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Low energy effective theory on a regularized brane in 6D gauged chiral supergravity

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We derive the low energy effective theory on a brane in six-dimensional chiral supergravity. The conical 3-brane singularities are resolved by introducing cylindrical codimension one 4-branes whose interiors are capped by a regular spacetime. The effective theory is described by the Brans-Dicke (BD) theory with the BD parameter given by $\omega_{\rm BD} = 1/2$. The BD field is originated from a modulus which is associated with the scaling symmetry of the system. If the dilaton potentials on the branes preserve the scaling symmetry, the scalar field has an exponential potential in the Einstein frame. We show that the time dependent solutions driven by the modulus in the four-dimensional effective theory can be lifted up to the six-dimensional exact solutions found in the literature. Based on the effective theory, we discuss a possible way to stabilize the modulus to recover standard cosmology and also study the implication for the cosmological constant problem.

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I. INTRODUCTION

Recently, much attention has been paid to six-dimensional supergravity [1, 2, 3]. The most intriguing property of six-dimensional supergravity is that the four-dimensional spacetime is always Minskowski even in the presence of branes with tension. A 3-brane with tension induces only a deficit angle in the six-dimensional spacetime and the tension does not curve the four-dimensional spacetime within the brane. This feature is called self-tuning and it may solve the cosmological constant problem [4, 5]. This is the basis of the supersymmetric large extra-dimension (SLED) proposal [6].

There have been several objections to the idea of self-tuning [7, 8]. The self-tuning relies on the classical scaling property of the model. The six-dimensional equations of motion are invariant under the constant rescaling $g_{MN} \to e^{\omega} g_{MN}$ and $e^{\phi} \to e^{\phi-\omega}$, where g_{MN} denotes the six-dimensional metric and ϕ is the dilaton field. Then there is a modulus associated with this scaling property. Ref. [8] derived an effective potential for this modulus. This modulus is shown to have an exponential potential. Then there must be a fine-tuning of parameters to ensure that the potential vanishes in order to have a static solution. This is the reason why the static solution always has vanishing cosmological constant. However, if this fine-tuning is broken, the modulus acquires a runaway potential and the four-dimensional spacetime becomes non-static. Non-static solutions in six-dimensional supergravity have been derived and they are supposed to correspond to the response of the bulk geometry to a change of tension of branes [9, 10, 11, 12].

However, it is difficult to deal with an arbitrary change of tension with a brane described by a pure conical singularity. This is because if we put matter on the brane other than cosmological constant, the metric diverges at the position of the brane. Recently, it was suggested that we can regularize the brane by resolving it by a codimension one cylindrical 4-brane [13, 14, 15, 16]. This type of models may be regarded as a variation of Kaluza-Klein/hybrid brane world [17]. Once the brane becomes a codimension one object, it is possible to put arbitrary matter on the brane without having the divergence of the metric. Then it becomes possible to study the effect of the change of tension on the four-dimensional geometry on the brane.

There is another interesting issue of whether it is possible to recover conventional cosmology at low energies in six-dimensional models. Recent works have shown that it is impossible to recover sensible cosmology if one derives cosmological solutions by considering a motion of branes in a given static bulk spacetime [18, 19]. It was concluded that the time-dependence of the bulk spacetime should be taken into account.

In this paper, we derive a four-dimensional effective theory for the modulus in six-dimensional supergravity with resolved 4-branes by extending the analysis of Ref. [20] which studied the low energy effective theory in the Einstein-

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Maxwell theory [21]. Arbitrary matter and potentials for the dilaton on 4-branes are allowed to exist. We use the gradient expansion technique to solve the six-dimensional geometry assuming that the deviation from the static solution is small [22, 23]. The gradient expansion method has been applied to various types of braneworlds [24]. Using this method, it is possible to solve the non-trivial dependence of the bulk geometry on the four-dimensional coordinates. By solving the effective four-dimensional equations, we can derive the time-dependent solutions and compare them with the exact six-dimensional time dependent solutions found in the literature [10, 11, 12]. It is also possible to study whether we can reproduce sensible cosmology at low energies or not. We also study the possibility to stabilize the modulus using the potentials for the dilaton on the branes along the line of Ref. [25].

The paper is organized as follows. In section II, basic equations are summarized. In section III, we solve the six-dimensional equations of motion using the gradient expansion method. In section IV, the effective theory on the regularized branes is derived by imposing junction conditions. Then we derive time dependent cosmological solutions in the effective theory and compare them with the exact six-dimensional solutions. The possible way to stabilize the modulus is discussed. Section V is devoted to conclusions.

II. BASIC EQUATIONS

The relevant part of the supergravity action we consider is

$$S = \int d^6x\sqrt{-g}\left[\frac{M^4}{2}R - \frac{M^4}{2}\left(\partial\phi\right)^2 - \frac{1}{4}F^2e^{-\phi} - \frac{M^4}{2L_I^2}e^{\phi}\right], \tag{1}$$

where ϕ is the dilaton, M is the fundamental scale of gravity, $(\partial\phi)^2 := g^{MN}\partial_M\phi\partial_N\phi$, $F^2 := F_{MN}F^{MN}$, and $F_{MN} = \partial_M A_N - \partial_N A_M$ is the field strength of the gauge field A_M. For the moment we are interested in solving the 6D bulk equations of motion. In Sec. IV we will add two 4-branes (at positions $y = y_\pm$) and L_I denotes the different bulk curvature scales on either sides of the branes, see Fig. 1. We start with the axisymmetric metric ansatz

$$g_{MN}dx^M dx^N = L_I^2 e^{2\lambda(x)}\frac{dy^2}{f(y)} + \ell^2 e^{2[\psi(y,x)-\lambda(x)]}f(y)d\theta^2 + 2\ell b_\mu(y,x)d\theta dx^\mu + a^2(y)\bar{h}_{\mu\nu}(y,x)dx^\mu dx^\nu, \tag{2}$$

where capital Latin indices numerate the 6D coordinates while the Greek indices are restricted to the 4D coordinates.

The evolution equations along the y-direction are given by

$$n^y\partial_y K_{\hat\mu}^{\hat\nu} + \hat{K}K_{\hat\mu}^{\hat\nu} = {}^5R_{\hat\mu}^{\hat\nu} - e^{-\lambda(x)}{}^5D_{\hat\mu}{}^5D^{\hat\nu}e^{\lambda(x)} - \partial_{\hat\mu}\phi\partial^{\hat\nu}\phi - \frac{1}{4L_I^2}e^{\phi}\delta_{\hat\mu}^{\hat\nu} - \frac{1}{M^4}\left(F_{\hat\mu M}F^{\hat\nu M} - \frac{1}{8}\delta_{\hat\mu}^{\hat\nu}F^2\right)e^{-\phi}, \tag{3}$$

where $n^y = e^{-\lambda}\sqrt{f}/L_I$, $K_{\hat\mu}^{\hat\nu}$ is the extrinsic curvature of $y=$ constant hypersurfaces, $\hat{K}$ is its 5D trace, ${}^5R_{\hat\mu}^{\hat\nu}$ is the 5D Ricci tensor and ${}^5D_{\hat\mu}$ is the covariant derivative with respect to the 5D metric. Here, $\hat\mu = \mu$ and θ. The Hamiltonian constraint is

$${}^5R + K_{\hat\mu}^{\hat\nu}K_{\hat\nu}^{\hat\mu} - \hat{K}^2 = -\frac{2}{M^4}\left(F_{yM}F^{yM} - \frac{1}{4}F^2\right)e^{-\phi} - 2(n^y\partial_y\phi)^2 + (\partial\phi)^2 + \frac{1}{L_I^2}e^{\phi}, \tag{4}$$

and the momentum constraints are

$${}^5D_{\hat\nu}\left(K_{\hat\mu}^{\hat\nu} - \delta_{\hat\mu}^{\hat\nu}\hat{K}\right) = \frac{1}{M^4}F_{\hat\mu M}F^{yM}n_y e^{-\phi} + D_{\hat\mu}\phi\, n^y\partial_y\phi, \tag{5}$$

where $n_y = e^{\lambda}L_I/\sqrt{f}$.

