Curvature perturbations from ekpyrotic collapse with multiple fields

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Abstract

A scale-invariant spectrum of isocurvature perturbations is generated during collapse in the ekpyrotic scaling solution in models where multiple fields have steep negative exponential potentials. The scale invariance of the spectrum is realized by a tachyonic instability in the isocurvature field. This instability drives the scaling solution to the late time attractor that is the old ekpyrotic collapse dominated by a single field. We show that the transition from the scaling solution to the single field dominated ekpyrotic collapse automatically converts the initial isocurvature perturbations about the scaling solution to comoving curvature perturbations about the late-time attractor. The final amplitude of the comoving curvature perturbation is determined by the Hubble scale at the transition.

1 Introduction

The existence of an almost scale-invariant spectrum of primordial curvature perturbations on large scales is one of the most important observations that any model for the early universe should explain. An inflationary expansion in the very early universe is most commonly assumed to achieve this, but it is important to consider whether there is any alternative model. In this paper we focus on the ekpyrotic scenario as an alternative [1] (see also [2, 3]). In the old ekpyrotic scenario, the large scale perturbations are supposed to be generated during a collapse driven by a single scalar field with a steep negative exponential potential. It was shown that the Newtonian potential acquires a scale-invariant spectrum, but the comoving curvature perturbation has a steep blue spectrum [4]. In this scenario we need a mechanism to convert contraction to expansion, and for a regular four-dimensional bounce, the scale-invariant Newtonian potential is matched to the decaying mode in an expanding universe, and the growing mode of curvature perturbations acquires a steep blue spectrum [5, 6]. It has been suggested that this conclusion might be altered by allowing a singular matching between collapse and expanion [7], but the general rule that the comoving curvature perturbation remains constant still holds for adiabatic perturbations on large scales [8]. In

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a braneworld context, the conversion from contraction to expansion might be accomplished by a collision of two branes where one of extra-dimensions disappears [9]. It was argued that the scale-invariant Newtonian potential can be transferred to the comoving curvature perturbations by this singular bounce [10]. However, without having a concrete theory to describe the singularity, it is difficult to have a definite conclusion on how perturbations pass through the singularity.

Recently, there has been some progress in generating a scale invariant spectrum for curvature perturbations in the ekpyrotic scenario [11, 12, 13], by considering non-adiabatic perturbations which has been suggested previously by Ref. [15]. In this case we require two or more fields. If these have steep exponential potentials then there exists a scaling solution where the energy density of the fields grow at the same rate during collapse [16, 17]. The isocurvature perturbations then have a scale-invariant spectrum [16]. These isocurvature perturbations can be converted to curvature perturbations if there is a sharp turn in the trajectory in field space [11, 12, 13]. For example Ref. [11] considered a situation where one of the fields changes its direction in field space, which corresponds to a time when a negative tension brane is reflected by a curvature singularity in the bulk, in the context of the heterotic M-theory. Refs. [12, 13] considered a regular bounce realized by a ghost condensate. One of the fields exits the ekpyrotic phase and hits the transition to the ghost condensate phase that creates a sharp turn in the trajectory in field space and curvature perturbations can be generated [12]. It is still necessary to match curvature perturbations in a contracting phase to those in an expanding universe, but it is shown that the comoving curvature perturbation is conserved on large scales resulting in an almost scale-invariant spectrum observed today for a regular bounce like a ghost condensate model [12].

The isocurvature perturbations behave like $\delta s \propto H$ on large scales. As the Hubble parameter is rapidly increasing in a collapsing universe, this signals an instability. In fact it is easy to see that we always require an instability of this form in order to generate a scale-invariant spectrum with a canonical scalar field during collapse[4]. As the amplitude of field perturbations at Hubble exit are of order H, we require the super-Hubble perturbations to grow at the same rate to maintain a scale-invariant spectrum. Ref. [19] studied this instability in a phase space analysis. The multi-field scaling solution was shown to be a saddle point in field space and the late-time attractor is the old ekpyrotic collapse dominated by a single-field. A tachyonic instability drives the scaling solution towards the late-time attractor (see also [11, 12]).

In Ref. [19], we pointed out that the natural turning point in the field space trajectory due to the instability of the scaling solution might itself offer the possibility of converting the scale invariant spectrum of isocurvature field perturbations into a scale invariant spectrum of curvature perturbations. In this paper, we confirm this expectation by explicitly solving the evolution equations for perturbations in a two field model. We find that the ratio between curvature perturbations and isocurvature perturbations at the final old ekpyrotic phase is solely determined by the ratio of exponents of the two exponential potentials, and the amplitude is set by the Hubble rate at the transition time.

[4]The only way to produce a scale-invariant spectrum without the presence of an instability seems to be due to non-canonical kinetic terms, as in the case of axion-type fields which can acquire scale-invariant perturbation spectra while remaining massless [18].

