

# Coupling of radial and non-radial oscillations of relativistic stars: gauge-invariant formalism

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Linear perturbation theory is appropriate to describe small oscillations of stars, while a mild non-linearity is still tractable perturbatively but requires to consider mode coupling, i.e. to take into account second order effects. It is natural to start to look at this problem by considering the coupling between linear radial and non-radial modes. A radial pulsation may be thought of as an important component of an overall mildly non-linear oscillation, e.g. of a proto-neutron star. Radial pulsations of a spherical compact objects do not *per se* emit gravitational waves but, if the coupling between the existing first order radial and non-radial modes is efficient in driving and possibly amplifying the non-radial oscillations, one may expect the appearance of non-linear harmonics, and gravitational radiation could then be produced to a significant level. More in general, mode coupling typically leads to an interesting phenomenology, thus it is worth investigating it in the context of star perturbations.

In this paper we develop the relativistic formalism to study the coupling of radial and non-radial first order perturbations of a compact spherical star. From a mathematical point of view, it is convenient to treat the two sets of perturbations as separately parametrized, using a 2-parameter perturbative expansion of the metric, the energy-momentum tensor and Einstein equations in which  $\lambda$  is associated with the radial modes,  $\epsilon$  with the non-radial perturbations, and the  $\lambda\epsilon$  terms describe the coupling. This approach provides a well-defined framework to consider the gauge dependence of perturbations, allowing us to use  $\epsilon$  order gauge-invariant non-radial variables on the static background and to define new second order  $\lambda\epsilon$  gauge-invariant variables representing the result of the non-linear coupling. We present the evolution and constraint equations for our variables outlining the setup for numerical computations, and briefly discuss the surface boundary conditions in terms of the second order  $\lambda\epsilon$  Lagrangian pressure perturbation.

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## I. INTRODUCTION

Since the discovery of pulsars in the sixties neutron stars have acquired a special status in physics: as supernovae remnants they are fundamental to our understanding of the final stages of evolution and fate of upper main sequence stars; as the most compact observed objects they are a test-bed for strong-field gravity, i.e. general relativity and its generalizations; they are a unique laboratory for fundamental physics such as nuclear interactions, superfluidity, superconductivity. Either as isolated objects or in binary systems neutron stars are important gravitational wave sources, and in the near future the analysis of gravitational radiation from neutron stars will open up a direct window on their interior, possibly revealing details on the equation of state of nuclear matter, the dynamics of the crust-mantle interaction and the inner superfluid/superconducting core. Early stages of neutron star formation during iron core collapse in supernovae and other possible transient dynamical phases are particularly interesting from the point of view of gravitational wave physics. In this case the gravitational radiation emitted should carry the signature of the yet not understood mechanism driving the explosion and that of the unknown phenomena responsible for the observed kick velocity of many pulsars, providing complementary information to that carried by neutrinos.

Physics of compact objects like supernovae core collapse or neutron stars may be studied with various approaches and approximations, from purely Newtonian, or Newtonian with relativistic corrections, to the full general relativistic treatment of numerical relativity. For many astrophysical problems even a purely Newtonian approach may be adequate. However, in order to study neutron stars as sources of gravitational waves, a full general relativistic treatment is more satisfactory because in this case the gravitational radiation is built-in in the calculations rather than calculated *a-posteriori* with the quadrupole formula, thus one can obtain more accurate results. As for other gravitational wave sources, in the long term the goal is a full numerical relativity treatment coupled with a detailed description of the matter physics. Meanwhile, relativistic perturbation methods remain a valid alternative: they are computationally far less intensive because, even in dealing with non-linear effects as we shall be doing here, one has to solve linear partial differential equations instead of the complete non-linear Einstein equations. Hence, accurate results

can be obtained at a relatively low cost. Moreover, even when numerical solutions of the complete field equations are available, analytic perturbative methods often help to shed light on the physical processes at work, clarifying the interpretation of the numerical results.

Linear perturbations and instabilities of neutron stars have been studied for long time [1, 2] but relatively little is known of non-linear dynamical effects (see e.g. [3, 4, 5, 6, 7]) and therefore second order studies may help to understand known problems and even reveal a new phenomenology. Non-linear effects are the rule rather than the exception in phenomena in all branches of physics, and have to be taken into account for accurate modeling. This should even be more natural when the mathematical modeling is not phenomenological, but rooted in a fundamental theory which is *per se* non-linear, as general relativity is. When we use perturbation theory, we generally expect that if the perturbations are very small, then second order effects should be negligible. On the other hand, while in order to treat strong non-linear dynamical effects a fully non-perturbative approach is required, one may expect that much of the interesting physics only involves a mild non-linearity for which a second order treatment should be perfectly suited. Furthermore, a proper parametrization of a problem may even lead to the unexpected result that a second order treatment appears to provide a good description of the physics even in a mildly non-linear regime where *a priori* one would expect the perturbative approach to fail. An example is given by the study of black hole collision in the close limit approximation [8]. With this in mind, it is reasonable to expect that a second order treatment of non-linear oscillations of neutron stars should be adequate in many circumstances of astrophysical and/or gravitational interest.

In this paper we develop the formalism to study a specific non-linear effect, focusing our investigation on the coupling of radial and non-radial first order perturbations of a relativistic star. Work is currently under way to apply this formalism and carry out the actual study, and will be presented elsewhere [9, 10]. The main goal is that of understanding whether this coupling can lead to new effects such as fluid instabilities and/or persistence of oscillations and amplifications that can produce a significant amount of gravitational radiation. Furthermore, this work may be a first step toward a more comprehensive study of second order perturbations of compact stars and mode coupling.

The physical picture we have in mind here is that of a star undergoing a phase of overall oscillation or wobbling that we want to describe going beyond the linear approximation. Purely radial modes are going to be a natural component of such dynamical phase; on the other hand, even leaving rotational effects aside (as we do here) it is hard to think of oscillations that don't have at least a tiny non-radial component. While purely radial and non-radial second order effects may also eventually become important in the non-linear phase, it is thus natural to first investigate the coupling of their linear components. A specific motivation for this is that the purely radial oscillations of a spherical star don't emit *per se* any gravitational waves but, if the coupling we aim to study is efficient in driving and possibly amplifying the non-spherical oscillations, one may expect the appearance of non-linear harmonics, and gravitational radiation could then be produced to a significant level, provided that at first order both the radial and the non-radial modes are non-vanishing. More in general, mode coupling typically leads to an interesting phenomenology, thus worth investigating. For example, a specific effect we can easily anticipate is that axial modes, decoupled from fluid perturbations at first order, can be sourced by radial oscillations. Finally, a further motivation is that there are a number of studies aiming at investigating if non-radial oscillations of stars can be excited by external sources (see e.g. [11, 12]). Instead, our general idea is to see if the non-radial oscillations can be driven or even amplified through coupling by an internal radial oscillation, regardless of the presence of an external source. These sort of non-linear processes could occur, for instance, in a proto-neutron star that is still pulsating. A mainly radial pulsation could for example drive the non-radial oscillations, either naturally present, or excited through fall-back accretion.

Our study can also be seen from a slightly different perspective: we are investigating linear non-radial perturbations of a radially oscillating star, where the radial oscillation is also treated perturbatively. From this point of view our problem is mathematically similar to that of studying linear perturbations of a slowly rotating star (see e.g. [13]) described by the Hartle-Thorne metric [14, 15, 16]: in the latter case the spacetime we perturb is itself a special perturbation - stationary and axisymmetric (typically up to second order) - of a spherical static star. Similarly, from this point of view we consider here perturbations of a spacetime which is itself a spherically symmetric time-dependent perturbation of a static spherical star. As we shall see, this point of view turns out to be of practical value, although it must be used with care because of the typical gauge issues of relativistic perturbation theory.

Mathematically is however more satisfactory to consider radial and non-radial perturbations as different first order perturbations of a static spherically symmetric background that couple at second order. This point of view renders transparent two crucial aspects of our problem: *i*) the perturbations are defined as fields on a static spherical background; *ii*) the two sets of perturbations are separately parametrized. Thus a well-defined framework for our study is provided by a multi-parameter relativistic perturbation formalism previously developed [17, 18]. This allows us to set up the formalism in a hopefully transparent way, properly considering the gauge dependence of perturbations. Fixing the gauge for radial perturbations, we borrow the formalism developed by Gundlach and Martín García [19, 20] (based on that of Gerlach and Sengupta [21]; we shall refer to the GSGM formalism in the following), which gives equations for gauge-invariant perturbations on a general time-dependent spherical background. This then allows us to: *i*) have gauge-invariant non-radial first order variables on our static background and *ii*) to define new second order variables,

describing the non-linear coupling of the the radial and non-radial linear perturbations, that are also gauge-invariant at second order. This higher order gauge invariance, attained by partially fixing the gauge at first order, is similar to that considered for example in [22] and [23], although in our case we deal with a 2-parameter expansion [17, 18] and we only need to fix the gauge for radial perturbations.

At first order most of the fluid perturbations appear in the polar part of the spectrum. Hence we expect that the effects of the radial non-radial coupling will predominantly manifest themselves through polar modes. Therefore in this paper we focus on polar perturbations of a perfect fluid star.

The plan of the paper is the following. In Section II we describe the general 2-parameter perturbative framework we are going to use. In Section III we briefly recall the GSGM formalism. Using this later, in Section IV we introduce the radial and non-radial first order perturbations of a static, spherically symmetric star. Section V introduces the second order perturbations that account for the coupling between the radial and non-radial first order ones; we prove the gauge invariance of such perturbations, and give the equations they fulfill. In Section VI we briefly discuss the problem of the boundary conditions, defining the second order Lagrangian pressure perturbations needed to fix the problem at the surface of the star. Finally in Section VII we draw our conclusions.

The conventions that we follow throughout this work are: Greek letters are used to denote spacetime indices; capital Latin letters are used for indices in the time-radial part of the metric; lower-case Latin indices are used for the spherical sector of the metric. We use physical units in which  $8\pi G = c = 1$ .

## II. PERTURBATIVE FRAMEWORK

In order to study the coupling of the first order radial and non-radial perturbations of a static spherically symmetric star it is convenient to use a 2-parameter perturbative approach, where the coupling will then appear at higher order. A natural framework for the study of this problem is therefore provided by the multi-parameter non-linear perturbative formalism introduced in [17, 18]. Generalising well known mathematical ideas at the basis of standard 1-parameter linear [24, 25] and non-linear [26, 27] perturbation theory, the basic underlying assumption in the construction of a multi-parameter relativistic non-linear formalism is the existence of a multi-parameter family of spacetime models that can be Taylor expanded around a *background* spacetime, representing an idealized situation. These spacetime models are labeled by a set of parameters that formally control the strength of the perturbations with respect to the background, and serve as book keeping. The crucial point to obtain a manageable theory is to choose a convenient background, in our case the static spherically symmetric star.

Let us denote the metric of this background with  $\mathbf{g}^{(0,0)}$ , i.e. a Tolman-Oppenheimer-Volkov (TOV) solution of the field equations. Denoting with  $\mathbf{g}$  the physical metric, we shall expand it in the two parameters  $\lambda$  and  $\epsilon$ , and we will use superscript indices  $(i, j)$  to denote perturbations of order  $i$  in  $\lambda$  and  $j$  in  $\epsilon$ .

We then have

$$\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\beta}^{(0,0)} + \lambda \mathbf{g}_{\alpha\beta}^{(1,0)} + \epsilon \mathbf{g}_{\alpha\beta}^{(0,1)} + \lambda\epsilon \mathbf{g}_{\alpha\beta}^{(1,1)} + O(\lambda^2, \epsilon^2), \quad (1)$$

where the terms  $\mathbf{g}^{(1,0)}$  and  $\mathbf{g}^{(0,1)}$  respectively represent first order radial and non-radial perturbations, and  $\mathbf{g}^{(1,1)}$  is the non-linear contribution due to the coupling, which is the new quantity that we want to compute. The other second order perturbations, i.e. the self-coupling terms of order  $\lambda^2$  and  $\epsilon^2$ , will not be considered in this work. Any field can be expanded as the metric in Eq. (1). In particular, we can expand in this way fluid variables like the energy density and the 4-velocity, and for the energy-momentum tensor  $\mathbf{T}$  we can formally write

$$T_{\alpha\beta} = T_{\alpha\beta}^{(0,0)} + \lambda T_{\alpha\beta}^{(1,0)} + \epsilon T_{\alpha\beta}^{(0,1)} + \lambda\epsilon T_{\alpha\beta}^{(1,1)} + O(\lambda^2, \epsilon^2), \quad (2)$$

where each term  $\mathbf{T}^{(i,j)}$  collects terms of the metric and fluid variables of the appropriate order. Let us now consider the structure of the perturbed field equations, following a standard procedure [24]. We start from the full Einstein equations:

$$\mathbf{E}[\mathbf{g}, \psi_A] = \mathbf{G}[\mathbf{g}] - \mathbf{T}[\mathbf{g}, \psi_A] = 0, \quad (3)$$

where  $\mathbf{G}$  denotes the Einstein tensor, and  $\psi_A$  ( $A=1, \dots$ ) the various fluid variables. If we introduce the perturbative expressions (1) and (2) into Eq. (3), we can expand the latter up to (1, 1) order, obtaining

$$\begin{aligned} \mathbf{E}[\mathbf{g}, \psi_A] = & \mathbf{E}^{(0,0)} \left[ \mathbf{g}^{(0,0)}, \psi_A^{(0,0)} \right] + \lambda \mathbf{E}^{(1,0)} \left[ \mathbf{g}^{(1,0)}, \psi_A^{(1,0)} \right] + \epsilon \mathbf{E}^{(0,1)} \left[ \mathbf{g}^{(0,1)}, \psi_A^{(0,1)} \right] \\ & + \lambda\epsilon \mathbf{E}^{(1,1)} \left[ \mathbf{g}^{(1,1)}, \psi_A^{(1,1)} \mid \mathbf{g}^{(1,0)} \otimes \mathbf{g}^{(0,1)}, \psi_A^{(1,0)} \otimes \psi_A^{(0,1)}, \mathbf{g}^{(1,0)} \otimes \psi_A^{(0,1)}, \psi_A^{(1,0)} \otimes \mathbf{g}^{(0,1)} \right] + O(\lambda^2, \epsilon^2) = 0. \end{aligned} \quad (4)$$

The previous equation is satisfied for arbitrary values of the two parameters if, and only if, each coefficient of the expansion vanishes. Therefore, setting each of these terms to zero,  $\mathbf{E}^{(0,0)} = 0$  represents the TOV equations (see, e.g., [28]), while each of the other  $\mathbf{E}^{(i,j)} = 0$  terms represent the perturbative equations of order  $(i, j)$ . As differential operators, the  $\mathbf{E}^{(i,j)}$  act linearly on each of the terms in square brackets, while they are non-linear functions of the background quantities  $\mathbf{g}^{(0,0)}$  and  $\psi_A^{(0,0)}$ .

At first order in  $\lambda$ , we obtain the equations describing the radial perturbations on the TOV background,

$$\mathbf{E}^{(1,0)} \left[ \mathbf{g}^{(1,0)}, \psi_A^{(1,0)} \right] = 0. \quad (5)$$

Linear radial perturbations have been extensively analyzed in the literature (see [28] and references therein, and [29] for more recent results).

The linearized equations for the non-radial perturbations come from the first order terms in  $\epsilon$ ,

$$\mathbf{E}^{(0,1)} \left[ \mathbf{g}^{(0,1)}, \psi_A^{(0,1)} \right] = 0, \quad (6)$$

and were first studied by Thorne and Campolattaro [30]. Later they became the subject of many investigations, see e.g. [31, 32, 33, 34, 35, 36].

