S}\left[W_N\left(V_{NS} + 2\chi\frac{d}{d\chi_S}V_{NS} - 2\chi V_{NS,0} - 2\chi_S\frac{d}{d\chi}V_N\right) + \right.\right.\\ & \left. +2W_N\chi_S V_{N,0} + 2\eth V_N \int_0^{\chi} d\chi' \frac{1}{\chi'}\bar{\eth} W_N + 2\bar{\eth} V_N \int_0^{\chi} d\chi' \frac{1}{\chi'}\eth W_N\right] + \\ & + \int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi_S}\left[-2\chi W_N\left(\frac{d^2}{d\chi^2}V_N - \frac{d}{d\chi}V_{N,0}\right) - \eth V_N \frac{1}{\chi}\bar{\eth} W_N\right. \\ & - \bar{\eth} V_N \frac{1}{\chi}\eth W_N - \eth V_{NS}\frac{1}{\chi}\bar{\eth} W_N - \bar{\eth} V_{NS}\frac{1}{\chi}\eth W_N - V_{NS}\frac{1}{\chi}\bar{\eth}\eth W_N + \\ & \left.\left. + V_{NS} W_{N,0}\right]\right\},\end{aligned} \tag{4.227}$$

and

$$\kappa^{(WW)} = \frac{1}{c^4}\left\{\int_0^{\chi_S} d\chi \frac{1}{\chi_S}\left[4W_N^2 - 4W_{NS}W_N + 8W_{N,0}\int_0^{\chi} d\chi' W_N + \right.\right.$$

$$
\begin{aligned}
& -10\eth W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi\chi'} \bar{\eth} W_N - 10\bar{\eth} W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi\chi'} \eth W_N + \\
& + 4W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi\chi'} \bar{\eth}\eth W_N + 2W_N \int_0^{\chi} d\chi' \frac{1}{\chi'} \bar{\eth}\eth W_N + \\
& +2\frac{1}{\chi}\eth\bar{\eth}\left(W_N \int_0^{\chi} d\chi' W_N\right)\Bigg] + \int_0^{\chi_S} d\chi \frac{\chi_S-\chi}{\chi_S}\Bigg[-4W_{N,0}^2 + \\
& -\bar{\eth}\eth W_N^2 \frac{1}{\chi} - 8W_N W_{N,0} + 2\left(\frac{d}{d\chi}W_N\right)^2 - 4W_{N,0}\frac{d}{d\chi}W_N + \\
& -4W_{N,00}\int_0^{\chi} d\chi' W_N - \frac{2}{\chi^2} W_N \int_0^{\chi} d\chi' \bar{\eth}\eth W_N + \\
& +6\eth W_N \int_0^{\chi} d\chi' \frac{1}{\chi\chi'} \bar{\eth} W_N + 6\bar{\eth} W_N \int_0^{\chi} d\chi' \frac{1}{\chi\chi'} \eth W_N + \\
& +8\eth W_{N,0} \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi\chi'} \bar{\eth} W_N + 8\bar{\eth} W_{N,0} \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi\chi'} \eth W_N + \\
& +2\eth W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi^2\chi'} \bar{\eth} W_N + 2\bar{\eth} W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi^2\chi'} \eth W_N + \\
& -2\bar{\eth}\eth\left(W_{N,0}\int_0^{\chi} d\chi' \frac{1}{\chi'} W_N\right) + 2\bar{\eth}\eth W_{N,0}\int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi\chi'} W_N + \\
& +4\frac{1}{\chi^2}\bar{\eth}\eth W_N \int_0^{\chi} d\chi' W_N - 4\frac{1}{\chi^2}\bar{\eth}\eth W_N \int_0^{\chi} d\chi' \left(\chi-\chi'\right) W_{N,0} \\
& -\frac{1}{2}\eth^2 W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi^2\chi'} \bar{\eth}^2 W_N - \frac{1}{2}\bar{\eth}^2 W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi^2\chi'} \eth^2 W_N + \\
& -2W_{N,0}\int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi\chi'} \bar{\eth}\eth W_N - \eth\left(\bar{\eth}\eth W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi^2\chi'} \bar{\eth} W_N\right) + \\
& -\eth\bar{\eth}^2 W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi^2\chi'} \eth W_N \Bigg]\Bigg\}.
\end{aligned}
\tag{4.228}
$$

The tensor potential h_{ij} contributes to the convergence in the following way:

$$
\begin{aligned}
\kappa^{(h)} =& \frac{1}{c^4}\frac{1}{4}\Bigg[h_{rr}(\chi_S) - \int_0^{\chi_S} d\chi \frac{1}{\chi}\left(\eth_{-1}h_r + \bar{\eth}_1 h_r - h_{rr}\right) + \\
& + \int_0^{\chi_S} d\chi \frac{\chi_S-\chi}{\chi_S\chi}\left(\eth\bar{\eth} h_{rr} - \chi h_{rr,0}\right)\Bigg].
\end{aligned}
\tag{4.229}
$$

Finally, the redshift perturbations are split into four different groups: the first group is denoted by $\kappa^{(\delta z)}$ and refers to the redshift perturbations independent of the peculiar velocity, while the other groups $\kappa^{(v)}$, $\kappa^{(v^2)}$, and $\kappa^{(v^4)}$ refer to the terms dependent on the peculiar velocity.

$$
\kappa^{(\delta z)} = \frac{1}{c^4}\Bigg\{\Bigg[\frac{1}{\chi_S}\int_0^{\chi_S} d\chi \left(2W_N - \left(\chi_S - \chi - \frac{c}{\mathscr{H}_s}\right)\frac{1}{\chi}\bar{\eth}\eth W_N\right) +
$$

$$
\begin{aligned}
&+\frac{c}{\chi_S\mathcal{H}_S}\left(\chi_S\frac{dV_{NS}}{d\chi_S}-2W_{NS}\right)\Bigg]\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}\right)+\\
&-\left(1-\frac{\mathcal{H}_S'c^2}{2\chi_S\mathcal{H}_S^3}-\frac{3c}{2\mathcal{H}_S\chi_S}\right)\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}\right)^2+\\
&+\left(\frac{c}{\mathcal{H}_S\chi_S}-1\right)\Bigg\{2U_{PS}-4\int_0^{\chi_S}d\chi W_{P,0}-\frac{5}{2}U_{NS}^2-\frac{1}{2}\int_0^{\chi_S}d\chi h_{rr,0}+\\
&-2\left(\int_0^{\chi_S}d\chi W_{N,0}\right)^2+4\left(U_{NS}\int_0^{\chi_S}d\chi W_{NS}\right)_{,0}+4W_{NS,0}\int_0^{\chi_S}d\chi W_N+\\
&+2\frac{1}{\chi_S}\bar{\eth}U_{NS}\int_0^{\chi_S}d\chi\left(\chi_S-\chi\right)\frac{1}{\chi}\eth W_N+2\frac{1}{\chi_S}\eth U_{NS}\int_0^{\chi_S}d\chi\left(\chi_S-\chi\right)\frac{1}{\chi}\bar{\eth}W_N+\\
&-4\frac{d}{\chi_S}U_{NS}\int_0^{\chi_S}d\chi W_N+V_{NS}\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}\right)+\\
&+\frac{c}{\mathcal{H}_S}\left(\frac{dU_{NS}}{d\chi_S}U_{NS}-2\frac{dU_{NS}}{d\chi_S}\int_0^{\chi_S}d\chi W_{N,0}-2W_{NS,0}U_{NS}+2W_{NS,0}2\int_0^{\chi_S}d\chi W_{N,0}\right)\\
&+\int_0^{\chi_S}d\chi\Bigg(4W_N\frac{d}{d\chi}U_N-4U_{N,0}W_N-4W_{N,0}W_N-4W_{N,00}\int_0^{\chi}d\chi' W_N+\\
&-2\eth U_N\int_0^{\chi}d\chi'\frac{1}{\chi\chi'}\bar{\eth}W_N-2\bar{\eth}U_N\int_0^{\chi}d\chi'\frac{1}{\chi\chi'}\eth W_N+\\
&-2\eth W_{N,0}\int_0^{\chi}d\chi\frac{\chi-\chi'}{\chi chi'}\bar{\eth}W_N-2\eth W_N\int_0^{\chi}d\chi\frac{\chi-\chi'}{\chi^2\chi'}\bar{\eth}W_N+\\
&-2\bar{\eth}W_{N,0}\int_0^{\chi}d\chi'\frac{\chi-\chi'}{\chi\chi'}\eth W_N-2\bar{\eth}W_N\int_0^{\chi}d\chi'\frac{\chi-\chi'}{\chi^2\chi'}\eth W_N\Bigg)\Bigg\}\Bigg\},
\end{aligned}
\tag{4.230}
$$

$$
\begin{aligned}
\kappa^{(v)}=&\frac{1}{c^4}\Bigg\{v_{S\parallel}\Bigg[\int_0^{\chi_S}d\chi\left(\frac{\chi_S-\chi}{\chi_S\chi}-\frac{c}{\chi_S\chi\mathcal{H}_S}\right)\frac{1}{2}\left(2\chi B_{Nr,0}-\frac{d}{d\chi}\left(\chi\eth_1 B_N\right)-\eth^2 B_{Nr}\right)+\\
&-\frac{c}{\chi_S\mathcal{H}_S}B_{NSr}+\int_0^{\chi_S}d\chi\left(B_{Nr}+\left(2-\frac{\mathcal{H}_S'c^2}{\chi_S\mathcal{H}_S^3}-\frac{3c}{\chi_S\mathcal{H}_S}\right)B_{Nr,0}\right)\Bigg]+\\
&-\left(\frac{c}{\mathcal{H}_S\chi_S}-1\right)\Bigg[{}_1v_S\left(\frac{1}{2}{}_{-1}B_{NS}+\frac{1}{2}\int_0^{\chi_S}d\chi\frac{1}{\chi}\left(\bar{\eth}B_{Nr}+{}_{-1}B_N\right)\right)+\\
&+{}_{-1}v_S\left(\frac{1}{2}{}_1B_{NS}+\frac{1}{2}\int_0^{\chi_S}d\chi\frac{1}{\chi}\left(\eth B_{Nr}+{}_1B_N\right)\right)\Bigg]\Bigg\},
\end{aligned}
\tag{4.231}
$$