2 Homogeneous field dynamics

We first review the background dynamics of the fields. During the ekpyrotic collapse the contraction of the universe is assumed to be described by a 4D Friedmann equation in the Einstein frame with scalar fields with negative exponential potentials

$$3H^2 = V + \frac{1}{2}\dot{\phi}_i^2 \,, \tag{1}$$

where

$$V = -\sum_i V_i e^{-c_i\phi_i} \,, \tag{2}$$

and we take $V_i > 0$ and set $8\pi G$ equal to unity.

The authors of [11] found a scaling solution (previously studied in [16, 17]) in which both fields roll down their potential as the universe approaches a big crunch singularity. In this ekpyrotic scaling collapse we find a power-law solution for the scale factor

$$a \propto (-t)^p \,, \quad \text{where } p = \sum_i \frac{2}{c_i^2} < \frac{1}{3} \,, \tag{3}$$

where

$$\frac{\dot{\phi}_i^2}{\dot{\phi}_j^2} = \frac{-V_i e^{-c_i\phi_i}}{-V_j e^{-c_j\phi_j}} = \frac{c_j^2}{c_i^2} \,. \tag{4}$$

As we will see in the next section, it is possible to generate scale-invariant isocurvature perturbations around this background. However, the ekpyrotic scaling solution (4) is unstable.

In addition to the scaling solution we have fixed points corresponding to any one of the original fields ϕ_i dominating the energy density where the other fields have negligible energy density. These correspond to the original ekpyrotic power-law solutions where

$$a \propto (-t)^{p_i} \,, \quad \text{where } p_i = \frac{2}{c_i^2} \,, \tag{5}$$

for $c_i^2 > 6$. We find that any of these single field dominated solutions is a stable local attractor at late times during collapse.

In Ref. [19], the ekpyrotic scaling solution (4) was shown to be a saddle point in the phase space. We briefly review the phase space analysis. Introducing phase space variables [20, 21, 17]

$$x_i = \frac{\dot{\phi}_i}{\sqrt{6}H}, \tag{6}$$

$$y_i = \frac{\sqrt{V_i e^{-c_i\phi_i}}}{\sqrt{3}H}. \tag{7}$$

the first order evolution equations for the phase space variables are given by

$$\frac{dx_i}{dN} = -3x_i(1 - \sum_j x_j^2) - c_i\sqrt{\frac{3}{2}}y_i^2, \tag{8}$$

$$\frac{dy_i}{dN} = y_i\left(3\sum x_j^2 - c_i\sqrt{\frac{3}{2}}x_i\right), \tag{9}$$

where $N = \log a$. The Friedmann equation gives a constraint

$$\sum_j x_j^2 - \sum_j y_j^2 = 1. \tag{10}$$

There are $(n+2)$ fixed points of the system where $dx_i/dN = dy_i/dN = 0$.

$$A: \qquad \sum_j x_j^2 = 1, \quad y_j = 0. \tag{11}$$

$$B_i: \qquad x_i = \frac{c_i}{\sqrt{6}}, \quad y_i = -\sqrt{\frac{c_i^2}{6} - 1}, \quad x_j = y_j = 0, \quad (\text{for} \quad j \neq i), \tag{12}$$

$$B: \qquad x_j = \frac{\sqrt{6}}{3p}\frac{1}{c_j}, \quad y_j = -\sqrt{\frac{2}{c_j^2 p}\left(\frac{1}{3p} - 1\right)}. \tag{13}$$

In this paper, we focus on the fixed points B and B_i assuming $c_i^2 > 6$ and $\sum c_i^{-2} < 1/6$. The linearized analysis shows that the multi-field scaling solution, B, always has one unstable mode. On the other hand, the single field dominated fixed points, B_i, are always stable.

From now on we concentrate our attention on two fields case. Then we have three fixed points B, B_1 and B_2. It is interesting to note that in the (x_1, x_2) plane, the fixed points B, B_1 and B_2 are connected by a straight line, which is given by

$$c_2 x_1 + c_1 x_2 = \frac{c_1 c_2}{\sqrt{6}}. \tag{14}$$

The eigenvector associated with the unstable mode around the scaling solution B lies in the same direction as the line (14). Thus this is an attractor trajectory, which all solutions near B approach. Fig. 1 shows numerical solutions for the evolution of x_1. Initial positions in the phase space are perturbed away from B along the line (14). The solutions go to B_1 or B_2, depending on the initial position in the phase space.

An important observation is that, as we follow phase space trajectories during the transition from the scaling solution B to the attractor solutions B_1 or B_2, the solutions obey the relation Eq. (14), even far away from the saddle point, B. Using the Friedmann equation and the field equations, we can show that

$$\left(H - \frac{\dot{\phi}_1}{c_1} - \frac{\dot{\phi}_2}{c_2}\right)^{\cdot} + 3H\left(H - \frac{\dot{\phi}_1}{c_1} - \frac{\dot{\phi}_2}{c_2}\right) = 0\,, \tag{15}$$

and hence

$$H - \frac{\dot{\phi}_1}{c_1} - \frac{\dot{\phi}_2}{c_2} = \frac{C}{a^3}\,, \tag{16}$$

where C is an integration constant. In terms of ϕ_1 and ϕ_2, Eq. (14) can be rewritten as

$$\frac{\dot{\phi}_1}{c_1} + \frac{\dot{\phi}_2}{c_2} = H. \tag{17}$$

and hence we see that for trajectories starting from point B we have $C = 0$, which is the late time attractor. Thus we see that Eq. (17) holds even during the transition caused by the tachyonic instability from point B to B_1 or B_2 and in the final single-field dominated phase. This fact will be important when we study perturbations.