Finally, the equations describing the radial non-radial coupling, the ones we shall focus on, have the following form

$$\mathbf{E}^{(1,1)} \left[ \mathbf{g}^{(1,1)}, \psi_A^{(1,1)} \mid \mathbf{g}^{(1,0)} \otimes \mathbf{g}^{(0,1)}, \psi_A^{(1,0)} \otimes \psi_A^{(0,1)}, \mathbf{g}^{(1,0)} \otimes \psi_A^{(0,1)}, \psi_A^{(1,0)} \otimes \mathbf{g}^{(0,1)} \right] = 0. \quad (7)$$

It is an intrinsic feature of perturbation theory that the procedure to solve the above equations is iterative. Thus, when we arrive at the stage of solving Eq. (7), the terms  $\mathbf{g}^{(1,0)}$  and  $\psi_A^{(1,0)}$  are assumed to be known from solving the radial linear equation (5), while  $\mathbf{g}^{(0,1)}$  and  $\psi_A^{(0,1)}$  are solutions of the non-radial linear perturbation equations (6). Hence, because of the linearity of the operator  $\mathbf{E}^{(1,1)}$  in acting on each of the terms in square brackets, these terms play the role of sources in Eq. (7). Taking into account the nature of the different sets of perturbations we are considering, it turns out that the operator  $\mathbf{E}^{(1,1)}$  acts on the pair  $\mathbf{g}^{(1,1)}, \psi_A^{(1,1)}$  in Eq. (7) in the same way that  $\mathbf{E}^{(0,1)}$  acts on  $\mathbf{g}^{(0,1)}, \psi_A^{(0,1)}$  in Eq. (6). The reason for this is that both operators  $\mathbf{E}^{(0,1)}$  and  $\mathbf{E}^{(1,1)}$  come from the linearization, around the static background, of the Einstein tensor operator acting on non-radial perturbations. Therefore, using again the linearity of  $\mathbf{E}^{(1,1)}$ , we can define

$$\mathbf{L}_{\text{NR}} [\cdot] \equiv \mathbf{E}^{(1,1)} [\cdot \mid \mathbf{0}] = \mathbf{E}^{(0,1)} [\cdot], \quad (8)$$

as the non-radial perturbation operator. Hence, we can re-write equation (7) in the final form

$$\mathbf{L}_{\text{NR}} \left[ \mathbf{g}^{(1,1)}, \psi_A^{(1,1)} \right] = \mathbf{S} \left[ \mathbf{g}^{(1,0)} \otimes \mathbf{g}^{(0,1)}, \psi_A^{(1,0)} \otimes \psi_A^{(0,1)}, \mathbf{g}^{(1,0)} \otimes \psi_A^{(0,1)}, \psi_A^{(1,0)} \otimes \mathbf{g}^{(0,1)} \right]. \quad (9)$$

The particular structure of these equations is very helpful in order to develop a numerical code for studying the coupling of radial and non-radial perturbations. We want to emphasize here that our goal is that of solving the perturbative equations in the time domain. To this end it is then very useful to rely on well known initial value formulations for the linear non-radial perturbations. Thus, assuming we have working numerical codes for the first order perturbations in  $\lambda$  and  $\epsilon$ , in order to implement a code for the coupling  $\lambda\epsilon$  variables we only need to modify the code for the  $\epsilon$  variables by adding the source terms to the right-hand side of the evolution algorithm. Then, at every time step in the evolution, and having fixed the background, we have to: *i*) evolve the equations to obtain the value of the first order radial and non-radial perturbations; *ii*) to use their values to evolve the coupling variables.

### III. SUMMARY OF THE GSGM FORMALISM

In this Section we briefly recall the GSGM formalism, introduced by Gerlach and Sengupta [21, 37] and further developed by Gundlach and Martín-García [19, 20, 38], to study first order gauge-invariant perturbations of a general time-dependent spherically symmetric stellar background. From the formal point of view of parameter expansion of Section II, here we are dealing with standard 1-parameter linear perturbations of the form

$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + \epsilon g_{\alpha\beta}^{(1)}, \quad (10)$$

where  $g^{(0)}$  is the time-dependent spherically symmetric background and the  $\epsilon$  perturbations are non-radial.

### A. The time dependent perfect fluid background

The background manifold is the warped product  $M^2 \times S^2$ , where  $S^2$  denotes the 2-sphere and  $M^2$  a two-dimensional Lorentzian manifold. The metric can be written as the semidirect product of a general Lorentzian metric on  $M^2$ ,  $g_{AB}$ , and the unit curvature metric on  $S^2$ , that we call  $\gamma_{ab}$ :

$$g_{\alpha\beta} = \begin{pmatrix} g_{AB} & 0 \\ 0 & r^2\gamma_{ab} \end{pmatrix}. \quad (11)$$

Hereafter,  $x^A$  denotes the coordinates on  $M^2$  and  $x^a$  the coordinates on  $S^2$ ;  $r = r(x^A)$  is a function on  $M^2$  that coincides with the invariantly defined radial (area) coordinate of spherically-symmetric spacetimes. A vertical bar is used to denote the covariant derivative on  $M^2$  and a semicolon to denote the one on  $S^2$ , thus we have  $g_{AB|C} = \gamma_{ab;c} = 0$ . One can introduce the completely antisymmetric covariant unit tensors on  $M^2$  and on  $S^2$ ,  $\epsilon_{AB}$  and  $\epsilon_{ab}$  respectively, in such a way they satisfy:  $\epsilon_{AB|C} = \epsilon_{ab;c} = 0$ ,  $\epsilon_{AC}\epsilon^{BC} = -g_A^B$ , and  $\epsilon_{ac}\epsilon^{bc} = -\gamma_a^c$ .

In this paper we consider a perfect-fluid description of the stellar matter, thus the energy-momentum tensor is

$$t_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta}, \quad (12)$$

where  $\rho$  and  $p$  are the energy density and pressure, and  $u_\alpha$  is the fluid velocity.

In the spherically symmetry background  $t_{\alpha\beta}$  has the same block diagonal structure than the metric,

$$t_{\alpha\beta} = \text{diag} ( t_{AB}, r^2 Q(x^C)\gamma_{ab} ), \quad (13)$$

and the fluid velocity takes the form  $u_\alpha = (u_A, 0)$ . An orthonormal frame on the submanifold  $M^2$  can be formed from  $u^A$  and the spacelike vector

$$n_A \equiv -\epsilon_{AB}u^B \quad \Rightarrow \quad n_A u^A = 0. \quad (14)$$

The metric  $g_{AB}$  and  $\epsilon_{AB}$  can be written in terms of these frame vectors as follows

$$g_{AB} = -u_A u_B + n_A n_B, \quad \epsilon_{AB} = n_A u_B - u_A n_B. \quad (15)$$

Then we have

$$t_{AB} = \rho u_A u_B + p n_A n_B, \quad Q = p. \quad (16)$$

In any given coordinate system for  $M^2$ ,  $\{x^A\}$ , one can define the following quantity:

$$v_A \equiv \frac{1}{r} r_{|A}. \quad (17)$$

Then, any covariant derivative on the spacetime can be written in terms of the covariant derivatives on  $M^2$  and  $S^2$ , plus some terms due to the warp factor  $r^2$ , which can be written in terms of  $v_A$ . Finally, the frame derivatives of a generic scalar function  $f$  are defined by

$$\dot{f} = u^A f_{|A}, \quad f' = n^A f_{|A}, \quad (18)$$

and we introduce the following background scalars:

$$\Omega = \ln \rho, \quad U = u^A v_A, \quad W = n^A v_A, \quad \mu = u^A_{|A}, \quad \nu = n^A_{|A}. \quad (19)$$

### B. Perturbations

Linear perturbations of a spherically-symmetric background can be decomposed in scalar, vector and tensor spherical harmonics. The scalar spherical harmonics  $Y^{lm}$  are eigenfunctions of the covariant Laplacian on the sphere:

$$\gamma^{ab} Y_{;ab}^{lm} = -l(l+1)Y^{lm}. \quad (20)$$

A basis of vector spherical harmonics (defined for  $l \geq 1$ ) is

$$Y_a^{lm} \equiv Y_{;a}^{lm}, \quad S_a^{lm} \equiv \epsilon_a^b Y_b^{lm}, \quad (21)$$

where the  $Y_a^{lm}$ 's have polar parity (they transform as  $(-1)^l$ , like the scalar harmonics, under parity transformations, and are also called even-parity type) and the  $S_a^{lm}$ 's have axial parity (they transform as  $(-1)^{l+1}$  under parity transformations, and are also called odd-parity type). A basis of tensor spherical harmonics (defined for  $l \geq 2$ ) is

$$Y_{ab}^{lm} \equiv Y^{lm} \gamma_{ab}, \quad Z_{ab}^{lm} \equiv Y_{:ab}^{lm} + \frac{l(l+1)}{2} Y^{lm} \gamma_{ab}, \quad S_{ab}^{lm} \equiv S_{a:b}^{lm} + S_{b:a}^{lm}, \quad (22)$$

where the  $Y_{ab}^{lm}$ ,  $Z_{ab}^{lm}$  have polar parity and the  $S_{ab}^{lm}$  have axial parity. In this paper we will only consider perturbations with polar parity.

The perturbations of the covariant metric and energy-momentum tensors can be expanded in this basis as

$$\delta g_{\alpha\beta} = \begin{pmatrix} h_{AB}^{lm} Y^{lm} & h_A^{lm} Y_a^{lm} \\ h_A^{lm} Y_a^{lm} & r^2 (K^{lm} \gamma_{ab} Y^{lm} + G^{lm} Y_{:ab}^{lm}) \end{pmatrix}, \quad (23)$$

$$\delta t_{\alpha\beta} = \begin{pmatrix} \delta t_{AB}^{lm} Y^{lm} & \delta t_A^{lm} Y_a^{lm} \\ \delta t_A^{lm} Y_a^{lm} & r^2 \delta t^{3lm} \gamma_{ab} Y^{lm} + \delta t^{2lm} Y_{:ab}^{lm} \end{pmatrix}. \quad (24)$$

Let  $X$  be an arbitrary tensor field on the background spacetime and  $\delta X$  its linear perturbation. It is well-known that under a first order gauge transformation, generated by a vector field  $\xi$  living on the background, the perturbation of  $X$  transforms as

$$\delta X \rightarrow \delta X + \mathcal{L}_\xi X. \quad (25)$$

Then, the perturbation  $\delta X$  is gauge-invariant if and only if the Lie derivative of the corresponding background quantity  $X$  with respect to an arbitrary vector field  $\xi$  vanishes:  $\mathcal{L}_\xi X = 0$  [25]. Using this well-known result, it is possible to show that a complete set of gauge-invariant variables, combinations of the perturbations  $h_{AB}$ ,  $h_A$ ,  $K$ ,  $G$ ,  $\delta t_{AB}$ ,  $\delta t_A$ ,  $\delta t^2$ ,  $\delta t^3$ , is given by the following quantities:

$$k_{AB} = h_{AB} - (p_{A|B} + p_{B|A}), \quad (26)$$

$$k = K - 2v^A p_A, \quad (27)$$

$$T_{AB} = \delta t_{AB} - t_{AB|C} p^C - t_{AC} p_{|B}^C - t_{BC} p_{|A}^C, \quad (28)$$

$$T^3 = \delta t^3 - p^C (Q_{|C} + 2Q v_C) + \frac{l(l+1)}{2} Q G, \quad (29)$$

$$T_A = \delta t_A - t_{AC} p^C - \frac{r^2}{2} Q G_{|A}, \quad (30)$$

$$T^2 = \delta t^2 - r^2 Q G, \quad (31)$$

where  $T_A$  is defined for  $l \geq 1$ ,  $T^2$  is defined for  $l \geq 2$ , and

$$p_A = h_A - \frac{1}{2} r^2 G_{|A}. \quad (32)$$

Therefore, any linear perturbation of the spherically-symmetric background (11) can be written as a linear combination of these gauge-invariant quantities. It is important to stress the fact that although we are considering a perfect fluid energy-momentum tensor, the gauge invariant quantities introduced above are defined for any energy-momentum tensor. The equation of state for a perfect fluid has the form  $p = p(\rho, s)$ ,  $s$  being the specific entropy. The corresponding sound speed,  $c_s$ , can then be defined through the thermodynamical derivative

$$c_s^2 = \left( \frac{\partial p}{\partial \rho} \right)_s. \quad (33)$$

Throughout this paper we will consider the particular case of a barotropic fluid (i.e. constant specific entropy), hence we will have  $p = p(\rho)$ .

The polar fluid perturbations are described as follows. The perturbations of the fluid velocity components can be written as

$$\delta u_\alpha = \left( \left[ \tilde{\gamma}^{lm} n_A + \frac{1}{2} h_{AB}^{lm} u^B \right] Y^{lm}, \tilde{\alpha}^{lm} Y_{:a}^{lm} \right), \quad (34)$$

where  $\tilde{\alpha}$  is defined for  $l \geq 1$ . The energy density and pressure perturbations can be cast in the following form (using the barotropic equation of state)

$$\delta\rho = \tilde{\omega}\rho Y^{lm}, \quad \delta p = c_s^2 \delta\rho. \quad (35)$$

In terms of these quantities it is possible to define a gauge-invariant set of fluid perturbations:

$$\alpha = \tilde{\alpha} - p^B u_B, \quad (36)$$

$$\gamma = \tilde{\gamma} - n^A \left[ p^B u_{A|B} + \frac{1}{2} u^B (p_{B|A} - p_{A|B}) \right], \quad (37)$$

$$\omega = \tilde{\omega} - p^A \Omega_{|A}. \quad (38)$$

The tensor  $k_{AB}$  can be decomposed in the frame  $\{u^A, n^A\}$ :

$$k_{AB} = \eta(-u_A u_B + n_A n_B) + \phi(u_A u_B + n_A n_B) + \psi(u_A n_B + n_A u_B), \quad (39)$$

where  $\eta$ ,  $\phi$  and  $\psi$  are scalars. It is useful to consider the following new scalar variable

$$\chi = \phi - k + \eta, \quad (40)$$

in the place of  $\phi$ . Then, combining Einstein equations with the energy-momentum equations we can obtain the following set of equations: for  $l \geq 2$ ,

$$\eta = 0, \quad (41)$$

for  $l \geq 1$ ,

$$-\ddot{\chi} + \chi'' + 2(\mu - U)\psi' = S_\chi, \quad (42)$$

$$-\ddot{k} + c_s^2 k'' - 2c_s^2 U\psi' = S_k, \quad (43)$$

$$-\dot{\psi} = S_\psi, \quad (44)$$

$$16\pi(\rho + p)\alpha = \psi' + C_\alpha, \quad (45)$$

$$-\dot{\alpha} = S_\alpha, \quad (46)$$

$$-\dot{\omega} - \left(1 + \frac{p}{\rho}\right) \gamma' = \bar{S}_\omega, \quad (47)$$

$$\left(1 + \frac{p}{\rho}\right) \dot{\gamma} + c_s^2 \omega' = \bar{S}_\gamma. \quad (48)$$

And finally, for  $l \geq 0$ ,

$$8\pi(\rho + p)\gamma = (\dot{k})' + C_\gamma, \quad (49)$$

$$8\pi\rho\omega = -k'' + 2U\psi' + C_\omega, \quad (50)$$

where the expressions of  $S_\chi$ ,  $S_\psi$ ,  $C_\alpha$ ,  $S_\alpha$ ,  $\bar{S}_\omega$ ,  $\bar{S}_\gamma$ ,  $C_\gamma$ ,  $C_\omega$  can be found in [19].

#### IV. RADIAL AND NON-RADIAL PERTURBATIONS OF STATIC RELATIVISTIC STARS

In what follows we summarize the first order perturbative analysis of the oscillations of a perfect-fluid static star. As anticipated in Section II, we consider separately the radial pulsations, parametrized by  $\lambda$ , and the non-radial oscillations, parametrized by  $\epsilon$ . Therefore, consistently with the notation of Section II, we will now explicitly use the indices  $(i, j)$ ; however, to simplify the notation, from now on we will use a bar to denote quantities associated with the static spacetime, the background of our 2-parameter perturbative formalism. Thus we have  $\bar{g}_{\alpha\beta} \equiv g_{\alpha\beta}^{(0,0)}$ , and in the same way  $\bar{\rho}$ ,  $\bar{p}$ , and  $\bar{u}_\alpha$ .

The equilibrium configuration is described by the static spherically-symmetric metric:

$$\bar{g}_{\alpha\beta} dx^\alpha dx^\beta = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (51)$$

with a perfect-fluid energy-momentum tensor

$$\bar{t}_{\alpha\beta} = (\bar{\rho} + \bar{p}) \bar{u}_\alpha \bar{u}_\beta + \bar{p} \bar{g}_{\alpha\beta}, \quad (52)$$

where  $\bar{u}_\alpha = (-e^\Phi, 0, 0, 0)$ . The mass function is introduced by means of the equality  $e^{-2\Lambda(r)} = 1 - 2M(r)/r$ . Then, the TOV equations are:

$$\Phi_{,r} = \frac{M + 4\pi\bar{p}r^3}{r(r - 2M)} = -\frac{\bar{p}_{,r}}{\bar{\rho} + \bar{p}}, \quad (53)$$

$$M_{,r} = 4\pi\bar{p}r^2. \quad (54)$$

Specifying the equation of state of the stellar matter one obtains a 1-parameter family of solutions of these equations, depending on the central density. For the barotropic equation of state we use,  $\bar{p} = \bar{p}(\bar{\rho})$ , the background sound speed is  $\bar{c}_s^2 = d\bar{p}/d\bar{\rho}$ .