$$
\begin{aligned}
\kappa^{(v^2)}=&\frac{1}{c^4}\Bigg\{\left(\frac{c}{\mathcal{H}_S\chi_S}-1\right)\Bigg\{v_{S\parallel}'^2\left(2\frac{c^2}{\mathcal{H}_S^2}\int_0^{\chi_S}d\chi W_{N,0}-\frac{c^2}{\mathcal{H}_S^2}U_{NS}\right)+\\
&+\left(v_Sv_S'+\frac{v_{S\parallel}c}{\mathcal{H}_S}v_{S\parallel}''\right)\frac{c}{\mathcal{H}_S}\left(U_{NS}-2\int_0^{\chi_S}d\chi W_{N,0}\right)+
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{c}{\mathcal{H}_S}v'_{S\|}\,{}_{-1}v_S\frac{1}{2}\int_0^{\chi_S}d\chi'\frac{1}{\chi'}\eth W_N-\frac{c}{\mathcal{H}_S}v'_{S\|}\,{}_{1}v_S\frac{1}{2}\int_0^{\chi_S}d\chi'\frac{1}{\chi'}\bar{\eth}W_N+\\
&+v_{S\|}\frac{c}{\mathcal{H}_S}\,{}_{-1}v'_S\frac{1}{2}\int_0^{\chi_S}d\chi'\frac{1}{\chi'}\eth W_N+v_{S\|}\frac{c}{\mathcal{H}_S}\,{}_{1}v'_S\frac{1}{2}\int_0^{\chi_S}d\chi'\frac{1}{\chi'}\bar{\eth}W_N+\\
&+v_{S\|}\,{}_{-1}v_S\frac{1}{2}\frac{1}{\chi_S}\eth W_{NS}+v_{S\|}\,{}_{1}v_S\frac{c}{2\mathcal{H}_S\chi_S}\bar{\eth}W_{NS}\ \}+\\
+&\left[\left(\frac{c}{\mathcal{H}_S\chi_S}-1\right)\left(\frac{c}{\mathcal{H}_S}\frac{dU_{NS}}{d\chi}-2\frac{c}{\mathcal{H}_S}W_{NS,0}\right)+\frac{2}{\chi_S}\int_0^{\chi_S}d\chi W_N-3V_{NS}+\right.\\
&+\frac{c}{\chi_S\mathcal{H}_S}\left(\chi_S\frac{dV_{NS}}{d\chi_S}+4W_{NS}\right)-\int_0^{\chi_S}d\chi\left(\frac{\chi_S-\chi}{\chi_S\chi}-\frac{c}{\chi_S\chi\mathcal{H}_s}\right)\bar{\eth}\eth W_N+\\
&\left.+\left(4-\frac{\mathcal{H}'_Sc^2}{\chi_S\mathcal{H}_S^3}-\frac{5c}{\mathcal{H}_S\chi_S}\right)2\int_0^{\chi_S}d\chi W_{N,0}-\left(1-\frac{\mathcal{H}'_Sc^2}{\chi_S\mathcal{H}_S^3}+\frac{c}{\mathcal{H}_S\chi_S}\right)U_{NS}\right]\frac{1}{2}v_S^2+\\
+&\left[\frac{2}{\chi_S}\int_0^{\chi_S}d\chi W_N-\int_0^{\chi_S}d\chi\left(\frac{\chi_S-\chi}{\chi_S\chi}-\frac{c}{\chi_S\chi\mathcal{H}_s}\right)\bar{\eth}\eth W_N+\right.\\
&+\frac{c}{\mathcal{H}_S}\frac{dV_{NS}}{d\chi_S}-V_{NS}+\left(-1+\frac{\mathcal{H}'_Sc^2}{\chi_S\mathcal{H}_S^3}+\frac{2c}{\chi_S\mathcal{H}_S}+\frac{\mathcal{H}'_Sc}{\mathcal{H}_S^2}\right)U_{NS}+\\
&\left.+\left(3-\frac{7c}{\mathcal{H}_S\chi_S}-\frac{2\mathcal{H}'_Sc^2}{\chi_S\mathcal{H}_S^3}-2\frac{\mathcal{H}'_Sc}{\mathcal{H}_S^2}\right)\int_0^{\chi_S}d\chi W_{N,0}\right]\left(\frac{c}{\mathcal{H}_S}v'_{S\|}v_{S\|}\right)+\\
&+\left[V_{NS}\frac{c}{2\mathcal{H}_S\chi_S}\left(1+\frac{\mathcal{H}'c}{\mathcal{H}^2}\right)-3V_{NS}+2W_{NS}\frac{c}{\mathcal{H}_S\chi_S}\left(4+\frac{\mathcal{H}'c}{\mathcal{H}^2}\right)++\right.\\
&+\frac{1}{2}\left(\chi_S\frac{d}{d\chi_S}V_{NS}\frac{c}{\mathcal{H}_S\chi_S}+\int_0^{\chi_S}d\chi\frac{1}{\chi}\bar{\eth}\eth W_N\frac{c}{\mathcal{H}_S\chi_S}\right)\left(1+\frac{\mathcal{H}'c}{\mathcal{H}^2}\right)+\\
&+\left(\frac{c}{\mathcal{H}_S\chi_S}-1\right)\left(5\frac{c}{\mathcal{H}_S}+\frac{\mathcal{H}'_Sc^2}{\mathcal{H}_S^3}-\frac{1}{2}\frac{c^2}{\mathcal{H}_S^2}\frac{d}{d\chi_S}\right)\frac{1}{2}\frac{dU_{NS}}{d\chi_S}+3\frac{c}{\mathcal{H}_S}\frac{dW_{NS}}{d\chi_S}+\\
&+\left(8\frac{c}{\mathcal{H}_S}+\frac{\mathcal{H}'_Sc^2}{\mathcal{H}_S^3}-\frac{1}{2}\frac{c^2}{\mathcal{H}_S^2}\frac{d}{d\chi_S}-5\frac{c^2}{\mathcal{H}_S^2\chi_S}-\frac{\mathcal{H}'_Sc^3}{\mathcal{H}_S^4\chi_S}+\frac{1}{2}\frac{c^3}{\mathcal{H}_S^3\chi_S}\frac{d}{d\chi_S}\right)W_{NS,0}+\\
&+\left(-1-\frac{7c}{2\chi_S\mathcal{H}_S}-\frac{\mathcal{H}''_Sc^3}{2\chi_S\mathcal{H}_S^4}+\frac{3\mathcal{H}'^2c^3}{2\chi_S\mathcal{H}_S^5}-\frac{\mathcal{H}'_Sc^2}{\chi_S\mathcal{H}_S^3}\right)U_{NS}+\\
&+\left(\frac{\mathcal{H}''_Sc^3}{\chi_S\mathcal{H}_S^4}+2-\frac{3\mathcal{H}'^2c^3}{\chi_S\mathcal{H}_S^5}-\frac{\mathcal{H}'_Sc^2}{\chi_S\mathcal{H}_S^3}-\frac{2c}{\mathcal{H}_S\chi_S}\right)\int_0^{\chi_S}d\chi W_{N,0}+\\
&\left.+\frac{c^2}{\mathcal{H}_S^2\chi_S}\left(-2\frac{d}{d\chi_S}W_{NS}+\frac{1}{2}\bar{\eth}\eth W_{NS}-\frac{d}{d\chi_S}V_{NS}-\frac{1}{2}\chi_S\frac{d^2}{d\chi_S^2}V_{NS}\right)\right]v_{S\|}^2+\\
&+\left(2-\frac{\mathcal{H}'_Sc^2}{\chi_S\mathcal{H}_S^3}-\frac{3c}{\mathcal{H}_S\chi_S}\right)v_{S\|}\left({}_{-1}v_S\frac{1}{2}\int_0^{\chi_S}d\chi'\frac{1}{\chi'}\eth W_N+{}_{1}v_S\frac{1}{2}\int_0^{\chi_S}d\chi'\frac{1}{\chi'}\bar{\eth}W_N\right)\Bigg\},
\end{aligned}
\tag{4.232}
$$

and

$$
\begin{aligned}
\kappa^{(v^4)} =&\frac{1}{c^4}\Bigg\{ \left(\frac{c}{\mathscr{H}_S\chi_S}-1\right)\Bigg[\frac{1}{2}\left(1+\frac{\mathscr{H}_S'c}{\mathscr{H}_S^2}\right)\frac{c}{\mathscr{H}_S}\left(v_{S\|}^2 v_S v_S' - v_{S\|}^3 v_{S\|}''\right)+\\
&+\frac{c}{2\mathscr{H}_S}v_S v_S' v_S^2 - \frac{c^2}{2\mathscr{H}_S^2}v_{S\|}^2 v_S v_S'' + \frac{c^3}{6\mathscr{H}_S^3}v_{S\|}^3 v_{S\|}''' - \frac{c^2}{1\mathscr{H}_S^2}v_{S\|}'^2 v_S^2 +\\
&-\frac{c^2}{\mathscr{H}_S^3}v_{S\|}'^3 v_{S\|} - \frac{1}{2}\frac{c^2}{\mathscr{H}_S^2}v_{S\|}^2 v_S'^2 + \frac{c^2}{2\mathscr{H}_S^2}v_{S\|}v_{S\|}'' v_S^2 + \frac{3c^2}{2\mathscr{H}_S^2}\frac{c}{\mathscr{H}_S}v_{S\|}^2 v_{S\|}' v_{S\|}'' +\\
&+v_{S\|}^2 v_{S\|}'^2\frac{c}{\mathscr{H}_S}\int_0^{\chi_S} d\chi W_{N,0}\Bigg] + \left(\frac{1}{2}+\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-\frac{\mathscr{H}_S'c^2}{2\mathscr{H}_S^2}\right)\frac{c}{\mathscr{H}_S}v_{S\|}' v_{S\|} v_S^2 +\\
&-\left(3-\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-\frac{4c}{\mathscr{H}_S\chi_S}\right)\frac{1}{8}v_S^4 - \left(\frac{3}{2}+\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-\frac{3\mathscr{H}_S'c}{2\mathscr{H}_S^2}-\frac{c}{\mathscr{H}_S\chi_S}\right)\frac{c^2}{\mathscr{H}_S^2}v_{S\|}'^2 v_{S\|}^2 +\\
&+\left(\frac{\mathscr{H}_S''c^3}{2\chi_S\mathscr{H}_S^4}-\frac{5c}{2\chi_S\mathscr{H}_S}-\frac{3\mathscr{H}'^2c^3}{2\chi_S\mathscr{H}_S^5}-\frac{3\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}\right)\frac{1}{2}v_S^2 v_{S\|}^2 +\\
&+\left(2-\frac{\mathscr{H}_S'c^2}{\chi_S\mathscr{H}_S^3}-\frac{3c}{\mathscr{H}_S\chi_S}\right)\frac{c}{\mathscr{H}_S}\Bigg[-v_{S\|}' v_{S\|}\frac{1}{2}v_S^2 + v_{S\|}'^2 v_{S\|}^2\frac{1}{\mathscr{H}_S} + v_{S\|}^2\frac{dU_{NS}}{d\chi} +\\
&-v_{S\|}^2 2W_{NS,0} + v_{S\|}^2 v_S v_S' - v_{S\|}^3\frac{c}{\mathscr{H}_S}v_{S\|}''\Bigg] + \Bigg(\frac{17c}{6\chi_S\mathscr{H}_S}-\frac{4}{3}-\frac{2\mathscr{H}_S''c^3}{3\chi_S\mathscr{H}_S^5}+\\
&+\frac{\mathscr{H}_S''c^2}{6\mathscr{H}_S^4}+\frac{\mathscr{H}'^2c^3}{3\chi_S\mathscr{H}_S^5}-\frac{\mathscr{H}_S'c^2}{2\chi_S\mathscr{H}_S^3}+\frac{3\mathscr{H}_S'c}{2\mathscr{H}_S^2}+\frac{\mathscr{H}_S'^2c^2}{6\mathscr{H}_S^4}\Bigg)\frac{c}{\mathscr{H}_S}v_{S\|}' v_{S\|}^3\\
&-\Bigg(\frac{\mathscr{H}_S'c^2}{24\chi_S\mathscr{H}_S^3}-\frac{c}{12\chi_S\mathscr{H}_S}-\frac{c^3\mathscr{H}_S''}{12\chi_S\mathscr{H}^4}+\frac{1\mathscr{H}'^2c^3}{4\chi_S\mathscr{H}_S^5}-\frac{\mathscr{H}_S'''c^4}{24\chi_S\mathscr{H}_S^5}+\\
&-\frac{15c^4\mathscr{H}_S'^3}{24\chi_S\mathscr{H}_S^7}+\frac{5c^4\mathscr{H}_S'\mathscr{H}_S''}{12\chi_S\mathscr{H}^6}\Bigg)v_{S\|}^4\Bigg\}.
\end{aligned}
\tag{4.233}
$$

