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Non-Gaussianities from ekpyrotic collapse with multiple fields

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Abstract

We compute the non-Gaussianity of the curvature perturbation generated by ekpyrotic collapse with multiple fields. The transition from the multi-field scaling solution to a single-field dominated regime converts initial isocurvature field perturbations to an almost scale-invariant comoving curvature perturbation. In the specific model of two fields, ϕ_1 and ϕ_2, with exponential potentials, $-V_i \exp(-c_i\phi_i)$, we calculate the bispectrum of the resulting curvature perturbation. We find that the non-Gaussianity is dominated by non-linear evolution on super-Hubble scales and hence is of the local form. The non-linear parameter of the curvature perturbation is given by $f_{NL} = -5c_j^2/12$, where c_j is the exponent of the potential for the field which becomes sub-dominant at late times. Since c_j^2 must be large, in order to generate an almost scale invariant spectrum, the non-Gaussianity is inevitably large. By combining the present observational constraints on $f_{\rm NL}$ and the scalar spectral index, the specific model studied in this paper is thus ruled out by current observational data.

1 Introduction

The existence of an almost scale-invariant spectrum of primordial curvature perturbations on large scales, with an approximately Gaussian distribution, is one of the most important observations that any model of the early universe should explain. The inflationary scenario offers a possible explanation in terms of vacuum fluctuations of scalar fields during an accelerated expansion preceding the standard hot big bang, though attempts to embed such a scenario within a fundamental theory such as superstring/M-theory may require some degree of fine-tuning (see [1] for a recent review).

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The ekpyrotic scenario [2] (see also [3, 4]) is one alternative approach where the large-scale perturbations are generated from vacuum fluctuations during a collapse phase driven by a scalar field with a steep, negative exponential potential. It was shown, however, that even though the Bardeen potential acquires a scale-invariant spectrum during ekpyrotic collapse, the comoving curvature perturbation has a steep blue spectrum in the original ekpyrotic scenario driven by a single scalar field [5]. If the contracting pre-big bang phase is connected to the expanding hot big bang phase through a regular four-dimensional bounce, then we expect the comoving curvature perturbation to remain constant for adiabatic perturbations on large scales [6, 7, 8], which means that the growing mode of curvature perturbations in the expanding phase also acquires a steep blue spectrum.

One way to avoid this is to consider non-adiabatic perturbations, which require two or more fields [9, 10]. Recently, there has been progress in generating a scale-invariant spectrum for curvature perturbations in the ekpyrotic scenario with more than one field, which we will refer to as the new ekpyrotic scenario [11, 12, 13]. If these fields have steep negative exponential potentials, there exists a scaling solution where the energy densities of the fields grow at the same rate during the collapse [14, 15]. In this multi-field scaling solution background, the isocurvature field perturbations have an almost scale-invariant spectrum [14], owing to a tachyonic instability in the isocurvature field. The multi-field scaling solution in the new ekpyrotic scenario can be shown to be an unstable saddle point in the phase space and the stable late-time attractor is the old ekpyrotic collapse dominated by a single field [16].

The existence of a tachyonic instability raises questions about initial conditions in the new ekpyrotic scenario [17], but the transition from the multi-field scaling solution to the single-field-dominated solution also provides a mechanism to automatically convert the initial isocurvature field perturbations about the multi-field scaling solution into comoving curvature perturbations about the late-time attractor [18]. In this case, the final amplitude of the comoving curvature perturbation is determined by the Hubble scale at the transition and if this parameter and the initial conditions are set appropriately, the prediction for the primordial curvature perturbations from this scenario is compatible with an almost scale-invariant spectrum (see [11, 12, 13, 19] for other mechanisms to convert the initial scale-invariant isocurvature perturbations into curvature perturbations).

Recently, the non-Gaussianity of the distribution of primordial curvature perturbations in the inflationary scenario has been extensively studied by many authors (see e.g. [20] for a review). Measurements of the non-Gaussianity already provide important constraints on specific models of the early universe and such measurements will continue to improve in the near future, for instance with the Planck satellite [21]. Since the non-Gaussian signal from single-field, slow-roll inflation is suppressed by slow-roll parameters to an undetectable level [22], if non-Gaussianities are detected then this would rule out many models of inflation.

Thus, as a natural extension of the study performed in [16, 18], in this paper we compute the non-Gaussianity of the primordial curvature perturbations generated from the contracting phase of the multi-field new ekpyrotic cosmology. We adopt the same specific model as the previous analysis of Refs. [16, 18] where the scale-invariant curvature perturbation is generated by the transition from a scaling solution, with two fields driving the collapse, to a single-field dominated regime. We assume that the ekpyrotic collapse is subsequently converted to expansion by a regular bounce, during which the comoving curvature perturbation is conserved on large scales and the curvature perturbation generated during the collapse is thus directly related to the amplitude of the observed primordial density perturbation.

This paper is organized as follows. In Sec. 2 we briefly review the background dynamics. In Sec. 3 we summarize the statistical properties of linear and non-linear perturbations. The δN-formalism is introduced which is used to compute the primordial curvature perturbation. In Sec. 4 we study linear perturbations while in Sec. 5 we generalize this study to non-linear perturbations and quantify the expected non-Gaussianity. In Sec. 6 we draw our conclusions. In three appendices we review previous results for the linear fluctuations of the isocurvature field during the ekpyrotic phase and calculate its intrinsic non-Gaussianity using the interaction Hamiltonian, as well as presenting a numerical check of our analytical results.

2 Homogeneous dynamics

We first review the background dynamics of the fields in the new ekpyrotic cosmology with multiple scalar fields. During the ekpyrotic collapse the contraction of the universe is assumed to be described by a 4D Friedmann equation in the Einstein frame with n scalar fields with negative exponential potentials

$$3H^2 = V + \sum_j^n \frac{1}{2}\dot{\phi}_j^2 \,, \tag{1}$$

where

$$V = -\sum_j^n V_j e^{-c_j \phi_j} \,, \tag{2}$$

and we take $V_i > 0$ and set $8\pi G$ equal to unity.

From now on, for simplicity, we concentrate our attention on the case of two fields. In this case, it will be easier to work in terms of new variables [16],

$$\varphi = \frac{c_2\phi_1 + c_1\phi_2}{\sqrt{c_1^2+c_2^2}} \,, \quad \chi = \frac{c_1\phi_1 - c_2\phi_2}{\sqrt{c_1^2+c_2^2}} \,, \tag{3}$$

corresponding to a fixed rotation in field space. The potential given by Eq. (2) can then be simply re-written as [14, 16, 23]

$$V = -U(\chi)\, e^{-c\varphi} \,, \tag{4}$$

where

$$\frac{1}{c^2} \equiv \sum_j \frac{1}{c_j^2} \,, \tag{5}$$

and the potential for the orthogonal field is given by

$$U(\chi) = V_1\, e^{-(c_1/c_2)c\chi} + V_2\, e^{(c_2/c_1)c\chi} \,, \tag{6}$$

which has a minimum at

$$\chi = \chi_0 \equiv \frac{1}{\sqrt{c_1^2+c_2^2}} \ln\left(\frac{c_1^2 V_1}{c_2^2 V_2}\right) \,. \tag{7}$$