### A. Non-radial perturbations in the GSGM formalism

The equations for the first order non-radial perturbations have been known for a long time [30, 31, 32]. In [39] they have been presented in the framework of the GSGM formalism. We do not write their explicit expressions here, because they can be obtained as a particular case of the equations that we will write in Section V C for the coupling perturbative terms of order (1, 1), like  $\mathbf{g}^{(1,1)}$ , considering the homogeneous part of those equations, i.e. simply neglecting the source terms. One can also obtain the same equations directly from the general GSGM equations (41-50), by considering the special case of a static background, represented by the following quantities,

$$\bar{u}^A = (e^{-\Phi}, 0), \quad \bar{n}^A = (0, e^{-\Lambda}), \quad (55)$$

$$\bar{\mu} = \bar{U} = 0, \quad \bar{\nu} = \Phi', \quad \bar{W} = \frac{e^{-\Lambda}}{r}, \quad (56)$$

and where the frame derivatives of a scalar function  $f$  take the special form  $f' = e^{-\Lambda} f_{,r}$  and  $\dot{f} = e^{-\Phi} f_{,t}$ . Boundary conditions have to be imposed at infinity, at the stellar origin and on the stellar surface. They can be found in the references cited above.

### B. Radial perturbations in the GSGM formalism

Radial perturbations can be seen as a particular subcase of non-radial ones, namely those corresponding to the harmonic  $l = 0$ . The fact that the GSGM quantities are not gauge-invariant in the  $l = 0$  case does not represent a problem for our study since, when we will consider the gauge invariance of our (1, 1) variables in Section V B, we will assume that the gauge of the perturbative order (1, 0) has been somehow fixed. In practice, we fix the radial gauge as in [19],

$$\psi^{(1,0)} = 0, \quad k^{(1,0)} = 0, \quad (57)$$

which considerably simplifies the equations. In this gauge, the metric perturbations have a diagonal form as in the background:

$$\mathbf{g}_{\alpha\beta}^{(1,0)} = \text{diag} \left( h_{AB}^{(1,0)}, 0 \right), \quad (58)$$

$$h_{AB}^{(1,0)} = \eta^{(1,0)} \bar{g}_{AB} + \phi^{(1,0)} (\bar{u}_A \bar{u}_B + \bar{n}_A \bar{n}_B). \quad (59)$$

Then, the components of  $h_{AB}^{(1,0)}$  are given by

$$h_{AB}^{(1,0)} = \text{diag} \left( e^{2\Phi} \left( \chi^{(1,0)} - 2\eta^{(1,0)} \right), e^{2\Lambda} \chi^{(1,0)} \right), \quad (60)$$

where  $\chi^{(1,0)} = \eta^{(1,0)} + \phi^{(1,0)}$ .



For the fluid velocity, from Eq. (34), we have

$$\delta u_A^{(1,0)} = \left( \left( \frac{\chi^{(1,0)}}{2} - \eta^{(1,0)} \right) e^\Phi, e^\Lambda \gamma^{(1,0)} \right), \quad \delta u_a^{(1,0)} = 0. \quad (61)$$

The other fluid perturbations are given by:

$$\delta \rho^{(1,0)} = \omega^{(1,0)} \bar{\rho}, \quad \delta p^{(1,0)} = \bar{c}_s^2 \delta \rho^{(1,0)}. \quad (62)$$

### C. Equations for the radial perturbations

The equations for the quantities  $\chi^{(1,0)}, \eta^{(1,0)}, \omega^{(1,0)}, \gamma^{(1,0)}$  describing the radial perturbations can be found in [19]. For our purposes it is more convenient to use a different set of variables. First, instead of using  $\omega^{(1,0)}$  we use the enthalpy perturbation  $H^{(1,0)}$ , which significantly simplifies the equations. The second change consists in replacing the metric perturbation  $\chi^{(1,0)}$  with the quantity  $S^{(1,0)}$ , in order to use a set of variables consistent with the one we will use for the non-radial perturbations in Section V C. The definitions of  $H^{(1,0)}$  and  $S^{(1,0)}$  are:

$$H^{(1,0)} \equiv \frac{\delta p^{(1,0)}}{\bar{\rho} + \bar{p}} = \frac{\bar{c}_s^2 \bar{\rho}}{\bar{\rho} + \bar{p}} \omega^{(1,0)}, \quad S^{(1,0)} \equiv \frac{\chi^{(1,0)}}{r}. \quad (63)$$

Furthermore, using equation (34) in [38] we end up with a set of evolution equations for  $S^{(1,0)}, H^{(1,0)}$  and  $\gamma^{(1,0)}$  which does not contain the quantity  $\eta^{(1,0)}$ ,

$$-\dot{H}^{(1,0)} = \bar{c}_s^2 \gamma^{(1,0)'} + \bar{c}_s^2 \left[ \left( 1 - \frac{1}{\bar{c}_s^2} \right) \bar{v} + 2\bar{W} - \frac{4\pi}{\bar{W}} (\bar{\rho} + \bar{p}) \right] \gamma^{(1,0)}, \quad (64)$$

$$\dot{\gamma}^{(1,0)} = -H^{(1,0)'} - \frac{4\pi}{\bar{W}} (\bar{\rho} + \bar{p}) H^{(1,0)} - \left( \bar{v} + \frac{\bar{W}}{2} \right) r S^{(1,0)}, \quad (65)$$

$$\dot{S}^{(1,0)} = -8\pi (\bar{\rho} + \bar{p}) \frac{1}{r\bar{W}} \gamma^{(1,0)}, \quad (66)$$

and with the Hamiltonian constraint

$$\bar{W} r S^{(1,0)'} = \left( 8\pi \bar{\rho} r - \frac{1}{r} - \bar{W} r' \right) S^{(1,0)} + 8\pi \frac{\bar{\rho} + \bar{p}}{\bar{c}_s^2} H^{(1,0)}. \quad (67)$$

The quantity  $\eta^{(1,0)}$  can be found, in terms of  $(S^{(1,0)}, H^{(1,0)}, \gamma^{(1,0)})$ , from the following equation:

$$\bar{W} \eta^{(1,0)'} = 4\pi (\bar{\rho} + \bar{p}) \left[ r S^{(1,0)} + \left( 1 + \frac{1}{\bar{c}_s^2} \right) H^{(1,0)} \right]. \quad (68)$$

The system of equations (64-67) is equivalent to the one used by Ruoff [40], cf. also [28].

### D. Boundary Conditions for radial perturbations

Boundary conditions must be fixed at the center and at the stellar surface  $r = R$ . The latter is given by the vanishing of the Lagrangian pressure perturbation,  $\Delta p = 0$ . Following [28], we can express  $\Delta p^{(1,0)}$  in terms of the *radial renormalized displacement function*  $\zeta$ :

$$r^2 \Delta p^{(1,0)} = -(\bar{\rho} + \bar{p}) \bar{c}_s^2 e^{-\Phi} \frac{\partial \zeta}{\partial r}. \quad (69)$$

Using the relation between  $\zeta$  and  $\gamma^{(1,0)}$

$$\zeta_{,t} = r^2 e^{-\Lambda} \gamma^{(1,0)} \quad (70)$$

we arrive at the following boundary condition on the surface:

$$(\bar{\rho} + \bar{p}) \bar{c}_s^2 e^{-\Phi} \left( r^2 e^{-\Lambda} \gamma^{(1,0)} \right)_{,r} \Big|_{r=R} = 0. \quad (71)$$

The behaviour of  $S^{(1,0)}$  and  $H^{(1,0)}$  on the surface can be found from the general evolution equations (66) and (64).

The analysis of the regularity conditions at the origin ( $r = 0$ ) leads to the following expressions:

$$(S^{(1,0)}, \eta^{(1,0)}, H^{(1,0)}, \gamma^{(1,0)}) \longrightarrow (S_o^{(1,0)}(t) r + O(r^3), \eta_o^{(1,0)}(t) + O(r^2), H_o^{(1,0)}(t) + O(r^2), \gamma_o^{(1,0)}(t) r + O(r^3)). \quad (72)$$

## V. COUPLING OF RADIAL AND NON-RADIAL PERTURBATIONS OF RELATIVISTIC STARS

At this point we have already specified the equations that determine the static background metric  $\mathbf{g}^{(0,0)}$ , the radial perturbations of order  $\lambda$ , and the non-radial perturbations of order  $\epsilon$ . The next crucial step is to find the equations for the perturbative coupling terms of order  $\lambda\epsilon$ , like the metric perturbations  $\mathbf{g}^{(1,1)}$ . To this end we can further expand the first order 1-parameter GSGM formalism of Section III as follows. The time dependent spherically symmetric variables can be split into a static and a radially oscillating parts, e.g. for the metric

$$\mathbf{g}_{\alpha\beta}^{(0)} = \mathbf{g}_{\alpha\beta}^{(0,0)} + \lambda \mathbf{g}_{\alpha\beta}^{(1,0)}. \quad (73)$$

The non-radial first order perturbations on this time dependent spacetime can be split into a part which is the non-radial perturbation of the static background and a further term that describes the coupling. For the metric we then have

$$\mathbf{g}_{\alpha\beta}^{(1)} = \mathbf{g}_{\alpha\beta}^{(0,1)} + \lambda \mathbf{g}_{\alpha\beta}^{(1,1)}. \quad (74)$$

Inserting (73) and (74) into Eq. (10) we re-obtain the 2-parameter expansion (1), and similarly for the fluid variables.

This approach, based on splitting the GSGM variables and their equations, is very convenient because it saves us from the need of computing directly a second order expansion of Einstein's equations. In addition, following this approach we shall show in Section VB how to build gauge-invariant quantities associated with the perturbations  $\mathbf{g}^{(1,1)}$  and the fluid variables of order (1, 1).

### A. GSGM formalism on a radially oscillating star

In order to implement the GSGM formalism on a radially oscillating star, which is itself treated perturbatively, we split the quantities  $u^A$ ,  $n^A$ ,  $U$ ,  $W$ ,  $\mu$ ,  $\nu$  in a static background part and a radial perturbation, as illustrated above. The frame vector fields are given by

$$u^A = \left( \left[ 1 - \lambda \left( \eta^{(1,0)} - \frac{\chi^{(1,0)}}{2} \right) \right] e^{-\Phi}, \lambda e^{-\Lambda} \gamma^{(1,0)} \right), \quad (75)$$

$$n^A = \left( \lambda e^{-\Phi} \gamma^{(1,0)}, \left( 1 - \lambda \frac{\chi^{(1,0)}}{2} \right) e^{-\Lambda} \right). \quad (76)$$

The frame derivatives of a scalar perturbation,  $f^{(1)} = f^{(0,1)} + \lambda f^{(1,1)}$ , on the radially oscillating star are:

$$\dot{f}^{(1)} = u^A f_{,A}^{(1)} = e^{-\Phi} f_{,t}^{(0,1)} + \lambda \left\{ e^{-\Phi} f_{,t}^{(1,1)} + e^{-\Lambda} \gamma^{(1,0)} f_{,r}^{(0,1)} - e^{-\Phi} \left( \eta^{(1,0)} - \frac{\chi^{(1,0)}}{2} \right) f_{,t}^{(0,1)} \right\}, \quad (77)$$

$$f^{(1)'} = n^A f_{,A}^{(1)} = e^{-\Lambda} f_{,r}^{(0,1)} + \lambda \left\{ e^{-\Lambda} f_{,r}^{(1,1)} + e^{-\Phi} \gamma^{(1,0)} f_{,t}^{(0,1)} - \frac{\chi^{(1,0)}}{2} e^{-\Lambda} f_{,r}^{(0,1)} \right\}. \quad (78)$$

The remaining quantities describing the spherical star are

$$U = u^A v_A = \lambda \frac{e^{-\Lambda}}{r} \gamma_{,t}^{(1,0)}, \quad (79)$$

$$W = n^A v_A = \left( \frac{1}{r} - \lambda \frac{\chi^{(1,0)}}{2r} \right) e^{-\Lambda}, \quad (80)$$

$$\mu = u^A|_A = \lambda \left( \gamma^{(1,0)} e^{-\Lambda} \right)_{,r}, \quad (81)$$

$$\nu = n^A|_A = \Phi_{,r} e^{-\Lambda} + \lambda \left\{ e^{-\Phi} \gamma_{,t}^{(1,0)} + e^{-\Lambda} \left[ \left( \eta_{,r}^{(1,0)} - \frac{1}{2} \chi_{,r}^{(1,0)} \right) - \frac{1}{2} \Phi_{,r} \chi^{(1,0)} \right] \right\}, \quad (82)$$

where in (81) we have used the (66).

### B. Gauge invariance of the $\lambda\epsilon$ coupling perturbations

In this Section we are going to construct a set of gauge-invariant quantities at the (1, 1) perturbative order. As we shall see, we can find them with the help of the GSGM formalism. The main idea behind our construction is to build

the (1, 1) gauge-invariant variables starting from the gauge-invariant quantities of the GSGM formalism, considering these latter as perturbations of a radially pulsating spacetime which is itself described as a perturbation of the static background, i.e.  $\mathbf{g}_{\alpha\beta}^{(0)} = \mathbf{g}_{\alpha\beta}^{(0,0)} + \lambda \mathbf{g}_{\alpha\beta}^{(1,0)}$ . More precisely, let  $\mathcal{G}^{(1)}$  be any of the gauge-invariant quantities in equations (26-31) and (36-38), but constructed as a metric and energy-momentum perturbation of the pulsating star. Then, we can expand  $\mathcal{G}^{(1)}$  in  $\lambda$  to get

$$\mathcal{G}^{(1)} = \mathcal{G}^{(0,1)} + \lambda \mathcal{G}^{(1,1)}. \quad (83)$$

It is important to remark here that the (1, 1) superscript refers not only to quantities constructed from the  $\mathbf{g}^{(1,1)}$  perturbations, but in general to any perturbative quantity of order  $\lambda\epsilon$ . In general  $\mathcal{G}^{(0,1)}$  and  $\mathcal{G}^{(1,1)}$  can be expressed as:

$$\mathcal{G}^{(0,1)} = \mathcal{H}^{(0,1)}, \quad (84)$$

$$\mathcal{G}^{(1,1)} = \mathcal{H}^{(1,1)} + \sum_{\sigma} \mathcal{I}_{\sigma}^{(1,0)} \mathcal{J}_{\sigma}^{(0,1)}, \quad (85)$$

where the objects  $\mathcal{H}^{(0,1)}$  and  $\mathcal{J}_{\sigma}^{(0,1)}$  are linear in the (0, 1) perturbations, while  $\mathcal{I}_{\sigma}^{(1,0)}$  and  $\mathcal{H}^{(1,1)}$  are respectively linear in the (1, 0) and the (1, 1) variables. It is clear that  $\mathcal{H}^{(0,1)}$  is nothing but any of the gauge-invariant quantities in (26-31) and (36-38) for the special case of a static background, as it must be for the (0, 1) perturbations. The quantity  $\mathcal{H}^{(1,1)}$  is constructed from (1, 1) quantities in the same way but, as we are going to see, it is not a gauge-invariant object at the order (1, 1): we have to add some extra terms of the form given in (85) and which come from the ansatz (83). In what follows we show that this gives the desired gauge-invariant (1, 1) quantities.

Gauge transformations and gauge invariance in 2-parameter perturbation theory have been studied in [17, 18]. The gauge transformation of a first order perturbation of order (0, 1) of a generic tensorial quantity  $\mathcal{T}$  is given, as in (25), by

$$\tilde{\mathcal{T}}^{(0,1)} = \mathcal{T}^{(0,1)} + \mathcal{L}_{\xi_{(0,1)}} \mathcal{T}^{(0,0)}, \quad (86)$$

whereas a second order perturbation of  $\mathcal{T}$ , in particular of order (1, 1), transforms according to

$$\tilde{\mathcal{T}}^{(1,1)} = \mathcal{T}^{(1,1)} + \mathcal{L}_{\xi_{(0,1)}} \mathcal{T}^{(1,0)} + \mathcal{L}_{\xi_{(1,0)}} \mathcal{T}^{(0,1)} + (\mathcal{L}_{\xi_{(1,1)}} + \{\mathcal{L}_{\xi_{(1,0)}}, \mathcal{L}_{\xi_{(0,1)}}\}) \mathcal{T}^{(0,0)}, \quad (87)$$

where  $\{, \}$  stands for the anti-commutator  $\{a, b\} = ab + ba$ .