We will study the behaviour of perturbations during the transition in the next section.

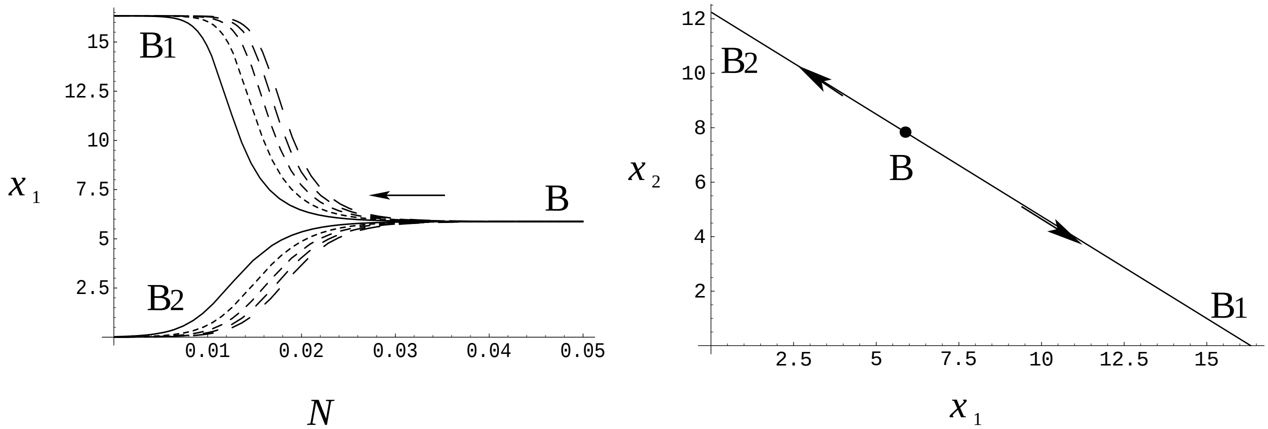

Figure 1: Left: Numerical solutions for $x_1(N)$. The horizontal axis is $N = \log a$ and we take $c_1 = 40$ and $c_2 = 30$. The initial time is $N = 0.05$. Note that N decreases towards the future in a collapsing universe. Right: The corresponding phase space trajectories in (x_1, x_2) plane.

3 Generation of quantum fluctuations

In this section, we consider inhomogeneous linear perturbations around the background solution. We consider the scalar field perturbations on spatially flat hypersurfaces. Then the scalar field perturbations are given by [22, 23, 24, 25]

$$\ddot{\delta\phi}_i + 3H\dot{\delta\phi}_i + \frac{k^2}{a^2}\delta\phi_i - c_i^2 V_i \exp(-c_i\phi_i)\delta\phi_i - \sum_j \frac{1}{a^3}\left(\frac{a^3}{H}\dot{\phi}_i\dot{\phi}_j\right)^{\cdot}\delta\phi_j = 0. \tag{18}$$

We can decompose the perturbations into the instantaneous adiabatic and entropy field perturbations as follows [24]:

$$\delta r = \frac{\dot{\phi}_1\delta\phi_1 + \dot{\phi}_2\delta\phi_2}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}}, \quad \delta s = \frac{\dot{\phi}_2\delta\phi_1 - \dot{\phi}_1\delta\phi_2}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}}. \tag{19}$$

The adiabatic field perturbation δr is the component of the two-field perturbation along the direction of the background fields' evolution while the entropy perturbation δs represents fluctuations orthogonal to the background classical trajectory. The adiabatic field perturbation leads to a perturbation in the comoving curavture perturbation:

$$\mathcal{R}_c = \frac{H\delta r}{\dot{r}}\,, \tag{20}$$

whereas the entropy field perturbations correspond to isocurvature perturbations.