If we expand $U(\chi)$ in Eq. (6) about its minimum we obtain

$$U(\chi) = U_0 \left[1 + \frac{c^2}{2}(\chi - \chi_0)^2 + \frac{\tilde{c}c^2}{6}(\chi - \chi_0)^3 + \ldots \right], \tag{8}$$

where

$$\tilde{c} \equiv \frac{c_2^2 - c_1^2}{\sqrt{c_1^2 + c_2^2}}. \tag{9}$$

The multi-field scaling solution corresponds to the classical solution along this minimum $\chi = \chi_0$, while φ is rolling down the exponential potential. The explicit form of the multi-field scaling solution is given as

$$a = (-t)^p \,, \tag{10}$$

$$\varphi = \frac{2}{c}\ln(-t) - \frac{1}{c}\ln\left(\frac{p(1-3p)}{U_0}\right) , \tag{11}$$

where $p = \sum_j 2/c_j^2 = 2/c^2$. The potential for χ has a negative mass-squared around $\chi = \chi_0$,

$$m_\chi^2 \equiv \frac{\partial^2 V}{\partial \chi^2} = c^2 V < 0 \,, \tag{12}$$

and thus χ represents the instability direction. Furthermore, the χ field evolution is nonlinear, with the cubic interaction being given by

$$V^{(3)} \equiv \frac{\partial^3 V}{\partial \chi^3} = \tilde{c} m_\chi^2 \,, \tag{13}$$

which becomes important when we consider the non-Gaussianity later in this paper.

If the initial condition for χ is slightly different from χ_0 or $\dot{\chi}$ is not zero, then χ starts rolling down the potential and the solution approaches a single-field-dominated scaling solution. The explicit form of the single-field-dominated scaling solution is given as

$$a = (-t)^{p_j} \,, \tag{14}$$

$$\phi_j = \frac{2}{c_j}\ln(-t) - \frac{1}{c_j}\ln\left(\frac{p_j(1-3p_j)}{V_j}\right) , \tag{15}$$

where $p_j = 2/c_j^2$. In this paper, we consider the case in which the background evolves from the multi-field scaling solution to the ϕ_2-dominated scaling solution without loss of generality.

3 Statistical correlators

Here we briefly summarize the statistical properties of the scalar field fluctuations during ekpyrotic collapse. Then, in order to link these to the observable primordial curvature perturbation we discuss the δN-formalism.

In the two-field new ekpyrotic cosmology, the isocurvature fluctuations acquired by the field χ during the multi-field scaling regime, play a crucial role to generate a scale-invariant spectrum of perturbations. On the other hand, the fluctuations of the field φ are negligible on large scales, because of its very blue spectral tilt [11, 12, 13]. Thus, in the following we neglect $\delta\varphi$ fluctuations.

In linear perturbation theory, when interactions are neglected, the free-field fluctuations $\delta\chi$ are Gaussian. Their power spectrum $\mathcal{P}_\chi$ can be defined by

$$\langle \delta\chi_{\mathbf{k}_1} \delta\chi_{\mathbf{k}_2} \rangle \equiv (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2) \frac{2\pi^2}{k_1^3} \mathcal{P}_\chi(k_1) \,, \tag{16}$$

where the angle brackets denote an ensemble average.

If the cubic self-interaction in Eq. (8) is taken into account $\delta\chi$ is no longer Gaussian and the first signal of non-Gaussianity comes from the three-point correlation function. Similarly to Eq. (16), the bispectrum of $\delta\chi$, B_χ, is defined by

$$\langle \delta\chi_{\mathbf{k}_1} \delta\chi_{\mathbf{k}_2} \delta\chi_{\mathbf{k}_3} \rangle \equiv (2\pi)^3 \delta^3(\sum_j \mathbf{k}_j) B_\chi(k_1, k_2, k_3) \,. \tag{17}$$

To characterize the bispectrum one can also define the nonlinear parameter of the field fluctuation, f_{NL}^χ, as

$$\frac{6}{5} f_{NL}^\chi \equiv \frac{\prod_j k_j^3}{\sum_j k_j^3} \frac{B_\chi}{4\pi^4 \mathcal{P}_\chi^2} \,. \tag{18}$$

When the intrinsic non-Gaussianity is local in real space this parameter is k independent and $\delta\chi$ can be written as

$$\delta\chi = \delta\chi_L + \frac{3}{5} f_{NL}^\chi \delta\chi_L^2 \,, \tag{19}$$

where $\delta\chi_L$ is the linear and Gaussian part of the field fluctuations.

To relate the non-Gaussianity of the scalar field fluctuations to observations, we need to calculate the three-point functions of the comoving curvature perturbation ζ. In order to do that, we can use the δN-formalism [24, 25, 26, 27, 28, 29]. In the δN-formalism, the comoving curvature perturbation ζ evaluated at some time $t = t_f$ coincides with the perturbed expansion integrated from an initial $flat$ hypersurface at $t = t_i$, to a final *uniform density* hypersurface at $t = t_f$, with respect to the background expansion, i.e.,

$$\zeta(t_f, \mathbf{x}) \simeq \delta N(t_f, t_i, \mathbf{x}) \equiv \mathcal{N}(t_f, t_i, \mathbf{x}) - N(t_f, t_i) \,, \tag{20}$$

with

$$\mathcal{N}(t_f, t_i, \mathbf{x}) \equiv \int_{t_i}^{t_f} \mathcal{H}(\mathbf{x}, t) dt \,, \quad N(t_f, t_i) \equiv \int_{t_i}^{t_f} H(t) dt \,, \tag{21}$$

where $\mathcal{H}(\mathbf{x}, t)$ is the inhomogeneous Hubble expansion. We can calculate δN on large scales using the homogeneous equations of motion, in the assumption that the local expansion on sufficiently large scales behaves like a locally homogeneous and isotropic universe, according to the so-called "separate universe" approach [30, 31, 6]. This allows us to compute the full nonlinear curvature perturbation in the large-scale limit. Note that we leave the initial time t_i unspecified and we are free to identifying it with the time of Hubble crossing $t = t_*$, i.e. the time when a mode k exits the Hubble radius during inflation, $k = aH$, or with a later time.

We will chose the initial time t_i to be *during* the multi-field scaling regime. Furthermore, since φ is unperturbed, δN can be expanded in series of the initial field fluctuations $\delta\chi_i$. Retaining only terms up to second order, we obtain

$$\delta N = N_{,\chi_i}\delta\chi_i + \frac{1}{2}N_{,\chi_i\chi_i}(\delta\chi_i)^2\,, \tag{22}$$

where $N_{,\chi}$ denotes the derivative of N with respect to χ.

Now we can convert the higher-order information about the initial field fluctuations into the statistical properties of the observed primordial curvature perturbations. The power spectrum of the comoving curvature perturbation ζ, $\mathcal{P}_\zeta$, is defined as

$$\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\rangle \equiv (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\frac{2\pi^2}{k_1^3}\mathcal{P}_\zeta(k_1)\,. \tag{23}$$

At lowest order, from Eqs. (16), (22) and (23), $\mathcal{P}_\zeta$ is expressed as

$$\mathcal{P}_\zeta = N_{,\chi_i}^2\mathcal{P}_{\chi_i}\,. \tag{24}$$

Note that $N_{,\chi_i}$ is independent of wavenumber k and hence the scale dependence of the primordial spectrum, $\Delta n \equiv d\ln\mathcal{P}_\zeta/d\ln k$, is given by the spectral tilt of the field fluctuations, Δn_χ, on the initial hypersurface [given in Eq. (69) in Appendix A].