In the present case, we have chosen to fix the gauge for the radial perturbations  $\mathbf{g}^{(1,0)}$ , see equation (57). This simplifies the previous transformation rule to

$$\tilde{\mathcal{T}}^{(1,1)} = \mathcal{T}^{(1,1)} + \mathcal{L}_{\xi_{(0,1)}} \mathcal{T}^{(1,0)} + \mathcal{L}_{\xi_{(1,1)}} \mathcal{T}^{(0,0)}. \quad (88)$$

We have assumed that  $\mathcal{G}^{(0,1)}$  is a gauge-invariant quantity at order (0, 1), therefore

$$\tilde{\mathcal{G}}^{(0,1)} - \mathcal{G}^{(0,1)} = \tilde{\mathcal{H}}^{(0,1)} - \mathcal{H}^{(0,1)} = 0. \quad (89)$$

From (85) and the fact that we have fixed the gauge for radial perturbations, we can write

$$\tilde{\mathcal{G}}^{(1,1)} - \mathcal{G}^{(1,1)} = \tilde{\mathcal{H}}^{(1,1)} - \mathcal{H}^{(1,1)} + \sum_{\sigma} \mathcal{I}_{\sigma}^{(1,0)} (\tilde{\mathcal{J}}_{\sigma}^{(0,1)} - \mathcal{J}_{\sigma}^{(0,1)}). \quad (90)$$

Furthermore, we note that every  $\mathcal{H}^{(1,1)}$  and  $\mathcal{J}_{\sigma}^{(0,1)}$  can be expressed as follows:

$$\mathcal{H}^{(1,1)} = \mathcal{A} [\mathbf{g}^{(1,1)}], \quad \mathcal{J}_{\sigma}^{(0,1)} = \mathcal{B}_{\sigma} [\mathbf{g}^{(0,1)}], \quad (91)$$

where  $\mathcal{A}$  and  $\mathcal{B}_{\sigma}$  are linear operators involving differentiation with respect to the coordinates of  $M^2$  and integration on  $S^2$ . These operators act on spacetime objects and return objects with indices on  $M^2$ . From now on, for the sake of simplicity, we only consider metric perturbations. The corresponding procedure for energy-momentum tensor perturbations follows the same lines and is given in Appendix C. Using the gauge transformations (86) and (88), the transformation rules for  $\mathcal{H}^{(1,1)}$  and  $\mathcal{J}_{\sigma}^{(0,1)}$  are given by

$$\tilde{\mathcal{H}}^{(1,1)} = \mathcal{H}^{(1,1)} + \mathcal{A} [\mathcal{L}_{\xi_{(0,1)}} \mathbf{g}^{(1,0)} + \mathcal{L}_{\xi_{(1,1)}} \mathbf{g}^{(0,0)}], \quad (92)$$

$$\tilde{\mathcal{J}}_{\sigma}^{(0,1)} = \mathcal{J}_{\sigma}^{(0,1)} + \mathcal{B}_{\sigma} [\mathcal{L}_{\xi_{(0,1)}} \mathbf{g}^{(0,0)}]. \quad (93)$$

Moreover, we know [21, 37] that the quantities in (26-31) are gauge-invariant as first order perturbations, hence  $\mathcal{A}[\mathcal{L}_\xi \mathbf{g}^{(0,0)}]$  must vanish for any vector field  $\xi$ . Therefore, (92) becomes

$$\tilde{\mathcal{H}}^{(1,1)} = \mathcal{H}^{(1,1)} + \mathcal{A}[\mathcal{L}_{\xi_{(0,1)}} \mathbf{g}^{(1,0)}], \quad (94)$$

and the gauge transformation (90) reduces to

$$\tilde{\mathcal{G}}^{(1,1)} = \mathcal{G}^{(1,1)} + \mathcal{A}[\mathcal{L}_{\xi_{(0,1)}} \mathbf{g}^{(1,0)}] + \sum_{\sigma} \mathcal{I}_{\sigma}^{(1,0)} \mathcal{B}_{\sigma}[\mathcal{L}_{\xi_{(0,1)}} \mathbf{g}^{(0,0)}]. \quad (95)$$

Using this expression we can show that the variables  $\mathcal{G}^{(1,1)}$  are gauge-invariant. For the sake of brevity we only give here the proof for the metric perturbation  $k_{AB}^{(1,1)}$  (26), the analysis for the other metric and fluid perturbations in (26-31) and (36-38) is given in Appendix C. To proceed with the proof we expand the generator of the gauge transformations associated with the non-radial perturbations in tensor harmonics,

$$\xi_{(0,1)\alpha} = (\hat{\xi}_A Y, r^2 \xi Y_a). \quad (96)$$

Since the metric perturbations  $\mathbf{g}^{(1,0)}$  do not depend on the coordinates of  $S^2$ , and taking into account the gauge choice (57), we have

$$\mathcal{L}_{\xi_{(0,1)}} \mathbf{g}_{AB}^{(1,0)} = \hat{\mathcal{L}}_{\hat{\xi}} h_{AB}^{(1,0)}, \quad \mathcal{L}_{\xi_{(0,1)}} \mathbf{g}_{Aa}^{(1,0)} = h_{AC}^{(1,0)} \hat{\xi}^C Y_{:a}, \quad \mathcal{L}_{\xi_{(0,1)}} \mathbf{g}_{ab}^{(1,0)} = 0, \quad (97)$$

where  $\mathcal{L}_{\hat{\xi}}$  is the Lie derivative acting on  $M^2$ . Now, we apply the ansatz described in (85) to  $k_{AB}$  and we get the following expressions

$$\mathcal{H}_{AB}^{(1,1)} = h_{AB}^{(1,1)} - \left( p_{A|B}^{(1,1)} + p_{B|A}^{(1,1)} \right), \quad \mathcal{I}_{AB}^{(1,0)C} = 2\Gamma_{AB}^{(1,0)C}, \quad \mathcal{J}_C^{(0,1)} = p_C^{(0,1)}, \quad (98)$$

where  $p_A$  is defined in (32) and  $\Gamma_{AB}^{(1,0)C}$  are the radial perturbations of the Christoffel symbols. From (94) and the analysis of [21, 37], we have

$$\tilde{\mathcal{H}}_{AB}^{(1,1)} = \mathcal{H}_{AB}^{(1,1)} + \hat{\mathcal{L}}_{\hat{\xi}} \left[ h_{AB}^{(1,0)} - \left( p_{A|B}^{(1,0)} + p_{B|A}^{(1,0)} \right) \right], \quad \tilde{\mathcal{J}}_C^{(0,1)} = \mathcal{J}_C^{(0,1)} + \tilde{p}_C^{(0,1)} - p_C^{(0,1)} = \mathcal{J}_C^{(0,1)} + \hat{\xi}_C, \quad (99)$$

where the explicit expression of the Lie derivatives is

$$\hat{\mathcal{L}}_{\hat{\xi}} h_{AB}^{(1,0)} = \hat{\xi}^C h_{AB|C}^{(1,0)} + h_{CB}^{(1,0)} \hat{\xi}_{|A}^C + h_{CA}^{(1,0)} \hat{\xi}_{|B}^C, \quad \hat{\mathcal{L}}_{\hat{\xi}} p_A^{(1,0)} = h_{AC}^{(1,0)} \hat{\xi}^C. \quad (100)$$

Finally, introducing all the expressions (98-100) into the gauge transformation law (95), we get the gauge invariance of  $k_{AB}$  at (1, 1) order:

$$\tilde{k}_{AB}^{(1,1)} = k_{AB}^{(1,1)}. \quad (101)$$

### C. Equations for the $\lambda\epsilon$ coupling perturbations

The explicit form of the equations that govern the behaviour of the coupling terms is obtained by introducing in equations (42-50) the following expressions: *i*) for the background quantities we will use the expressions of the GSGM quantities describing the radially oscillating spacetime (the static background plus radial perturbations), given by equations (75-82); *ii*) for the perturbative quantities we use the corrections to the radially oscillating star, that is, the quantities that come from perturbative terms like  $g_{\alpha\beta}^{(1)} = g_{\alpha\beta}^{(0,1)} + \lambda g_{\alpha\beta}^{(1,1)}$ . Once we have introduced all these quantities, expanded the equations and extracted the  $\lambda\epsilon$  part, we get a set of equations that can be expressed as a linear non-radial operator  $\mathbf{L}_{\text{NR}}$  acting on the (1, 1) variables, and a source term  $\mathcal{S}$  built from the (1, 0), (0, 1) quantities, see Eq. (9).

As we explained in Section II, this particular structure of the (1, 1) equations is quite convenient in order to build an initial-boundary value problem and solve it numerically by using time-domain methods. The basic idea is that given a numerical algorithm capable of evolving linear non-radial perturbations, we can build an algorithm for our (1, 1) perturbations by just adding source terms to the original algorithm.

The time evolution of non-radial perturbations of a static star has been successfully analyzed by numerically integrating different systems of perturbation equations [33, 34, 39]. Taking into account the main features of our formulation, the scheme introduced in [39, 41] seems to be adequate for the purpose of implementing a numerical code to solve our perturbation equations. One of the main points in the scheme introduced in [39, 41] is the fact that the Hamiltonian constraint is not just an error estimator for the evolution equations, as it is usually done in many free evolution schemes. In the scheme of [39, 41], the Hamiltonian constraint is part of the system of equations and it is solved at every time step for the perturbative quantity  $k$ , Eq. (27). This provides some control of the errors induced by constraint violation. As a consequence, the resulting numerical code [39] is able to evolve non-radial perturbations for long times and is capable to estimate the damping time and mode frequencies with an accuracy comparable to frequency domain calculations.

The main idea of our present ongoing work [10] on the numerical solution of our perturbative equations is to follow the scheme of [39, 41]. Taking into account our discussion in Section II and above about the general structure of the perturbative equations, in particular the same differential structure of the perturbative equations at the orders (0, 1) and (1, 1), it is clear that this scheme is easily portable to our problem. To that end, it is very important the fact that the Hamiltonian constraint is solved for one of the perturbative quantities since at every time step we need to evolve the equations for the (0, 1) and (1, 1) perturbations. This means that if we do not solve the Hamiltonian constraint, the errors accumulated from constraint violation would be double than in a standard computation of non-radial perturbations. Therefore, the use of the scheme of [39, 41] is a key ingredient in trying to obtain accurate long term evolutions. We expect that the resulting numerical code would allow us to investigate the non-linear effects of coupling. In particular, we are interested in looking for non-linear harmonics, possible resonances, parameter amplification, and/or changes in the damping time of non-radial perturbations.

In the stellar interior, we evolve the (hyperbolic) equations for the metric perturbation  $\chi^{(1,1)}$  and for the fluid perturbation  $H^{(1,1)}$ , which in some particular gauges coincides with the enthalpy perturbation. The Hamiltonian constraint provides us the metric perturbation  $k^{(1,1)}$ . Subsequently, all the other metric  $\psi^{(1,1)}$  and fluid ( $\gamma^{(1,1)}$ ,  $\alpha^{(1,1)}$ ) perturbations can be obtained from the perturbative equations (44), (49) and (45).

The wave equation for  $\chi^{(1,1)}$  and the Hamiltonian constraint are given by (42) and (50) respectively, while the sound wave equation for  $H^{(1,1)}$  has to be determined. We define the fluid perturbation  $H^{(1)}$  (see Appendix C for a proof of the gauge invariant character of this quantity) as

$$H^{(1)} \equiv \frac{c_s^{2(0)} \rho^{(0)}}{\rho^{(0)} + p^{(0)}} \omega^{(1)}, \quad (102)$$

where the superscripts (0) and (1) have the meaning already explained in Section II. The sound speed in the radially pulsating spacetime can be split as follows

$$c_s^{2(0)} = \bar{c}_s^2 + \lambda \frac{d\bar{c}_s^2}{d\rho} \delta\rho^{(1,0)}. \quad (103)$$

In particular gauges, the Regge-Wheeler [42] one for instance, the gauge-invariant quantity  $\omega^{(1)}$  coincides with the gauge dependent perturbation  $\tilde{\omega}^{(1)}$  [see Eq. (38)], and  $H^{(1)}$  describes the enthalpy perturbation,

$$H^{(1)} \equiv \frac{\delta p^{(1)}}{\rho^{(0)} + p^{(0)}}, \quad (104)$$

where  $\delta p^{(1)}$  is defined by (35). The wave equation for  $H^{(1,1)}$  is obtained as a linear combination of the time frame derivative of equation (47) and the spatial frame derivative of (48). After having introduced the equations (44, 46, 48, 43, 50) to reduce the number of perturbative unknowns and the transformation (102), we have the following wave equation (written in the GSGM formalism):

$$-\ddot{H} + c_s^2 H'' + \mathcal{F}_H = 0, \quad (105)$$

where  $\mathcal{F}_H$  contains all the remaining terms (with derivatives of lower order). The complete equation has been written in Appendix B. The wave equation (105) is valid in the GSGM framework for barotropic non-radial perturbations on a time dependent background. In case of a static background, provided the introduction of the background quantities (55-56), it reduces to an equation well known in the literature (see i.e. [33], [34], [39]).

We can now write the perturbative equations for the stellar interior. We consider instead of the perturbative quantity  $\chi^{(1,1)}$ , which diverges like  $r$  as we approach spatial infinity, the perturbation variable  $S^{(1,1)} = \chi^{(1,1)}/r$  which of course is well behaved at infinity. This quantity satisfies the following *gravitational wave* equation:

$$-S_{,tt}^{(1,1)} + e^{2(\Phi-\Lambda)} S_{,rr}^{(1,1)} + e^{2(\Phi-\Lambda)} \left[ (5\Phi_{,r} - \Lambda_{,r}) S_{,r}^{(1,1)} + \frac{4}{r} \left( \frac{1-e^{2\Lambda}}{r^2} + \Phi_{,r}^2 + \frac{\Lambda_{,r}}{r} \right) k^{(1,1)} \right]$$

$$+ \frac{1}{r} \left( \Phi_{,r} (5 + 4\Phi_{,rr}) + 3\Lambda_{,r} + \frac{2 - (l(l+1) + 2) e^{2\Lambda}}{r} \right) S^{(1,1)} \Big] = e^{2\Phi} \mathcal{S}_S. \quad (106)$$

where  $\mathcal{S}_S$  denotes the source term for this wave equation. The source terms in our (1, 1) perturbative equations have the following pattern

$$\mathcal{S}^{(1,1)} = \sum_I \mathcal{C}_I^{(1,0)} \mathcal{Q}_I^{(0,1)}, \quad (107)$$

which shows how the source terms introduce the coupling between radial and non-radial perturbations in the (1, 1) equations. In particular, the source term in the gravitational-wave equation,  $\mathcal{S}_S$  has the following form

$$\mathcal{S}_S = a_1 S_{,rr}^{(0,1)} + a_2 S_{,r}^{(0,1)} + a_3 S_{,t}^{(0,1)} + a_4 S^{(0,1)} + a_5 \left( \psi_{,r}^{(0,1)} - 2e^{\Lambda-\Phi} k_{,t}^{(0,1)} \right) + a_6 k^{(0,1)} + a_7 \psi^{(0,1)}, \quad (108)$$

where the coefficients  $a_i$  are just linear combinations of radial perturbations with coefficients constructed from background quantities. Their explicit form is given in Appendix A.

The perturbative fluid variable  $H^{(1,1)}$  also satisfies a wave equation, but with a different propagation speed. We call this equation the *sound wave equation*. It has the following form:

$$\begin{aligned} -H_{,tt}^{(1,1)} + \bar{c}_s^2 e^{2(\Phi-\Lambda)} H_{,rr}^{(1,1)} + e^{2(\Phi-\Lambda)} \left\{ \left[ \left( \frac{2}{r} + 2\Phi_{,r} - \Lambda_{,r} \right) \bar{c}_s^2 - \Phi_{,r} \right] H_{,r}^{(1,1)} \right. \\ \left. + \frac{1}{r} \left[ (1 + 3\bar{c}_s^2) (\Lambda_{,r} + \Phi_{,r}) - \bar{c}_s^2 \frac{l(l+1)}{r} e^{2\Lambda} \right] H^{(1,1)} - \frac{1 - \bar{c}_s^2}{2} \Phi_{,r} \left[ \left( rS^{(1,1)} \right)_{,r} - k_{,r}^{(1,1)} \right] \right. \\ \left. + \left[ -2\Phi_{,r}^2 + [(3\Phi_{,r} + \Lambda_{,r})r + 1 - e^{2\Lambda}] \frac{\bar{c}_s^2}{r^2} \right] \left( rS^{(1,1)} + k^{(1,1)} \right) \right\} = e^{2\Phi} \mathcal{S}_H, \quad (109) \end{aligned}$$

and the source term can be written as

$$\begin{aligned} \mathcal{S}_H = b_1 H_{,rr}^{(0,1)} + b_2 H_{,tr}^{(0,1)} + b_3 H_{,t}^{(0,1)} + b_4 H_{,r}^{(0,1)} + b_5 H^{(0,1)} + b_6 k_{,t}^{(0,1)} + b_7 r S_{,t}^{(0,1)} + b_8 \left[ k_{,r}^{(0,1)} - \left( rS^{(0,1)} \right)_{,r} \right] \\ + b_9 \left( rS^{(0,1)} + k^{(0,1)} \right) + b_{10} \gamma_{,r}^{(0,1)} + b_{11} \gamma^{(0,1)} + b_{12} \psi_{,r}^{(0,1)} + b_{13} \psi^{(0,1)} + b_{14} \alpha^{(0,1)}, \quad (110) \end{aligned}$$

where the coefficients  $b_i$  have the same structure as the  $a_i$  coefficients in (108). Their explicit expressions can be found in the Appendix A.