Their evolution equations are given by [24]

$$\ddot{\delta r} + 3H\dot{\delta r} + \frac{k^2}{a^2}\delta r + \left[V_{,rr} - \dot{\theta}^2 - \frac{1}{a^3}\left(\frac{a^3\dot{r}^2}{H}\right)^{\cdot}\right]\delta r = 2\dot{\theta}\dot{\delta s} + 2\left[\ddot{\theta} - \left(\frac{V_{,r}}{\dot{r}} + \frac{\dot{H}}{H}\right)\dot{\theta}\right]\delta s, \tag{21}$$

$$\ddot{\delta s} + 3H\dot{\delta s} + \frac{k^2}{a^2}\delta s + (V_{,ss} - \dot{\theta}^2)\delta s = -2\frac{\dot{\theta}}{\dot{r}}\left[\dot{r}\dot{\delta r} - \left(\frac{\dot{r}^3}{2H} + \ddot{r}\right)\delta r\right], \tag{22}$$

where the angle θ is defined as

$$\cos\theta = \frac{\dot{\phi}_2}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}}, \quad \sin\theta = \frac{\dot{\phi}_1}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}}, \tag{23}$$

such that

$$\begin{aligned} \dot{r} &= (\cos\theta)\dot{\phi}_2 + (\sin\theta)\dot{\phi}_1, & (24)\\ \dot{\theta} &= -\frac{V_{,s}}{\dot{r}}, & (25) \end{aligned}$$

and

$$\begin{aligned} V_{,r} &= (\sin\theta)c_1V_1\exp(-c_1\phi_1) + (\cos\theta)c_2V_2\exp(-c_2\phi_2), & (26)\\ V_{,s} &= (\cos\theta)c_1V_1\exp(-c_1\phi_1) - (\sin\theta)c_2V_2\exp(-c_2\phi_2), & (27)\\ V_{,rr} &= -(\sin\theta)^2c_1^2V_1\exp(-c_1\phi_1) - (\cos\theta)^2c_2^2V_2\exp(-c_2\phi_2), & (28)\\ V_{,ss} &= -(\sin\theta)^2c_2^2V_2\exp(-c_2\phi_2) - (\cos\theta)^2c_1^2V_1\exp(-c_1\phi_1). & (29) \end{aligned}$$

For the multi-field scaling solution, B, we have

$$\theta = \arctan\frac{c_2}{c_1}, \tag{30}$$

and for the single field scaling solutions we have

$$\theta = \frac{\pi}{2} \quad (B_1), \quad \theta = 0 \quad (B_2). \tag{31}$$

Thus we have $\theta =$constant for the fixed points and the adiabatic and the entropy fields are decoupled. This allows us to quantise the independent fluctuations in the two fields.

For the multi-field scaling solution B, the spectrum of quantum fluctuations of the entropy field is given on large scales ($k \ll aH$) by

$$\mathcal{P}_{\delta s} \equiv \frac{k^3}{2\pi^2}|\delta s^2| = C_\nu^2\frac{k^2}{a^2}(-k\tau)^{1-2\nu}, \tag{32}$$

where $\tau < 0$ is conformal time, and

$$\nu^2 = \frac{9}{4} - \frac{3\epsilon}{(\epsilon-1)^2}, \quad \epsilon \equiv -\dot{H}/H^2 = 1/p, \tag{33}$$

and $C_\nu = 2^{\nu-3/2}\Gamma(\nu)/\pi^{3/2}$ [16]. The spectral tilt is given by

$$\Delta n_{\delta\chi} \simeq \frac{2}{\epsilon}, \tag{34}$$

to leading order in a fast-roll expansion ($\epsilon \gg 1$) [11, 12, 13]. In this limit, the spectrum (32) can be written as

$$\mathcal{P}_{\delta s}^{1/2} = \epsilon\left|\frac{H}{2\pi}\right|. \tag{35}$$

Note that $|H|$ is rapidly increasing and thus δs is also growing on super-Hubble scales due to the tachyonic instability. This instability is essential in order to realize the scale invariance of the spectrum (35). The amplitude of field perturbations at Hubble exit is of order H and thus we require the super-Hubble perturbations to grow at the same rate in order to maintain a scale-invariant spectrum.

Note that in the simplest model, the spectrum is slightly blue [11, 12, 13]. However, any deviations from an exponential potential for adiabatic field can introduce the corrections to the spectral tilt and thus it becomes model dependent [11, 12].

The spectrum of quantum fluctuations in the adiabatic field about the scaling solution has the same power-law form on large scales

$$\mathcal{P}_{\delta r} = C_\mu^2 \frac{k^2}{a^2} (-k\tau)^{1-2\mu} \,, \tag{36}$$

where to $\mu \simeq 1/2$ to leading order in $1/\epsilon$. Thus the adiabatic field perturbations become constant in the large scale limit and the spectral tilt is given by

$$\Delta n_{\delta r} \simeq 2 \,. \tag{37}$$

Thus we have

$$\frac{\mathcal{P}_{\delta r}}{\mathcal{P}_{\delta s}} \propto (-k\tau)^2 \,, \tag{38}$$

and hence in what follows we can neglect the adiabatic field fluctuations in the large-scale limit.

By contrast, for the single field dominated scaling solutions, the adiabatic and entropy field perturbations are both frozen on super-Hubble scales:

$$\delta s \,,\, \delta r = \text{const.} \tag{39}$$

These perturbations both have a steep blue spectrum if they cross the horizon when the background solutions are described by the single field dominated solution [19].