The bispectrum of the curvature perturbation ζ, which includes the first signal of non-Gaussianity, is defined as

$$\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle \equiv (2\pi)^3\delta^{(3)}(\sum_j \mathbf{k}_j)B_\zeta(k_1,k_2,k_3)\,, \tag{25}$$

where the left hand side of Eq. (25) can be evaluated by the δN-formalism using Wick's theorem,

$$\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = N_{,\chi_i}^3\langle\delta\chi_{i\mathbf{k}_1}\delta\chi_{i\mathbf{k}_2}\delta\chi_{i\mathbf{k}_3}\rangle + \frac{1}{2}N_{,\chi_i}^2N_{,\chi_i\chi_i}\langle\delta\chi_{i\mathbf{k}_1}\delta\chi_{i\mathbf{k}_2}(\delta\chi_i\star\delta\chi_i)_{\mathbf{k}_3}\rangle + \text{perms}\,. \tag{26}$$

In the above equation, a star $\star$ denotes the convolution and we have neglected correlators higher than the four-point.

Observational limits on the non-Gaussianity of the primordial curvature perturbations are usually given on the nonlinear parameter f_{NL} defined by [22]

$$\frac{6}{5}f_{NL} \equiv \frac{\prod_j k_j^3}{\sum_j k_j^3}\frac{B_\zeta}{4\pi^4\mathcal{P}_\zeta^2}\,. \tag{27}$$

If the non-Gaussianity is local, one can write ζ as

$$\delta N = \zeta_L + \frac{3}{5}f_{NL}\zeta_L^2\,, \tag{28}$$

where ζ_L is a Gaussian variable.

To compute the bispectrum of the curvature perturbation one can use the δN-formalism and after some manipulations, from Eqs. (25) and (26) one finds

$$B_\zeta(k_1,k_2,k_3) = N^3_{,\chi_i} B_{\chi_i}(k_1,k_2,k_3) + 4\pi^4 \mathcal{P}^2_\zeta \frac{\sum_j k_j^3}{\prod_j k_j^3} \frac{N_{,\chi_i\chi_i}}{N^2_{,\chi_i}}\,, \tag{29}$$

where $B_{\chi_i}(k_1,k_2,k_3)$ is the bispectrum of the scalar field evaluated at $t = t_i$. Together with Eq. (27) the nonlinear parameter f_{NL} becomes

$$f_{NL} = \frac{f^{\chi_i}_{NL}}{N_{,\chi_i}} + \frac{5}{6}\frac{N_{,\chi_i\chi_i}}{N^2_{,\chi_i}}\,, \tag{30}$$

where we have used Eqs. (19) and (27). The first term on the right hand side of Eq. (30),

$$f^{(3)}_{NL} \equiv \frac{f^{\chi_i}_{NL}}{N_{,\chi_i}}\,, \tag{31}$$

comes from the intrinsic three-point correlation functions of the field fluctuations and contains also the non-Gaussianity of *quantum* origin generated on small-scales, i.e., *inside* the Hubble radius during the ekpyrotic collapse described by the multi-field scaling solutions. The second term on the right hand side of Eq. (30),

$$f^{(4)}_{NL} \equiv \frac{5}{6}\frac{N_{,\chi_i\chi_i}}{N^2_{,\chi_i}}\,, \tag{32}$$

is completely momentum independent and *local* in real space, because it is due to the evolution of nonlinearities *outside* the Hubble radius during the multi-field ekpyrotic collapse. Note that the splitting in $f^{(3)}_{NL}$ and $f^{(4)}_{NL}$ depends on the time t_i.

4 Linear curvature perturbation

The power spectrum of χ is given by (see [18] and Eq. (70) in Appendix A)

$$\mathcal{P}_\chi(k) = \epsilon^2 \left(\frac{H}{2\pi}\right)^2\,, \tag{33}$$

in the limit of large ϵ, where the fast-roll parameter ϵ is defined as $\epsilon \equiv -\dot{H}/H^2 = c^2/2 = 1/p$, and it is constant for the multi-field scaling solution. Thus, in this limit, $\mathcal{P}_\chi$ has a scale invariant spectrum.

On the other hand, as shown in [18] in the fast-roll limit and assuming an *instantaneous transition* from the multi-field scaling solution to the single-field ϕ_2- dominated scaling solution, the power spectrum of the final (after the transition) curvature perturbations becomes

$$\mathcal{P}_\zeta = \frac{\epsilon^2}{c_1^2 + c_2^2}\left|\frac{H_T}{2\pi}\right|^2\,, \tag{34}$$

where H_T is the Hubble parameter evaluated at the transition time $t = t_T$. The scalar spectral index of ζ is

$$\Delta n = 4\left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right) . \tag{35}$$

Note that it is always positive, i.e., the power spectrum is always blue

Let us now interpret the result above in the light of the δN-formalism. We consider the situation in which χ_i is perturbed on the $t = t_i$ hypersurface, while H_i assumes on this hypersurface a constant value. This is justified by the fact that the $t = t_i$ hypersurface is flat and since χ is an isocurvature field its fluctuations do not affect the local Hubble expansion. Furthermore, we assume that the transition into the single-field-dominated scaling solution at the time $t = t_T$, happens *instantaneously* on the hypersurface $\chi = \chi_T = \text{const.}$, where H_T is perturbed.

Under these assumptions, the expansion N defined by Eq. (21) can be split into

$$N = \int_{t_i}^{t_T} H dt + \int_{t_T}^{t_f} H dt \, , \tag{36}$$

where t_f is set sufficiently later than the transition time t_T. In Eq. (36), the first integral is over the multi-field scaling evolution and the last integral is over the ϕ_2-dominated phase.

Since the multi-field scaling solution is characterized by Eq. (10), the first term on the right hand side of Eq. (36) can be expressed as $(1/\epsilon)\ln(H_i/H_T)$. Similarly, since the single-field dominated scaling solution is characterized by Eq. (14), the second term becomes $(1/\epsilon_2)\ln(H_T/H_f)$, where $\epsilon_2 = c_2^2/2$. Then, for a fixed t_i and t_f, the expansion N can be expressed as

$$N = -\frac{2}{c_1^2}\ln|H_T| + \text{const.}, \tag{37}$$

which depends only on the parameter c_1, besides the transition time t_T.