For the last perturbative variable, the metric perturbation  $k^{(1,1)}$ , we will use the *Hamiltonian constraint* instead of an evolution equation. After some calculations we get:

$$\begin{aligned} k_{,rr}^{(1,1)} - S_{,r}^{(1,1)} + \left( \frac{2}{r} - \Lambda_{,r} \right) k_{,r}^{(1,1)} + \frac{2}{r\bar{c}_s^2} (\Lambda_{,r} + \Phi_{,r}) H^{(1,1)} + \frac{1}{r^2} [(1 - l(l+1)) e^{2\Lambda} + 2\Lambda_{,r}r - 1] k^{(1,1)} \\ - \frac{1}{2r} [l(l+1)e^{2\Lambda} + 4 - 4\Lambda_{,r}r] S^{(1,1)} = \mathcal{S}_{Hamil}, \quad (111) \end{aligned}$$

where  $\mathcal{S}_{Hamil}$  is the source term for the Hamiltonian constraint. As in the previous equations, it follows the pattern (107). The precise form of  $\mathcal{S}_{Hamil}$  is:

$$\begin{aligned} \mathcal{S}_{Hamil} = c_1 \left( k_{,rr}^{(0,1)} - S_{,r}^{(0,1)} \right) + c_2 k_{,r}^{(0,1)} + c_3 k_{,t}^{(0,1)} + c_4 S^{(0,1)} + c_5 k^{(0,1)} + c_6 H^{(0,1)} + c_7 \psi_{,r}^{(0,1)} \\ + c_8 \psi^{(0,1)} + c_9 \gamma^{(0,1)}. \quad (112) \end{aligned}$$

The coefficients  $c_i$ , in the same way as the coefficients  $a_i$  and  $b_i$  only contain radial perturbations  $g^{(1,0)}$  and quantities associated with the static background. They are also given in Appendix A. It is worth to remark that the polar non-radial perturbation equations on a static background are obtained from equations (106,109,111) by discarding the source terms and replacing all the (1, 1) perturbations with the corresponding non-radial (0, 1). The sources are determined from first order perturbations. The radial perturbations from the equations (64-68), and the non-radial perturbations (described by the quantities  $S^{(0,1)}$ ,  $k^{(0,1)}$ , and  $H^{(0,1)}$ ) from the first order analogues of the above system (see [39]), and the equations (44, 49, 45) adapted to a static background to get the  $\psi^{(0,1)}$ ,  $\gamma^{(0,1)}$  and  $\alpha^{(0,1)}$ .

The *stellar exterior* is described by a Schwarzschild spacetime on which gravitational waves carry away some energy of the stellar oscillations. All fluid perturbations are not defined outside the star and the radial perturbations vanish

because of Birkhoff's theorem. Therefore, the source terms in our perturbation equations vanish. Only the metric perturbations survive, and they satisfy the gravitational wave equation (106) and the Hamiltonian constraint (111), which take the following form:

$$-S_{,tt}^{(1,1)} + e^{2(\Phi-\Lambda)} S_{,rr}^{(1,1)} + e^{2\Phi} \left[ \frac{6M}{r^2} S_{,r}^{(1,1)} - \left[ \frac{2M}{r^3} \left( 1 - \frac{2M}{r} e^{2\Lambda} \right) + \frac{l(l+1)}{r^2} \right] S^{(1,1)} - \frac{4M}{r^4} \left( 3 - \frac{M}{r} e^{2\Lambda} \right) k^{(1,1)} \right] = 0, \quad (113)$$

$$e^{-2\Lambda} \left( k_{,rr}^{(1,1)} - S_{,r}^{(1,1)} \right) + \left( \frac{2}{r} - \frac{3M}{r^2} \right) k_{,r}^{(1,1)} - \frac{l(l+1)}{r^2} k^{(1,1)} - \left( \frac{2}{r} - \frac{2M}{r^2} + \frac{l(l+1)}{2r} \right) S^{(1,1)} = 0. \quad (114)$$

It is worth to mention that the above equations coincide with the equations for non-radial perturbations of a static stellar background outside the star, as expected.

On the other hand, Zerilli showed that the even-parity perturbations of a Schwarzschild background have just one degree of freedom, and therefore can be described by just one variable, the Zerilli function, satisfying a wave equation. At order (1, 1) the Zerilli function can be built from the two metric perturbations  $S^{(1,1)}$  and  $k^{(1,1)}$  and their derivatives, as at first order [43], and is given by

$$Z^{(1,1)} = \frac{2r^2 e^{-2\Phi}}{(l+2)(l-1)r + 6M} \left[ r S^{(1,1)} + \frac{1}{2} \left( l(l+1) + \frac{2M}{r} \right) e^{2\Phi} k^{(1,1)} - r k_{,r}^{(1,1)} \right]. \quad (115)$$

It satisfies the Zerilli equation [44, 45]

$$-Z_{,tt}^{(1,1)} + e^{2(\Phi-\Lambda)} Z_{,rr}^{(1,1)} + \frac{M}{r^2} e^{2\Phi} Z_{,r}^{(1,1)} - V(r) Z^{(1,1)} = 0, \quad (116)$$

where  $V(r)$  is the Zerilli potential [45]:

$$V(r) = - \left( 1 - \frac{2M}{r} \right) \frac{n_l(n_l - 2)^2 r^3 + 6(n_l - 2)^2 M r^2 + 36(n_l - 2) M^2 r + 72 M^3}{r^3 [(n_l - 2)r + 6M]^2}, \quad (117)$$

where the quantity  $n_l = l(l+1)$  has been introduced to simplify the expression.

Finally, we can determine the power of the gravitational radiation emission at infinity by using the following expression [46]

$$\frac{dE}{dt} = \frac{1}{64\pi} \sum_{l,m} \frac{(l+2)!}{(l-2)!} |\dot{Z}_{lm}|^2. \quad (118)$$

## VI. BOUNDARY CONDITIONS

In this Section we discuss the boundary conditions at the origin and at the stellar surface for the (1, 1) perturbations describing the coupling of radial and non-radial modes. With regard to the outer boundary, by locating it far enough we can use the well-known Sömmersfeld outgoing boundary conditions on our fields.

At the origin, the boundary conditions are just regularity conditions on the perturbative fields, which can be obtained by a careful analysis of the equations that they satisfy. The analysis of Taylor expansions of the differential operators that appear in our equations near the origin leads to the following behaviour for the non-radial perturbations  $g^{(0,1)}$  and  $g^{(1,1)}$  [19]:

$$l \geq 0, \quad S^{(1)} \sim r^{l+1} \quad k^{(1)} \sim r^l \quad \psi^{(1)} \sim r^{l+1}, \quad (119)$$

$$l \geq 1, \quad \gamma^{(1)} \sim r^{l-1} \quad H^{(1)} \sim r^l \quad \alpha \sim r^l. \quad (120)$$

By using the behaviour given by these expressions and also the behaviour given in expressions (72) for the radial perturbations it can be proved the regularity of the source terms at the origin.

At the surface of the star, we have to consider a boundary condition for the matter variable  $H$ , which vanishes in the spacetime region outside the star. For the metric perturbation variables  $S$  and  $k$ , since they must be continuous through the stellar surface, we can just use the junction conditions at the surface to determine them.

To that end, let  $\bar{\Sigma}$  be the surface of the static unperturbed star (i.e.  $r = R_s$ ). The surface of the perturbed star can then be described in the following way

$$\Sigma \equiv \left\{ x + \lambda \xi^{(1,0)} + \epsilon \xi^{(0,1)} + \lambda \epsilon \xi^{(1,1)} : x \in \bar{\Sigma} \right\}, \quad (121)$$

where  $\xi^{(i,j)}$  is a vector field that denotes the Lagrangian displacement of a fluid element due to the action of perturbations of the order  $(i, j)$ . A physical requirement that follows from junction conditions is the vanishing of the unperturbed pressure  $\bar{p}$  at the unperturbed surface  $\bar{\Sigma}$ . In the same way, the corresponding boundary condition for the perturbed spacetime is the vanishing of the total pressure  $\bar{p} + \lambda \delta p^{(1,0)} + \epsilon \delta p^{(0,1)} + \lambda \epsilon \delta p^{(1,1)}$  at the perturbed surface  $\Sigma$ . This condition turns out to be equivalent to the vanishing of the Lagrangian pressure perturbations on  $\bar{\Sigma}$ , the unperturbed surface, at every order. The Lagrangian pressure perturbations are given by:

$$\Delta p^{(1,0)} = \delta p^{(1,0)} + \mathcal{L}_{\xi_{(1,0)}} \bar{p}, \quad (122)$$

$$\Delta p^{(0,1)} = \delta p^{(0,1)} + \mathcal{L}_{\xi_{(0,1)}} \bar{p}, \quad (123)$$

$$\begin{aligned} \Delta p^{(1,1)} &= \delta p^{(1,1)} + \left( \mathcal{L}_{\xi_{(1,1)}} + \frac{1}{2} \{ \mathcal{L}_{\xi_{(1,0)}}, \mathcal{L}_{\xi_{(0,1)}} \} \right) \bar{p} + \mathcal{L}_{\xi_{(0,1)}} \delta p^{(1,0)} + \mathcal{L}_{\xi_{(1,0)}} \delta p^{(0,1)} \\ &= \delta p^{(1,1)} + \left( \mathcal{L}_{\xi_{(1,1)}} - \frac{1}{2} \{ \mathcal{L}_{\xi_{(1,0)}}, \mathcal{L}_{\xi_{(0,1)}} \} \right) \bar{p}, \end{aligned} \quad (124)$$

where  $\delta$  and  $\Delta$  denote the Eulerian and Lagrangian perturbations respectively, and we have used the lower order boundary conditions  $\Delta p^{(1,0)} = \Delta p^{(0,1)} = 0$  in order to simplify the condition (124).

From this analysis we can conclude that the boundary conditions for the fluid perturbations are described by the set of expressions given in (122-124). However, in practice, in many applications of first order perturbation theory, *dynamical boundary conditions* either for density or enthalpy perturbations have been considered. This alternative boundary conditions follow from the analysis of the time derivative of the condition (123) (see [33] for more details). In our current development of the numerical implementation of the perturbative equations we are considering both types of boundary conditions with the perspective of analyzing which type works best for our formulation.

Finally, the junction conditions for the metric perturbations can be determined by imposing continuity of first and second fundamental differential forms and their perturbations at the surface [47, 48, 49, 50]. The explicit form of these conditions has been presented in [20] for first order perturbations of a time-dependent stellar background. Alternatively, one may use the ‘‘extraction formulas’’ [20] that relate the Zerilli function with metric perturbations at the stellar boundary.

## VII. REMARKS AND CONCLUSIONS

Non linearity is the rule rather than the exception in dynamical phenomena in all branches of physics. The modeling of compact objects such as neutron stars and supernovae core must ultimately be rooted in general relativity (or some of its generalisations), where non-linearity represents a fundamental physical character of the theory, i.e. the self-interaction of the gravitational field, and not just corrections to an underlying linear modeling of gravitational phenomena. In relativistic theories of gravity, gravitational radiation is the typical outcome of dynamical phases in the life of sources such as binary systems and supernovae, and major experimental efforts are currently under way to detect this most elusive prediction of Einstein gravity for the first time. This will eventually lead to the development of a whole new branch of astronomy, based on observing gravitational radiation, much in the same way it has been in the past for x-rays and other parts of the electromagnetic spectrum outside the visible band. In this context the accurate theoretical modeling of sources is crucial to the final end of providing templates in this game of looking for a needle - the signal - in the haystack, the noise. While ultimately a full numerical relativity description of gravitational wave sources is needed to model the most non-linear dynamical phases, much interesting physics can be understood by using approximate methods. Furthermore, a semi-analytical approach typically helps to shed light on the physical processes, thus complementing the numerical work.

Relativistic perturbation theory is ideal for those cases where a known solution of the field equations is explicitly known, as for black holes, or can easily be obtained, as is the case for compact stars. An advantage of the relativistic perturbative approach is that it directly incorporates gravitational waves. For smaller perturbations, linear theory suffices. If one wants to consider mildly non-linear oscillations of a compact object, second order effects and mode coupling have to be taken into account. For black holes, many studies already exists in this direction (see e.g. [8, 23, 51]). In the case of neutron stars, while linear perturbations and instabilities have been studied for long time [1, 2], relatively little is known of non-linear dynamical effects, mostly through numerical studies (see e.g.



[3, 4, 5, 6, 7, 52, 53]). A second order perturbative approach is therefore timely and may help to understand known problems and even reveal a new phenomenology.

In this paper we have developed the relativistic formalism to study a particular second order effect, the coupling of radial and non-radial first order perturbations of a compact spherical star. From a mathematical point of view it is very convenient to treat the two sets of perturbations, radial and non-radial, as separately parametrized, using the multi-parameter perturbative formalism developed in [17, 18]. Then we have considered the expansion of the metric, the energy-momentum tensor and Einstein equations in terms of two parameters  $\lambda$  and  $\epsilon$ , where  $\lambda$  parametrizes the radial modes,  $\epsilon$  the non-radial perturbations, and the  $\lambda\epsilon$  terms describe the coupling. This approach provides a well-defined framework to consider the gauge dependence of perturbations. In this mathematical context we have imported the formalism of Gundlach and Martín García [19, 20] and Gerlach and Sengupta [21], describing gauge-invariant perturbations of a general time-dependent spherical spacetime, expanding the latter in a static background and a radial perturbation. Fixing the gauge for radial perturbations allows us to: *i*) use the GSGM gauge-invariant non-radial  $\epsilon$  variables on the static background; *ii*) define new second order  $\lambda\epsilon$  variables, describing the non-linear coupling of the radial and non-radial linear perturbations, that are also gauge-invariant at the  $\lambda\epsilon$  second order. This higher order gauge invariance, attained by partially fixing the gauge at first order, is similar to that considered for example in [22] and [23]. In our case however we use a 2-parameter  $\lambda - \epsilon$  expansion [17, 18], so that we only need to fix the gauge for radial perturbations. Assuming a barotropic perfect fluid, we have derived the evolution and constraint equations for our variables, in particular those for the coupling terms of order  $\lambda\epsilon$ , focusing on polar perturbations. We leave for future studies the implementation of more realistic equations of state, such as the non-isentropic one used in [52, 53]. As expected, in the interior the  $\lambda\epsilon$  variables satisfy inhomogeneous linear equations where the homogeneous part is governed by the same linear operator acting on the first order  $\epsilon$  non-radial perturbations, while the source terms are quadratic and made of products of  $\lambda$  and  $\epsilon$  terms. In the exterior there is no direct coupling, and the whole dynamics is embodied in the  $\lambda\epsilon$  order Zerilli function. Thus the effect of the coupling is transmitted from the interior to the exterior through the junction conditions at the surface of the star. Finally, we have given a brief discussion of the boundary conditions, focusing on those on the surface. These are typically expressed in terms of the Lagrangian pressure perturbation, therefore we have defined a  $\lambda\epsilon$  second order Lagrangian displacement and a corresponding  $\lambda\epsilon$  Lagrangian pressure perturbation, appropriately related to the Eulerian perturbation. Thanks to the vanishing of the first order  $\lambda$  and  $\epsilon$  Lagrangian pressure perturbations on the surface, this relation turns out to be linear.

Work is currently under way, numerically implementing the formalism presented here, in order to provide a first analysis of the possible effects of the coupling between radial and polar non-radial perturbations [10]. Some of these effects are easily anticipated for the case of axial oscillations. These are decoupled from fluid perturbations at first order, but are driven by the radial pulsations at the  $\lambda\epsilon$  order [9]. Eventually these studies may possibly even lead to discover new unexpected effects of mode coupling. Surely we expect to find non-linear harmonics arising from the radial non-radial coupling, similar to those between various radial modes found in [54] for tori around black holes, a prediction that appears to be confirmed by a numerical relativity study of neutron stars in the conformally-flat spacetime approximation [55].

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## APPENDIX A: SOURCE TERMS FOR THE (1, 1) PERTURBATION EQUATIONS

In this Section we give the expressions of the coefficients appearing in the source terms of the equations (106-112) for the (1, 1) perturbations.