The Maxwell equations are given by

$$\nabla_M\left(e^{-\phi}F^{MN}\right) = 0, \tag{6}$$

where ∇_M is the covariant derivative with respect to the 6D metric. The dilaton equation of motion is

$$\nabla_M\nabla^M\phi + \frac{1}{4M^4}F^2e^{-\phi} - \frac{1}{2L_I^2}e^{\phi} = 0. \tag{7}$$

III. GRADIENT EXPANSION APPROACH

In this section we will use the gradient expansion method [22, 23] to solve the 6D bulk equations. We assume that the length scale ℓ is of the same order of L_I. The small expansion parameter is the ratio of the bulk curvature scale to the 4D intrinsic curvature scale,

$$\varepsilon = \ell^2 |R|.$$

We expand the various quantities as

$$\begin{aligned} \bar{h}_{\mu\nu} = h_{\mu\nu}(x) + \varepsilon h^{(1)}_{\mu\nu}(y,x) + \cdots, \quad \psi = \psi^{(0)} + \varepsilon \psi^{(1)} + \cdots, \quad \phi = \phi^{(0)} + \varepsilon \phi^{(1)} + \cdots, \\ K_\mu^{\ \nu} = \overset{(0)}{K}{}_\mu^{\ \nu} + \varepsilon \overset{(1)}{K}{}_\mu^{\ \nu} + \cdots, \quad K_\theta^{\ \theta} = \overset{(0)}{K}{}_\theta^{\ \theta} + \varepsilon \overset{(1)}{K}{}_\theta^{\ \theta} + \cdots, \quad F_{y\theta} = \overset{(0)}{F}_{y\theta} + \varepsilon \overset{(1)}{F}_{y\theta} + \cdots. \end{aligned} \tag{8}$$

As to the other quantities, we follow [20] and first assume

$$\begin{aligned} b_\mu = \varepsilon^{1/2} b^{(1/2)}_\mu + \cdots, \quad K_\theta^{\ \nu} = \varepsilon^{1/2} \overset{(1/2)}{K}{}_\theta^{\ \nu} + \cdots, \\ F^{\mu y} = \varepsilon^{1/2} \overset{(1/2)}{F}{}^{\mu y} + \cdots, \quad F_{\mu\nu} = \varepsilon \overset{(1)}{F}_{\mu\nu} + \cdots, \end{aligned} \tag{9}$$

and then will show that all the $\mathcal{O}(\varepsilon^{1/2})$ quantities in fact vanish. Since $\partial_\mu A_\theta \sim \varepsilon^{1/2} \partial_y A_\theta$, we have $F_{\mu\theta} = \varepsilon^{1/2} \overset{(1/2)}{F}{}^{\mu\theta} + \cdots$. We will show that this $\mathcal{O}(\varepsilon^{1/2})$ term in $F_{\mu\theta}$ also vanishes. The bulk energy-momentum tensor contains terms like $F_{\mu\lambda}F^{\nu\lambda}$ but these do not contribute to the low energy effective theory as they are higher order in the gradient expansion. The 5D Ricci tensor is given by

$$ {}^5R_\mu^{\ \nu} = \varepsilon \frac{1}{a^2} \left(R_\mu^{\ \nu}[h] - \mathcal{D}_\mu \mathcal{D}^\nu \tilde{\psi} - \mathcal{D}_\mu \tilde{\psi} \mathcal{D}^\nu \tilde{\psi} \right) + \cdots, \tag{10}$$

$$ {}^5R_\theta^{\ \theta} = -\varepsilon \frac{1}{a^2} \left(\mathcal{D}_\lambda \mathcal{D}^\lambda \tilde{\psi} + \mathcal{D}_\lambda \tilde{\psi} \mathcal{D}^\lambda \tilde{\psi} \right) + \cdots, \tag{11}$$

and ${}^5R_\theta^{\ \mu} = \mathcal{O}(\varepsilon^{3/2})$, where $\tilde{\psi} := \psi^{(0)} - \lambda$. $R_\mu^{\ \nu}[h]$ and $\mathcal{D}_\mu$ are respectively the Ricci tensor and the covariant derivative constructed from $h_{\mu\nu}(x)$.

A. Zeroth order equations

The θ component of the Maxwell equations at zeroth order reads

$$\partial_y \left(a^4 e^{-\phi^{(0)} + \psi^{(0)}} \overset{(0)}{F}{}^{y\theta} \right) = 0, \tag{12}$$

while the equation of motion for the dilaton at zeroth order is given by

$$\frac{1}{a^4} \partial_y \left(a^4 f e^{\psi^{(0)}} \partial_y \phi^{(0)} \right) + \frac{1}{2} \left(\frac{1}{M^2 \ell} \overset{(0)}{F}_{y\theta} \right)^2 e^{2\lambda - \phi^{(0)} - \psi^{(0)}} - \frac{1}{2} e^{2\lambda + \phi^{(0)} + \psi^{(0)}} = 0. \tag{13}$$

The $(\mu\nu)$ and $(\theta\theta)$ components of the evolution equations are given respectively by

$$\begin{aligned} & f \left[\partial_y \left(\frac{\partial_y a}{a} \right) + \left(4 \frac{\partial_y a}{a} + \frac{\partial_y f}{f} + \partial_y \psi^{(0)} \right) \frac{\partial_y a}{a} \right] \\ & \qquad = \frac{1}{4} \left(\frac{1}{M^2 \ell} \overset{(0)}{F}_{y\theta} \right)^2 e^{2\lambda - \phi^{(0)} - 2\psi^{(0)}} - \frac{1}{4} e^{2\lambda + \phi^{(0)}}, \end{aligned} \tag{14}$$

$$\begin{aligned} & f \left[\partial_y \left(\frac{\partial_y f}{2f} + \partial_y \psi^{(0)} \right) + \left(4 \frac{\partial_y a}{a} + \frac{\partial_y f}{f} + \partial_y \psi^{(0)} \right) \left(\frac{\partial_y f}{2f} + \partial_y \psi^{(0)} \right) \right] \\ & \qquad = -\frac{3}{4} \left(\frac{1}{M^2 \ell} \overset{(0)}{F}_{y\theta} \right)^2 e^{2\lambda - \phi^{(0)} - 2\psi^{(0)}} - \frac{1}{4} e^{2\lambda + \phi^{(0)}}, \end{aligned} \tag{15}$$

and the Hamiltonian constraint becomes

$$4f\left[3\left(\frac{\partial_y a}{a}\right)^2+\frac{\partial_y a}{a}\left(\frac{\partial_y f}{f}+2\partial_y\psi^{(0)}\right)\right]=\left(\frac{1}{M^2\ell}\overset{(0)}{F}_{y\theta}\right)^2 e^{2\lambda-\phi^{(0)}-2\psi^{(0)}}+f\left(\partial_y\phi^{(0)}\right)^2-e^{2\lambda+\phi^{(0)}}. \tag{16}$$