The convergence at second-order in standard perturbation theory has been computed in [106]. We expect some of the terms in our formalism to be equivalent to the SPT result, while others will be different, due to the different counting of perturbations.

4.3.4 The rotation ω

The rotation ω is related to the imaginary part of the spin-0 component ${}_0\tilde{\mathscr{D}}$ via eq. (4.16), which is proportional to the anti-symmetric part of $\tilde{\mathscr{D}}_{ab}$, see eq. (4.14) . From eqs. (4.98), (4.101), (4.176), (4.177) and (4.178) we see that up to order $\mathscr{O}\left(\frac{1}{c^3}\right)$, $\tilde{\mathscr{D}}_{ab}$ is symmetric and that there is therefore no rotation at those orders. At order $\mathscr{O}\left(\frac{1}{c^4}\right)$ on the other hand, there is a anti-symmetric contribution generated by the coupling $\mathscr{R}_{ac}^{(2)}\mathscr{D}_{cb}^{(2)}$ in eq. (4.112). Using ${}_0\tilde{\mathscr{D}}^{(4)}_{\mathrm{I}}$ in equation (4.214), we obtain

$$\omega^{(4)} = \frac{1}{2\chi_S c^4}\int_0^{\chi_S} d\chi (\chi_S-\chi)\frac{1}{\chi^2}\left(\eth^2 W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\bar{\eth}^2 W_N \right.$$
$$\left. -\bar{\eth}^2 W_N \int_0^{\chi} d\chi' \frac{\chi-\chi'}{\chi'}\eth^2 W_N\right). \tag{4.234}$$

We see that the only terms that contribute to the rotation at order $\mathscr{O}\left(\frac{1}{c^4}\right)$ are the lens-lens coupling and the post-Born correction, i.e. the terms with four transverse derivatives, which dominate at small scales. The rotation contributes in principle to the ellipticity orientation, as discussed in [17]. However, since the shear is at least of order $\mathscr{O}\left(\frac{1}{c^2}\right)$ and the rotation is of order $\mathscr{O}\left(\frac{1}{c^4}\right)$, the contribution to the ellipticity is of order $\mathscr{O}\left(\frac{1}{c^6}\right)$. This represents a very small contribution to the ellipticity B-mode.

A thorough analysis and discussion of the result can be found in the paper [54], which is based on this chapter.

4.4 Comparison with Standard Perturbation Theory

In this section, we want to compare our results to those using SPT, e.g. [17]. In [80] it was shown that the first-order SPT can be recovered from the PF approach: first "resummed variables" ϕ_P and ψ_P were introduced, which are composed of different orders of the PF formalism. Secondly, the Einstein field equations were linearised and the new variables substituted. The outcome yields the first order field equations of SPT[6].The resummed variables ϕ_P and ψ_P read

$$\phi_P = -\left(U_N + \frac{2}{c^2}U_P\right) \quad \text{and} \quad \psi_P = -\left(V_N + \frac{2}{c^2}V_P\right). \tag{4.235}$$

The lensing potential in terms of the resummed variables is defined as

$$\Psi_P \equiv \frac{1}{2}\left(\phi_P + \psi_P\right). \tag{4.236}$$

We linearise κ and g and substitute (4.235), which yields

$$\kappa = \int_0^{\chi_S} d\chi \frac{\chi_S-\chi}{\chi\chi_S}\bar{\eth}\eth\Psi_P + \psi_{PS} - \frac{2}{\chi_S}\int_0^{\chi_S} d\chi \Psi_P \tag{4.237}$$

[6] We omitted any contributions of the redshift perturbations.

and

$$g = -\int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi} \eth^2 \Psi_P. \tag{4.238}$$

Our outcome matches with the results of [17, 23].

4.5 Conclusion

Weak gravitational lensing is becoming a powerful tool to map the Universe. It could help to constrain both modified gravity or the equation of state and could give us a thorough insight into the distribution of matter regardless of its composition. Most of the analysis done on weak lensing uses standard perturbation theory, which is valid on large scales, where the perturbations are assumed to be small. However, weak lensing is a gravitational effect that connects large to small scales by integrating along the line of sight. In addition, future surveys will deliver high-precision data from small, nonlinear scales.

In this chapter we compute the convergence κ, the shear γ, and the rotation ω up to order $\mathscr{O}\left(\frac{1}{c^4}\right)$ in the Post-Friedmann formalism. Our results provide a systematic and consistent description of weak lensing observables, including scalar, vector and tensor modes as well as galaxies' peculiar velocities. We choose a spherical screen space, whereby we are able to go beyond the small-angle or thin-lens approximation. The PF formalism is especially advantageous for weak lensing analysis because of its validity on all scales [80]: from small, nonlinear scales, where the density contrast is large, to large, linear scales well within the relativistic regime. The metric of the PF approximation consists of scalar potentials associated with Newtonian dynamics and at higher orders scalar potentials that represent relativistic corrections. Furthermore, it includes vector and tensor potentials, where the lower order vector potential is sourced by Newtonian quantities and will therefore be associated with the Newtonian regime. At the orders considered in this work, the tensor potential is non-dynamical and does not represent gravitational waves. We choose the Poisson gauge; our Newtonian and relativistic scalar potentials are U_N and V_N, and U_P and V_P, respectively. The vector potential B^i is split into a "Newtonian" contribution B^i_N and a relativistic correction B^i_P, which are divergence free. The tensor potential h_{ij} is transverse and tracefree. Furthermore, we denote the Weyl potential as $W_X = \frac{1}{2}(U_X + V_X)$ with $X = N, P$. In derivation of κ, γ and ω, we have not picked a set of field equations and we therefore assume that the Newtonian scalar potentials U_N and V_N not necessarily coincide. In General Relativity, one obtains at leading order via the Einstein field equations that

$U_N = V_N$, but in order to keep our result as general as possible, we refrained from assuming the Newtonian potentials to be equal. This leaves the possiblity to use our analysis to test modified gravity models.

Following [17, 23, 21], we compute κ, γ and ω in terms of spin-0 and spin-2 operators, respectively, on a sphere. The spin-weighted operators represent the spherical symmetry that is associated with κ, γ, and ω. By using these operators,κ, γ, and ω can be decomposed into the sum of spin-weighted spherical harmonics. We express our result in terms of the redshift z instead of an affine parameter λ or χ, because neither are measurable quantities, but the redshift z is. By introducing the redshift z as variable, we need take redshift perturbations δz into account. The redshift is affected by the peculiar velocities, which modify the apparent distance between the source and the observer, and consequently the apparent galaxy size. E.g. at the lowest order, $\mathcal{O}\left(\frac{1}{c}\right)$, the peculiar velocities affect the convergence in eq. (4.220). This effect, called Doppler magnification [21, 6, 22]. By choosing a spherical screen space, we avoid the restrictions of the thin-lens approximation. In the thin-lens or small-angle approximation relativistic correction and derivatives along the line of sight can be neglected. In our result, we keep all coupling terms.

The null geodesic is conformally invariant and one would expect that our result can be expressed only in terms of the Weyl potential W_X, which remains the metric potential after the (quasi) conformal transformation of the metric in equation (4.91). However, there occur contributions and coupling terms involving the potentials U_X and V_X that come in over various performed perturbations such as perturbing the path of the geodesic in order to go beyond the Born approximation or the redshift perturbations. Even before, we see that the convergence comprises terms of the scalar potential V_X evaluated at the source. These terms origin from the fact that the parallel transport of the basis n_a^μ is not conformally invariant [17].

We extend our analysis up to order $\mathcal{O}\left(\frac{1}{c^4}\right)$, where we have to take perturbations of the photon geodesic into account and hence go beyond the Born approximation. We express our results in terms of the redshift z instead of the parameters λ and χ. The next step in this project is to compute the correlation function for the shear and convergence and compare the results to observational data. Therefore, it is convenient to use the redshift z, because it a measurable quantity, but the parameters λ and χ are not. However, the redshift is a perturbative quantity and we need to take the redshift perturbations δz into account. δz contributes lens-source couplings and perurbations involving the peculiar velocity of the source.