In the case of the *gravitational wave* equation (108) the source term has the form:

$$\mathcal{S}_S = a_1 S_{,rr}^{(0,1)} + a_2 S_{,r}^{(0,1)} + a_3 S_{,t}^{(0,1)} + a_4 S^{(0,1)} + a_5 \left( \psi_{,r}^{(0,1)} - 2e^{\Lambda-\Phi} k_{,t}^{(0,1)} \right) + a_6 k^{(0,1)} + a_7 \psi^{(0,1)}, \quad (\text{A1})$$

where the coefficients  $a_i$  are given by

$$a_1 = 2 \left( r S^{(1,0)} - \eta^{(1,0)} \right) e^{-2\Lambda}, \quad (\text{A2})$$

$$a_2 = \left[ 2(\Lambda_{,r} - 5\Phi_{,r})\eta^{(1,0)} - ((\Lambda_{,r} - 5\Phi_{,r})r + 3)S^{(1,0)} - (\Lambda_{,r} + \Phi_{,r})(5 - \bar{c}_s^{-2})H^{(1,0)} \right] e^{-2\Lambda} - 4\gamma_{,t}^{(1,0)}e^{-\Phi-\Lambda}, \quad (\text{A3})$$

$$a_3 = -4(\Lambda_{,r} + \Phi_{,r})\gamma^{(1,0)}e^{-\Lambda-\Phi} - e^{-2\Phi}\eta_{,t}^{(1,0)} + \frac{2}{r}\left(r\gamma^{(1,0)}e^{-\Phi}\right)_{,r}e^{-\Lambda}, \quad (\text{A4})$$

$$a_4 = -\left\{ \frac{4}{r}(1 + 2r\Phi_{,r})\gamma_{,t}^{(1,0)}e^{-\Phi+\Lambda} + 2\left[ 2\Phi_{,r}\left(\frac{1}{r} + 2\Phi_{,r}\right) + \frac{2 - (l(l+1) + 2)e^{2\Lambda}}{r^2} + 3\frac{\Lambda_{,r} + \Phi_{,r}}{r} \right] \eta^{(1,0)} \right. \\ \left. - \left[ \Phi_{,r} + 3\Lambda_{,r} + \frac{3 - (l(l+1) + 2)e^{2\Lambda}}{r} \right] S^{(1,0)} + (\Lambda_{,r} + \Phi_{,r}) \left[ \left(5 + \frac{3}{\bar{c}_s^2}\right)\frac{1}{r} + 8\Phi_{,r} \right] H^{(1,0)} \right\} e^{-2\Lambda}, \quad (\text{A5})$$

$$a_5 = -2\left[ \left(\frac{e^\Phi}{r}\gamma^{(1,0)}\right)_{,r} - (\Lambda_{,r} + \Phi_{,r})\left(\frac{e^\Phi}{r}\gamma^{(1,0)}\right) \right] e^{-2\Lambda-\Phi}, \quad (\text{A6})$$

$$a_6 = -\left[ \frac{2}{r^2}(-5 + 2r(\Phi_{,r} - \Lambda_{,r}) + 2e^{2\Lambda})S^{(1,0)} + \frac{8}{r^3}(1 + r^2\Phi_{,r}^2 + r\Lambda_{,r} - e^{2\Lambda})\eta^{(1,0)} \right. \\ \left. + \frac{4}{r^2}(\Lambda_{,r} + \Phi_{,r})\left(2r\Phi_{,r} + \frac{1}{\bar{c}_s^2}\right)H^{(1,0)} + \frac{8}{r}\Phi_{,r}\gamma_{,t}^{(1,0)}e^{-\Phi+\Lambda} \right] e^{-2\Lambda}, \quad (\text{A7})$$

$$a_7 = -\frac{2}{r}\left\{ (1 - \bar{c}_s^2)r\left(\frac{e^\Phi}{r}\gamma^{(1,0)}\right)_{,rr} + \left(\frac{e^\Phi}{r}\gamma^{(1,0)}\right)_{,r} [r(\Phi_{,r} - 2\Lambda_{,r}) + (2\Lambda_{,r}r + \Phi_{,r}r - 4)\bar{c}_s^2 \right. \\ \left. + \frac{\Phi_{,r}\Lambda_{,r} + \Phi_{,r}d\bar{c}_s^2}{4\pi\bar{c}_s^2}e^{-2\Lambda} \right] + [(2 - 2r\Phi_{,r} - 3r\Lambda_{,r})\Phi_{,r} - (1 + r\Lambda_{,r})\Lambda_{,r} + (r(\Lambda_{,r}^2 - \Phi_{,r}^2) + 2\Phi_{,r} + 5\Lambda_{,r})\bar{c}_s^2 \\ \left. + \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2}\Phi_{,r} \left[ r + \frac{e^{-2\Lambda}}{4\pi r} (3 - (\Lambda_{,r} + \Phi_{,r})r) \frac{d\bar{c}_s^2}{d\bar{\rho}} \right] \frac{e^\Phi}{r}\gamma^{(1,0)} \right\} e^{-\Phi-2\Lambda}. \quad (\text{A8})$$

For the *sound wave* equation the source term is given by (110):

$$\mathcal{S}_H = b_1H_{,rr}^{(0,1)} + b_2H_{,tr}^{(0,1)} + b_3H_{,t}^{(0,1)} + b_4H_{,r}^{(0,1)} + b_5H^{(0,1)} + b_6k_{,t}^{(0,1)} + b_7rS_{,t}^{(0,1)} + b_8\left[k_{,r}^{(0,1)} - (rS^{(0,1)})_{,r}\right] \\ + b_9\left(rS^{(0,1)} + k^{(0,1)}\right) + b_{10}\gamma_{,r}^{(0,1)} + b_{11}\gamma^{(0,1)} + b_{12}\psi_{,r}^{(0,1)} + b_{13}\psi^{(0,1)} + b_{14}\alpha^{(0,1)}, \quad (\text{A9})$$

and the expression of the coefficients  $b_i$  is the following

$$b_1 = -\left[ 2\left(\eta^{(1,0)} - rS^{(1,0)}\right)\bar{c}_s^2 + \frac{e^{-2\Lambda}}{4\pi r}\frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2}\frac{d\bar{c}_s^2}{d\bar{\rho}}H^{(1,0)} \right] e^{-2\Lambda}, \quad (\text{A10})$$

$$b_2 = 2(1 - \bar{c}_s^2)e^{-\Phi-\Lambda}\gamma^{(1,0)}, \quad (\text{A11})$$

$$b_3 = -\left\{ \frac{e^{-\Lambda}}{r^3}\left(-\frac{e^{-2\Lambda}}{2\pi}\frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2}\frac{d\bar{c}_s^2}{d\bar{\rho}} + \frac{2\bar{c}_s^2\bar{\rho} - \bar{p}}{\bar{\rho}}r\right)\left(r^2\gamma^{(1,0)}e^\Phi\right)_{,r} \right. \\ \left. + (\Lambda_{,r} + \Phi_{,r})\left(\frac{e^{-2\Lambda}}{2\pi r}\frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2}\frac{d\bar{c}_s^2}{d\bar{\rho}} - \bar{c}_s^2 + 1 + \frac{\bar{p}}{\bar{\rho}}\right)\gamma^{(1,0)}e^{\Phi-\Lambda} + \eta_{,t}^{(1,0)} \right\} e^{-2\Phi}, \quad (\text{A12})$$

$$b_4 = \frac{1}{4\pi r}\left[\Lambda_{,r}\left(\Lambda_{,r} - \frac{1}{r}\right) - \left(\frac{1}{r} + \Phi_{,r}\right)(2\Phi_{,r} + \Lambda_{,r})\right]\frac{1}{\bar{c}_s^2}\frac{d\bar{c}_s^2}{d\bar{\rho}}H^{(1,0)}e^{-4\Lambda} \\ - \left\{ 2(1 - \bar{c}_s^2)H_{,r}^{(1,0)} + 2\left[\left(2\Phi_{,r} - \Lambda_{,r} + \frac{2}{r}\right)\bar{c}_s^2 - \Phi_{,r}\right]\eta^{(1,0)} \right. \\ \left. + \left[3\Phi_{,r} + \frac{1}{2r} + \left(\Lambda_{,r} - 4\Phi_{,r} - \frac{7}{2r}\right)\bar{c}_s^2\right]rS^{(1,0)} \right\} e^{-2\Lambda}, \quad (\text{A13})$$

$$\begin{aligned}
b_5 = & - \left\{ \left[ \frac{e^{-2\Lambda}}{4\pi r^2} \left( 1 - e^{2\Lambda} + \Phi_{,r} \Lambda_{,r} r^2 + \left( \Lambda_{,r} + \frac{5}{2} \Phi_{,r} \right) r \right) \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2} \frac{d\bar{c}_s^2}{d\bar{\rho}} - (\Lambda_{,r} + \Phi_{,r}) (1 + 3\bar{c}_s^2) \right. \right. \\
& + \left. \frac{l(l+1)}{r} \bar{c}_s^2 e^{2\Lambda} \right] S^{(1,0)} + \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2 r} \left[ \frac{1}{4\pi r} \left( (6\bar{c}_s^2 + 1 + r\Phi_{,r}) \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2} e^{-2\Lambda} - \frac{l(l+1)}{r} \right) \frac{d\bar{c}_s^2}{d\bar{\rho}} \right. \\
& + \left. 3\bar{c}_s^4 + 4\bar{c}_s^2 + 1 \right] H^{(1,0)} + \frac{e^{-2\Lambda}}{4\pi r} \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2} \frac{d\bar{c}_s^2}{d\bar{\rho}} \left[ \left( \frac{2}{r} - \Lambda_{,r} + 2\Phi_{,r} \right) H_{,r}^{(1,0)} + H_{,rr}^{(1,0)} \right] \\
& \left. + \frac{2}{r} \left[ (\Lambda_{,r} + \Phi_{,r}) (1 + 3\bar{c}_s^2) - \frac{l(l+1)}{r} \bar{c}_s^2 e^{2\Lambda} \right] \eta^{(1,0)} \right\} e^{-2\Lambda}, \tag{A14}
\end{aligned}$$

$$b_6 = \left\{ \left[ (\Lambda_{,r} + \Phi_{,r}) \left( 1 + \bar{c}_s^2 - \frac{\bar{p}}{\rho} \right) \bar{c}_s^2 - (1 - \bar{c}_s^2) \Phi_{,r} \right] \gamma^{(1,0)} e^\Phi - \frac{\bar{c}_s^2}{r^2} \left( 1 + \bar{c}_s^2 - \frac{\bar{p}}{\rho} \right) \left( r^2 \gamma^{(1,0)} e^\Phi \right)_{,r} \right\} e^{-2\Phi - \Lambda}, \tag{A15}$$

$$b_7 = \left\{ \left[ \left( \Lambda_{,r} + \frac{2}{r} \right) \bar{c}_s^2 + \frac{\Phi_{,r}}{2} (1 + \bar{c}_s^2) \right] \gamma^{(1,0)} e^\Phi - \frac{\bar{c}_s^2}{r^2} \left( r^2 \gamma^{(1,0)} e^\Phi \right)_{,r} \right\} e^{-2\Phi - \Lambda}, \tag{A16}$$

$$b_8 = - \left[ \frac{\Phi_{,r}}{8\pi r} \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2} \frac{d\bar{c}_s^2}{d\bar{\rho}} H^{(1,0)} e^{-2\Lambda} + (1 - \bar{c}_s^2) \left( \Phi_{,r} \eta^{(1,0)} + \frac{1}{2} H_{,r}^{(1,0)} \right) \right] e^{-2\Lambda}, \tag{A17}$$

$$\begin{aligned}
b_9 = & - \left\{ \left[ \frac{2}{r^2} (1 - e^{2\Lambda} + 3r\Phi_{,r} + r\Lambda_{,r}) \bar{c}_s^2 - 4\Phi_{,r}^2 \right] \eta^{(1,0)} \right. \\
& + \left[ \left[ 3 \frac{e^{2\Lambda} - 1}{r} - 2\Phi_{,r} (4 + r\Lambda_{,r}) - 3\Lambda_{,r} \right] \bar{c}_s^2 + 4r\Phi_{,r}^2 \right] S^{(1,0)} \\
& - \frac{\Lambda_{,r} + \Phi_{,r}}{r} \left[ 1 + 2\Phi_{,r} r + 3\bar{c}_s^2 - \frac{e^{-2\Lambda}}{4\pi r^2} \frac{1}{\bar{c}_s^2} \frac{d\bar{c}_s^2}{d\bar{\rho}} \left( (3\Phi_{,r} + \Lambda_{,r}) r + 1 - e^{2\Lambda} \right) \right] H^{(1,0)} \\
& \left. + 2 \left[ 2\Phi_{,r} + \left( \Lambda_{,r} - 2\Phi_{,r} - \frac{2}{r} \right) \bar{c}_s^2 \right] H_{,r}^{(1,0)} - 2\bar{c}_s^2 H_{,rr}^{(1,0)} \right\} e^{-2\Lambda}, \tag{A18}
\end{aligned}$$

$$b_{10} = \frac{2\bar{c}_s^2}{r^2} \left\{ \left[ \left( 1 - \frac{\bar{p}}{2\rho} \right) (\Lambda_{,r} + \Phi_{,r}) + \frac{2}{r} \right] \left( r^2 \gamma^{(1,0)} e^\Phi \right) - \left( 1 - \frac{\bar{p}}{2\rho} \right) \left( r^2 \gamma^{(1,0)} e^\Phi \right)_{,r} \right\} e^{-\Phi - 2\Lambda}, \tag{A19}$$

$$\begin{aligned}
b_{11} = & - \frac{1}{r^3} \left\{ 2r\bar{c}_s^2 (1 - \bar{c}_s^2) \left( r^2 \gamma^{(1,0)} e^\Phi \right)_{,rr} + \left[ \frac{\Phi_{,r}}{2\pi} (\Lambda_{,r} + \Phi_{,r}) \frac{d\bar{c}_s^2}{d\bar{\rho}} e^{-2\Lambda} + 2 \left( \Phi_{,r} + 2\Lambda_{,r} + \frac{2}{r} \right) r\bar{c}_s^4 \right. \right. \\
& - \left. \left( 4(1 + r\Lambda_{,r}) + \frac{\bar{p}}{\rho} (2 + r\Phi_{,r}) \right) \bar{c}_s^2 - \left( 2 - \frac{\bar{p}}{\rho} \right) r\Phi_{,r} \right] \left( r^2 \gamma^{(1,0)} e^\Phi \right)_{,r} + [-2(\Lambda_{,r} + \Phi_{,r}) (1 + (\Phi_{,r} - \Lambda_{,r}) r) \bar{c}_s^4 \\
& + \left( \frac{5}{r} - \frac{\Phi_{,r}}{2} (2\Phi_{,r} r + 7 + 8r\Lambda_{,r}) - 2\Lambda_{,r} (r\Lambda_{,r} - 1) + \frac{\bar{p}}{\rho} \frac{\Phi_{,r} r (2\Phi_{,r} r + 5) + 2}{2r} - \frac{e^{2\Lambda}}{2r} \left( 1 + \frac{\bar{p}}{\rho} \right) (2 + \Phi_{,r} r) \right] \bar{c}_s^2 \\
& \left. + \left( \left( 4 - \frac{\bar{p}}{\rho} \right) (\Lambda_{,r} + \Phi_{,r}) r + 4 \right) \Phi_{,r} - \frac{e^{-2\Lambda}}{2\pi} (\Lambda_{,r} + \Phi_{,r})^2 \Phi_{,r} \frac{d\bar{c}_s^2}{d\bar{\rho}} \right] \left( r^2 \gamma^{(1,0)} e^\Phi \right) \right\} e^{-\Phi - 2\Lambda}, \tag{A20}
\end{aligned}$$

$$b_{12} = \frac{\bar{c}_s^2}{2r^2} \left\{ \left[ (\Lambda_{,r} + \Phi_{,r}) (1 + \bar{c}_s^2) + \frac{4}{r} \right] \left( r^2 \gamma^{(1,0)} e^\Phi \right) - (1 + \bar{c}_s^2) \left( r^2 \gamma^{(1,0)} e^\Phi \right)_{,r} \right\} e^{-\Phi - 2\Lambda}, \tag{A21}$$

