Evolution of gravitational waves in Randall-Sundrum cosmology

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We investigate the evolution of gravitational wave perturbations about a brane cosmology embedded in a five-dimensional anti-de Sitter bulk. During slow-roll inflation in a Randall-Sundrum brane-world, the zero mode of the 5-dimensional graviton is generated, while the massive modes remain in their vacuum state. When the zero mode re-enters the Hubble radius during radiation domination, massive modes are generated. We show that modes decouple in the low-energy/near-brane limit and develop perturbative techniques to calculate the mode-mixing at finite energy.

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I. INTRODUCTION

The Randall-Sundrum (RS) scenario [1, 2] is a simple and novel way of realizing the idea that our observable universe could be a 4-brane embedded in a higher-dimensional bulk spacetime, with interactions of the standard model confined to the brane, while gravitational interactions access the bulk. The original RS model considered a Minkowski brane in an anti de Sitter (AdS) bulk, and this was generalized to a Friedmann-Robertson-Walker (FRW) brane in a Schwarzschild-AdS bulk [3, 4].

The massless 5-dimensional graviton has a massless (zero) mode when projected onto the brane, but it also has a tower of massive Kaluza-Klein (KK) modes, i.e. modes which have an effective mass from a brane viewpoint. These massive modes introduce new features into the generation and evolution of cosmological gravitational waves, and the observational implications of these features provide in principle constraints on the RS scenario. It has been shown [6] (see also Refs. [8–10]) that during high-energy de Sitter inflation on the brane (in an AdS bulk), the zero mode is generated with a scale-invariant spectrum, at an amplitude which can be much greater than the corresponding general relativity value:

$$\mathcal{P}_T = \left[\frac{8}{M_P^2}\left(\frac{H}{2\pi}\right)^2\right] \times\left\{\sqrt{1+\ell^2H^2}+\ell^2H^2\ln\left(\frac{\ell H}{1+\sqrt{1+\ell^2H^2}}\right)\right\}^{-1} \quad (1)$$

Here the general relativity expression is in square brackets, and the braneworld modification is in curly brackets, with ℓ being the curvature scale of the AdS bulk. (Table-top experiments to test deviations from Newton's law currently impose the constraint $\ell < 0.1\,$mm.) At high energies, $\ell H \gg 1$, the correction factor is large ($\to \frac{3}{2}\ell H$).

The massive modes are not excited during inflation [6], and the zero mode remains frozen after inflation while it is beyond the Hubble radius. However, when the zero mode enters the Hubble radius during radiation or matter domination, the separation between the massless zero mode and massive bulk modes no longer holds. A changing Hubble parameter induces mode-mixing, and an initial zero mode will generate massive modes. An estimate of this effect [8], based on an instantaneous transition from a de Sitter to a Minkowski brane, indicates that the effect will be very small. This estimate has been refined by considering the sharp transition to be from one de Sitter brane to another with slightly smaller Hubble rate [10].

In this paper we investigate the evolution of tensor metric perturbations about an AdS bulk in Gaussian normal coordinates defined with respect to an FRW brane. We first review the analytic results for perturbations about the original RS solution with two Minkowski branes embedded in AdS. We go on to develop a perturbative analytic approximation (valid at low energies, near the brane) to study the generation of massive modes with an FRW cosmology on the brane.

II. GRAVITATIONAL WAVE EQUATION AND ITS EXACT SOLUTIONS

The generic 5D bulk metric that allows a spatially flat FRW metric on the brane, can be written in Gaussian normal coordinates as

$$^{(5)}d\bar{s}^2 = -N^2(t,y)dt^2 + A^2(t,y)d\vec{x}^{\,2} + dy^2\,, \quad (2)$$

where the brane is located at $y=0$. Tensor metric perturbations [5–10] are given by

$$^{(5)}ds^2 = -N^2(t,y)dt^2 + A^2(t,y)\left[\delta_{ij}+h_{ij}\right]dx^idx^j + dy^2\,, \quad (3)$$

where h_{ij} is 3-transverse ($\partial^i h_{ij}=0$) and 3-tracefree ($\delta^{ij}h_{ij}=0$). We will treat one Fourier mode at a time, so that

$$h_{ij} = F(t,y)\,\hat{e}_{ij}(\mathbf{x})\,, \quad (4)$$

with $\hat{e}_{ij}$ a transverse and tracefree polarisation tensor which is an eigenfunction of the spatial Laplacian ($\partial^k\partial_k\hat{e}_{ij} = -k^2\hat{e}_{ij}$).