In summary, we extended the work of [17, 23] using the post-Friedmann approximation scheme. We computed the convergence and reduced shear up to order $\mathcal{O}\left(\frac{1}{c^4}\right)$ and expressed our results in terms of spin weighted operators and in terms of the redshift z. The post-Friedmann formalism comprises scalar, vector, and tensor potentials. The vector potential, a relativistic contribution, is sourced by Newtonian quantities and we showed how it contributes to the convergence and reduced shear at the lower order $\mathcal{O}\left(\frac{1}{c^3}\right)$.

Chapter 5

Conclusion

5.1 Summary of the work so far

In this thesis I present my work on nonlinear approximation schemes in relativistic Cosmology. In the course of my PhD studies I focused on two main projects: the first project deals with non-Gaussian contributions in the density field and the mixing of $f_{\rm NL}$ and $g_{\rm NL}$ at higher orders [55]. The second project comprises the analysis of weak lensing using the post-Friedmann approach [54]. The two projects were carried out using various approximation schemes, which are valid on different scales.

In the Introduction I give a general overview of General Relativity and differential geometry and introduced the quantities that will be used in the later chapters. Furthermore, I give a general overview of the current state of theoretical and observational cosmology. Chapter 2 is dedicated to relativistic approximation schemes. I discuss perturbation theory and gauge transformations and subsequently introduce the three different approximation schemes that will be used in the chapters 3 and 4, namely standard perturbation theory, the gradient expansion, and the post-Friedmann approximation.

In chapter 3 I present the work based on the paper [55]. We investigate the evolution of the density contrast on very large scales. We extend the work of [27] and [26] up to fourth order in standard perturbation theory within the leading order of the gradient expansion, which is up to order $\mathscr{O}\left(\nabla^2\right)$. We recovered the second order results of [26], [105], and [109], while the present third and fourth order results are new solutions published in [55]. As discussed in the chapter 2 in section 2.3, the gradient expansion is an approximation scheme that has a spatially homogeneous and isotropic space-time as background and expands in terms of spatial gradients. Therefore, at low orders such as $\mathscr{O}\left(\nabla^2\right)$ we consider very large scales of the order of the Hubble radius. The continuity equation (3.5) and the energy constraint (3.7) show that the density contrast

δ, the expansion ϑ, and the Ricci scalar R are of order $\mathcal{O}\left(\nabla^2\right)$ in the gradient expansion. Therefore, it follows that the evolution equation for the density contrast (3.15) is linear in δ and ϑ at this order of the gradient expansion and takes the same form as the evolution equation at first order using standard perturbation theory. Furthermore, the Ricci scalar R remains constant at these scales. We chose the synchronous-comoving gauge and expressed the spatial metric perturbations in terms of the curvature perturbation ζ. At order $\mathcal{O}\left(\nabla^2\right)$ of the gradient expansion the spatial part of the metric can be assumed to be conformally flat, i.e. neglect anisotropic metric perturbations. As a metric perturbation we introduce the curvature perturbation ζ, which occurs in the exponent of the conformal factor. We solve the evolution equation for the density contrast δ up to fourth order in standard perturbation theory and impose primordial non-Gaussianity up to the required order as initial conditions. At third and fourth order of standard perturbation theory, we show how non-Gaussianity contributes to the density contrast. In particular, we show that at third order terms involving $g_{\rm NL}$ as well as $f_{\rm NL}$ occur. This is a consequence of the fact that at this order we obtain terms of order $\mathcal{O}(3)$, $\mathcal{O}(2)\mathcal{O}(1)$, and $\mathcal{O}(1)\mathcal{O}(1)\mathcal{O}(1)$. Naturally, the terms of order $\mathcal{O}(3)$ and $\mathcal{O}(2)\mathcal{O}(1)$ will contain $g_{\rm NL}$ and $f_{\rm NL}$, respectively. At fourth order, we find terms of order $\mathcal{O}(4)$, $\mathcal{O}(3)\mathcal{O}(1)$, $\mathcal{O}(2)\mathcal{O}(2)$, $\mathcal{O}(2)\mathcal{O}(1)\mathcal{O}(1)$, and $\mathcal{O}(1)\mathcal{O}(1)\mathcal{O}(1)\mathcal{O}(1)$. Therefore, the density contrast displays terms involving $h_{\rm NL}$, $g_{\rm NL}$, $f_{\rm NL}^2$, and $f_{\rm NL}$. In summary, our main result is that the nonlinear nature of General Relativity affects the density contrast by contributing non-Gaussian terms and by mixing primordial non-Gaussian parameters $f_{\rm NL}$, $g_{\rm NL}$, and $h_{\rm NL}$ at higher orders.

In chapter 4 I present the work [54] on weak lensing using the post-Friedmann approximation scheme. The incentive to use this approximation scheme is its validity on all scales. The PF approximation unites the relativistic treatment of SPT on large scales with the nonlinear Newtonian dynamics on small scales [80]. In weak lensing, we integrate along the line of sight and thereby couple large to small scales. Furthermore, we choose spherical spin-weighed coordinates to go beyond the thin-lens or small-angle approximation following the work of [17]. The PF formalism is a post-Newtonian-type approximation in a cosmological setting. We expand in inverse powers of the speed of light c. Order $\mathcal{O}\left(\frac{1}{c^2}\right)$ and $\mathcal{O}\left(\frac{1}{c^3}\right)$ are sourced by Newtonian quantities such as the scalar potentials U_N and V_N, whereas at higher orders relativistic corrections are added. In chapter 4, I compute the convergence κ, the reduced shear g, and the rotation ω up to $\mathcal{O}\left(\frac{1}{c^4}\right)$ including scalar, vector and tensor perturbations. For our analysis we chose the Poisson gauge. At orders considered in this work, the tensor contributions are non-dynamical. Vector perturbations on the other hand contribute already at order

$\mathcal{O}\left(\frac{1}{c^3}\right)$ to the convergence and shear. In the PF formalism, the vector potential B^i_N at order $\mathcal{O}\left(\frac{1}{c^3}\right)$ is sourced by Newtonian quantities, but is a relativistic contribution. The vector perturbation does not influence the matter dynamics but affects the light path and therefore the weak lensing analysis [103]. As we compute the Jacobi mapping $\mathscr{D}_{ab}$ up to order $\mathcal{O}\left(\frac{1}{c^4}\right)$, we need to go beyond the Born approximation and introduce perturbations of the photon geodesic. We express $\mathscr{D}_{ab}$ in terms of the redshift z instead of the parameter λ or χ, because unlike λ and χ, the redshift z is a measurable quantity. The change to redshift introduces further perturbations, in particular perturbations involving the peculiar velocity of the source. In the PF formalism, the peculiar velocity adds a factor $1/c$, which affects the order of the approximation. In order to perform a full sky weak lensing analysis, we chose a spherical screen space. It is standard to project the Jacobi mapping $\mathscr{D}_{ab}$ onto a flat two dimensional subspace. Thereby, we would limit our analysis to the small-angle or thin-lens approximation. In this approximation, for example, derivatives along the line of sight are neglected. With a spherical screen space, we capture the full sky and therefore more terms and couplings contribute to $\mathscr{D}_{ab}$. Following the work of [17], we introduce spherical spin functions. The spin s of a function reflects the spherical symmetry. Moreover, we introduce derivatives $\eth$ and $\bar{\eth}$, which increase and lower the spin s by 1, respectively. In the harmonic space, the differential operators $\eth$ and $\bar{\eth}$ are linked to an l factor. Thus, terms with higher order derivatives of these differential operators would be dominant in the thin-lens approximation. I compute the reduced shear g and the convergence κ up to order $\mathcal{O}\left(\frac{1}{c^4}\right)$ in terms of the spherical spin derivatives and functions. At the lowest order $\mathcal{O}\left(\frac{1}{c}\right)$ the convergence is affected by the Doppler effect and displays a term solely dependent on the scale factor a and the peculiar velocity at the source v_S. At order $\mathcal{O}\left(\frac{1}{c^2}\right)$ the reduced shear coincides mathematically with the results of the first order standard perturbation theory [17], yet the physical meaning of the results diverge. In the PF formalism the lowest orders are built up by Newtonian quantities, whereas the first order in standard perturbation theory contains relativistic corrections. The reduced shear (from order $\mathcal{O}\left(\frac{1}{c^3}\right)$) and the convergence (from order $\mathcal{O}\left(\frac{1}{c^2}\right)$ experience redshift perturbations in terms of a Doppler effect with contributions involving the velocity v_S as well as contributions that couple the source to the lens. At order $\mathcal{O}\left(\frac{1}{c^3}\right)$ the vector field B^i_N contributes to both the reduced shear and the convergence. At order $\mathcal{O}\left(\frac{1}{c^4}\right)$ tensor and vector perturbations occur, whereas the latter ones are introduced over the velocity dependent redshift perturbations. In both the reduced shear and the convergence, we find various coupling terms including a huge number of lens-lens coulings of the lensing potential $W_N = \frac{1}{2}\left(U_N + V_N\right)$. In summary, I present the full-sky, all scales weak lensing analysis in terms of the redshift z up to higher orders using the post-Friedmann approximation scheme. In particular, the main result was to show how the gravimagnetic potential contributes to the convergence and reduced shear in an approximation that is valid inter alia on the small, nonlinear scales.

5.2 Future directions and work in progress

After concluding the second project by submitting the paper [54] to a peer-reviewed journal, I intend to continue the project in collaboration with Camille Bonvin, and my supervisors Marco Bruni and David Bacon to examine the results using well-established statistics in order to compare them to observational data. Following the work [16] I will compute the correlation function of the shear and convergence up to higher orders using the post-Friedmann approximation. In order to cover the full sky, we choose spherical coordinates and spin weighted operators analogously to the previous project [54] and to [16, 17]. The shear correlation function can be obtained directly from observed ellipticities and therefore can be compared to observational data. Our aim, in particular, is to compute the angular power spectrum of the terms involving the gravimagnetic potential in the shear and convergence.