$$\begin{aligned}
b_{13} = & - \frac{1}{r^2} \left\{ \bar{c}_s^2 (1 + \bar{c}_s^2) \left( r^2 \gamma^{(1,0)} e^\Phi \right)_{,rr} + \frac{e^{-2\Lambda}}{4\pi r} (\Lambda_{,r} + \Phi_{,r}) \Phi_{,r} \left[ (\Lambda_{,r} + \Phi_{,r}) \left( r^2 \gamma^{(1,0)} e^\Phi \right) - \left( r^2 \gamma^{(1,0)} e^\Phi \right)_{,r} \right] \frac{d\bar{c}_s^2}{d\bar{\rho}} \right. \\
& - \left[ (2\Lambda_{,r} + \Phi_{,r}) \bar{c}_s^4 + \left( \frac{2}{r} + 2\Lambda_{,r} + 3\Phi_{,r} + \frac{\bar{p}}{\rho} \left( \frac{1}{r} - \frac{\Phi_{,r}}{2} \right) \right) \bar{c}_s^2 + \left( 2 - \frac{\bar{p}}{2\rho} \right) \Phi_{,r} \right] \left( r^2 \gamma^{(1,0)} e^\Phi \right)_{,r} \\
& + \left[ \left( \frac{4}{r} + \frac{\Lambda_{,r} + \Phi_{,r}}{2} \left( 6 - \frac{\bar{p}}{\rho} \right) \right) \Phi_{,r} - (\Lambda_{,r} + \Phi_{,r}) \left( \Lambda_{,r} - \Phi_{,r} + \frac{1}{r} \right) \bar{c}_s^4 \right. \\
& + \left[ \frac{1}{2} \left( \Phi_{,r} + \frac{1}{2r} \right) \left( 6\Lambda_{,r} + 7\Phi_{,r} - \frac{\bar{p}}{\rho} \left( \Phi_{,r} - \frac{2}{r} \right) \right) + \frac{2}{r^2} - \left( \Lambda_{,r} - \frac{1}{2r} \right) \left( \Lambda_{,r} + \frac{1}{r} \right) \right. \\
& \left. \left. + \frac{1}{4r^2} \left( 1 + \frac{\bar{p}}{\rho} \right) (r\Phi_{,r} - 2) e^{2\Lambda} \right] \bar{c}_s^2 \right] r^2 \gamma^{(1,0)} e^\Phi \right\} e^{-\Phi - 2\Lambda}, \tag{A22}
\end{aligned}$$

$$\begin{aligned}
b_{14} = & -\frac{\bar{c}_s^2}{r^3} \left\{ \left[ 2(\Lambda_{,r} + \Phi_{,r}) \left( 1 + \bar{c}_s^2 - \frac{\bar{p}}{\rho} \right) + \frac{\bar{p} l(l+1)}{\bar{\rho} r} e^{2\Lambda} \right] \left[ \left( r^2 \gamma^{(1,0)} e^\Phi \right)_{,r} - (\Lambda_{,r} + \Phi_{,r}) r^2 \gamma^{(1,0)} e^\Phi \right] e^{-2\Lambda} \right. \\
& \left. - 2l(l+1) \gamma^{(1,0)} e^\Phi \right\} e^{-\Lambda - \Phi}. \tag{A23}
\end{aligned}$$

Finally, in the case of the *Hamiltonian constraint*, the equation we have considered for  $k^{(1,1)}$ , the source term is given by (112):

$$\begin{aligned}
\mathcal{S}_{\text{Hamil}} = & c_1 \left( k_{,rr}^{(0,1)} - S_{,r}^{(0,1)} \right) + c_2 k_{,r}^{(0,1)} + c_3 k_{,t}^{(0,1)} + c_4 S^{(0,1)} + c_5 k^{(0,1)} + c_6 H^{(0,1)} + c_7 \psi_{,r}^{(0,1)} + \\
& + c_8 \psi^{(0,1)} + c_9 \gamma^{(0,1)}, \tag{A24}
\end{aligned}$$

where the coefficients  $c_i$  are given by

$$c_1 = r S^{(1,0)}, \tag{A25}$$

$$c_2 = \left( \frac{3}{2} S^{(1,0)} + \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2} H^{(1,0)} \right), \tag{A26}$$

$$c_3 = -(\Lambda_{,r} + \Phi_{,r}) e^{\Lambda - \Phi} \gamma^{(1,0)}, \tag{A27}$$

$$c_4 = -\left( S^{(1,0)} + 2 \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2} H^{(1,0)} \right), \tag{A28}$$

$$c_5 = -\frac{2}{r} \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2} H^{(1,0)}, \tag{A29}$$

$$c_6 = -\frac{2}{r} \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^2} \left[ 1 + \frac{1}{\bar{c}_s^2} - \frac{e^{-2\Lambda}}{4\pi r} \frac{\Lambda_{,r} + \Phi_{,r}}{\bar{c}_s^4} \frac{d\bar{c}_s^2}{d\bar{\rho}} \right] H^{(1,0)}, \tag{A30}$$

$$c_7 = \frac{2}{r} \gamma^{(1,0)}, \tag{A31}$$

$$c_8 = \frac{1}{r^2} \left[ (2 - 4\Lambda_{,r} r + l(l+1) e^{2\Lambda}) \gamma^{(1,0)} + 2r \gamma_{,r}^{(1,0)} \right], \tag{A32}$$

$$c_9 = -\frac{4}{r} (\Lambda_{,r} + \Phi_{,r}) \gamma^{(1,0)}. \tag{A33}$$

## APPENDIX B: SOUND WAVE EQUATION

We write here the complete sound wave equation (105) of a generic barotropic fluid perturbation on a time-dependent background in terms of the GSGM quantities,

$$\begin{aligned}
& -\ddot{H} + c_s^2 H'' + (\mu + 2U) \left( c_s^2 - \frac{p}{\rho} - \frac{2}{c_s^2} (\rho + p) \frac{dc_s^2}{d\rho} \right) \dot{H} + ((2c_s^2 - 1)\nu + 2c_s^2 W) H' \\
& + \left\{ (\rho + p) \left[ (\rho + p) (\mu + 2U)^2 \frac{1}{c_s^2} \left( \frac{d^2 c_s^2}{d\rho^2} - \frac{2}{c_s^2} \frac{dc_s^2}{d\rho} \right) + \left[ (\mu + 2U)^2 \left( 2 + \frac{\rho - p}{\rho c_s^2} \right) \right. \right. \right. \\
& + \left. \left. \frac{1}{c_s^2} \left( 3U^2 - \dot{\mu} - (2\nu + W)W + 8\pi p + \frac{1}{r^2} \right) \right] \frac{dc_s^2}{d\rho} + 4\pi (1 + 3c_s^2) \right] - \frac{l(l+1)}{r^2} c_s^2 \left. \right\} H \\
& + \frac{1}{2} (c_s^2 - 1) \nu (\chi' - k') + c_s^2 \mu \dot{\chi} + \frac{c_s^2}{2} (\mu + 2U) \left( 1 + c_s^2 - \frac{p}{\rho} \right) \dot{k} + \{ c_s^2 [2(2\nu + W)W + 4\pi(\rho - p)] \\
& + 2\dot{\mu} - \frac{2}{r^2} + \left( 1 + \frac{p}{\rho} - c_s^2 \right) \mu^2 - 2 \left( 1 + c_s^2 - \frac{p}{\rho} \right) \mu U - 2U^2 \} - 2\nu^2 \left. \right\} (\chi + k) \\
& + \frac{c_s^2}{2} \left( (1 + c_s^2) \mu - 2(1 - c_s^2) U \right) \psi' + \left\{ \frac{1}{2} (1 + c_s^2) (c_s^2 \mu' + \dot{\nu}) - \frac{1}{2} (\rho + p) \frac{dc_s^2}{d\rho} \left( 1 - \frac{1}{c_s^2} \right) (\mu + 2U) \nu \right. \\
& + \left[ c_s^2 \left( 1 - \frac{2p}{\rho} + 3c_s^2 \right) W + \left[ c_s^2 \left( c_s^2 + \frac{p}{\rho} - 3 \right) + \frac{p}{\rho} \right] \nu \right] U + \left[ c_s^2 \left( 1 - \frac{p}{\rho} + 3c_s^2 \right) W \right. \\
& + \left. \left. \frac{1}{2} \left[ c_s^2 (c_s^2 - 2) + \frac{p}{\rho} (1 + c_s^2) - 3 \right] \nu \right] \mu \right\} \psi + c_s^2 \left[ 2\mu - \frac{p}{\rho} (\mu + U) \right] \gamma' + \{ (1 - c_s^2) (c_s^2 \mu' + \dot{\nu}) \}
\end{aligned}$$

$$\begin{aligned}
& + (\rho + p) \left(1 + \frac{1}{c_s^2}\right) (\mu + 2U) \nu \frac{dc_s^2}{d\rho} + \left[2c_s^2 \left(1 + c_s^2 - 2\frac{p}{\rho}\right) W - 2 \left[c_s^2 \left(1 + c_s^2 + \frac{p}{\rho}\right) - \frac{p}{\rho}\right] \nu\right] U \\
& + \left[2c_s^2 \left(1 - c_s^2 - \frac{p}{\rho}\right) W + \left[c_s^2 (2 - c_s^2) + \frac{p}{\rho} (1 - c_s^2) - 1\right] \nu\right] \mu \Big\} \gamma \\
& + \left\{ \left[8\pi (\rho + p) c_s^4 + \left(\frac{l(l+1)p}{r^2} + 8\pi \frac{\rho^2 - p^2}{\rho}\right) c_s^2\right] (\mu + 2U) - 2\frac{l(l+1)}{r^3} c_s^2 \dot{r} \right\} \alpha = 0. \tag{B1}
\end{aligned}$$

### APPENDIX C: GAUGE INVARIANCE

In this Section we show the gauge-invariant character of the perturbative quantities (27-31) and (36-38) at the perturbative order (1,1). To that end we follow the procedure described in Section V B. Namely, we determine, for each quantity, the corresponding term  $\sum_{\sigma} \mathcal{I}_{\sigma}^{(1,0)} \mathcal{J}_{\sigma}^{(0,1)}$  in the expansion (85) and the gauge transformation of  $\mathcal{H}^{(1,1)}$  [see Eq. (94)]. Then, considering the gauge transformations for the non-radial perturbations  $\mathcal{J}^{(0,1)}$  and introducing all the terms in (95), we will prove the gauge invariance of our perturbations.

The non-radial gauge transformations (which we will use later) are

$$\tilde{p}_A^{(0,1)} = p_A^{(0,1)} + \hat{\xi}_A, \quad \tilde{G}^{(0,1)} = G^{(0,1)} + 2\xi. \tag{C1}$$

*Metric perturbations.* In Section V we have shown the gauge invariance of the metric perturbation  $k_{AB}^{(1,1)}$ . Here we prove that of the metric perturbation  $k^{(1,1)}$ . From (97) we find that,

$$\sum_{\sigma} \mathcal{I}_{\sigma}^{(1,0)} \mathcal{J}_{\sigma}^{(0,1)} = 2g^{(1,0)AB} v_B p_A^{(0,1)}, \tag{C2}$$

$$\tilde{\mathcal{H}}^{(1,1)} = \mathcal{H}^{(1,1)} - 2v^A g_{AC}^{(1,0)} \xi^C, \tag{C3}$$

where  $v_A = r_{|A}/r$ . Therefore using (C1) we find from (95) that  $\tilde{k}^{(1,1)} = k^{(1,1)}$ .

*Stress-energy tensor perturbations.* They are described by the seven quantities (28-31). The corresponding terms  $\sum_{\sigma} \mathcal{I}_{\sigma}^{(1,0)} \mathcal{J}_{\sigma}^{(0,1)}$  are:

$$\begin{aligned}
T_{AB} : \quad & p^{(0,1)C} t_{AB|C}^{(1,0)} + t_{AC}^{(1,0)} p_{|B}^{(0,1)C} + t_{CB}^{(1,0)} p_{|A}^{(0,1)C} - \bar{t}_{AB|C} g^{(1,0)CD} p_D^{(0,1)} \\
& - \bar{t}_{AC} \left( g_{|B}^{(1,0)CD} p_D^{(0,1)} + g^{(1,0)CD} p_{D|B}^{(0,1)} \right) - \bar{t}_{BC} \left( g_{|A}^{(1,0)CD} p_D^{(0,1)} + g^{(1,0)CD} p_{D|A}^{(0,1)} \right), \tag{C4}
\end{aligned}$$

$$T^3 : \quad (\bar{Q}_{|A} + 2\bar{Q}v_A) g^{(1,0)AB} p_B^{(0,1)} - \left( Q_{|A}^{(1,0)} + 2Q^{(1,0)}v_A \right) \bar{g}^{AB} p_B^{(0,1)} + \frac{l(l+1)}{2} Q^{(1,0)} G^{(0,1)}, \tag{C5}$$

$$T_A : \quad \left( t_A^{(1,0)C} - \bar{t}_{AB} g^{(1,0)BC} \right) p_C^{(0,1)} - \frac{r^2}{2} Q^{(1,0)} G_{|A}^{(0,1)}, \tag{C6}$$

$$T^2 : \quad r^2 Q^{(1,0)} G^{(0,1)}. \tag{C7}$$

where  $Q$  is the pressure. The gauge transformation of the  $\mathcal{H}^{(1,1)}$  part of the quantities under discussion is given by  $\mathcal{L}_{\xi(0,1)} t_{\alpha\beta}^{(1,0)}$ , where the energy-momentum tensor for radial perturbations has a block diagonal form

$$t_{\alpha\beta}^{(1,0)} = \text{diag} \left( t_{AB}^{(1,0)}; r^2 Q^{(1,0)} \gamma_{ab} \right). \tag{C8}$$

Then, we find

$$\mathcal{L}_{\xi(0,1)} t_{AB}^{(1,0)} = \mathcal{L}_{\xi} t_{AB}^{(1,0)}, \tag{C9}$$

$$\mathcal{L}_{\xi(0,1)} g_{Aa}^{(1,0)} = \left( t_{AC}^{(1,0)} \hat{\xi}^C + r^2 Q^{(1,0)} \xi_{|A} \right) Y_a, \tag{C10}$$

$$\mathcal{L}_{\xi(0,1)} t_{ab}^{(1,0)} = r^2 \left( Q_{|C}^{(1,0)} \hat{\xi}^C - l(l+1) Q^{(1,0)} \xi + 2v_C Q^{(1,0)} \hat{\xi}^C \right) Y \gamma_{ab} + \left( 2r^2 Q^{(1,0)} \xi \right) Z_{ab}. \tag{C11}$$

Therefore the gauge transformations of the  $\mathcal{H}^{(1,1)}$  terms for the above quantities are:

$$T_{AB} : \quad \hat{\xi}^C t_{AB|C}^{(1,0)} + t_{CB}^{(1,0)} \hat{\xi}_{|A}^C + t_{AC}^{(1,0)} \hat{\xi}_{|B}^C - \bar{t}_{AB|C} g_D^{(1,0)C} \hat{\xi}^D - \bar{t}^C{}_A \left( g_{CD|B}^{(1,0)} \hat{\xi}^D + g_{CD}^{(1,0)} \hat{\xi}_{|B}^D \right) - \bar{t}^C{}_B \left( g_{CD|A}^{(1,0)} \hat{\xi}^D + g_{CD}^{(1,0)} \hat{\xi}_{|A}^D \right), \quad (C12)$$

$$T^3 : \quad \left( Q_{|A}^{(1,0)} + 2Q^{(1,0)} v_A \right) \hat{\xi}^A - l(l+1)Q^{(1,0)} \xi - \left( \bar{Q}_{|A} + 2\bar{Q} v_A \right) g_B^{(1,0)A} \hat{\xi}^B, \quad (C13)$$

$$T_A : \quad t_{AB}^{(1,0)} \hat{\xi}^B + r^2 Q^{(1,0)} \hat{\xi}_{|A} - \bar{t}_{AB} g^{(1,0)BC} p_C^{(0,1)}, \quad (C14)$$

$$T^2 : \quad 2r^2 Q^{(1,0)} \xi. \quad (C15)$$

Combining all this terms into the relation (95), we finish the proof of the gauge invariance of (28-31) at the perturbative order (1,1).