The solutions for the above equations are obtained as

$$a(y)=\sqrt{y},\qquad f(y)=\frac{1}{4}\left(-y+\frac{\mu}{y}-\frac{q^2}{y^3}\right),\qquad \lambda(x)=\frac{1}{2}\Phi(x), \tag{17}$$

and

$$\psi^{(0)}(y,x)=\Phi(x)+\sigma(x),\qquad \phi^{(0)}(y,x)=-\ln y-\Phi(x),\qquad \overset{(0)}{F}_{y\theta}=M^2\ell\frac{q}{a^4}e^{\phi^{(0)}+\psi^{(0)}}=M^2\ell\frac{q}{y^3}e^{\sigma(x)}, \tag{18}$$

where μ and q are integration constants. The momentum constraint implies $\partial_\mu\sigma=0$, and therefore $\sigma=$ constant. This immediately leads to $\overset{(1/2)}{F}_{\mu\theta}=0$ and hence $F_{\mu\theta}=\mathcal{O}(\varepsilon^{3/2})$. In the following, we put $\sigma=0$ without loss of generality. The 6D metric at the zeroth order is given by

$$g_{MN}dx^M dx^N=e^{\Phi(x)}\left[L_I^2\frac{dy^2}{f}+\ell^2 f d\theta^2\right]+a^2(y)h_{\mu\nu}(x)dx^\mu dx^\nu. \tag{19}$$

Then we can see that $\Phi(x)$ is associated with the scaling symmetry $g_{MN}\to e^\omega g_{MN}$ and $e^\phi\to e^{\phi-\omega}$. In fact, we will find that a solution for $h_{\mu\nu}$ is given by $h_{\mu\nu}=e^\Phi\eta_{\mu\nu}$ if the brane preserves the scaling symmetry, where $\eta_{\mu\nu}$ denotes the 4D Minkowski metric.

B. First order equations

At first order, the $(\mu\nu)$ component of the evolution equations is given by

$$\begin{aligned}\frac{\sqrt{f}}{L_I}e^{-\Phi/2}\left[\partial_y \overset{(1)}{K}{}_\mu^{\ \nu}+\left(\frac{2}{y}+\frac{\partial_y f}{2f}\right)\overset{(1)}{K}{}_\mu^{\ \nu}+\frac{1}{2y}\left(\overset{(1)}{K}{}_\lambda^{\ \lambda}+\overset{(1)}{K}{}_\theta^{\ \theta}\right)\delta_\mu^{\ \nu}\right] &= \frac{1}{y}\left(R_\mu^{\ \nu}-\mathcal{D}_\mu\mathcal{D}^\nu\Phi-\frac{3}{2}\mathcal{D}_\mu\Phi\mathcal{D}^\nu\Phi\right)\\ &\quad -\frac{1}{4L_I^2}e^{\phi^{(0)}}\phi^{(1)}\delta_\mu^{\ \nu}+\frac{1}{4}\mathcal{F}\delta_\mu^{\ \nu},\end{aligned} \tag{20}$$

where

$$\mathcal{F}:=\frac{1}{M^4}\left(\overset{(0)}{F}_{y\theta}\overset{(1)}{F}{}^{y\theta}+\overset{(1)}{F}_{y\theta}\overset{(0)}{F}{}^{y\theta}\right)e^{-\phi^{(0)}}-\frac{1}{M^4}\overset{(0)}{F}_{y\theta}\overset{(0)}{F}{}^{y\theta}e^{-\phi^{(0)}}\phi^{(1)}. \tag{21}$$

The 4D Ricci tensor $R_\mu^{\ \nu}$ does not depend on y because it is computed from $h_{\mu\nu}$ which is a function of x^μ only and the index is raised by $h_{\mu\nu}$.

The 4D traceless part of Eq. (20) is found to be

$$\partial_y\left(y^2\sqrt{f}\,\mathbb{K}_\mu^{\ \nu}\right)=e^{\Phi/2}yL_I\mathbb{R}_\mu^{\ \nu}, \tag{22}$$

where we defined $\mathbb{K}_\mu^{\ \nu}:=\overset{(1)}{K}{}_\mu^{\ \nu}-(1/4)\delta_\mu^{\ \nu}\overset{(1)}{K}{}_\lambda^{\ \lambda}$ and

$$\mathbb{R}_\mu^{\ \nu}:=R_\mu^{\ \nu}-\frac{1}{4}\delta_\mu^{\ \nu}R-\left(\mathcal{D}_\mu\mathcal{D}^\nu\Phi-\frac{1}{4}\delta_\mu^{\ \nu}\mathcal{D}^2\Phi\right)-\frac{3}{2}\left[\mathcal{D}_\mu\Phi\mathcal{D}^\nu\Phi-\frac{1}{4}\delta_\mu^{\ \nu}(\mathcal{D}\Phi)^2\right], \tag{23}$$

where $\mathcal{D}^2\Phi:=h^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu\Phi$ and $(\mathcal{D}\Phi)^2:=h^{\mu\nu}\mathcal{D}_\mu\Phi\mathcal{D}_\nu\Phi$. The general solution to the above equation is given by

$$\mathbb{K}_\mu^{\ \nu}=\frac{e^{\Phi/2}}{2\sqrt{f}}L_I\mathbb{R}_\mu^{\ \nu}+\frac{1}{y^2\sqrt{f}}\mathbb{C}_\mu^{\ \nu}(x), \tag{24}$$

where the traceless tensor $\mathbb{C}_\mu^{\ \nu}(x)$ is the integration "constant" to be fixed by the boundary conditions.

The 4D trace part of the evolution equations is

$$\frac{\sqrt{f}}{L_I}e^{-\Phi/2}\left[\partial_y \overset{(1)}{K}{}_\lambda^{\lambda} + \left(\frac{4}{y}+\frac{\partial_y f}{2f}\right)\overset{(1)}{K}{}_\lambda^{\lambda} + \frac{2}{y}\overset{(1)}{K}{}_\theta^{\theta}\right] = \frac{1}{y}\left[R - \mathcal{D}^2\Phi - \frac{3}{2}(\mathcal{D}\Phi)^2\right] + \mathcal{F} - \frac{1}{L_I^2}\frac{e^{-\Phi}}{y}\phi^{(1)}, \tag{25}$$

and the $(\theta\theta)$ component of the evolution equations is

$$\frac{\sqrt{f}}{L_I}e^{-\Phi/2}\left[\partial_y \overset{(1)}{K}{}_\theta^{\theta} + \left(\frac{2}{y}+\frac{\partial_y f}{f}\right)\overset{(1)}{K}{}_\theta^{\theta} + \frac{\partial_y f}{2f}\overset{(1)}{K}{}_\lambda^{\lambda}\right] = -\frac{1}{2y}\left[\mathcal{D}^2\Phi + (\mathcal{D}\Phi)^2\right] - \frac{3}{4}\mathcal{F} - \frac{1}{4L_I^2}\frac{e^{-\Phi}}{y}\phi^{(1)}. \tag{26}$$