During the multi-field scaling regime, the linear evolution equation of χ on large scales is given by

$$\ddot{\chi} + 3H\dot{\chi} + m_\chi^2\chi = 0 \, , \tag{38}$$

where the mass of χ is defined in Eq. (12). During the multi-field scaling the evolution of χ is dominated by the tachyonic mass and thus $3H\dot{\chi} \ll m_\chi^2\chi$. With $m_\chi^2 = -2/t^2$ (see the appendix, Sec. A), one finds $\chi \propto 1/t \propto H$, and thus

$$H_T = H_i\frac{\chi_T}{\chi_i}. \tag{39}$$

Using this relation one can derive N with respect to χ_i and obtain

$$N_{,\chi_i} = \frac{2}{c_1^2\chi_i} = \frac{2}{c_1^2\chi_T}\frac{H_T}{H_i}. \tag{40}$$

In particular, by using this equation and comparing Eq. (24) with Eq. (34), one obtains the value of χ_T, i.e.,

$$\chi_T = \frac{2\sqrt{c_1^2 + c_2^2}}{c_1^2}. \tag{41}$$

In appendix C, we have calculated $N_{,\chi_i}$ numerically and checked Eq. (41).

5 Non-Gaussianities

In this section we compute the non-Gaussianity of the primordial curvature perturbation generated by the nonlinear dynamics during multi-field ekpyrotic collapse. We will use the δN-formalism to calculate $\delta N(\chi)$ in the instantaneous transition approximation. We will thus consider quadratic terms in $\delta\chi_i$ in the δN-expansion (22) and, in contrast with Sec. 4, we also allow for non-linear evolution of χ on large scales. For another example of the multi-field calculation of non-Gaussianity from fields with exponential potential see [32], even though in the context of assisted inflation.

Including the cubic self-interaction $V^{(3)}$ given in Eq. (13), the large scale evolution equation for χ in the multi-field scaling regime becomes

$$\ddot{\chi} + 3H\dot{\chi} + m_\chi^2\chi = -\frac{1}{2}\tilde{c}m_\chi^2\chi^2. \tag{42}$$

The above evolution equation can be solved perturbatively. Given the solution to the linear equation (38), i.e., $\chi_L \propto H$, the growing-mode solution for χ is

$$\chi = \chi_L + \frac{1}{4}\tilde{c}\chi_L^2 = \alpha H + \frac{1}{4}\tilde{c}\alpha^2 H^2\,, \tag{43}$$

where α is a constant parameter whose value distinguishes the different trajectories. Note that this is a *perturbative* result, i.e., it is valid only as long as $\tilde{c}\chi \ll 1$. However, since χ grows during the collapse, unless prevented by a bouncing phase, eventually this condition is violated.

For perturbations in the value of χ on a hypersurface of uniform H we have

$$\delta\chi = \left(1 + \frac{1}{2}\tilde{c}\alpha H\right) H\delta\alpha + \frac{1}{4}\tilde{c}H^2(\delta\alpha)^2\,. \tag{44}$$

The non-linear self-interaction of χ in Eq. (41) grows in time on super-Hubble scales, and thus the intrinsic non-Gaussianity of χ increases. Therefore, unless we take $t_{\rm i}$ sufficiently early that we can neglect these nonlinearities, we cannot naively use Eq. (39), which is only valid at linear order, and its derivative with respect to χ_i, to estimate the non-Gaussianity of the curvature perturbation. We need to work with a variable that is as close as possible to a Gaussian. It turns out that it is convenient to choose α as such a variable. Indeed, if we assume that $\delta\chi_L$, the perturbation of the solution of the linear equation (38), is Gaussian, then also $\delta\alpha$ is a Gaussian random variable because $\delta\alpha = \delta\chi_L/H$. Comparing this equation with Eq. (19) we find that the intrinsic non-Gaussianity of χ is of local form and time independent, and it is given by

$$f_{NL}^{\chi} = \frac{5}{12}\tilde{c}. \tag{45}$$

In Sec. B of the appendix, we have checked that this result agrees with the one obtained using the approach of Maldacena [22] with an interaction Hamiltonian containing $V^{(3)}$ [13], and confirms that we can consistently assume $\delta\chi_L$, and thus $\delta\alpha$, to be Gaussian. The reason for this is that the classical nonlinear evolution on large scales completely dominates over the sub-Hubble nonlinear interactions of quantum nature.

We will now compute the non-Gaussianity of the curvature perturbation using Eqs. (30–32). The final result for f_{NL} will be manifestly independent of the time we choose to evaluate $\delta\chi_i$ although the splitting into $f_{NL}^{(3)}$ and $f_{NL}^{(4)}$ is dependent on t_i.

Since $\delta\alpha$ can be assumed to be Gaussian, the simplest way to compute f_{NL} is to calculate the δN corresponding to the fluctuation $\delta\alpha$, i.e.,

$$\delta N = N_{,\alpha}\delta\alpha + \frac{1}{2}N_{,\alpha\alpha}(\delta\alpha)^2 \,. \tag{46}$$

In order to compute $N_{,\alpha}$ and $N_{,\alpha\alpha}$ we want to use Eq. (37), and for this we need to know how H_T varies as a function of α at the transition from multi-field scaling to single-field ϕ_2-dominated scaling solution. Inverting Eq. (43) (to leading order in $\tilde{c}\chi$) gives

$$\alpha = \frac{\chi}{H}\left(1 - \frac{1}{4}\tilde{c}\chi\right) \,. \tag{47}$$

Assuming as in the linear case that the transition corresponds to a critical value of the tachyon field $\chi = \chi_T$, on the transition surface (constant χ_T) we have from (47) that $\alpha \propto H_T^{-1}$ and hence we find

$$\delta N = \frac{2}{c_1^2}\frac{\delta\alpha}{\alpha} - \frac{1}{c_1^2}\left(\frac{\delta\alpha}{\alpha}\right)^2 \,, \tag{48}$$

which means

$$N_{,\alpha} = \frac{2}{c_1^2}\frac{1}{\alpha}, \qquad N_{,\alpha\alpha} = -\frac{2}{c_1^2}\frac{1}{\alpha^2} \,. \tag{49}$$

Taking $\delta\alpha$ to be a Gaussian random variable and comparing with Eq. (28) with $\zeta_L = -2\delta\alpha/(c_1^2\alpha)$ we obtain the nonlinear parameter for the curvature perturbation after the transition:

$$f_{NL} = \frac{5}{6}\frac{N_{,\alpha\alpha}}{N_{,\alpha}^2} = -\frac{5}{12}c_1^2 \,. \tag{50}$$

This is our main result. The non-Gaussianity is given in terms of c_1, where $-V_1\exp(-c_1\phi_1)$ is the potential of the field ϕ_1 which remains subdominant after the transition from the multi-field scaling to the single-field dominated regime. The non-Gaussianity is of local form. This is due to the fact that it is generated by the nonlinear super-Hubble evolution.

Equation (50) includes also the non-linear growth of the tachyon field on large scales due to its self-interaction. In order to see this, we can compute $f_{NL}^{(3)}$ and $f_{NL}^{(4)}$ defined in Eqs. (31) and (32). Thus, we have to identify the linear and non-linear dependence of δN on the field values $\delta\chi_i$ on the initial hypersurface.

Replacing Eq. (45) in Eq. (31) we obtain

$$f_{NL}^{(3)} = \frac{5}{12}\frac{\tilde{c}}{N_{,\chi_i}} \,. \tag{51}$$

Furthermore, from Eqs. (47) and (48) we have

$$N_{,\chi_i} = \frac{dN}{d\alpha}\frac{d\alpha}{d\chi_i} = \frac{2}{c_1^2\chi_i}\left(1 - \frac{1}{4}\tilde{c}\chi_i\right) \,, \tag{52}$$

and hence

$$f_{NL}^{(3)} = \frac{5}{24}c_1^2\tilde{c}\chi_i \tag{53}$$

$$= \frac{5}{12}(c_2^2 - c_1^2)\frac{H_i}{H_T}\,, \tag{54}$$

where for the last equality we have replaced $\tilde{c}$ using its definition, Eq. (9), expressed χ_i in terms of χ_T using the linear relation Eq. (39), and replace χ_T using Eq. (41).