The fluid perturbations for a barotropic fluid that have to be considered are the two components of the velocity (36-37) and the energy density (38). Applying the same procedure we find the corresponding  $\sum_\sigma \mathcal{I}_\sigma^{(1,0)} J_\sigma^{(0,1)}$  terms

$$\alpha : \quad g_{AB}^{(1,0)} p^{(0,1)A} \bar{u}^B - p_A^{(0,1)} u^{(1,0)A}, \quad (C16)$$

$$\gamma : \quad \left( n^{(1,0)A} - g^{(1,0)AC} \bar{n}_C \right) \left( \delta u_A^{(0,1)} - \frac{1}{2} h_{AB}^{(0,1)} \bar{u}^B - \bar{u}_{A|B} p^{(0,1)B} - p_{[B|A]}^{(0,1)} \bar{u}^B \right) - \bar{n}^A \left[ \left( u^{(1,0)B} - g^{(1,0)BC} \bar{u}_C \right) \left( p_{[B|A]}^{(0,1)} + \frac{1}{2} h_{AB}^{(0,1)} \right) + \left( u_{A|B}^{(1,0)} - \Gamma_{AB}^{(1,0)D} \bar{u}_D \right) p^{(0,1)B} - g^{(1,0)BD} \bar{u}_{A|B} p_D^{(0,1)} \right], \quad (C17)$$

$$\omega : \quad g^{(1,0)AB} p_B^{(0,1)} \bar{\Omega}_{|A} - p^{(0,1)A} \Omega_{|A}^{(1,0)}. \quad (C18)$$

The gauge transformations (94) for  $\mathcal{H}^{(1,1)}$  are now determined by the Lie derivative of the radial perturbations of the fluid velocity and energy density

$$u_\alpha^{(1,0)} \equiv \left( u_A^{(1,0)}, 0 \right), \quad \Omega^{(1,0)} \equiv \Omega^{(1,0)}(t, r), \quad (C19)$$

where  $\Omega \equiv \ln \rho$ . We obtain

$$\mathcal{L}_{\xi(0,1)} u_A^{(1,0)} = \hat{\xi}^B u_{A|B}^{(1,0)} + u_B^{(1,0)} \hat{\xi}_{|A}^B, \quad (C20)$$

$$\mathcal{L}_{\xi(0,1)} u_a^{(1,0)} = u_A^{(1,0)} \hat{\xi}^A Y_a, \quad (C21)$$

$$\mathcal{L}_{\xi(0,1)} \Omega^{(1,0)} = \hat{\xi}^A \Omega_{|A}. \quad (C22)$$

The corresponding  $\mathcal{H}^{(1,1)}$  transformations are

$$\alpha : \quad u_A^{(1,0)} \hat{\xi}^A - g_{AB}^{(1,0)} \hat{\xi}^A \bar{u}^B, \quad (C23)$$

$$\gamma : \quad \bar{n}^A \left\{ \hat{\xi}^B u_{A|B}^{(1,0)} + \left( u^{(1,0)B} - g^{(1,0)BC} \bar{u}_C \right) \hat{\xi}_{B|A} - g^{(1,0)BC} \bar{u}_{A|B} \hat{\xi}_C - \Gamma_{AC}^{(1,0)B} \hat{\xi}^C \bar{u}_B \right\}, \quad (C24)$$

$$\omega : \quad \hat{\xi}^A \Omega_{|A}^{(1,0)} - g^{(1,0)AB} \hat{\xi}_A \bar{\Omega}_B, \quad (C25)$$

It is again easy to verify the gauge invariance of these perturbations by bringing all terms into the relation (95). The fluid perturbation  $H^{(1)}$ , defined in Eq. (102), can be expanded like the previous quantities,

$$H^{(0,1)} = \mathcal{H}^{(0,1)} = \frac{\bar{c}_s^2 \bar{\rho}}{\bar{\rho} + \bar{p}} \omega^{(0,1)}, \quad (C26)$$

$$H^{(1,1)} = \mathcal{H}^{(1,1)} + \sum_\sigma \mathcal{I}_\sigma^{(1,0)} J_\sigma^{(0,1)}, \quad (C27)$$

where

$$\mathcal{H}^{(1,1)} = \frac{\bar{c}_s^2 \bar{\rho}}{\bar{\rho} + \bar{p}} \omega^{(1,1)}, \quad (C28)$$

$$\mathcal{I}_\sigma^{(1,0)} = \left[ \bar{c}_s^2 + \bar{\rho} \left( \frac{d\bar{c}_s^2}{d\bar{\rho}} - (1 + \bar{c}_s^2) \frac{\bar{c}_s^2}{\bar{\rho} + \bar{p}} \right) \right] \frac{\bar{\rho}}{\bar{\rho} + \bar{p}} \omega^{(1,0)}, \quad (C29)$$

$$J^{(0,1)} = \omega^{(0,1)}. \quad (C30)$$

Therefore, the gauge-invariant character of  $H^{(0,1)}$  and  $H^{(1,1)}$ , having fixed the gauge for radial perturbations, follows from the gauge invariance of  $\omega^{(0,1)}$  and  $\omega^{(1,1)}$ , which has already been proved previously.

#### APPENDIX D: CONNECTION TO THE REGGE-WHEELER METRIC VARIABLES

The perturbations of spherical stars and Black Holes are commonly studied in the Regge-Wheeler (RW) gauge [42]. Therefore, in this Appendix we provide the relations between the perturbative variables used in this work and those of RW. In the RW gauge the linear polar non-radial perturbations assume the following expansion in spherical tensor harmonics,

$$g_{\alpha\beta}^{(0,1)} \Big|_{RW} = \begin{pmatrix} H_{0,lm}^{(0,1)} e^{2\Phi} & H_{1,lm}^{(0,1)} & 0 & 0 \\ H_{1,lm}^{(0,1)} & H_{2,lm}^{(0,1)} e^{2\Lambda} & 0 & 0 \\ 0 & 0 & r^2 K_{lm}^{(0,1)} & 0 \\ 0 & 0 & 0 & r^2 K_{lm}^{(0,1)} \sin^2\theta \end{pmatrix} Y^{lm}, \quad (D1)$$

where  $H_{0,lm}^{(0,1)}$ ,  $H_{1,lm}^{(0,1)}$ ,  $H_{2,lm}^{(0,1)}$ ,  $K_{lm}^{(0,1)}$  are functions of  $(t, r)$ . The Einstein equations for a spherical star imply that  $H_{0,lm}^{(0,1)} = H_{2,lm}^{(0,1)}$ . We can also choose the RW gauge for the perturbative variables describing the coupling by enforcing the RW form of the polar metric perturbations at the  $(1, 1)$  perturbative order. Then, we impose the following form of  $g_{\alpha\beta}^{(1,1)}$ :

$$g_{\alpha\beta}^{(1,1)} \Big|_{RW} = \begin{pmatrix} H_{0,lm}^{(1,1)} e^{2\Phi} & H_{1,lm}^{(1,1)} & 0 & 0 \\ H_{1,lm}^{(1,1)} & H_{2,lm}^{(1,1)} e^{2\Lambda} & 0 & 0 \\ 0 & 0 & r^2 K_{lm}^{(1,1)} & 0 \\ 0 & 0 & 0 & r^2 K_{lm}^{(1,1)} \sin^2\theta \end{pmatrix} Y^{lm}, \quad (D2)$$

where also  $H_{0,lm}^{(1,1)}$ ,  $H_{1,lm}^{(1,1)}$ ,  $H_{2,lm}^{(1,1)}$ ,  $K_{lm}^{(1,1)}$  are functions of  $(t, r)$ .

The expansion of the linear non-radial and the coupling metric perturbations in terms of the GSGM variables can be derived from (23) by applying the 2-parameter expansion of GSGM formalism described in section V. First of all, we must first apply the RW gauge at first and second perturbation order, i.e.,

$$h_{A,lm}^{(0,1)} = G_{lm}^{(0,1)} = h_{A,lm}^{(1,1)} = G_{lm}^{(1,1)} = 0.$$

Then, we take the perturbative expansion of the gauge-invariant tensor  $k_{AB}$  (39), written in the basis of  $M^2$  spanned by the vectors (75,76), and use the definition (40) for the  $\chi$  perturbation. Finally, taking into account Einstein's equation (41), we find the following relations: (i) For the  $(0, 1)$  linear perturbations,

$$H_{0,lm}^{(0,1)} = H_{2,lm}^{(0,1)} = \chi_{lm}^{(0,1)} + k_{lm}^{(0,1)}, \quad H_{1,lm}^{(0,1)} = -\psi_{lm}^{(0,1)} e^{\Phi+\Lambda}, \quad K_{lm}^{(0,1)} = k_{lm}^{(0,1)}. \quad (D3)$$

(ii) For the  $(1, 1)$  coupling perturbations,

$$H_{0,lm}^{(1,1)} = \chi_{lm}^{(1,1)} + k_{lm}^{(1,1)} + \left(2\eta^{(1,0)} - \chi^{(1,0)}\right) \left(\chi_{lm}^{(0,1)} + k_{lm}^{(0,1)}\right) + 2\gamma^{(1,0)}\psi_{lm}^{(0,1)}, \quad (D4)$$

$$H_{1,lm}^{(1,1)} = -\left[\psi_{lm}^{(1,1)} + 2\gamma^{(1,0)} \left(\chi_{lm}^{(0,1)} + k_{lm}^{(0,1)}\right) + \eta^{(1,0)}\psi_{lm}^{(0,1)}\right] e^{\Phi+\Lambda}, \quad (D5)$$

$$H_{2,lm}^{(1,1)} = \chi_{lm}^{(1,1)} + k_{lm}^{(1,1)} + \chi^{(1,0)} \left(\chi_{lm}^{(0,1)} + k_{lm}^{(0,1)}\right) + 2\gamma^{(1,0)}\psi_{lm}^{(0,1)}, \quad (D6)$$

$$K_{lm}^{(1,1)} = k_{lm}^{(1,1)}. \quad (D7)$$

Before concluding this section we also give here the relations even between the quantities used in this paper and those considered by Allen et al. [33] and Ruoff [34] at linear order:

$$\chi^{(0,1)} = \begin{cases} r e^{-2\Phi} S & \text{in [33],} \\ r S & \text{in [34],} \end{cases} \quad k^{(0,1)} = \begin{cases} F/r & \text{in [33],} \\ T/r & \text{in [34].} \end{cases} \quad (\text{D8})$$

For more details about the connection between the GSGM and other formalisms see the appendix in [19].

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- [1] N. Andersson, *Class. Quant. Grav.* **20**, R105 (2003), astro-ph/0211057.  
[2] K. D. Kokkotas and J. Ruoff (2002), gr-qc/0212105.  
[3] U. Sperhake, P. Papadopoulos, and N. Andersson (2001), astro-ph/0110487.  
[4] A. K. Schenk, P. Arras, É. É. Flanagan, S. A. Teukolsky, and I. Wasserman, *Phys. Rev. D* **65**, 024001 (2002).  
[5] S. M. Morsink, *Astrophys. J.* **571**, 435 (2002).  
[6] L. Lindblom, J. E. Tohline, and M. Vallisneri, *Phys. Rev. D* **65**, 084039 (2002).  
[7] P. Arras, É. É. Flanagan, S. M. Morsink, A. K. Schenk, S. A. Teukolsky, and I. Wasserman, *Astrophys. J.* **591**, 1129 (2003).  
[8] R. H. Price, in *Black Holes, Gravitational Radiation, and the Universe: Essays in Honor of C.V. Vishveshwara* (Kluwer Academic Publishers, Dordrecht, 1998), p. 351.  
[9] A. Passamonti, A. Nagar, M. Bruni, L. Gualtieri, and C. F. Sopuerta, *in preparation*. (2004).  
[10] A. Passamonti, A. Nagar, M. Bruni, L. Gualtieri, and C. F. Sopuerta, *in preparation*. (2005).  
[11] L. Gualtieri, E. Berti, J. A. Pons, G. Miniutti, and V. Ferrari, *Phys. Rev. D* **64**, 104007 (2001), gr-qc/0107046.  
[12] E. Poisson, *Phys. Rev. D* **47**, 1497 (1993).  
[13] J. Ruoff, A. Stavridis, and K. D. Kokkotas, *Mon. Not. Roy. Astron. Soc.* **332**, 676 (2002), gr-qc/0109065.  
[14] J. B. Hartle, *Astrophys. J.* **150**, 1005 (1967).  
[15] J. B. Hartle and K. S. Thorne, *Astrophys. J.* **153**, 807 (1968).  
[16] E. Berti, F. White, A. Maniopoulou, and M. Bruni (2004), gr-qc/0405146.  
[17] M. Bruni, L. Gualtieri, and C. F. Sopuerta, *Class. Quant. Grav.* **20**, 535 (2003), gr-qc/0207105.  
[18] C. F. Sopuerta, M. Bruni, and L. Gualtieri, *to appear in Phys. Rev. D* (2004), gr-qc/0306027.  
[19] C. Gundlach and J. M. Martín-García, *Phys. Rev. D* **61**, 084024 (2000), gr-qc/9906068.  
[20] J. M. Martín-García and C. Gundlach, *Phys. Rev. D* **64**, 024012 (2001), gr-qc/0012056.  
[21] U. H. Gerlach and U. K. Sengupta, *Phys. Rev. D* **19**, 2268 (1979).  
[22] C. T. Cunningham, R. H. Price, and V. Moncrief, *Astrophys. J.* **236**, 674 (1980).  
[23] R. J. Gleiser, C. O. Nicasio, R. H. Price, and J. Pullin, *Class. Quant. Grav.* **13**, L117 (1996), gr-qc/9510049.  
[24] R. Wald, *General Relativity* (The University of Chicago press, Chicago, 1984).  
[25] J. M. Stewart and M. Walker, *Royal Society of London Proceedings Series A* **341**, 49 (1974).  
[26] M. Bruni, S. Matarrese, S. Mollerach, and S. Sonego, *Class. Quant. Grav.* **14**, 2585 (1997), gr-qc/9609040.  
[27] S. Sonego and M. Bruni, *Commun. Math. Phys.* **193**, 209 (1998), gr-qc/9708068.  
[28] C. W. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman & Co., San Francisco, 1973).  
[29] K. D. Kokkotas and J. Ruoff, *Astron. Astrophys.* **366**, 565 (2001), gr-qc/0011093.  
[30] K. S. Thorne and A. Campolattaro, *Astrophys. J.* **149**, 591 (1967).  
[31] S. Detweiler and L. Lindblom, *Astrophys. J.* **292**, 12 (1985).  
[32] S. Chandrasekhar and V. Ferrari, *Royal Society of London Proceedings Series A* **432**, 247 (1991).  
[33] G. Allen, N. Andersson, K. D. Kokkotas, and B. F. Schutz, *Phys. Rev. D* **58**, 124012 (1998), gr-qc/9704023.  
[34] J. Ruoff, *Phys. Rev. D* **63**, 064018 (2001).  
[35] E. Seidel and T. Moore, *Phys. Rev. D* **35**, 2287 (1987).  
[36] E. Seidel, *Phys. Rev. D* **42**, 1884 (1990).  
[37] U. H. Gerlach and U. K. Sengupta, *Phys. Rev. D* **22**, 1300 (1980).  
[38] J. M. Martín-García and C. Gundlach, *Phys. Rev. D* **59**, 064031 (1999), gr-qc/9809059.  
[39] A. Nagar, G. Díaz, J. A. Pons, and J. A. Font, *Phys. Rev. D* **69**, 124028 (2004), gr-qc/0403077.  
[40] J. O. Ruoff, Ph.D. thesis, Universitaet Tuebingen (2000), gr-qc/0010041.  
[41] A. Nagar and G. Díaz, in *Proceedings of the 27th Spanish Relativity Meeting. Gravitational Radiation* (Editorial Services of the University of Alicante, Alicante, 2004).  
[42] T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).  
[43] V. Moncrief, *Ann. Phys. (N.Y.)* **88**, 323 (1974).  
[44] F. J. Zerilli, *Phys. Rev. D* **2**, 2141 (1970).  
[45] F. J. Zerilli, *Phys. Rev. Lett.* **24**, 737 (1970).  
[46] C. T. Cunningham, R. H. Price, and V. Moncrief, *Astrophys. J.* **224**, 643 (1978).  
[47] G. Darmon, in *Mémoires des Sciences Mathématiques* (Gauthier-Villars, Paris, 1927), vol. XXV, chap. V.  
[48] A. Lichnerowicz, *C. R. Acad. Sci.* **273**, 528 (1971).  
[49] S. O'Brien and J. L. Singe, *Proc. Dublin Inst. Adv. Stud.* **A9**, 1 (1952).



- [50] W. Israel, *Nuovo Cimento* **B44**, 1 (1966).
- [51] M. Campanelli and C. O. Lousto, *Phys. Rev.* **D59**, 124022 (1999), gr-qc/9811019.
- [52] J. A. Font et al., *Phys. Rev.* **D65**, 084024 (2002), gr-qc/0110047.
- [53] N. Stergioulas, T. A. Apostolatos, and J. A. Font, *Mon. Not. Roy. Astron. Soc.* **352**, 1089 (2004), astro-ph/0312648.
- [54] O. Zanotti, J. A. Font, L. Rezzolla, and P. J. Montero (2004), astro-ph/0411116.
- [55] H. Dimmelmeier, N. Stergioulas, and J. A. Font, *MNRAS*, *submitted* (2004).