5.2.1 Angular power spectrum

In chapter 4 and [54] we expressed the shear and convergence in terms of the spatial basis $\left\{\bar{k}^i, e^i_+, e^i_-\right\}$. Let $\mathscr{A}$ be a complex field, which transforms as $\mathscr{A} \to e^{is\alpha}\mathscr{A}$, where α denotes the angle of a rotation about $\bar{k}^i$. Expanded into spin-weighted spherical harmonics, $\mathscr{A}$ and its complex conjugate $\mathscr{A}^*$ yield

$$\mathscr{A} = \sum_{lm} {}_s a_{lm\, s} Y_{lm} \quad \text{and} \tag{5.1}$$

$$\mathscr{A}^* = \sum_{lm} {}_{-s} a_{lm\, -s} Y_{lm}, \tag{5.2}$$

respectively. The coefficients ${}_{\pm s} a_{lm}$ read

$${}_s a_{lm} = \not{l}(l,-s) \int d\hat{n} Y^*_{lm} \bar{\eth}^s \mathscr{A} \quad \text{and} \tag{5.3}$$

$${}_{-s} a_{lm} = \not{l}(l,s) \int d\hat{n} Y^*_{lm} \eth^s \mathscr{A}^* \tag{5.4}$$

with $\not{l} \equiv \sqrt{\frac{(l+s)!}{(l-s)!}}$. We use $\hat{n} = \bar{k}^i$ to denote the spatial direction of the geodesic.

The angular power spectrum C_l is defined via the two-point correlation function

$$\langle a_{lm} a_{l'm'} \rangle \equiv C_l \delta_{ll'} \delta_{mm'}. \tag{5.5}$$

5.2.2 Contribution of the gravimagnetic potential to the angular power spectrum

At order $\mathscr{O}\left(\frac{1}{c^3}\right)$ the gravimagnetic potential B_N^i contributes to both the convergence κ and the reduced shear g. We start our analysis with the correlation function for the convergence κ, which is a spin 0 field. We start with the terms from ${}_0\tilde{\mathscr{D}}$ (4.199)

$$D = -\frac{1}{2c^3} \int_0^{\chi_S} d\chi \left(\bar{\eth}_1 B_N + \eth_{-1} B_N - 4 B_{Nr} \right), \tag{5.6}$$

which contributes to the convergence κ (4.222). Then, the coefficient a_{lm} reads

$$a_{lm} = \not{l}(l,0) \int dk Y_{lm}^* D. \tag{5.7}$$

We perform a Fourier transformation on D in (5.7) and obtain

$$\begin{aligned} a_{lm} =& \not{l}(l,0) \int dn Y_{lm}^* \int \frac{d^3\mathbf{k}}{(2\pi)^3} D(\mathbf{k},z) e^{i\mathbf{k}^i \hat{n}_i} \\ =& \int dn Y_{lm}^* \int \frac{d^3\mathbf{k}}{(2\pi)^3} D(\mathbf{k},z) 4\pi \sum_{l'm'} i^{l'} j_{l'}(\mathbf{k}z) Y_{l'm'}(\hat{n}) Y_{l'm'}^*(\mathbf{k}) \qquad (5.8) \\ =& \sum_{lm} i^l \int \frac{d^3\mathbf{k}}{(2\pi)^3} D(\mathbf{k},z) j_l(\mathbf{k}z) Y_{lm}^*(\mathbf{k}). \qquad (5.9) \end{aligned}$$

We now substitute a_{lm} in (5.9) into the two point correlation function (5.5) and obtain

$$\langle a_{lm} a_{l'm'}^* \rangle = i^{l-l'} \int \frac{d^3\mathbf{k}}{(2\pi)^3} j_l(\mathbf{k}z) Y_{lm}^*(\mathbf{k}) \int \frac{d^3\mathbf{k}'}{(2\pi)^3} j_{l'}(\mathbf{k}'z') Y_{l'm'}^*(\mathbf{k}') \langle D(\mathbf{k},z) D^*(\mathbf{k},z) \rangle. \tag{5.10}$$

The aim of the project is to compute the angular power spectrum and compare the outcome to observational data. Work [33, 103] on the vector potential B_N^i using the PF formalism has shown that although the magnitude of the vector potential is small, it is not negligible. Any effects caused by the vector potential are purely relativistic. Hence, this offers the possibility to test General Relativity using current and future observational data.

Bibliography

[1] T. M. C. Abbott et al. Dark Energy Survey year 1 results: Cosmological constraints from galaxy clustering and weak lensing. *Phys. Rev.*, D98(4):043526, 2018.

[2] Julian Adamek, David Daverio, Ruth Durrer, and Martin Kunz. gevolution: a cosmological N-body code based on General Relativity. *JCAP*, 1607(07):053, 2016.

[3] P. A. R. Ade et al. Planck 2015 results. XVII. Constraints on primordial non-Gaussianity. *Astron. Astrophys.*, 594:A17, 2016.

[4] Adam Amara and Alexandre Refregier. Systematic Bias in Cosmic Shear: Beyond the Fisher Matrix. *Mon. Not. Roy. Astron. Soc.*, 391:228–236, 2008.

[5] Sambatra Andrianomena, Camille Bonvin, David Bacon, Philip Bull, Chris Clarkson, Roy Maartens, and Teboho Moloi. Testing General Relativity with the Doppler magnification effect. 2018.

[6] David J. Bacon, Sambatra Andrianomena, Chris Clarkson, Krzysztof Bolejko, and Roy Maartens. Cosmology with Doppler Lensing. *Mon. Not. Roy. Astron. Soc.*, 443(3):1900–1915, 2014.

[7] David J. Bacon, Alexandre R. Refregier, and Richard S. Ellis. Detection of weak gravitational lensing by large-scale structure. *Mon. Not. Roy. Astron. Soc.*, 318:625, 2000.

[8] James M. Bardeen. Gauge-invariant cosmological perturbations. *Phys. Rev. D*, 22:1882–1905, Oct 1980.

[9] Matthias Bartelmann and Peter Schneider. Weak gravitational lensing. *Phys. Rept.*, 340:291–472, 2001.

[10] N. Bartolo, E. Komatsu, Sabino Matarrese, and A. Riotto. Non-Gaussianity from inflation: Theory and observations. *Phys. Rept.*, 402:103–266, 2004.

[11] N. Bartolo, S. Matarrese, O. Pantano, and A. Riotto. Second-order matter perturbations in a ΛCDM cosmology and non-Gaussianity. *Classical and Quantum Gravity*, 27(12):124009, June 2010.

[12] Nicola Bartolo, Daniele Bertacca, Marco Bruni, Kazuya Koyama, Roy Maartens, Sabino Matarrese, Misao Sasaki, Licia Verde, and David Wands. A relativistic signature in large-scale structure. *Phys. Dark Univ.*, 13:30–34, 2016.

[13] Nicola Bartolo, Sabino Matarrese, and Antonio Riotto. Non-Gaussianity of Large-Scale Cosmic Microwave Background Anisotropies beyond Perturbation Theory. *JCAP*, 0508:010, 2005.

[14] Eloisa Bentivegna. An automatically generated code for relativistic inhomogeneous cosmologies. *Phys. Rev.*, D95(4):044046, 2017.

[15] Eloisa Bentivegna and Marco Bruni. Effects of nonlinear inhomogeneity on the cosmic expansion with numerical relativity. *Phys. Rev. Lett.*, 116:251302, Jun 2016.

[16] F. Bernardeau, C. Bonvin, N. Van de Rijt, and F. Vernizzi. Cosmic shear bispectrum from second-order perturbations in general relativity. *Physical Review D*, 86(2):023001, July 2012.

[17] F. Bernardeau, C. Bonvin, and F. Vernizzi. Full-sky lensing shear at second order. *Physical Review D*, 81(8):083002, April 2010.

[18] F. Bernardeau, S. Colombi, E. Gaztanaga, and R. Scoccimarro. Large scale structure of the universe and cosmological perturbation theory. *Phys. Rept.*, 367:1–248, 2002.

[19] Francis Bernardeau, Ludovic Van Waerbeke, and Y. Mellier. Weak lensing statistics as a probe of Omega and power spectrum. *Astron. Astrophys.*, 322:1–18, 1997.

[20] R. D. Blandford, A. B. Saust, T. G. Brainerd, and J. V. Villumsen. The distortion of distant galaxy images by large-scale structure. *Mon. Not. R. Astr. Soc.*, 251:600–627, August 1991.

[21] Camille Bonvin. Effect of Peculiar Motion in Weak Lensing. *Phys. Rev.*, D78:123530, 2008.

[22] Camille Bonvin, Sambatra Andrianomena, David Bacon, Chris Clarkson, Roy Maartens, Teboho Moloi, and Philip Bull. Dipolar modulation in the size of galaxies: The effect of Doppler magnification. *Mon. Not. Roy. Astron. Soc.*, 472(4):3936–3951, 2017.

[23] Camille Bonvin, Ruth Durrer, and M. Alice Gasparini. Fluctuations of the luminosity distance. *Phys. Rev.*, D73:023523, 2006. [Erratum: Phys. Rev.D85,029901(2012)].

[24] Marco Bruni, Robert Crittenden, Kazuya Koyama, Roy Maartens, Cyril Pitrou, and David Wands. Disentangling non-Gaussianity, bias and GR effects in the galaxy distribution. *Phys. Rev.*, D85:041301, 2012.

[25] Marco Bruni, Peter K. S. Dunsby, and George F. R. Ellis. Cosmological perturbations and the physical meaning of gauge invariant variables. *Astrophys. J.*, 395:34–53, 1992.

[26] Marco Bruni, Juan Carlos Hidalgo, Nikolai Meures, and David Wands. Non-gaussian initial conditions in ΛCDM: Newtonian, relativistic, and primordial contributions. *The Astrophysical Journal*, 785(1):2, 2014.

[27] Marco Bruni, Juan Carlos Hidalgo, and David Wands. Einstein's signature in cosmological large-scale structure. *The Astrophysical Journal Letters*, 794(1):L11, 2014.

[28] Marco Bruni, Sabino Matarrese, Silvia Mollerach, and Sebastiano Sonego. On relativistic perturbations of second and higher order. In *General relativity and gravitational physics. Proceedings, 12th Italian Conference, Rome, Italy, September 23-27, 1996*, pages 389–393, 1996.

[29] Marco Bruni, Sabino Matarrese, Silvia Mollerach, and Sebastiano Sonego. Perturbations of space-time: Gauge transformations and gauge invariance at second order and beyond. *Class. Quant. Grav.*, 14:2585–2606, 1997.

[30] Marco Bruni, Sabino Matarrese, and Ornella Pantano. Dynamics of silent universes. *Astrophys. J.*, 445:958–977, 1995.