The Hamiltonian constraint at first order reduces to

$$\frac{1}{y}\left[R - \mathcal{D}^2\Phi - \frac{3}{2}(\mathcal{D}\Phi)^2\right] + \mathcal{F} = 2\frac{\sqrt{f}}{L_I}e^{-\Phi/2}\left[\left(\frac{3}{2y}+\frac{\partial_y f}{2f}\right)\overset{(1)}{K}{}_\lambda^{\lambda} + \frac{2}{y}\overset{(1)}{K}{}_\theta^{\theta}\right] + \frac{1}{L_I^2}\frac{e^{-\Phi}}{y}\phi^{(1)} + \frac{2f}{L_I^2}\frac{e^{-\Phi}}{y}\partial_y\phi^{(1)}. \tag{27}$$

The dilaton equation of motion at first order reads

$$\frac{f}{L_I^2}e^{-\Phi}\left[\partial_y^2\phi^{(1)} + \left(\frac{2}{y}+\frac{\partial_y f}{f}\right)\partial_y\phi^{(1)}\right] - \frac{\sqrt{f}}{L_I}\frac{e^{-\Phi/2}}{y}\left(\overset{(1)}{K}{}_\lambda^{\lambda} + \overset{(1)}{K}{}_\theta^{\theta}\right)$$
$$-\frac{1}{y}\left[\mathcal{D}^2\Phi + (\mathcal{D}\Phi)^2\right] - \frac{1}{2L_I^2}\frac{e^{-\Phi}}{y}\phi^{(1)} + \frac{1}{2}\mathcal{F} = 0. \tag{28}$$

Now we define convenient quantities

$$\mathcal{J} := n^y\partial_y\phi^{(1)} + \frac{1}{2}\overset{(1)}{K}{}_\lambda^{\lambda} \tag{29}$$

and

$$\mathcal{K} := \frac{3}{4}\overset{(1)}{K}{}_\lambda^{\lambda} + \overset{(1)}{K}{}_\theta^{\theta} + \frac{\sqrt{f}}{L_I}e^{-\Phi/2}\left(\frac{\partial_y f}{2f} - \frac{1}{2y}\right)\psi^{(1)} + \frac{y}{M^4\ell^2 L_I\sqrt{f}}\overset{(0)}{F}_{y\theta}\, e^{-\Phi/2}A_\theta^{(1)} - \frac{\sqrt{f}}{L_I}\frac{e^{-\Phi/2}}{y}\phi^{(1)}. \tag{30}$$

The evolution equations for these variables can be derived using Eqs. (25)–(28). With some manipulation one arrives at

$$\partial_y\left(y^2\sqrt{f}\mathcal{J}\right) = \frac{1}{2}e^{\Phi/2}yL_I\left[R + \mathcal{D}^2\Phi + \frac{1}{2}(\mathcal{D}\Phi)^2\right], \tag{31}$$

$$\partial_y\left(y^2\sqrt{f}\mathcal{K}\right) = \frac{1}{4}e^{\Phi/2}yL_I\left[R - 3\mathcal{D}^2\Phi - \frac{7}{2}(\mathcal{D}\Phi)^2\right]. \tag{32}$$

The two equations have the same structure as that of Eq. (22). The general solution for each evolution equation contains one integration "constant" which will be determined by the boundary conditions.

In terms of the above variables, the momentum constraint equations are simplified to

$$\mathcal{D}_\nu\left(e^{\Phi/2}\mathbb{K}_\mu^{\ \nu}\right) - \mathcal{D}_\mu\left(e^{\Phi/2}\mathcal{K}\right) + e^{\Phi/2}\mathcal{J}\mathcal{D}_\mu\Phi = 0. \tag{33}$$

IV. JUNCTION CONDITIONS AND EFFECTIVE THEORY ON A REGULARIZED BRANE

Our choice of parameters μ, q implies that $f(y)$ vanishes at y_N and y_S. These points are conical singularities that are sourced by 3-branes. In order to accommodate usual matter on the branes we need to resolve these singularities. We will use the regularization scheme of [13, 25]. The conical branes are replaced with cylindrical codimension-one branes at $y = y_\pm$ and their interiors are filled with regular caps. See figure 1 for a sketch of the model. The geometry of the caps and the central bulk is described by the 6D solutions found in the previous section, with different curvature scales L_+ (L_-) for the north (south) cap and L_0 for the central bulk.

The action of each brane is taken to be

$$S_{\rm brane} = -\int d^5x\sqrt{-q}\left[V(\phi) + \frac{1}{2}U(\phi)(\partial_{\hat\mu}\Sigma - {\rm e}A_{\hat\mu})(\partial^{\hat\mu}\Sigma - {\rm e}A^{\hat\mu})\right] + \int d^5x\sqrt{-q}\,\mathcal{L}_{\rm m}, \tag{34}$$

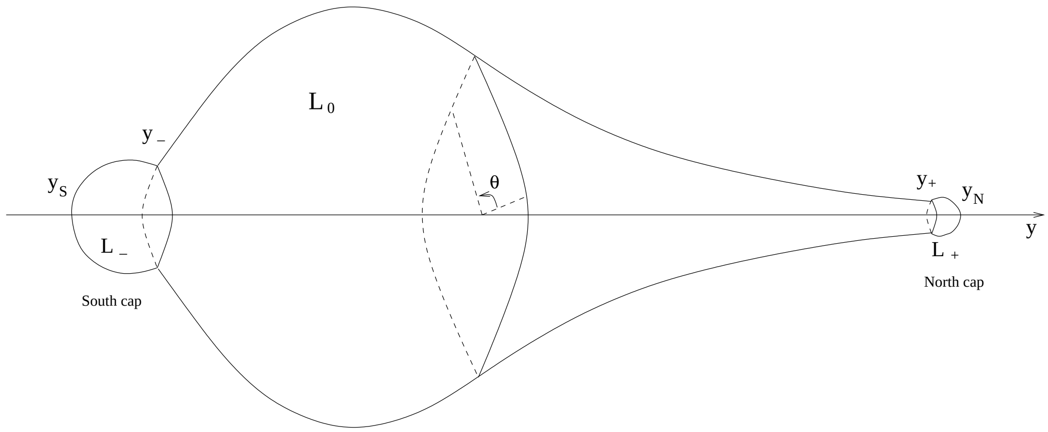

FIG. 1: Schematic representation of the bulk spacetime with two regularized caps.

where $q_{\hat{\mu}\hat{\nu}}$ is the induced metric on the 4-brane, $V(\phi)$ and $U(\phi)$ are the couplings to the dilaton, and $\mathcal{L}_{\rm m}$ is the Lagrangian of usual matter localized on the brane. At this stage we assume that the brane matter $\mathcal{L}_{\rm m}$ does not couple to the dilaton field. We introduce a Stueckelberg field Σ, which is obtained by integrating out the massive radial mode of a brane Higgs field. The equation of motion for Σ gives the gradient expansion form of the solution as [20]

$$\Sigma(\theta, x) = n\theta + c^{(0)}(x) + \varepsilon c^{(1)}(x) + \cdots , \tag{35}$$

where n must be an integer because of the periodicity $\theta \simeq \theta + 2\pi$.