Secondly we have the contribution due to

$$N_{,\chi_i\chi_i} = \frac{d^2N}{d\alpha^2}\left(\frac{d\alpha}{d\chi_i}\right)^2 + \frac{dN}{d\alpha}\frac{d^2\alpha}{d\chi_i^2} = -\frac{2}{c_1^2\chi_i^2}\,. \tag{55}$$

Note that this relation is valid to linear order in $\tilde{c}\chi_i$, and thus contains also the nonlinear self-interaction of χ generating non-Gaussianities after t_i. Using Eq. (32), this gives

$$f_{NL}^{(4)} = -\frac{5}{12}c_1^2\left(1 + \frac{1}{2}\tilde{c}\chi_i\right) \tag{56}$$

$$= -\frac{5}{12}c_1^2 - \frac{5}{12}(c_2^2 - c_1^2)\frac{H_i}{H_T}\,. \tag{57}$$

Both $f_{NL}^{(3)}$ and $f_{NL}^{(4)}$ depend upon the choice of the initial hypersurface and thus of χ_i. However the total f_{NL} comes from the sum of the two terms, it is independent of t_i, and is given by Eq. (50).

In appendix C, we calculated $N_{,\alpha}$ and $N_{,\alpha\alpha}$ numerically and confirm that the result Eq. (50) obtained by the instantaneous transition and the fast roll approximations is satisfied with good accuracy.

6 Conclusion

In this paper we have studied the nonlinear evolution of perturbations in the multi-field new ekpyrotic cosmology. If one sets the model parameters and initial conditions appropriately, then the prediction for the power spectrum of curvature perturbations produced in multi-field ekpyrotic collapse can be constrained by present observations [11, 12, 13, 18]. In order to distinguish this model from other early universe scenarios, such as inflation, by future observations, it is important to estimate the non-Gaussianity of the curvature perturbations.

We have studied the simplest model based on two fields with exponential potentials and considered the specific scenario in which the nearly scale-invariant comoving curvature perturbation is generated by the transition from the multi-field scaling solution to the single-field dominated attractor solution. We have applied the δN-formalism, which is widely adopted to study the non-linearity of the primordial curvature perturbation from inflation [28], to ekpyrotic cosmology. We identify the non-linear curvature perturbation on uniform-density hypersurfaces at late times with the perturbed local expansion, δN, with respect to an initial spatially flat hypersurface. The primordial non-Gaussianity parameter f_{NL} is a sum of two contributions: $f_{NL}^{(3)}$, defined in Eq. (31), comes from the intrinsic three-point function of the isocurvature field perturbation $\delta\chi$ on an initial hypersurface, and $f_{NL}^{(4)}$, defined in Eq. (32), originates from the nonlinear relation between the primordial curvature perturbation and the isocurvature field perturbations during multi-field ekpyrotic collapse. It should be emphasized that although the decomposition of f_{NL} into $f_{NL}^{(3)}$ and $f_{NL}^{(4)}$ may be convenient, it is unphysical and depends upon the choice of the initial time t_i. However, we show that the physical quantity, the total f_{NL}, is independent of t_i.

Both the multi-field and single-field ekpyrotic solutions are power-law solutions. We find a general result that in case of a sudden transition between two power-law solutions the local expansion is only a function of the Hubble rate at the transition, H_T. Thus the calculation of the primordial curvature perturbation in our model reduces to finding the perturbation of H_T for different trajectories in phase-space, and hence the local value of the isocurvature field χ on the initial spatially flat hypersurface.

We find that after the transition to the single-field attractor solution the non-Gaussian parameter $f_{NL} = -5c_1^2/12$, where $-V_1 \exp(-c_1\phi_1)$ is the potential of the field ϕ_1 which becomes subdominant at late time. Since the non-Gaussianity is mainly generated by the nonlinear super-Hubble evolution, it is of the local form, and the nonlinear parameter is k independent. We show in appendix B that the contribution of the intrinsic non-Gaussianity of the isocurvature field perturbations on sub-Hubble scales is subdominant.

We have checked our analytical results by calculating the expansion, N, numerically in appendix C. We confirmed that when the fast-roll parameter satisfies $400 > \epsilon > 25$ the analytic estimation is accurate within at a few % level. The discrepancy arises from the breakdown of both the fast-roll approximation and the sudden transition approximation.

A negative value of $f_{\rm NL}$ is much more tightly constrained by current observations than a positive value. For instance, if we choose $c_1 = 5$ we obtain a nonlinear parameter $f_{\rm NL} \approx -10$ that is marginally consistent with current constraints on the non-Gaussianity of the primordial density perturbation [33, 34]. However such a small value of c_1 leads to too large a value of the spectral index (35), $\Delta n > +0.16$, which is excluded by observations [33].

Corrections to the exact exponential potentials that we have studied in this paper have been proposed [11, 12] to produce a red spectrum of perturbations. Indeed corrections are also required to stabilise the multi-field scaling solution at early times (before observables scales exit the Hubble scale) [17]. We assume that the classical background solution starts close to the multi-field scaling solution. However, as the multi-field scaling solution is a saddle point in the phase space, we need some preceding phase that initially drives the classical background solution to the saddle point. It is important to check whether such corrections will also affect the non-linear evolution of the field perturbations during the collapse phase, which could lead to additional sources of primordial non-Gaussianity.

Modifications to the effective action are certainly required at high energies to turn contraction to expansion before the collapse phase reaches the big crunch singularity [35, 11, 12, 13]. In this paper, we assumed that the comoving curvature perturbation is conserved on super-Hubble scales through such a non-singular bounce, neglecting the effect of non-adiabatic perturbations which are expected to rapidly decay about the single-field attractor solution. In this case the power spectrum and the non-Gaussian parameter Eq. (50) that we have calculated in the single-field dominated collapse are directly related to those observed perturbations in the expanding, hot big bang phase. It is certainly possible to convert the initial isocurvature field perturbations to curvature perturbations through a different mechanism, e.g., via the bounce [11, 12, 13, 17]. In this case the non-linear dependence of the expansion, δN, upon the initial field perturbations may be different, but the non-linear growth on super-Hubble scales of the isocurvature field perturbations about the multi-field scaling solution is still expected to lead to a large non-Gaussianity [13, 17]. Further work is required to quantify the non-Gaussianity predicted in these models.

Note added: This arXiv version of our article includes corrections made after publication of the journal article. We give the correct expressions for several equations that have an incorrect sign. This incorrect sign is due to a sign error in the integrated expansion N in Eq. (37), which then propagates into other equations. As a consequence, the nonlinear parameter $f_{\rm NL}$ in this new ekpyrotic scenario turns out to be negative and it is much more tightly constrained by current data than it would have been with a positive sign. By combining the present observational constraints on $f_{\rm NL}$ and the scalar spectral index, the specific model studied in this paper is thus ruled out by current observational data.