4 Generation of curvature perturbations

Now let us consider the evolution of perturbations in a situation where the classical solution starts from near the saddle point, B. As emphasized in Ref. [19] this requires an additional preceding mechanism that drives the classical background solution to the unstable saddle point throughout our observable part of the universe. In this paper, we will not discuss the mechanism required to bring the classical solution to the saddle point and we just assume that the classical solution stays near the saddle point for long enough to ensure that a scale-invariant spectrum of isocurvature perturbations is generated over the relevant scales for the observed large scale structure of our Universe.

Then the initial conditions for the adiabatic and the entropy field perturbations can be set from the amplitude of quantum fluctuation as described in the previous section:

$$\delta r = 0, \quad \delta s = \epsilon \left| \frac{H}{2\pi} \right| \,, \tag{40}$$

on sufficiently large scales and for $1/\epsilon \ll 1$.

Unless the spatially homogeneous background solution is located exactly at the fixed point, the tachyonic instability drives the background solution away from the multi-field scaling solution, B, to one of the single field dominated solutions, B_1 or B_2, depending on the initial conditions. During the transition θ is not constant and the adiabatic and entropy field perturbations mix, so it is possible to generate perturbations in the adiabatic field, and hence comoving curvature perturbations (20), from initial fluctuations in the entropy field.

We can solve the evolution equations (21) and (22) numerically for any given classical background. Figure 2 shows the behaviour of δr and δs. Due to the coupling between δr and δs during the transition, curvature perturbations are generated during the transition. On the other hand, δs shows a tachyonic instability according to Eq. (40) close to the scaling solution, but when the background solution goes to B_i, the entropy field perturbation becomes constant. The final amplitude of δr depends on when the transition from B to B_i occurs, but, interestingly, the final ratio between δr and δs does not depend on the details of the transition. We find that the ratio is determined solely by the parameters c_1 and c_2 as

$$\frac{\delta r}{\delta s} = \frac{c_1}{c_2}, \quad \text{at } B_1, \tag{41}$$

$$\frac{\delta r}{\delta s} = -\frac{c_2}{c_1}, \quad \text{at } B_2, \tag{42}$$

as is shown in Fig. 2. We will explain later why such a simple result is found.

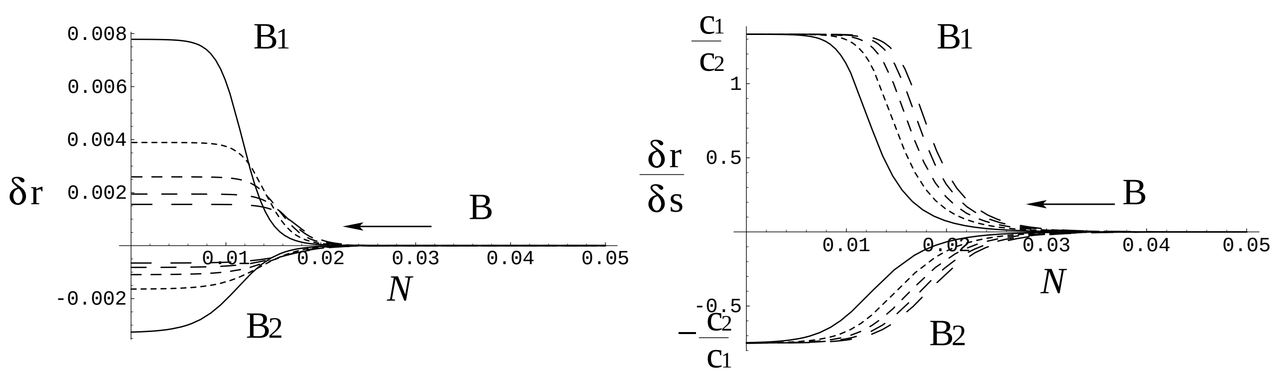

Figure 2: Left: Solutions for $\delta r(N)$, using the same parameters as in Figure 1. The corresponding background solutions are shown in Figure 1. Right: The ratio between δr and δs. The ratio approaches a constant given by Eqs. (41) and (42). In this case 1.3333 for B_1 and -0.75 for B_2.

The resulting curvature perturbation on a comoving hypersurface in the final single-field dominated phase is thus given by

$$\mathcal{R}_c = \frac{H}{\dot{\phi}_1}\delta r = \frac{1}{c_1}\delta r, \quad \text{at } B_1, \tag{43}$$

$$\mathcal{R}_c = \frac{H}{\dot{\phi}_2}\delta r = \frac{1}{c_2}\delta r, \quad \text{at } B_2. \tag{44}$$

The equation for the entropy field perturbation (22) can be rewritten as [24]

$$\ddot{\delta s} + 3H\dot{\delta s} + \left(\frac{k^2}{a^2} + V_{ss} + 3\dot{\theta}^2\right)\delta s = 4\frac{\dot{\theta}}{\dot{r}}\frac{k^2}{a^2}\Psi\,, \tag{45}$$

where Ψ is the curvature perturbations in Newtonian gauge. The change of the curvature perturbations is determined by [24]

$$\dot{\mathcal{R}}_c = \frac{H}{\dot{H}}\frac{k^2}{a^2}\Psi + \frac{2H}{\dot{r}}\dot{\theta}\delta s. \tag{46}$$

On large scales, we can neglect $(k^2/a^2)\Psi$ and it is possible to reproduce the previous results from Eq. (46).