The tensor amplitude F obeys the 5D wave equation [6]

$$\frac{1}{N^2}\left[\ddot{F}+\left(3\frac{\dot{A}}{A}-\frac{\dot{N}}{N}\right)\dot{F}\right]+\frac{k^2}{A^2}F = F''+\left(3\frac{A'}{A}+\frac{N'}{N}\right)F'\,. \quad (5)$$

In the RS scenario, the bulk is an orbifold with Z_2-symmetry about two fixed points, $y=0$ and $y=L$. This identifies points $y\leftrightarrow -y$ and $y+L\leftrightarrow L-y$. We will assume that we have branes at the two orbifold fixed points with just the right surface energy-momentum tensors to satisfy the Israel junction conditions. In the case of Minkowski branes, this simply implies that the branes have constant brane tension. For a generic FRW brane at $y=0$ this puts a strong physical restriction [12, 15] on the required equation of state on the second FRW brane at $y=L$, which we will discuss in Sec. III A.

Perturbations must also satisfy the Z_2-symmetry. For a Z_2-symmetric background this will be true, so long as the initial conditions are Z_2-symmetric. The boundary conditions for the metric perturbations at the branes, in the absence of anisotropic pressure perturbations on the branes (which we will assume) are

$$F'|_{y=0}=0\,,\qquad F'|_{y=L}=0\,. \quad (6)$$

This should be imposed on any initial conditions, and is then preserved by the subsequent evolution.

In the special case when the FRW metric reduces to a static Minkowski brane, the anti-de Sitter bulk metric Eq. (2) is given by [1]

$$N=A=\exp(-\mu y)\,,\quad \mu=\frac{1}{\ell}\,, \quad (7)$$

where μ is the mass scale corresponding to the anti-de Sitter curvature scale ℓ. Here and in the following, the expressions hold for $0\le y\le L$, and we consider the Z_2-symmetry as implicitly imposed for other values of y. The wave equation (5) reduces to the simple separable form,

$$e^{2\mu y}\left(\ddot{F}+k^2F\right)=F''-4\mu F'\,. \quad (8)$$

Separating variables as

$$F(t,y)=\sum_m \varphi_m(t)F_m(y)\,, \quad (9)$$

we find that

$$\ddot{\varphi}_m+\left(m^2+k^2\right)\varphi_m = 0\,, \quad (10)$$

$$F_m''-4\mu F_m'+m^2e^{2\mu y}F = 0\,. \quad (11)$$

The general solutions are [1]

$$\varphi_m = c_+e^{+i\sqrt{m^2+k^2}t}+c_-e^{-i\sqrt{m^2+k^2}t}\,, \quad (12)$$

$$F_0 = c_1+c_2e^{4\mu y}\,, \quad (13)$$

$$F_m = e^{2\mu y}B_2\left(\frac{m}{\mu}\,e^{\mu y}\right)\quad (m>0)\,, \quad (14)$$

where $c_\pm, c_1, c_2$ are constants and B_2 is a linear combination of Bessel functions of order two. We only want the KK modes that satisfy the boundary conditions (6). For the zero mode, this requires

$$c_2=0\,. \quad (15)$$

For the massive modes, the first condition (at $y=0$) requires

$$\frac{m}{\mu}B_2'\left(\frac{m}{\mu}\right)+2B_2\left(\frac{m}{\mu}\right)=0\,, \quad (16)$$

where the prime here is the derivative with respect to the argument of the Bessel function. This chooses the appropriate linear combination of J_2 and Y_2. The second (at $y=L$) requires

$$\frac{m}{\mu}\,e^{\mu L}B_2'\left(\frac{m}{\mu}\,e^{\mu L}\right)+2B_2\left(\frac{m}{\mu}\,e^{\mu L}\right)=0\,. \quad (17)$$