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A Linear field fluctuations

Here, we briefly summarise the results obtained in the previous works [11, 12, 13, 14, 16] about the generation and the evolution of linear fluctuations of the χ field about the background multi-field scaling solution.

In the multi-field scaling solution $\delta\chi$ coincides with entropy field perturbation, which is automatically gauge-invariant and does not couple with gravity at first order. Since there is no coupling between the adiabatic and the entropy field perturbations, neglecting the non-linear self-interactions of the entropy field which are negligible at sufficiently early times, the equation of motion for $\delta\chi$ becomes simply the equation of a free massive field in an unperturbed FRW metric,

$$\ddot{\delta\chi} + 3H\dot{\delta\chi} + \left(\frac{k^2}{a^2} + m_\chi^2\right)\delta\chi = 0\,, \tag{58}$$

where m_χ^2 has been defined in Eq. (12). Introducing the rescaled field $v \equiv a\delta\chi$, and writing the wave equation in terms of conformal time $\tau \equiv \int dt/a$, we have

$$v_{,\tau\tau} + \left[k^2 - \frac{a_{,\tau\tau}}{a} + m_\chi^2 a^2\right] v = 0\,, \tag{59}$$

where $(\ldots)_{,\tau}$ denotes the derivative with respect to τ.

For the multi-field scaling solutions, we can show that the following relations hold,

$$aH = \frac{1}{(\epsilon - 1)\tau}, \tag{60}$$

$$\frac{a_{,\tau\tau}}{a} = -(\epsilon - 2)a^2 H^2, \tag{61}$$

$$m_\chi^2 = 2\epsilon(3-\epsilon)H^2\,. \tag{62}$$

Thus, Eq. (58) becomes

$$v_{,\tau\tau} + \left[k^2 + \frac{\epsilon + 3\eta_\chi - 2}{(\epsilon-1)^2\tau^2}\right] v = 0\,, \tag{63}$$

where $\eta_\chi \equiv m_\chi^2/(3H^2)$.

Using the usual Bunch-Davies vacuum state to normalise the amplitude of the fluctuations on small scales, we obtain

$$v = \frac{\sqrt{\pi}}{2}\frac{e^{i(\nu+1/2)\frac{\pi}{2}}}{k^{1/2}}(-k\tau)^{1/2} H_\nu^{(1)}(-k\tau)\,, \tag{64}$$

where the order of the Hankel function is given by

$$\nu^2 = \frac{9}{4} - \frac{3\epsilon}{(\epsilon-1)^2}\,. \tag{65}$$

At late times, $-k\tau \to 0$, making use of the asymptotic form of the Hankel function,

$$H_\nu^{(1)}(-k\tau) \to -i\frac{\Gamma(\nu)}{\pi}\left(\frac{-k\tau}{2}\right)^{-\nu}\,, \tag{66}$$

the power spectrum of $\delta\chi$ in this limit becomes

$$\mathcal{P}_\chi = C_\nu^2 \frac{k^2}{a^2}(-k\tau)^{1-2\nu}\,, \tag{67}$$

where $C_\nu \equiv 2^{\nu-3/2}\Gamma(\nu)/\pi^{3/2}$. Then the spectral tilt of the generated fluctuations is

$$\Delta n_\chi \equiv \frac{d\ln \mathcal{P}_\chi}{d\ln k} = 3 - 2\nu\,, \tag{68}$$

and to leading order in a fast-roll expansion,

$$\Delta n_\chi \simeq \frac{2}{\epsilon}\,. \tag{69}$$

Thus, for a steep exponential potential we obtain a slightly blue spectrum, becoming scale-invariant as $\epsilon \to \infty$.

Approximating $\nu \simeq 3/2$, on large scales we can relate the amplitude of $\delta\chi$ to H as

$$\mathcal{P}_\chi^{1/2} = \epsilon\frac{|H|}{2\pi}\,. \tag{70}$$

B Isocurvature field bispectrum

In this section we calculate the intrinsic non-Gaussianity of the field χ, using the approach of Maldacena [22], which also includes the contribution from sub-Hubble nonlinear interactions. Following [22], the three-point correlator is given by

$$\langle \delta\chi^3(t) \rangle = -i \int_{-\infty}^{t} dt \langle [\delta\chi^3(t), H_{int}(t')] \rangle \,, \tag{71}$$

where H_{int} is the interaction Hamiltonian. Here we consider only the cubic interaction, which is the dominant one for the three-point function,

$$H_{int}(t') = \int d^3x a^3 \frac{V^{(3)}}{3!} \delta\chi^3 \,, \tag{72}$$

where $V^{(3)}$ is given in Eq. (13).

Writing Eq. (71) in Fourier space,

$$\begin{aligned} \langle \delta\chi_{\mathbf{k}_1}(t)\delta\chi_{\mathbf{k}_2}(t)\delta\chi_{\mathbf{k}_3}(t) \rangle &= (2\pi)^3 \delta(\sum_j \mathbf{k}_j) \times \\ 2Re\left(-i\delta\chi_{\mathbf{k}_1}(t)\delta\chi_{\mathbf{k}_2}(t)\delta\chi_{\mathbf{k}_3}(t) \int_{-\infty}^{t} dt' a^3 V^{(3)} \delta\chi^*_{\mathbf{k}_1}(t')\delta\chi^*_{\mathbf{k}_2}(t')\delta\chi^*_{\mathbf{k}_3}(t') \right) \,, \end{aligned} \tag{73}$$

and using the normalised free field solution

$$\delta\chi_{\mathbf{k}} = \frac{1}{a\sqrt{2k}} e^{-ik\tau} \left(1 - \frac{i}{k\tau} \right) \tag{74}$$

valid in the limit $p \ll 1$, we get

$$\begin{aligned} \langle \delta\chi_{\mathbf{k}_1}(t)\delta\chi_{\mathbf{k}_2}(t)\delta\chi_{\mathbf{k}_3}(t) \rangle &= -(2\pi)^3 \delta(\sum_j \mathbf{k}_j) \frac{V^{(3)}}{4H^2} \frac{\epsilon^2 H^4}{\prod_j k_j^3} \times \\ Re\Big[i\left(1 + ik_1\tau\right)\left(1 + ik_2\tau\right)\left(1 + ik_3\tau\right) e^{-i\sum_j k_j \tau} \\ \tau \int_{-\infty}^{\tau} \frac{d\tau'}{\tau'^5} \left(1 - ik_1\tau'\right)\left(1 - ik_2\tau'\right)\left(1 - ik_3\tau'\right) e^{i\sum_j k_j\tau'} \Big] \,, \end{aligned} \tag{75}$$

where we have taken out $V^{(3)}$ from the time integral using that $V^{(3)}/H^2$ is constant.