Although the physical meaning of the instantaneous adiabatic and entropy field perturbations is clear, the dynamics of the perturbations during the transition are rather complicated in this basis. We find it is much easier to work in terms of new variables [19]

$$\varphi = \frac{c_2\phi_1 + c_1\phi_2}{\sqrt{c_1^2+c_2^2}}\,, \quad \chi = \frac{c_1\phi_1 - c_2\phi_2}{\sqrt{c_1^2+c_2^2}}\,, \tag{47}$$

corresponding to a fixed rotation in field space. The potential Eq. (2) can then be simply re-written as [29, 16, 19]

$$V = -U(\chi)\,e^{-c\varphi}\,, \tag{48}$$

where

$$\frac{1}{c^2} = \sum_i \frac{1}{c_i^2}\,, \tag{49}$$

and the potential for the orthogonal field is given by

$$-U(\chi) = -V_1\,e^{-(c_1/c_2)c\chi} - V_2\,e^{(c_2/c_1)c\chi}\,, \tag{50}$$

which has a maximum at

$$\chi = \chi_0 = \frac{1}{\sqrt{c_1^2+c_2^2}}\ln\left(\frac{c_1^2V_1}{c_2^2V_2}\right). \tag{51}$$

The multi-field scaling solution corresponds to $\chi = \chi_0$, while φ is rolling down the exponential potential. The potential for χ has a negative mass-squared around $\chi = \chi_0$, and thus χ represents the instability direction. If the initial condition for χ is slightly different from χ_0 or $\dot{\chi}$ is not zero, then χ starts rolling down the potential and the solution approaches a single-field dominated solution.

Note that perturbations $\delta\varphi$ and $\delta\chi$ coincide with the instantaeous adiabatic and entropy field perturbations respectively, defined in Eq. (19), at the scaling solution, B. Thus we can use the initial perturbations (40) due to vacuum fluctuations about the scaling solution previously calculated. However as we follow the evolution away from this saddle point we can no longer identify φ and χ with the adiabatic and entropy perturbations. Nevertheless, the dynamics of perturbations turns out to be much simpler using these fields.

In terms of φ and χ the equations for perturbations are given by

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} + \frac{k^2}{a^2}\delta\varphi + M_{\varphi\varphi}\delta\varphi + M_{\varphi\chi}\delta\chi = 0, \tag{52}$$

$$\ddot{\delta\chi} + 3H\dot{\delta\chi} + \frac{k^2}{a^2}\delta\chi + M_{\chi\chi}\delta\chi + M_{\varphi\chi}\delta\varphi = 0, \tag{53}$$

where

$$M_{\varphi\varphi} = V_{,\varphi\varphi} - \frac{1}{a^3}\left(\frac{a^3}{H}\dot{\varphi}^2\right)^{\cdot}, \tag{54}$$

$$M_{\varphi\chi} = V_{,\varphi\chi} - \frac{1}{a^3}\left(\frac{a^3}{H}\dot{\varphi}\dot{\chi}\right)^{\cdot}, \tag{55}$$

$$M_{\chi\chi} = V_{,\chi\chi} - \frac{1}{a^3}\left(\frac{a^3}{H}\dot{\chi}^2\right)^{\cdot}. \tag{56}$$

A key observation is that the phase space trajectory of the background fields during the transition, Eq. (17), can be re-written as

$$\frac{\dot{\varphi}}{H} = c. \tag{57}$$

Thus even away from the multi-field scaling solution, φ obeys a scaling relation. We can then show that two of the effective mass terms become

$$M_{\varphi\varphi} = V_{,\varphi\varphi} + cV_{,\varphi} = 0\,, \tag{58}$$

$$M_{\varphi\chi} = V_{,\varphi\chi} + cV_{,\chi} = 0\,, \tag{59}$$

independently of the form of $U(\chi)$ since $V \propto \exp(-c\varphi)$.

Thus on large scales $\delta\varphi$ is constant. As we take $\delta\varphi = 0$ as our initial condition, this remains so even during the transition and in the final single field dominated phase. Thus from Eq. (47) we have a relation between $\delta\phi_1$ and $\delta\phi_2$,

$$c_2\delta\phi_1 + c_1\delta\phi_2 = 0, \tag{60}$$

and $\delta\phi_1$ and $\delta\phi_2$ are given by

$$\delta\phi_1 = \frac{c_1}{\sqrt{c_1^2+c_2^2}}\delta\chi, \tag{61}$$

$$\delta\phi_2 = -\frac{c_2}{\sqrt{c_1^2+c_2^2}}\delta\chi. \tag{62}$$

In the single field dominated solutions, φ is no longer the adiabatic field. The adiabatic field is simply ϕ_1 (or ϕ_2) in B_1 (or B_2) and the entropy field is $-\phi_2$ (or ϕ_1) in B_1 (or B_2). Thus at the late-time attractor we have

$$\delta r = \delta\phi_1, \quad \delta s = -\delta\phi_2, \quad \text{at } B_1, \tag{63}$$

$$\delta r = \delta\phi_2, \quad \delta s = \delta\phi_1, \quad \text{at } B_2. \tag{64}$$

Thus the ratio between δr and δs is determined by the ratio between $\delta\phi_1$ and $\delta\phi_2$ and we can easily find the ratio $\delta r/\delta s$ as given in Eqs.(41) and (42).