The jump conditions for the Maxwell field are

$$\big[\big[n^M F_{MN} e^{-\phi}\big]\big] = -{\rm e}U(\partial_N \Sigma - {\rm e}A_N), \tag{36}$$

while for the dilaton field we have

$$\big[\big[n^M \partial_M \phi\big]\big] = \frac{1}{M^4}\left[\frac{dV}{d\phi} + \frac{1}{2}\frac{dU}{d\phi}(\partial_{\hat{\lambda}}\Sigma - {\rm e}A_{\hat{\lambda}})(\partial^{\hat{\lambda}}\Sigma - {\rm e}A^{\hat{\lambda}})\right], \tag{37}$$

where $[[F]]_{y_{\rm b}} := \lim_{\epsilon\to 0}\left(F|_{y_{\rm b}+\epsilon} - F|_{y_{\rm b}-\epsilon}\right)$. Here and hereafter in this section all the quantities are evaluated at the position of the brane under consideration. The Israel conditions are given by

$$\left[\left[K_{\hat{\mu}}^{\hat{\nu}} - \delta_{\hat{\mu}}^{\hat{\nu}}\hat{K}\right]\right] = -\frac{1}{M^4}T_{\hat{\mu}({\rm tot})}^{\hat{\nu}} \tag{38}$$

where

$$T_{\hat{\mu}({\rm tot})}^{\hat{\nu}} = -V\delta_{\hat{\mu}}^{\hat{\nu}} + U\left[(\partial_{\hat{\mu}}\Sigma - {\rm e}A_{\hat{\mu}})(\partial^{\hat{\nu}}\Sigma - {\rm e}A^{\hat{\nu}}) - \frac{1}{2}\delta_{\hat{\mu}}^{\hat{\nu}}(\partial_{\hat{\lambda}}\Sigma - {\rm e}A_{\hat{\lambda}})(\partial^{\hat{\lambda}}\Sigma - {\rm e}A^{\hat{\lambda}})\right] + T_{\hat{\mu}}^{\hat{\nu}}, \tag{39}$$

and $T_{\hat{\mu}}^{\hat{\nu}}$ represents the matter energy-momentum tensor.

A. Zeroth order

At zeroth order in the gradient expansion the junction conditions (36)–(38) are written as

$$\text{Maxwell:} \quad \left[\left[\frac{\sqrt{f}}{L_I}\overset{(0)}{F}_{y\theta}\, y\, e^{\Phi/2}\right]\right] = -{\rm e}U^{(0)}\left(n - {\rm e}A_\theta^{(0)}\right), \tag{40}$$

$$\text{Dilaton:} \quad \left[\left[\frac{\sqrt{f}}{L_I}\frac{1}{y}e^{-\Phi/2}\right]\right] = -\frac{1}{M^4}\left[\frac{dV^{(0)}}{d\phi^{(0)}} + \frac{1}{2}\frac{dU^{(0)}}{d\phi^{(0)}}\frac{e^{-\Phi}}{\ell^2 f}\left(n - {\rm e}A_\theta^{(0)}\right)^2\right], \tag{41}$$

$$\text{Israel } (\mu\nu): \quad \left[\left[\frac{\sqrt{f}}{L_I}\left(\frac{3}{2y} + \frac{\partial_y f}{2f}\right)e^{-\Phi/2}\right]\right] = -\frac{1}{M^4}\left[V^{(0)} + \frac{1}{2}U^{(0)}\frac{e^{-\Phi}}{\ell^2 f}\left(n - {\rm e}A_\theta^{(0)}\right)^2\right], \tag{42}$$

$$\text{Israel } (\theta\theta): \quad \left[\left[\frac{\sqrt{f}}{L_I}\frac{2}{y}e^{-\Phi/2}\right]\right] = -\frac{1}{M^4}\left[V^{(0)} - \frac{1}{2}U^{(0)}\frac{e^{-\Phi}}{\ell^2 f}\left(n - {\rm e}A_\theta^{(0)}\right)^2\right]. \tag{43}$$

The above conditions relate several parameters with each other, and the detail of the parameter counting of the configuration is found in Ref. [25]. In particular, the dilaton jump condition (41) and the Israel condition (43) imply

$$\frac{V^{(0)}}{2}-\frac{dV^{(0)}}{d\phi^{(0)}}-\frac{1}{2}\frac{e^{-\Phi}}{\ell^2 f}\left(\frac{U^{(0)}}{2}+\frac{dU^{(0)}}{d\phi^{(0)}}\right)\left(n-\mathrm{e}A_\theta^{(0)}\right)^2=0. \tag{44}$$

The classical scaling symmetry is preserved by the special choice of the potentials [2, 25]

$$V(\phi)=ve^{\phi/2},\qquad U(\phi)=ue^{-\phi/2}. \tag{45}$$

With these potentials the junction conditions (40)–(43) put no constraints on $\Phi(x)$ and Eq. (44) is trivially satisfied. In this case the first order analysis will provide the equation of motion for $\Phi(x)$, as will be seen in the next subsection. In the following, we assume that at the zeroth order, the potentials are given by (45), that is, $U^{(0)}(\phi^{(0)})=u^{(0)}e^{-\phi^{(0)}/2}$ and $V^{(0)}(\phi^{(0)})=v^{(0)}e^{\phi^{(0)}/2}$. Then we expand the potentials as follows:

$$V(\phi) \;=\; V^{(0)}(\phi^{(0)})+\varepsilon\left(V^{(1)}(\phi^{(0)})+\frac{dV^{(0)}}{d\phi^{(0)}}\phi^{(1)}\right), \tag{46}$$

$$U(\phi) \;=\; U^{(0)}(\phi^{(0)})+\varepsilon\left(U^{(1)}(\phi^{(0)})+\frac{dU^{(0)}}{d\phi^{(0)}}\phi^{(1)}\right), \tag{47}$$

where $V^{(1)}(\phi^{(0)})$ and $U^{(1)}(\phi^{(0)})$ stand for the deviations from the zeroth order potentials.

B. First order

The 4D traceless part of the Israel conditions at first order is given by

$$[[\mathbb{K}_\mu^{\ \nu}]]=-\frac{1}{M^4}\mathbb{T}_\mu^{\ \nu}, \tag{48}$$

where $\mathbb{T}_\mu^{\ \nu}:=T_\mu^{\ \nu}-(1/4)\delta_\mu^{\ \nu}T_\lambda^{\ \lambda}$. The 4D trace part of the Israel conditions reduces to

$$\left[\left[\frac{3}{4}\overset{(1)}{K}{}_\lambda^{\ \lambda}+\overset{(1)}{K}{}_\theta^{\ \theta}\right]\right]=\frac{1}{4M^4}T_\lambda^{\ \lambda}-\frac{1}{M^4}\Delta V+\frac{U^{(0)}}{M^4}\Delta-\frac{1}{M^4}\left[\frac{dV^{(0)}}{d\phi^{(0)}}+\frac{1}{2}\frac{dU^{(0)}}{d\phi^{(0)}}\frac{e^{-\Phi}}{\ell^2 f}\left(n-\mathrm{e}A_\theta^{(0)}\right)^2\right]\phi^{(1)}, \tag{49}$$

where we defined

$$\Delta V=V^{(1)}(\phi^{(0)})+\frac{1}{2}U^{(1)}(\phi^{(0)})\frac{e^{-\Phi}}{\ell^2 f}\left(n-\mathrm{e}A_\theta^{(0)}\right)^2, \tag{50}$$

and

$$\Delta:=\frac{e^{-\Phi}}{\ell^2 f}\left(n-\mathrm{e}A_\theta^{(0)}\right)\left[\mathrm{e}A_\theta^{(1)}+\left(n-\mathrm{e}A_\theta^{(0)}\right)\psi^{(1)}\right]. \tag{51}$$