[31] Marco Bruni and Sebastiano Sonego. Observables and gauge invariance in the theory of nonlinear space-time perturbations: Letter to the editor. *Class. Quant. Grav.*, 16:L29–L36, 1999.

[32] Marco Bruni and Carlos F Sopuerta. Covariant fluid dynamics: a long wavelength approximation. *Classical and Quantum Gravity*, 20(24):5275, 2003.

[33] Marco Bruni, Daniel B. Thomas, and David Wands. Computing General Relativistic effects from Newtonian N-body simulations: Frame dragging in the post-Friedmann approach. *Phys.Rev.*, D89(4):044010, 2014.

[34] Christian T. Byrnes and Ki-Young Choi. Review of local non-Gaussianity from multi-field inflation. *Adv. Astron.*, 2010:724525, 2010.

[35] G. L. Comer, Nathalie Deruelle, David Langlois, and Joe Parry. Growth or decay of cosmological inhomogeneities as a function of their equation of state. *Phys. Rev. D*, 49:2759–2768, Mar 1994.

[36] Asantha Cooray and Wayne Hu. Second order corrections to weak lensing by large scale structure. *Astrophys. J.*, 574:19, 2002.

[37] Liang Dai, Enrico Pajer, and Fabian Schmidt. On Separate Universes. *JCAP*, 1510(10):059, 2015.

[38] N. Dalal, O. Doré, D. Huterer, and A. Shirokov. Imprints of primordial non-Gaussianities on large-scale structure: Scale-dependent bias and abundance of virialized objects. *Physical Review D*, 77(12):123514, June 2008.

[39] David Daverio, Yves Dirian, and Ermis Mitsou. A numerical relativity scheme for cosmological simulations. *Class. Quant. Grav.*, 34(23):237001, 2017.

[40] Roland de Putter, Olivier Doré, and Daniel Green. Is There Scale-Dependent Bias in Single-Field Inflation? *JCAP*, 1510(10):024, 2015.

[41] Nathalie Deruelle and David Langlois. Long wavelength iteration of einstein's equations near a spacetime singularity. *Phys. Rev. D*, 52:2007–2019, Aug 1995.

[42] Vincent Desjacques, Donghui Jeong, and Fabian Schmidt. Large-Scale Galaxy Bias. *Phys. Rept.*, 733:1–193, 2018.

[43] Scott Dodelson, Charles Shapiro, and Martin J. White. Reduced shear power spectrum. *Phys. Rev.*, D73:023009, 2006.

[44] William E. East, Radosław Wojtak, and Tom Abel. Comparing Fully General Relativistic and Newtonian Calculations of Structure Formation. *Phys. Rev.*, D97(4):043509, 2018.

[45] G. F. R. Ellis and M. Bruni. Covariant and gauge-invariant approach to cosmological density fluctuations. *Phys. Rev. D*, 40:1804–1818, Sep 1989.

[46] G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum. *Relativistic Cosmology*. Cambridge University Press, Cambridge, UK, March 2012.

[47] GeorgeF.R. Ellis. Republication of: Relativistic cosmology. *General Relativity and Gravitation*, 41(3):581–660, 2009.

[48] C. W. F. Everitt et al. Gravity Probe B: Final Results of a Space Experiment to Test General Relativity. *Phys. Rev. Lett.*, 106:221101, 2011.

[49] Christian Fidler, Thomas Tram, Cornelius Rampf, Robert Crittenden, Kazuya Koyama, and David Wands. Relativistic initial conditions for N-body simulations. *JCAP*, 1706(06):043, 2017.

[50] John T. Giblin, James B. Mertens, and Glenn D. Starkman. Departures from the friedmann-lemaitre-robertston-walker cosmological model in an inhomogeneous universe: A numerical examination. *Phys. Rev. Lett.*, 116:251301, Jun 2016.

[51] John T. Giblin, James B. Mertens, and Glenn D. Starkman. A cosmologically motivated reference formulation of numerical relativity. *Class. Quant. Grav.*, 34(21):214001, 2017.

[52] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, and E. C G Sudarshan. Spin-s spherical harmonics and o. *Journal of Mathematical Physics*, 8(11):2155–2161, 1967.

[53] Sophia R. Goldberg, Timothy Clifton, and Karim A. Malik. Cosmology on all scales: a two-parameter perturbation expansion. *Phys. Rev.*, D95(4):043503, 2017.

[54] Hedda A. Gressel, Camille Bonvin, Marco Bruni, and David Bacon. Full-sky weak lensing: a nonlinear post-Friedmann treatment. 2019. arXiv: 1902.00059.

[55] Hedda A. Gressel and Marco Bruni. fNL gNL mixing in the matter density field at higher orders. *JCAP*, 1806(06):016, 2018.

[56] Nastassia Grimm and Jaiyul Yoo. Jacobi Mapping Approach for a Precise Cosmological Weak Lensing Formalism. *JCAP*, 1807(07):067, 2018.

[57] Lam Hui and Edmund Bertschinger. Local approximations to the gravitational collapse of cold matter. *Astrophys. J.*, 471:1, 1996.

[58] J.-c. Hwang, H. Noh, and J.-O. Gong. Second-order Solutions of Cosmological Perturbation in the Matter-dominated Era. *The Astrophysical Journal*, 752:50, June 2012.

[59] Jai-chan Hwang and Hyerim Noh. Third order perturbations of a zero-pressure cosmological medium: Pure general relativistic nonlinear effects. *Phys. Rev.*, D72:044012, 2005.

[60] Andrew Johnson et al. The 6dF Galaxy Survey: cosmological constraints from the velocity power spectrum. *Mon. Not. Roy. Astron. Soc.*, 444(4):3926–3947, 2014.

[61] N. Kaiser. Weak gravitational lensing of distant galaxies. *Astrophys. J.*, 388:272–286, April 1992.

[62] Nick Kaiser, Gillian Wilson, and Gerard A. Luppino. Large scale cosmic shear measurements. 2000. arXiv: astro-ph/0003338.

[63] Martin Kilbinger. Cosmology with cosmic shear observations: a review. *Rept. Prog. Phys.*, 78:086901, 2015.

[64] H. Kodama and M. Sasaki. Cosmological Perturbation Theory. *Progress of Theoretical Physics Supplement*, 78:1, 1984.

[65] Eiichiro Komatsu and David N. Spergel. Acoustic signatures in the primary microwave background bispectrum. *Phys. Rev.*, D63:063002, 2001.

[66] Kazuya Koyama. Non-Gaussianity of quantum fields during inflation. *Class. Quant. Grav.*, 27:124001, 2010.

[67] L.D. Landau and E.M. Lifshitz. *The Classical Theory of Fields*. Oxford:Pergamon, 1975.

[68] G. Lemaître. L'Univers en expansion. *Annales de la Société Scientifique de Bruxelles*, 53, 1933.

[69] A. Lewis and A. Challinor. Weak gravitational lensing of the CMB. *Physics Reports*, 429:1–65, June 2006.

[70] E M Lifshitz and I M Khalatnikov. Problems of relativistic cosmology. *Soviet Physics Uspekhi*, 6(4):495, 1964.

[71] D. H. Lyth. Large-scale energy-density perturbations and inflation. *Phys. Rev. D*, 31:1792–1798, Apr 1985.

[72] David H. Lyth and Yeinzon Rodriguez. The Inflationary prediction for primordial non-Gaussianity. *Phys. Rev. Lett.*, 95:121302, 2005.

[73] D.H. Lyth and A.R. Liddle. *The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure*. Cambridge University Press, 2009.

[74] Hayley J. Macpherson, Paul D. Lasky, and Daniel J. Price. Inhomogeneous Cosmology with Numerical Relativity. *Phys. Rev.*, D95(6):064028, 2017.

[75] Karim A. Malik and David R. Matravers. A Concise Introduction to Perturbation Theory in Cosmology. *Class. Quant. Grav.*, 25:193001, 2008.

[76] Karim A. Malik and David Wands. Cosmological perturbations. *Phys. Rept.*, 475:1–51, 2009.

[77] Jerome Martin and Dominik J. Schwarz. The Influence of cosmological transitions on the evolution of density perturbations. *Phys. Rev.*, D57:3302–3316, 1998.

[78] Sabino Matarrese, Silvia Mollerach, and Marco Bruni. Second order perturbations of the Einstein-de Sitter universe. *Phys. Rev.*, D58:043504, 1998.

[79] Sabino Matarrese and Antonio Riotto. Large - scale curvature perturbations with spatial and time variations of the inflaton decay rate. *JCAP*, 0308:007, 2003.

[80] Irene Milillo, Daniele Bertacca, Marco Bruni, and Andrea Maselli. Missing link: A nonlinear post-Friedmann framework for small and large scales. *Phys. Rev.*, D92(2):023519, 2015.

[81] Irene Milillo, Daniele Bertacca, Marco Bruni, and Andrea Maselli. Missing link: A nonlinear post-Friedmann framework for small and large scales. *Phys. Rev.*, D92(2):023519, 2015.

[82] J. Miralda-Escude. The correlation function of galaxy ellipticities produced by gravitational lensing. *Astrophys. J.*, 380:1–8, October 1991.

[83] E. T. Newman and R. Penrose. Note on the Bondi-Metzner-Sachs group. *J. Math. Phys.*, 7:863–870, 1966.

[84] E. T. Newman and R. Penrose. Note on the Bondi-Metzner-Sachs Group. *Journal of Mathematical Physics*, 7:863–870, May 1966.

[85] Patrick Peter and Jean-Philippe Uzan. *Primordial cosmology*. Oxford Graduate Texts. Oxford Univ. Press, Oxford, 2009.

[86] E. Poisson and C.M. Will. *Gravity: Newtonian, Post-Newtonian, Relativistic*. Cambridge University Press, 2014.

[87] E. Poisson and C.M. Will. *Gravity: Newtonian, Post-Newtonian, Relativistic*. Cambridge University Press, 2014.

[88] Cornelius Rampf and Gerasimos Rigopoulos. Initial conditions for cold dark matter particles and General Relativity. *Phys. Rev.*, D87:123525, 2013.

[89] Cornelius Rampf, Eleonora Villa, Daniele Bertacca, and Marco Bruni. Lagrangian theory for cosmic structure formation with vorticity: Newtonian and post-Friedmann approximations. *Phys. Rev.*, D94(8):083515, 2016.