The amplitude of the curvature perturbations can be estimated from $\delta\chi$. In the initial stage, $\delta\chi$ coincides with the entropy field perturbations δs, and thus its initial amplitude on super-Hubble scales is given by

$$\delta\chi = \frac{c^2}{2}\left|\frac{H}{2\pi}\right|. \tag{65}$$

where we used $\epsilon = c^2/2$ in Eq. (40) and c^2 is given by Eq. (49). After the transition, $\delta\chi$ becomes massless and the amplitude becomes frozen on large scales. In the single field dominated solutions, the comoving curvature perturbation $\mathcal{R}_c$ is given by

$$|\mathcal{R}_c| = \frac{1}{\sqrt{c_1^2+c_2^2}}\delta\chi. \tag{66}$$

Assuming the transition occurs suddenly, the final amplitude of the comoving curvature perturbation is given by

$$|\mathcal{R}_c| = \frac{c^2}{2\sqrt{c_1^2+c_2^2}}\left|\frac{H}{2\pi}\right|_T, \tag{67}$$

where the subscript T means that the quantity is evaluated at the transition time. On the other hand the amplitude of the entropy field perturbation is given by

$$\delta s = \frac{c_2 c^2}{2\sqrt{c_1^2+c_2^2}}\left|\frac{H}{2\pi}\right|_T, \quad \text{at } B_1, \tag{68}$$

$$\delta s = \frac{c_1 c^2}{2\sqrt{c_1^2+c_2^2}}\left|\frac{H}{2\pi}\right|_T, \quad \text{at } B_2. \tag{69}$$

From the numerical solutions for δr and δs, we can reconstruct H_T. We confirm that H_T constructed in this way agrees with the Hubble scale at the transition as is shown in Figure 3.

5 Conclusion

In this paper we have studied the generation of curvature perturbations during an ekpyrotic collapse with multiple fields. We must assume that the classical background solution starts from a state very close to a saddle point in the phase space that corresponds to the multi-field ekpyrotic scaling solution. If the solution stays at this fixed point for long enough, a scale-invariant spectrum of isocurvature perturbations is generated over the range of scales that is relevant for large scale structure in the Universe. Even if the background solution deviates only slightly from the multi-field scaling solution initially, the tachyonic instability eventually drives the solution to the old ekpyrotic collapse dominated by a single field. During this transition, the initial isocurvature field perturbations generate a scale-invariant spectrum of comoving curvature perturbations.

First we studied the perturbations by decomposing them into the instantaneous adiabatic and the entropy field perturbations. These fields are decoupled at the fixed points. The adiabatic field perturbations are effectively massless around both the multi-field scaling solutions

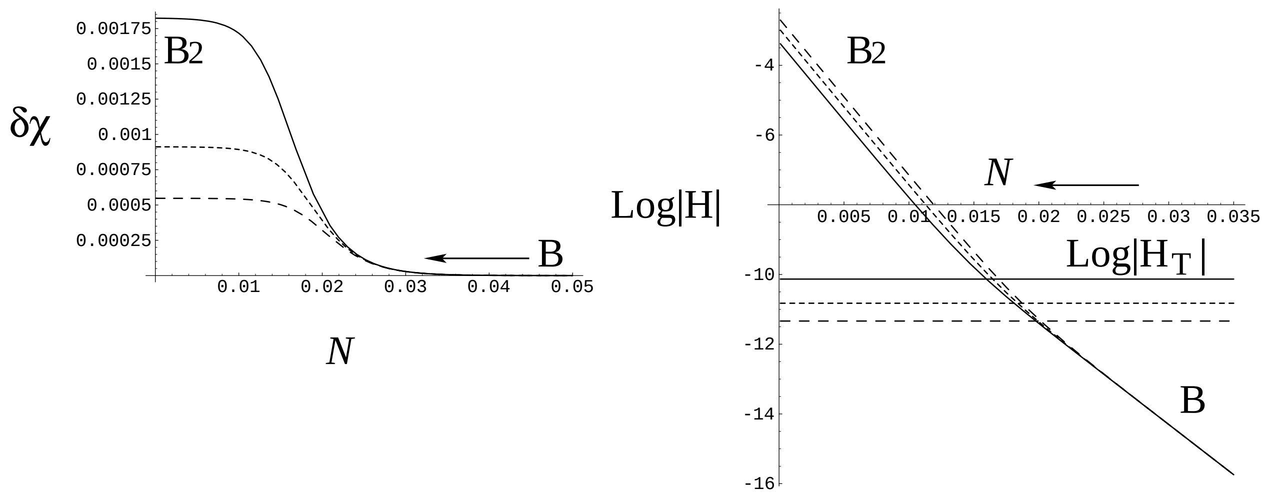

Figure 3: Left: Solutions for $\delta\chi(N)$. We used the same parameters as Figures 1 and 2. Right: Solutions for $\log|H|$ with three different background solutions. We also show $\log|H_T|$ that is determined from the numerical solutions for $\delta\chi$.