Taking into account only the leading contribution in $-k_j\tau \ll 1$, the last two lines of the above equation yield $\sum_j k_j^3/4$. Furthermore, using also Eqs. (13) and (62) to replace $V^{(3)}$, we finally find

$$\langle \delta\chi_{\mathbf{k}_1}(t)\delta\chi_{\mathbf{k}_2}(t)\delta\chi_{\mathbf{k}_3}(t) \rangle = (2\pi)^3 \delta(\sum_j \mathbf{k}_j) \frac{\sum_j k_j^3}{\prod_j k_j^3} \frac{\tilde{c}}{8} \epsilon^4 H^4 \,. \tag{76}$$

The intrinsic non-Gaussianity after Hubble-exit is thus of the local form, as it is dominated by the super-Hubble evolution. Note that using $\epsilon = 1/p$ and $H = -p/t$ we can rewrite the last term in the equation above as

$$\frac{\tilde{c}}{8} \epsilon^4 H^4 = \frac{\tilde{c}}{8t^4}. \tag{77}$$

Thus, for $\tilde{c} = 1/M$ we recover the result of [13]. With the definition of the power spectrum for $\delta\chi$, Eq. (33), we can rewrite Eq. (76) as

$$\langle \delta\chi_{\mathbf{k}_1}(t)\delta\chi_{\mathbf{k}_2}(t)\delta\chi_{\mathbf{k}_3}(t)\rangle = (2\pi)^3\delta(\sum_j \mathbf{k}_j)\frac{\sum_j k_j^3}{\prod_j k_j^3}2\pi^4\tilde{c}\mathcal{P}_\chi^2\,, \tag{78}$$

which yields, using Eq. (17),

$$B_\chi(k_1,k_2,k_3) = \frac{\sum_j k_j^3}{\prod_j k_j^3}2\pi^4\tilde{c}\mathcal{P}_\chi^2\,. \tag{79}$$

This equation is equivalent to Eq. (45), confirming the result found from the large scale nonlinear evolution in Sec. 5. Thus non-linearities on sub-Hubble scales are completely subdominant with respect to the nonlinear classical super-Hubble evolution.

C Numerical results

In this appendix, we check the validity of the analysis of Secs. 4 and 5 by numerically calculating the expansion δN.

C.1 Phase space variables

For numerical calculations, it is more convenient to adopt phase space variables. Thus, first we summarise the background dynamics in terms of the phase space variables. These are defined as [15, 36, 37]

$$x_j = \frac{\dot{\phi}_j}{\sqrt{6}H}\,, \tag{80}$$

$$y_j = \frac{\sqrt{V_j e^{-c_j\phi_j}}}{\sqrt{3}H}\,, \tag{81}$$

and their evolution equations are given by

$$\frac{dx_j}{dN} = -3x_j(1-\sum_k x_k^2) - c_j\sqrt{\frac{3}{2}}y_j^2\,, \tag{82}$$

$$\frac{dy_j}{dN} = y_j\left(3\sum_k x_k^2 - c_j\sqrt{\frac{3}{2}}x_j\right)\,, \tag{83}$$

where $N = \log a$ and $j = 1, 2$ for two fields case. The Friedmann equation gives a constraint,

$$\sum_j x_j^2 - \sum_j y_j^2 = 1\,. \tag{84}$$

Assuming $\sum_j c_j^{-2} < 1/6$, there are 4 fixed points of the system where $dx_j/dN = dy_j/dN = 0$,

$$A: \qquad \sum_k x_k^2 = 1\,, \quad y_j = 0\,. \tag{85}$$

$$B_j: \qquad x_j = \frac{c_j}{\sqrt{6}}\,, \quad y_j = -\sqrt{\frac{c_j^2}{6} - 1}\,, \quad x_k = y_k = 0\,, \ \text{(for} \quad j \neq k)\,, \tag{86}$$

$$B: \qquad x_j = \frac{\sqrt{6}}{3p}\frac{1}{c_j}\,, \quad y_j = -\sqrt{\frac{2}{c_j^2 p}\left(\frac{1}{3p} - 1\right)}\,, \tag{87}$$

where $p = \sum_j 2/c_j^2$. A linearized stability analysis shows that the multi-field scaling solution, B, always has one unstable mode. On the other hand, the single-field-dominated scaling solutions, B_j, are always stable.

In the (x_1, x_2) plane, the three fixed points B, B_1 and B_2 are connected by a straight line, which is given by

$$c_2 x_1 + c_1 x_2 = \frac{c_1 c_2}{\sqrt{6}}\,. \tag{88}$$

Solutions which start close to the saddle point B evolve along this line to one of the single-field dominated solutions, B_1 or B_2. As in the main text, we concentrate on the case in which the background evolves from the multi-field scaling solution B to the single-field ϕ_2-dominated scaling solution B_2 for simplicity.

C.2 Numerical scheme

We first briefly summarise the numerical scheme. For the phase space variables x_j, y_j defined by Eqs. (80) and (81), we solve the evolution equations (82) and (83), numerically. Since we concentrate on the case in which the background evolves from the multi-field scaling solution, initial conditions are characterised by a point close to B in the phase space. As in the main text, in the multi-field scaling solution background, the fluctuations of the field φ are negligible.

Thus, for the phase space variables, we choose the initial conditions

$$\frac{\dot{\varphi}}{\sqrt{6}H} = \frac{c_2 x_1 + c_1 x_2}{\sqrt{c_1^2 + c_2^2}} = \frac{c}{\sqrt{6}}\,, \tag{89}$$

$$\frac{-\dot{\chi}}{\sqrt{6}H} = \frac{c_2 x_2 - c_1 x_1}{\sqrt{c_1^2 + c_2^2}} \equiv z\,, \tag{90}$$

where Eq. (89) is given by the background scaling solution in Eqs. (10) and (11). This guarantees that (x_1, x_2) lies on the line given by Eq. (88).

Equation (90) defines the quantity z, which characterises the deviation along the instability direction from B. Once we fix a model, i.e. the values of c_1 and c_2, the initial values of x_1 and x_2 are determined from Eqs. (89) and (90) for a given initial value of z.

For this set of (x_1, x_2), y_1 is fixed so that (x_1, x_2, y_1) is in the instability direction from B, and then y_2 is fixed from the constraint equation (84). Using this set of initial values (x_1, x_2, y_1, y_2), we can specify a background trajectory by solving the evolution equation.

The left panel of Fig. 1 shows an example of the background solution. For this background, we evaluate an expansion δN from the hypersurface characterised by H_i to that by H_f.

From Eqs. (43), (47) and (90), we can see that a different choice of z corresponds to that of α and χ as

$$z = \frac{\epsilon H}{\sqrt{6}}\left(1+\frac{1}{2}\tilde{c}\alpha H\right)\alpha\,, \tag{91}$$

$$= \frac{\epsilon}{\sqrt{6}}\left(1+\frac{1}{4}\tilde{c}\chi\right)\chi\,. \tag{92}$$

Then z_i is related with χ_i and it determines the Hubble parameter at the transition H_T in this scheme. In order to calculate the derivatives of $N(z_i)$ with respect to z_i, we change z_i slightly and calculate the difference of the expansion $N(z_i)$. The right panel of Fig. 1 shows the effect of the change in z_i on the Hubble parameter.