[90] Helmut Rumpf. Relativitätstheorie und Kosmologie II. *Lecture notes*.

[91] Lewis Ryder. *Introduction to General Relativity*. Cambridge University Press, 2009.

[92] R. Sachs. Gravitational Waves in General Relativity. VI. The Outgoing Radiation Condition. *Proceedings of the Royal Society of London Series A*, 264:309–338, November 1961.

[93] D. S. Salopek and J. R. Bond. Nonlinear evolution of long-wavelength metric fluctuations in inflationary models. *Phys. Rev. D*, 42:3936–3962, Dec 1990.

[94] Viraj A. A. Sanghai and Timothy Clifton. Post-Newtonian Cosmological Modelling. *Phys. Rev.*, D91:103532, 2015. [Erratum: Phys. Rev.D93,no.8,089903(2016)].

[95] P. Schneider, J. Ehlers, and E. E. Falco. *Gravitational Lenses*. 1992.

[96] S. Seitz, P. Schneider, and J. Ehlers. Light propagation in arbitrary spacetimes and the gravitational lens approximation. *Classical and Quantum Gravity*, 11:2345–2373, September 1994.

[97] R.U. Sexl and H.K. Urbantke. *Gravitation und Kosmologie*. Bibliographisches Inst., 1975.

[98] Sebastiano Sonego and Marco Bruni. Gauge dependence in the theory of nonlinear space-time perturbations. *Commun. Math. Phys.*, 193:209–218, 1998.

[99] J. M. Stewart. Perturbations of Friedmann-Robertson-Walker cosmological models. *Class. Quant. Grav.*, 7:1169–1180, 1990.

[100] Ryuichi Takahashi. Third Order Density Perturbation and One-loop Power Spectrum in a Dark Energy Dominated Universe. *Prog. Theor. Phys.*, 120:549–559, 2008.

[101] Daniel B Thomas, Marco Bruni, Kazuya Koyama, Baojiu Li, and Gong-Bo Zhao. f(R) gravity on non-linear scales: The post-Friedmann expansion and the vector potential. *JCAP*, 1507(07):051, 2015.

[102] Daniel B. Thomas, Marco Bruni, and David Wands. Relativistic weak lensing from a fully non-linear cosmological density field. *JCAP*, 1509(09):021, 2015.

[103] Daniel B. Thomas, Marco Bruni, and David Wands. The fully non-linear post-Friedmann frame-dragging vector potential: Magnitude and time evolution from N-body simulations. *Mon. Not. Roy. Astron. Soc.*, 452(2):1727–1742, 2015.

[104] Kenji Tomita. Evolution of irregularities in a chaotic early universe. *Progress of Theoretical Physics*, 54(3):730–739, 1975.

[105] Claes Uggla and John Wainwright. Second order density perturbations for dust cosmologies. *Phys. Rev.*, D90(4):043511, 2014.

[106] Obinna Umeh, Chris Clarkson, and Roy Maartens. Nonlinear relativistic corrections to cosmological distances, redshift and gravitational lensing magnification. II - Derivation. *Class. Quant. Grav.*, 31:205001, 2014.

[107] J.-P. Uzan and F. Bernardeau. Cosmic strings lens phenomenology: General properties of distortion fields. *Physical Review D*, 63(2):023004, January 2001.

[108] Ludovic van Waerbeke et al. Detection of correlated galaxy ellipticities on CFHT data: First evidence for gravitational lensing by large scale structures. *Astron. Astrophys.*, 358:30–44, 2000.

[109] Eleonora Villa and Cornelius Rampf. Relativistic perturbations in ΛCDM: Eulerian & Lagrangian approaches. *JCAP*, 1601(01):030, 2016.

[110] Christian Wagner, Fabian Schmidt, Chi-Ting Chiang, and Eiichiro Komatsu. Separate Universe Simulations. *Mon. Not. Roy. Astron. Soc.*, 448(1):L11–L15, 2015.

[111] R. M. Wald. *General Relativity*. The University of Chicago Press, 1984.

[112] David Wands. Local non-Gaussianity from inflation. *Class. Quant. Grav.*, 27:124002, 2010.

[113] David Wands, Karim A. Malik, David H. Lyth, and Andrew R. Liddle. A New approach to the evolution of cosmological perturbations on large scales. *Phys. Rev.*, D62:043527, 2000.

[114] Richard Watkins, Hume A. Feldman, and Michael J. Hudson. Consistently Large Cosmic Flows on Scales of 100 Mpc/h: a Challenge for the Standard LCDM Cosmology. *Mon. Not. Roy. Astron. Soc.*, 392:743–756, 2009.

[115] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. July 1972.

[116] David M. Wittman, J. Anthony Tyson, David Kirkman, Ian Dell'Antonio, and Gary Bernstein. Detection of weak gravitational lensing distortions of distant galaxies by cosmic dark matter at large scales. *Nature*, 405:143–149, 2000.

[117] Jaiyul Yoo and Ruth Durrer. Gauge-Transformation Properties of Cosmological Observables and its Application to the Light-Cone Average. *JCAP*, 1709(09):016, 2017.

[118] Jaiyul Yoo and Jinn-Ouk Gong. Exact analytic solution for non-linear density fluctuation in a ΛCDM universe. *JCAP*, 1607(07):017, 2016.

Appendix A

Spin weighted functions

A.1 Real and imaginary contributions using spherical spin operators

In (4.16) and (4.17), the convergence κ, the shear γ, and the rotation ω have been defined via the real and imaginary part of ${}_0\mathscr{D}$ and ${}_2\mathscr{D}$. In order to split ${}_0\mathscr{D}$ and ${}_2\mathscr{D}$ into $\Re({}_0\mathscr{D})$ and $\Im({}_0\mathscr{D})$, and into $\Re({}_2\mathscr{D})$ and $\Im({}_2\mathscr{D})$, respectively, we need to examine the derivatives $\eth$ and $\bar{\eth}$, and its application on scalars, vectors, and tensors. We start with various combinations of the derivatives applied on the scalar functions X and Y:

$$\bar{\eth}\eth X = \left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi + \cot\theta\right)\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right)X = \left(\partial_\theta^2 + \cot\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right)X \tag{A.1}$$

$$= \Delta_{\theta\phi}X \in \mathbb{R} \tag{A.2}$$

$$\eth X\bar{\eth}X = \left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right)X\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right)X = \partial_\theta X\partial_\theta X + \frac{1}{\sin^2\theta}\partial_\phi X\partial_\phi X \in \mathbb{R} \tag{A.3}$$

$$\eth X\bar{\eth}Y + \eth Y\bar{\eth}X = 2\left(\partial_\theta X\partial_\theta Y + \frac{1}{\sin^2\theta}\partial_\phi X\partial_\phi Y\right) \in \mathbb{R} \tag{A.4}$$

$$\eth\eth X = \left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi - \cot\theta\right)\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right)X \tag{A.5}$$

$$= \left[\sin\theta\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\right) - \frac{1}{\sin^2\theta}\partial_\phi\partial_\phi + i2\partial_\theta\left(\frac{1}{\sin\theta}\partial_\phi\right)\right]X \in \mathbb{C} \tag{A.6}$$

$$\eth X\eth Y = \partial_\theta X\partial_\theta Y - \frac{1}{\sin^2\theta}\partial_\phi X\partial_\phi Y + \frac{i}{\sin\theta}\left(\partial_\theta X\partial_\phi Y + \partial_\theta Y\partial_\phi X\right) \in \mathbb{C} \tag{A.7}$$

We will find combinations like (A.1) - (A.4) in the expression for ${}_0\tilde{\mathscr{D}}^{(2)}$ and ${}_0\tilde{\mathscr{D}}^{(4)}$ in (4.198) and (4.200), respectively. In sections 4.2 and 4.3.3 we discuss the physical interpretations of the real and the imaginary part of ${}_0\tilde{\mathscr{D}}$ and ${}_2\tilde{\mathscr{D}}$. While $\Re({}_2\tilde{\mathscr{D}})$ and $\Im({}_2\tilde{\mathscr{D}})$ are both related to the shear (4.17), $\Re({}_0\tilde{\mathscr{D}})$ and $\Im({}_0\tilde{\mathscr{D}})$ are proportional to the convergence κ and rotation ω (4.16), respectively. Because (A.1) - (A.4) are $\in \mathbb{R}$, ${}_0\tilde{\mathscr{D}}^{(2)}$ and ${}_0\tilde{\mathscr{D}}^{(4)}$ will solely contribute to the convergence κ. Therefore, for the order $\mathscr{O}\left(\frac{1}{c^2}\right)$ and $\mathscr{O}\left(\frac{1}{c^4}\right)$ the rotation ω, which is related to the imaginary part of ${}_0\tilde{\mathscr{D}}^{(2)}$ and ${}_0\tilde{\mathscr{D}}^{(4)}$, is zero.

Next we discuss the slashed derivatives applied to vectors. In ${}_0\tilde{\mathscr{D}}^{(3)}$ (4.199) and ${}_2\tilde{\mathscr{D}}^{(3)}$ (4.189) we find the following combinations:

$$\bar{\eth}_1 B = -\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi + \cot\theta\right)\left(B_\theta + iB_\phi\right) \tag{A.8}$$

$$= -\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta B_\theta\right) + \frac{1}{\sin\theta}\partial_\phi B_\phi\right] + i\left[-\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta B_\phi\right) + \frac{1}{\sin\theta}\partial_\phi B_\theta\right] \in \mathbb{C} \tag{A.9}$$

$$\eth_{-1} B = -\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi + \cot\theta\right)\left(B_\theta - iB_\phi\right) \tag{A.10}$$

$$= -\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta B_\theta\right) + \frac{1}{\sin\theta}\partial_\phi B_\phi\right] + i\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta B_\phi\right) - \frac{1}{\sin\theta}\partial_\phi B_\theta\right] \in \mathbb{C} \tag{A.11}$$

$$\bar{\eth}_1 B + \eth_{-1} B = -2\left(\partial_\theta B_\theta + \frac{1}{\sin\theta}\partial_\phi B_\phi\right) \in \mathbb{R} \tag{A.12}$$

$$\eth_1 B = -\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi - \cot\theta\right)\left(B_\theta + iB_\phi\right) \tag{A.13}$$

$$= -\left[\sin\theta\partial_\theta\left(\frac{1}{\sin\theta}B_\theta\right) - \frac{1}{\sin\theta}\partial_\phi B_\phi\right] - i\left[\sin\theta\partial_\theta\left(\frac{1}{\sin\theta}B_\phi\right) + \frac{1}{\sin\theta}\partial_\phi B_\theta\right] \in \mathbb{C} \tag{A.14}$$

The only combination occurring in ${}_0\tilde{\mathscr{D}}^{(3)}$ is the real contribution (A.12). Therefore, we conclude that ${}_0\tilde{\mathscr{D}}^{(3)}$ only contributes to the convergence κ and not to the rotation ω.