and the single field dominated solution, so become constant on large scales. On the other hand, the entropy field perturbations have a tachyonic mass around the multi-field scaling solution, where they grow like $\delta s \propto H$ on large scales, but they are effectively massless around the single field dominated solution. We set initial conditions for these perturbations from quantum fluctuations about the multi-field scaling solution. As the adiabatic field perturbations have a blue spectrum they can be neglected compared with the entropy field perturbations on large scales. However, during the transition to the single-field attractor, adiabatic field perturbations are generated from the entropy field perturbations.

It turns out to be more convenient to use new fields φ and χ defined in Eqs. (47) to follow the perturbations through the transition. These fields coincide with the adiabatic and entropy fields around the multi-field scaling solution but not during the transition or at the final single field dominated phase. In terms of φ and χ, the potential is given by a product of the potential for χ, $U(\chi)$, and an exponential potential for φ. $U(\chi)$ has an extremum which corresponds to the multi-field scaling solution. A crucial observation is that even during the transition and in the final phase, φ satisfies the scaling relation Eq. (57). We can then show that the field perturbations for φ and χ are always decoupled. Since $\delta\varphi = 0$ on large scales during the ekpyrotic scaling solution, this remains so. This determines the ratio between the adiabatic and entropy field perturbations at the final phase. On the other hand, $\delta\chi$ grows during the scaling solution and then becomes constant during the single field dominated solution.

Our final results are Eqs. (67), (68) and (69) for the amplitude of comoving curvature and isocurvature field perturbations during the single field ekpyrotic collapse phase. The amplitude of the comoving curvature perturbation is determined by the Hubble scale at the transition.

We still need to convert the ekpyrotic collapse to expansion (see, for instance, [12, 13]) and see how this curvature perturbation is matched to that in an expanding universe. For a regular bounce the comoving curvature perturbation is conserved for adiabatic perturbations and thus Eq. (67) is directly related to the amplitude of the observed primordial density perturbation. If the radiation and matter content in today's universe comes solely from the single field that dominates the final ekpyrotic phase, we will have no isocurvaure perturbations in an expanding universe.

It is interesting to compare this multi-field model with a single field model that gives a scale-invariant spectrum of comoving curvature perturbations during collapse [14, 26, 27]. The single-field model requires the correct exponent for a relatively flat and positive exponential potential in order to obtain $a \propto |t|^{2/3}$, whereas the ekyprotic model only requires sufficiently steep, negative exponential potentials to obtain $a \propto |t|^{1/\epsilon}$ with $\epsilon \gg 1$. On the other hand both models require fine-tuned initial conditions as it is the existence of an instability that gives rise to the scale-invariant perturbation spectrum during collapse. In both models the amplitude of tensor perturbations is determined by the Hubble scale when the perturbations leave the horizon as tensor perturbations are then frozen on super-horizon scales. In the single field model the tensor metric perturbations thus acquire the same almost scale-invariant spectrum [28] as the comoving curvature perturbation, with a similar amplitude, giving rise to a dangerously large tensor-scalar ratio which severely constrains the model [27]. But in the ekpyrotic scaling solution the tensor perturbations have a steep blue spectrum and are completely negligible at scales relevant for the cosmic microwave anisotropies.

In summary, a simple ekpyrotic model with two steep, negative exponential potentials is capable of generating a scale-invariant spectrum of comoving curvature perturbations. A key ingredient is the instability of the multi-field scaling solution [16, 11, 12, 19]. This instability generates a scale-invariant spectrum of isocurvature field perturbations from vacuum fluctuations about the scaling solution, and converts them to a scale-invariant spectrum of comoving curvature perturbations. We should emphasize that this conversion occurs automatically due to the dynamics of the fields in our simple model and does not require any change in the shape of the potential or any additional dynamics [11, 12, 13].

Thus ekpyrotic collapse with multiple fields can generate a scale-invariant spectrum for curvature perturbations in several different ways (see also [30] for a different idea). In all these models, we need some preceding phase that initially drives the classical background solution to the unstable multi-field scaling solution throughout our observable region of space. This problem cannot be solved within the simplest model with multiple exponential potentials that we considered in this paper and we would need to appeal to a more ambitious framework for the model such as the cyclic scenario [31, 32] to address this problem.

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