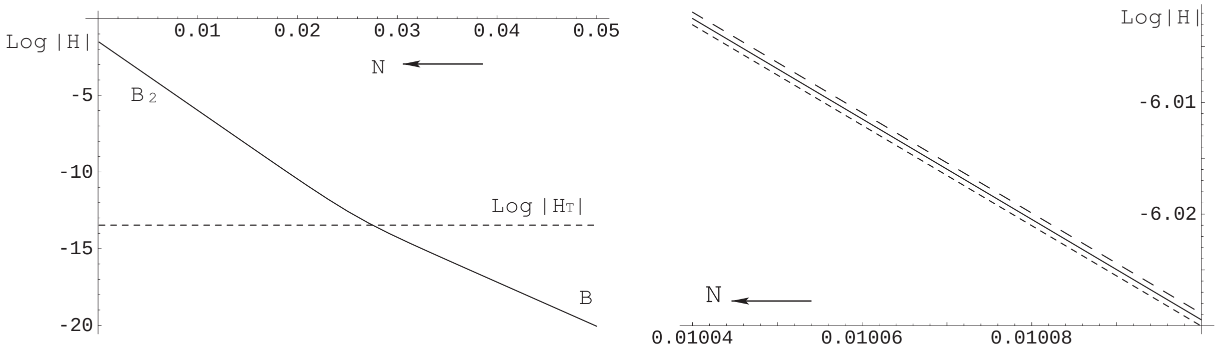

Figure 1: Left: An example of background solution for $\log|H|$ with initial condition $z = z_i$. We also show $\log|H_T|$ which is determined from the amplitude of $\delta\chi$. Right: The same background solution shown around $N \sim 0.01005$ which is much later than the transition. We also show the solutions with slightly different initial condition, $z = z_i + \delta z_i$, $z = z_i - \delta z_i$, with dashed lines. After the transition, this difference of initial z generates the curvature perturbations which can be evaluated as the difference of N at $H = H_f$.

C.3 Power spectrum and non-Gaussian parameter

In the following we first check numerically the linear relation Eq. (34). Note that from Eqs. (91) and (92), $\delta z \propto \delta\alpha \propto \delta\chi$ holds at the linearised level. From Eqs. (24) and (33), the power spectrum of the curvature perturbation is given by

$$\mathcal{P}_\zeta = N_{,\chi_i}^2 \epsilon^2 \left|\frac{H_i}{2\pi}\right|^2 \,. \tag{93}$$

In our numerical scheme, from Eq. (92) we evaluate $N_{,\chi_i}$ as

$$N_{,\chi_i} = \frac{\epsilon}{\sqrt{6}} N_{,z_i} = \frac{\epsilon}{\sqrt{6}} \frac{N(z_i+\delta z_i) - N(z_i - \delta z_i)}{2\delta z_i}\,, \tag{94}$$

which is valid for sufficiently small δz_i. From Eq. (34), the power spectrum of the curvature perturbations can also be expressed in terms of the scalar field fluctuation $\delta\chi$ as

$$\mathcal{P}_\zeta = \frac{\epsilon^2}{c_1^2+c_2^2}\left|\frac{H_T}{2\pi}\right|^2 = \frac{\epsilon^2}{c_1^2+c_2^2}\frac{|\delta\chi|^2_{B_2}}{|\delta\chi|^2_B}\left|\frac{H_i}{2\pi}\right|^2 . \tag{95}$$

As Eq. (95) and Eq. (93) should agree, we examine whether a quantity q defined by

$$q \equiv \frac{N^2_{,\chi_i}(c_1^2+c_2^2)|\delta\chi|^2_B}{|\delta\chi|^2_{B_2}} , \tag{96}$$

becomes close enough to 1. We find that $q=1$ is satisfied within about 1 % accuracy in our numerical simulations.

Next, we compute numerically f_{NL}. From Eq. (91), we can write down $N_{,\alpha}$ and $N_{,\alpha\alpha}$ in terms of z:

$$N_{,\alpha} = N_{,z}\frac{dz}{d\alpha} = \frac{\epsilon}{\sqrt{6}}H_i\left(1+\frac{\sqrt{6}\tilde{c}}{\epsilon}z_i\right)N_{,z} , \tag{97}$$

$$N_{,\alpha\alpha} = N_{,zz}\left(\frac{dz}{d\alpha}\right)^2 + N_{,z}\frac{d^2z}{d\alpha^2} = \frac{\epsilon^2}{6}H_i^2\left(1+\frac{2\sqrt{6}\tilde{c}}{\epsilon}z_i\right)N_{,zz} + \frac{\epsilon\tilde{c}H_i^2}{\sqrt{6}}N_{,z} . \tag{98}$$

In our numerical scheme, we evaluate $N_{,z_iz_i}$ as

$$N_{,z_iz_i} = \frac{N(z_i+\delta z_i) - 2N(z_i) + N(z_i-\delta z_i)}{(\delta z_i)^2} . \tag{99}$$

If the instantaneous transition approximation is valid, the non-Gaussianity parameter f_{NL} is given by

$$f_{NL} = \frac{5}{6}\frac{N_{,\alpha\alpha}}{N^2_{,\alpha}} . \tag{100}$$

Thus, in order to verify the accuracy of the analytical result $f_{NL} = -5c_1^2/12$, we examine whether a quantity r defined by

$$r \equiv -\frac{2N_{,\alpha\alpha}}{c_1^2 N^2_{,\alpha}} , \tag{101}$$

becomes close to 1 or not.

Since we checked that the choice of δz_i does not affect the results for sufficiently small values, we use $\delta z_i = 10^{-5}$. It was also verified that the time evolution of $\delta\chi$ can be described by that of the multi-field scaling solution up to $z_i = 0.01$; thus, we take $z_i = 0.01$. Using these initial conditions, we calculate f_{NL} for various combinations of c_1 and c_2. It is worth noting that in terms of c_1 and c_2 the fast-roll parameter is expressed as $\epsilon = c_1^2c_2^2/2(c_1^2+c_2^2)$. For the examples we show in Table 1, the range of the value of ϵ varies from 25 ($c_1=c_2=10$) to 400 ($c_1=c_2=40$). The results for f_{NL} and r are summarised in Table 1.

From Table 1 we can see that for these parameters, the results for f_{NL} are accurately described by $f_{NL} = -5c_1^2/12$ which agrees with the result obtained analytically based on the instantaneous transition approximation and the fast roll approximation in the main text.

Table 1: c_1 and c_2 dependence of the nonlinear parameter. We also compare with the results obtained by the instantaneous transition and the fast-roll approximations. The deviation from $r = 1$ denotes the error of these approximations. We adopt $z_i = 0.01$ and $\delta z_i = 10^{-5}$.

c_1	c_2	f_{NL}	r
10	10	−40.6165	0.974797
10	20	−40.9285	0.982283
10	40	−41.5317	0.996761
15	10	−90.4979	0.965311
15	20	−92.6425	0.988187
15	40	−93.5294	0.997647
20	10	−161.014	0.966082
20	20	−164.976	0.989855
20	40	−169.365	1.01619
40	10	−639.909	0.959865
40	20	−659.744	0.989617
40	40	−664.996	0.997494

We can see that the analytical results deviate from the numerical results for smaller ϵ (compare the cases with $c_1 = c_2$). This is because the fast-roll approximation becomes worse for smaller ϵ. We can also see that, for a fixed c_1, the analytical results again deviate from the numerical results for smaller c_2. This is because the potential around B becomes flatter along the direction to B_2 for smaller c_2, and the instantaneous transition approximation becomes worse.

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