Using the zeroth order junction conditions, Eq. (49) simply gives

$$[[\mathcal{K}]]=\frac{1}{4M^4}T_\lambda^{\ \lambda}-\frac{1}{M^4}\Delta V. \tag{52}$$

The $(\theta\theta)$ component of the Israel conditions is

$$\begin{aligned}\left[\left[\overset{(1)}{K}{}_\lambda^{\ \lambda}\right]\right] \;=\; &\frac{T_\theta^{\ \theta}}{M^4}-\frac{U^{(0)}}{M^4}\Delta-\frac{1}{M^4}\left[\frac{dV^{(0)}}{d\phi^{(0)}}-\frac{1}{2}\frac{dU^{(0)}}{d\phi^{(0)}}\frac{e^{-\Phi}}{\ell^2 f}\left(n-\mathrm{e}A_\theta^{(0)}\right)^2\right]\phi^{(1)}\\ &-\frac{1}{M^4}\Delta V+U^{(1)}(\phi^{(0)})\frac{e^{-\Phi}}{M^4\ell^2 f}\left(n-\mathrm{e}A_\theta^{(0)}\right)^2,\end{aligned} \tag{53}$$

and the dilaton jump condition is

$$\begin{aligned}\left[\left[n^y\partial_y\phi^{(1)}\right]\right] \;=\; &-\frac{1}{M^4}\frac{dU^{(0)}}{d\phi^{(0)}}\Delta+\frac{1}{M^4}\left[\frac{d^2V^{(0)}}{d\phi^{(0)2}}+\frac{1}{2}\frac{d^2U^{(0)}}{d\phi^{(0)2}}\frac{e^{-\Phi}}{\ell^2 f}\left(n-\mathrm{e}A_\theta^{(0)}\right)^2\right]\phi^{(1)}\\ &-\frac{1}{M^4}\frac{d}{d\Phi}(\Delta V)-\frac{1}{2}U^{(1)}(\phi^{(0)})\frac{e^{-\Phi}}{M^4\ell^2 f}\left(n-\mathrm{e}A_\theta^{(0)}\right)^2.\end{aligned} \tag{54}$$

Using the fact that the zeroth order potential have the scale invariant forms (45), the above two conditions are combined to give

$$[[\mathcal{J}]] = \frac{1}{2M^4}T_\theta^{\ \theta} - \frac{1}{M^4}\frac{d}{d\Phi}(\Delta V) - \frac{1}{2M^4}\Delta V. \tag{55}$$

Therefore, the momentum constraints become

$$\mathcal{D}_\nu\left(e^{\Phi/2}T_\mu^{\ \nu} - e^{\Phi/2}\Delta V\delta_\mu^{\ \nu}\right) = \left(\frac{1}{2}T_\theta^{\ \theta} - \frac{d}{d\Phi}(\Delta V) - \frac{1}{2}\Delta V\right)e^{\Phi/2}\mathcal{D}_\mu\Phi. \tag{56}$$

In terms of the energy-momentum tensor integrated along the θ-direction,

$$\overline{T}_{\hat{\mu}}^{\ \hat{\nu}} := 2\pi\ell\sqrt{f}e^{\Phi/2}T_{\hat{\mu}}^{\ \hat{\nu}}, \tag{57}$$

this can be rewritten as

$$\mathcal{D}_\nu\overline{T}_\mu^{\ \nu} = \frac{1}{2}\overline{T}_\theta^{\ \theta}\mathcal{D}_\mu\Phi. \tag{58}$$

To fix the integration constants completely, we need the boundary conditions at the north and south poles. Near a pole with the coordinate $y = y_p$, where $p = \{N, S\}$, we have $f \sim y - y_p$. In order for the evolution equations (22), (31), and (32) to be regular at the poles, we require

$$\mathbb{K}_\mu^{\ \nu},\ \mathcal{K},\ \mathcal{J} \lesssim |y - y_p|^{1/2} \to 0. \tag{59}$$

Now we can determine all the integration constants included in the general solutions for $\mathbb{K}_\mu^{\ \nu}$, $\mathcal{K}$ and $\mathcal{J}$. Since the structure of the evolution equations and boundary conditions are identical for these three variables, we summarize the procedure to fix the integration constants in Appendix A, and here we focus on the resulting effective theory on the brane.

Using Eqs. (48) and (52) together with the solution for $\mathbb{K}_\mu^{\ \nu}$ and $\mathcal{K}$ in terms of R and Φ, we end up with the effective equations

$$e^\Phi\left(R_\mu^{\ \nu}[q^+] - \frac{1}{2}\delta_\mu^{\ \nu}R[q^+] - \Phi_{;\mu}^{;\nu} + \delta_\mu^{\ \nu}\Phi_{;\lambda}^{;\lambda} - \frac{3}{2}\Phi_{;\mu}\Phi^{;\nu} + \frac{5}{4}\delta_\mu^{\ \nu}\Phi_{;\lambda}\Phi^{;\lambda}\right) = \kappa_+^2\left(\overline{T}_\mu^{+\nu} - \overline{\Delta V}^+\delta_\mu^\nu\right) + \frac{a_-^2}{a_+^2}\kappa_-^2\left(\overline{T}_\mu^{-\nu} - \overline{\Delta V}^-\delta_\mu^\nu\right), \tag{60}$$

where the 4D gravitational couplings are defined as

$$\kappa_\pm^2 := \frac{a_\pm^2}{2\pi\ell_*^2 M^4}, \quad \text{with} \quad \ell_*^2 = \ell\int_{y_S}^{y_N} L_I y dy, \tag{61}$$

; denotes a covariant derivative with respect to the induced metric $q_{\mu\nu}^+ = a_+^2 h_{\mu\nu}$, $R_{\mu\nu}[q^+]$ is Ricci tensor computed from $q_{\mu\nu}^+$ and the potential integrated along the θ-direction is defined as

$$\overline{\Delta V} = 2\pi\ell\sqrt{f}e^{\Phi/2}\Delta V. \tag{62}$$

The first order equations for $\mathcal{J}$ give the equation of motion for Φ:

$$\left(e^\Phi\right)_{;\mu}^{;\mu} = \frac{\kappa_+^2}{4}\left(\overline{T}_\lambda^{+\lambda} - \overline{T}_\theta^{+\theta} + 2\frac{d}{d\Phi}(\overline{\Delta V}^+) - 4\overline{\Delta V}^+\right) + \frac{a_-^2}{a_+^2}\frac{\kappa_-^2}{4}\left(\overline{T}_\lambda^{-\lambda} - \overline{T}_\theta^{-\theta} + 2\frac{d}{d\Phi}(\overline{\Delta V}^-) - 4\overline{\Delta V}^-\right). \tag{63}$$

For simplicity let us ignore the matter energy-momentum tensor and the potential on the south brane: $\overline{T}_{\hat{\mu}}^{-\hat{\nu}} = \overline{\Delta V}^- = 0$. In the absence of the $(\theta\theta)$ component of the energy momentum tensor on the north brane, the 4D effective equations can be deduced from the action

$$S_{\rm eff} = \int d^4x\sqrt{-q^+}\left[\frac{e^\Phi}{2\kappa_+^2}\left(R[q^+] - \omega_{\rm BD}\Phi_{;\mu}\Phi^{;\mu}\right) - \overline{\Delta V}^+ + \overline{\mathcal{L}}_{\rm m}^+\right], \tag{64}$$

with the Brans-Dicke parameter $\omega_{\rm BD} = 1/2$ (see also Appendix B of Ref. [11]).