There are slashed derivatives of the tensor potential in both ${}_0\tilde{\mathscr{D}}$ (4.200) and ${}_2\tilde{\mathscr{D}}$ (4.190). However, the combinations only involve h_{rr}, which is a scalar function of spin-0 like X and Y in (A.1) - (A.7), and ${}_{\pm 1}h_r$[1], which is a spin-± 1 function such as

[1] ${}_{\pm 1}h_r$ expressed in terms of spherical coordinates yields ${}_{\pm 1}h_r = h_{r\theta} \pm i h_{r\theta}$ analogously to ${}_{\pm 1}B = B_\theta \pm iB_\phi$.

${}_{\pm 1}B$ in (A.8) - (A.14). Thus, we can use the above relations to compute the real and imaginary part.

A.2 Useful relations

In this subsection, we list useful relations using the spin-weighted formalism. First we look at derivatives of the basis vectors e^i_+, e^i_-, and e^i_r.

$$e^i_+ \bar{k}^j_{,i} = \frac{1}{\chi} e^j_+ \tag{A.15}$$

$$e^i_- \bar{k}^j_{,i} = \frac{1}{\chi} e^j_- \tag{A.16}$$

$$e^i_+ e^j_{-,i} = e^i_- e^j_{+,i} = -\frac{2}{\chi} \bar{k}^j \tag{A.17}$$

$$e^i_+ e^j_+ e^m_{r,ij} = 0 \tag{A.18}$$

For the scalar function X we find the following relations useful:

$$e^i_+ e^j_+ X_{,ij} = \frac{1}{\chi^2} \eth^2 X \tag{A.19}$$

$$e^i_+ \bar{k}^j X_{,ij} = -\frac{1}{\chi} \eth X_{,r} + \frac{1}{\chi^2} \eth X \tag{A.20}$$

$$e^i_- e^j_- X_{,ij} = \frac{1}{\chi^2} \bar{\eth}^2 X \tag{A.21}$$

$$e^i_+ e^j_- X_{,ij} = e^i_- e^j_+ X = \frac{1}{\chi^2} \bar{\eth}\eth X + \frac{2}{\chi} X_{,r} \tag{A.22}$$

$$e^i_+ e^j_+ \bar{k}^m X_{,ijm} = \frac{1}{\chi^2} \eth^2 X_{,r} - \frac{2}{\chi^3} \eth^2 X \tag{A.23}$$

$$e^i_+ e^j_- e^m_+ X_{,ijm} = e^i_+ e^j_+ e^m_- X_{,ijm} = -\frac{1}{\chi^3} \eth\bar{\eth}\eth X - \frac{4}{\chi^2} \eth X_{,r} + \frac{2}{\chi^3} \eth X \tag{A.24}$$

$$e^i_- e^j_+ e^m_- X_{,ijm} = e^i_- e^j_- e^m_+ X_{,ijm} = -\frac{1}{\chi^3} \bar{\eth}\eth\bar{\eth} X - \frac{4}{\chi^2} \bar{\eth} X_{,r} + \frac{2}{\chi^3} \bar{\eth} X \tag{A.25}$$

$$e^i_+ e^j_- \bar{k}^m X_{,ijm} = e^i_- e^j_+ \bar{k}^m X_{,ijm} = \frac{1}{\chi^2} \eth\bar{\eth} X_{,r} - \frac{2}{\chi^3} \eth\bar{\eth} X + 2\frac{1}{\chi} X_{,rr} - 2\frac{1}{\chi^2} X_{,r} \tag{A.26}$$

$$\begin{aligned} X^{,i} X_{,i} =& \left(\bar{k}^i \bar{k}^j + \frac{1}{2} e^i_+ e^j_- + \frac{1}{2} e^i_- e^j_+ \right) X_{,i} X_{,j} \\ =& X_{,r} X_{,r} + \frac{1}{\chi^2} \eth X \bar{\eth} X. \end{aligned} \tag{A.27}$$

Let Y^i be a vector field. We can express Y^i in terms of the basis $\left\{\bar{k}^i, e^i_+, e^i_-\right\}$ as $Y^i = Y_r\bar{k}^i + \frac{1}{2}{}_{-1}Ye^i_+ + \frac{1}{2}{}_1Ye^i_-$. Then, the following relations can be found:

$$e^i_+e^j_-\bar{k}^m Y_{m,ij} = \frac{1}{\chi^2}\left(\bar{\eth}\eth Y_r - 2\chi Y_{r,r} - 2Y_r + \bar{\eth}_1 Y + \eth_{-1}Y\right) \tag{A.28}$$

$$e^i_-e^j_+Y_{i,j} = -\frac{1}{\chi}\bar{\eth}_1 Y + \frac{1}{\chi}2Y_r \tag{A.29}$$

$$e^i_+e^j_-Y_{i,j} = -\frac{1}{\chi}\eth_{-1}Y + \frac{1}{\chi}2Y_r \tag{A.30}$$

$$\begin{aligned} Y^iY_i &= \left(\bar{k}^i\bar{k}^j + \frac{1}{2}e^i_+e^j_- + \frac{1}{2}e^i_-e^j_+\right)Y_iY_j \\ &= Y_rY_r + {}_{-1}Y{}_1Y. \end{aligned} \tag{A.31}$$

For a tensor field Z_{ij}, which can be expressed as $Z^{ij} = Z_{rr}\left[\bar{k}^i\bar{k}^j - \frac{1}{4}\left(e^i_+e^j_- + e^i_-e^j_+\right)\right] + {}_{-1}Z_r\frac{1}{2}\left(e^i_+\bar{k}^j + \bar{k}^ie^j_+\right) + {}_1Z_r\frac{1}{2}\left(e^i_-\bar{k}^j + \bar{k}^ie^j_-\right) + \frac{1}{4}{}_{-2}Ze^i_+e^j_+ + \frac{1}{4}{}_2Ze^i_-e^j_-$ the following relations hold:

$$e^i_+e^j_+Z_{ij} = {}_2Z \tag{A.32}$$

$$e^i_+e^j_-Z_{ij} = -Z_{rr} \tag{A.33}$$

$$e^i_+e^j_+\bar{k}^m Z_{im,j} = -\frac{1}{\chi}\eth_1 Z_r - \frac{1}{\chi}{}_2Z \tag{A.34}$$

$$e^i_+e^j_-\bar{k}^m Z_{im,j} = -\frac{1}{\chi}\bar{\eth}_1 Z_r + \frac{3}{\chi}Z_{rr} \tag{A.35}$$

$$e^i_-e^j_+\bar{k}^m Z_{im,j} = -\frac{1}{\chi}\eth_1 Z_r + \frac{3}{\chi}Z_{rr} \tag{A.36}$$

$$e^i_+e^j_+\bar{k}^m\bar{k}^n Z_{mn,ij} = \frac{1}{\chi^2}\eth^2 Z_{rr} + 4\frac{1}{\chi^2}\eth_1 Z_r + \frac{1}{\chi^2}{}_2Z \tag{A.37}$$

$$e^i_+e^j_-\bar{k}^m\bar{k}^n Z_{mn,ij} = \frac{2}{\chi^2}\left(\frac{1}{2}\eth\bar{\eth} - \chi\partial_r - 3\right)Z_{rr} + \frac{2}{\chi^2}\left(\eth_{-1}Z_r + \bar{\eth}_1 Z_r\right). \tag{A.38}$$

Let ${}_sT$ be a function of spin s. The operators $\eth$ and $\bar{\eth}$ obey the following commutation rule

$$\left(\bar{\eth}\eth - \eth\bar{\eth}\right)_s T = 2s_sT. \tag{A.39}$$

FORM UPR16

Research Ethics Review Checklist

Please include this completed form as an appendix to your thesis (see the Research Degrees Operational Handbook for more information

UNIVERSITY OF PORTSMOUTH

Postgraduate Research Student (PGRS) Information			**Student ID:**	753439
PGRS Name:	Hedda Alice Gressel			
Department:	ICG/Tech	**First Supervisor:**	Dr. Marco Bruni	
Start Date: (or progression date for Prof Doc students)		**01/10/2014**		

Study Mode and Route:	Part-time ☐	MPhil ☐	MD ☐
	Full-time ☒	PhD ☒	Professional Doctorate ☐

Title of Thesis:	Nonlinear Approximation Schemes in Relativistic Cosmology
Thesis Word Count: (excluding ancillary data)	44288

If you are unsure about any of the following, please contact the local representative on your Faculty Ethics Committee for advice. Please note that it is your responsibility to follow the University's Ethics Policy and any relevant University, academic or professional guidelines in the conduct of your study

Although the Ethics Committee may have given your study a favourable opinion, the final responsibility for the ethical conduct of this work lies with the researcher(s).

UKRIO Finished Research Checklist:
(If you would like to know more about the checklist, please see your Faculty or Departmental Ethics Committee rep or see the online version of the full checklist at: http://www.ukrio.org/what-we-do/code-of-practice-for-research/)

a) Have all of your research and findings been reported accurately, honestly and within a reasonable time frame?	YES ☒ NO ☐
b) Have all contributions to knowledge been acknowledged?	YES ☒ NO ☐
c) Have you complied with all agreements relating to intellectual property, publication and authorship?	YES ☒ NO ☐
d) Has your research data been retained in a secure and accessible form and will it remain so for the required duration?	YES ☒ NO ☐
e) Does your research comply with all legal, ethical, and contractual requirements?	YES ☒ NO ☐

Candidate Statement:

I have considered the ethical dimensions of the above named research project, and have successfully obtained the necessary ethical approval(s)

Ethical review number(s) from Faculty Ethics Committee (or from NRES/SCREC):	C605-5556-E28E-1339-6DB0-4B2A-D209-4687

If you have *not* submitted your work for ethical review, and/or you have answered 'No' to one or more of questions a) to e), please explain below why this is so:

Signed ***(PGRS)*:**		**Date:** 03/04/2019