C. The exact time-dependent solutions in the 4D effective theory

We now consider cosmological solutions in the 4D effective theory and compare them with the known solutions to the full 6D field equations [10, 11, 12].

Let us assume $\overline{T}_{\mu}^{\pm\nu} = \overline{\Delta V}^{-} = 0$ and $\overline{T}_{\theta}^{\pm\theta} = 0$. We consider the case where the first order potential is scale invariant form. Then $\Delta V^{+} \propto e^{-\Phi/2}$ and $\overline{\Delta V}^{+} = \text{const.} := \Lambda/\kappa_{+}^{2}$. We go to the Einstein frame defined by

$$\tilde{h}_{\mu\nu} = e^{\Phi} q_{\mu\nu}^{+}, \tag{65}$$

and then the equations of motion become

$$\tilde{R}_{\mu}^{\ \nu}[\tilde{h}] - \frac{1}{2}\delta_{\mu}^{\ \nu}\tilde{R}[\tilde{h}] \ = \ -\Lambda e^{-2\Phi}\delta_{\mu}^{\ \nu} + 2\Phi_{|\mu}\Phi^{|\nu} - \Phi_{|\lambda}\Phi^{|\lambda}\delta_{\mu}^{\ \nu}, \tag{66}$$

$$\Phi_{|\mu}^{\ |\mu} \ = \ -\Lambda e^{-2\Phi}, \tag{67}$$

where $|$ stands for the covariant derivative with respect to the Einstein frame metric $\tilde{h}_{\mu\nu}$.

Taking $\tilde{h}_{\mu\nu}$ to be a flat Friedman-Robertson-Walker metric, $\tilde{h}_{\mu\nu}dx^{\mu}dx^{\nu} = A^{2}(\tau)(-d\tau^{2} + d\mathbf{x}^{2})$, the equations of motion reduce to

$$\frac{A''}{A} = \frac{2}{3}\Lambda A^{2} e^{-2\Phi} - \frac{1}{3}\Phi'^{2}, \quad \left(\frac{A'}{A}\right)^{2} = \frac{1}{3}\Lambda A^{2} e^{-2\Phi} + \frac{1}{3}\Phi'^{2}, \tag{68}$$

and

$$\Phi'' + 2\frac{A'}{A}\Phi' = \Lambda A^{2} e^{-2\Phi}, \tag{69}$$

where $' := d/d\tau$. For $\Lambda = 0$ a solution of the above equations is

$$A^{2}(\tau) = A_{1}\tau + A_{2}, \quad \Phi(\tau) = \pm\sqrt{3}\ln A(\tau) + A_{3}, \tag{70}$$

where A_{i} $(i = 1, 2, 3)$ are integration constants. For $\Lambda \neq 0$ a solution is

$$A(\tau) = e^{C\tau}, \quad \Phi(\tau) = \ln A(\tau), \tag{71}$$

where $C^{2} = \Lambda/2$. This indicates that $q_{\mu\nu}^{+} = e^{\Phi}\eta_{\mu\nu}$ which is expected from the scaling symmetry.

The brane scale factor $a_{\rm b}$ and the "radion" Ψ are given by

$$a_{\rm b} = e^{-\Phi/2}A, \quad \Psi = e^{\Phi}. \tag{72}$$

The solution (70) gives the same 4D observables (the brane scale factor and radion) as an exact 6D solution found by Copeland and Seto (equation (70) of [12]). In the same way the solution (71) reproduces the 4D quantities of their equation (76). (The latter solution was first found by Tolley et al. [10].) Thus we show that the solutions to the full 6D equations are reproduced by our 4D effective theory on a regularized brane with the scale invariant potential and with or without additional "tension" Λ.

At the zeroth order, the amplitudes $v^{(0)}$ and $u^{(0)}$ of the potentials are fine-tuned. If we change the amplitude of the scale invariant potentials, which is equivalent to add a cosmological constant in the 4D effective theory, we get a runaway potential for Φ and the 4D spacetime becomes non-static.

D. Breaking the scale invariance

Finally, let us consider the case where the first order potential breaks the scale invariance. As the BD parameter is given by 1/2, this model violates the constraints coming from the solar system experiments unless the BD scalar Φ is stabilized. It is suggested that the potential on a brane can naturally stabilize the modulus. For example, if we consider potentials $V_{1}(\phi^{(0)}) = v^{(1)}e^{s\phi^{(0)}}$ and $U_{1}(\phi^{(0)}) = u^{(1)}e^{t\phi^{(0)}}$, the effective potential is given by

$$\overline{\Delta V} = 2\pi\ell\sqrt{f}e^{\Phi/2}\left(\frac{v^{(1)}}{y_{+}^{s}}e^{-s\Phi} + \frac{u^{(1)}}{2\ell^{2}fy_{+}^{t}}e^{-(t+1)\Phi}(n - \mathrm{e}A_{\theta}^{(0)})^{2}\right). \tag{73}$$

As we saw in the previous subsection, if we take $s = 1/2$ and $t = -1/2$, $\overline{\Delta V}$ is independent of Φ. However, in general, it is possible to have a potential with a minimum by choosing $s, t, v^{(1)}$ and $u^{(1)}$ appropriately [25]. Then Φ can be stabilized and general relativity (GR) is recovered.

V. CONCLUSIONS

In this paper, we derived the low energy effective theory in the six-dimensional supergravity with resolved 4-branes. The gradient expansion method is used to solve the bulk geometry. The resultant effective theory is a Brans-Dicke theory with the Brans-Dicke parameter given by $\omega_{\rm BD} = 1/2$. If we choose the dilaton potentials on the branes so that they keep the scaling symmetry in the bulk and if we tune their amplitudes then there is no potential in the effective theory and the modulus is massless. Thus the static four-dimensional spacetime has vanishing cosmological constant. It is also possible to obtain time-dependent solutions due to the dynamics of the modulus field and we showed that they are identified with the six-dimensional exact time dependent solutions found in [12].

Even if the potentials preserve the scaling symmetry, it was found that there appears an effective cosmological constant in the four-dimensional effective theory by changing the amplitude of the potentials. Then in the Einstein frame, the modulus field acquires an exponential potential and the static solution is no longer allowed. Again, we showed that the cosmological solutions obtained in the effective theory can be identified with the six-dimensional exact time dependent solutions found in [10, 11, 12].

Our effective theory allows us to discuss cosmology with arbitrary matter on the brane. As the BD parameter is given by 1/2, it is impossible to reproduce realistic cosmology without stabilizing the modulus field. As it was suggested by Ref. [25], it is easy to generate a potential for the modulus Φ with a minimum by breaking the scaling symmetry from the dilaton potentials on the branes. Then it is possible to reproduce GR at low energies. However, once we stabilize the modulus, the cosmological constant on a brane curves the four-dimensional spacetime in the same way as in GR.

Our result would indicate that it is possible to reproduce sensible cosmology in this six-dimensional supergravity model at low energies but it would be difficult to address the cosmological constant problem in this set-up. However, we should mention that our effective theory is valid only up to the energy scale determined by inverse of the size of extra-dimensions. This condition is roughly given by $H\ell_* < 1$ where H is the Hubble parameter. If we consider scales smaller than ℓ_* gravity becomes six-dimensional and from the table-top experiments, ℓ_* is smaller than a few μm. Then for $H > 10^{-2}$ eV, our Universe becomes six-dimensional and it is impossible to use the four-dimensional effective theory. In order to address the behaviour of the universe at high energies, we should deal with time-dependent solutions directly in six-dimensional spacetime. This remains an open question.

Finally, we briefly make a comment on the limit where the codimension one branes are shrunk to codimension two objects. Our effective theory shows no pathological behaviour in this limit as long as the four-dimensional energy-momentum tensor integrated along the θ-direction remains finite. However, in this limit, the first order extrinsic curvature $\overset{(1)}{K}_{\mu\nu}$ diverges and then the first order correction to the four-dimensional metric diverges. Then it is not clear whether there is a physical meaning in this limit. This is related to a deep issue of whether it is possible to put ordinary matter on codimension 2 objects [26] and we also leave this problem as an open question.

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APPENDIX A: SOLVING THE BULK EVOLUTION EQUATIONS

All of the key evolution equations in the main text have the form of

$$\partial_y \left[y^2 \sqrt{f} K(y,x) \right] = e^{\Phi(x)/2} y L_I R(x), \tag{A1}$$

subject to the boundary conditions

$$K(y_N, x) = K(y_S, x) = 0, \tag{A2}$$

$$[[K]]|_{y=y_\pm} = T^\pm(x). \tag{A3}$$

In the south cap we have the solution

$$K = \frac{y^2 - y_S^2}{2y^2\sqrt{f}} e^{\Phi/2} L_- R. \tag{A4}$$

In the bulk the solution can be written as

$$K = \frac{e^{\Phi/2}\left[y^2 L_0 R + C(x)\right]}{2y^2\sqrt{f}}, \tag{A5}$$

where the integration constant $C(x)$ is determined by the condition (A3) as

$$C(x) = \left[-y_-^2 L_0 + (y_-^2 - y_S^2)L_-\right] R + 2y_-^2\sqrt{f_-}e^{-\Phi/2}T^-. \tag{A6}$$

In the north cap we have the solution

$$K = \frac{y^2 - y_N^2}{2y^2\sqrt{f}} e^{\Phi/2} L_+ R, \tag{A7}$$

and the boundary condition (A3) requires

$$\left[(y_N^2 - y_+^2)L_+ + y_+^2 L_0\right] R + C = -2y_+^2\sqrt{f_+}e^{-\Phi/2}T^+. \tag{A8}$$

In the above we defined $f_\pm := f(y_\pm)$. Substituting Eq. (A6) into Eq. (A8) we obtain

$$\left(\int_{y_S}^{y_N} L_I y dy\right)\cdot R = -\sum_{i=\pm} y_i^2\sqrt{f_i}e^{-\Phi/2}T^i. \tag{A9}$$

APPENDIX B: $\mathcal{O}(\varepsilon^{1/2})$ QUANTITIES

The μ component of the $\mathcal{O}(\varepsilon^{1/2})$ Maxwell equations reads

$$\partial_y\left(e^{-\phi^{(0)}} a^4 \overset{(1/2)}{F^{y\mu}}\right) = 0, \tag{B1}$$

and thus we have

$$\overset{(1/2)}{F^{\mu y}} = M^2 \frac{C_1^\mu(x)}{y^4}. \tag{B2}$$

The $\mathcal{O}(\varepsilon^{1/2})$ evolution equation reduces to

$$\begin{aligned} \frac{e^{-\Phi/2}}{L_I}\frac{1}{y^2}\partial_y\left(y^2\sqrt{f}\,\overset{(1/2)}{K_\theta^{\ \nu}}\right) &= -\frac{1}{M^4}\overset{(0)}{F_{\theta y}}\overset{(1/2)}{F^{\nu y}} \\ &= \ell q\frac{C_1^\nu}{y^6}, \end{aligned} \tag{B3}$$

which can be integrated to give

$$\overset{(1/2)}{K_\theta^{\ \nu}} = \frac{e^{\Phi/2}}{y^2\sqrt{f}}\left[-\frac{L_I \ell q}{3}\frac{C_1^\nu(x)}{y^3} + C_2^\nu(x)\right]. \tag{B4}$$

The $\mathcal{O}(\varepsilon)$ evolution equations contain terms like $\overset{(1/2)}{F_{\mu y}}\overset{(1/2)}{F^{\nu y}} \propto h_{\mu\lambda}C_1^\lambda C_1^\nu/f(y)$. In the cap regions, we thus require $C_1^\nu = 0$ to avoid the singular behavior at the poles. Further, the regularity of $\overset{(1/2)}{K_\theta^{\ \nu}}$ at the poles imposes $C_2^\nu = 0$ in the cap regions. To fix the integration constants in the central bulk, we use the Maxwell jump conditions and Israel conditions at each brane:

$$\left[\left[n_y \overset{(1/2)}{F^{y\mu}} e^{-\phi^{(0)}}\right]\right] = -\mathrm{e}U\left(\partial^\mu\Sigma - \mathrm{e}A^\mu\right)^{(1/2)}, \tag{B5}$$

$$\left[\left[\overset{(1/2)}{K_\theta^{\ \nu}}\right]\right] = -\frac{U}{M^4}\left(n - \mathrm{e}A_\theta^{(0)}\right)\left(\partial^\mu\Sigma - \mathrm{e}A^\mu\right)^{(1/2)}. \tag{B6}$$

Combining these two equations and noting that $\overset{(1/2)}{F^{y\mu}} = 0 = \overset{(1/2)}{K_\theta^{\ \nu}}$ in each cap, we obtain two linear algebraic equations for the bulk values of C_1^ν and C_2^ν:

$$\left[\overset{(1/2)}{K_\theta^{\ \nu}} - \frac{1}{\mathrm{e}M^4}\left(n - \mathrm{e}A_\theta^{(0)}\right) n_y \overset{(1/2)}{F^{y\mu}} e^{-\phi^{(0)}}\right]\Bigg|_{y_\pm \mp \epsilon} = 0. \tag{B7}$$

Therefore, $C_1^\nu = C_2^\nu = 0$ in the bulk. Now we also see that

$$(\partial^\mu \Sigma - \mathrm{e}A^\mu)^{(1/2)} = 0 \quad \text{on the branes,} \tag{B8}$$

and $b_\mu = \mathcal{O}(\varepsilon^{3/2})$.

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