These two conditions can be rewritten, using the recurrence relation for Bessel functions $xB_n'+nB_n=xB_{n-1}$, in the simpler form

$$B_1\left(\frac{m}{\mu}\right)=0=B_1\left(\frac{m}{\mu}\,e^{\mu L}\right)\,, \quad (18)$$

where B_1 is the same linear combination as B_2 but with J_1 and Y_1 instead of J_2 and Y_2.

The second condition can only be satisfied by particular values of m^2, which selects a discrete set of KK eigenmodes in the two-brane scenario. The allowed values of m are the solutions of

$$J_1\left(\frac{m}{\mu}\,e^{\mu L}\right)Y_1\left(\frac{m}{\mu}\right)-Y_1\left(\frac{m}{\mu}\,e^{\mu L}\right)J_1\left(\frac{m}{\mu}\right)=0\,. \quad (19)$$

The only case other than Minkowski branes for which the 5D wave equation (5) is separable in Gaussian normal coordinates is the case of a (anti-)de Sitter brane, which reduces to the Minkowski brane as a limiting case. For completeness we give the de Sitter mode functions in an appendix.

III. COSMOLOGICAL BACKGROUND SOLUTION

The 5D bulk metric, Eq. (2), for a general FRW brane at $y=0$ in an anti-de Sitter bulk is given explicitly by [3]

$$A(t,y)=a(t)\left\{\cosh\mu y-[1+r(t)]\sinh\mu y\right\}\,, \quad (20)$$

$$N(t,y)=\frac{A(t,y)}{a(t)}+3\,[1+w(t)]\,r(t)\sinh\mu y\,, \quad (21)$$

where

$$r\equiv\frac{\rho}{\lambda}\,,\ w\equiv\frac{p}{\rho}\,,\ \mu=\frac{\kappa^2\lambda}{6}\,. \quad (22)$$

Here λ is the tension for a Minkowski brane, so that the high-energy regime is $\rho > \lambda$, and $\kappa^2 = 8\pi/M_5^3$, where M_5 is the fundamental Planck scale. The effective Planck scale on the brane is $M_P = \sqrt{M_5^3/\mu}$.

The two unknown functions of time, $a(t)$ and $r(t)$, are obtained by solving the Friedmann and energy conservation equations on the brane,

$$H = \frac{\dot{a}}{a} = \sqrt{\mu^2 r(2+r)}\,, \tag{23}$$

$$\dot{r} = -3H(1+w)r\,. \tag{24}$$

For $r \ll 1$, one recovers the standard Friedmann equation. For $r > 1$, the evolution is unconventional.

For a barotropic fluid with constant w we have [3]

$$r = \left(\frac{a_*}{a}\right)^{3(1+w)}\,, \tag{25}$$

$$a^{3(1+w)} = a_*^{3(1+w)}\left[1 + \frac{3(1+w)\mu t}{2}\right]3(1+w)\mu t, \tag{26}$$

where $a = a_*$ when $r = 1$, so that a_* marks the transition between high- and low-energy regimes. In the case of radiation domination, $w = \frac{1}{3}$, we have

$$r = \left(\frac{a_*}{a}\right)^4\,, \tag{27}$$

$$a = a_*\,(1+2\mu t)^{1/4}\,(4\mu t)^{1/4}\,, \tag{28}$$

recovering the standard evolution, $a \propto t^{1/2}$, for $t \gg \mu^{-1}$.

The coordinate singularity y_c, defined by $A(t, y_c) = 0$, and the event horizon y_h, defined by $N(t, y_h) = 0$, are given by

$$\coth \mu y_c = 1 + r\,, \quad \coth \mu y_h = 1 - (2+3w)\,r\,. \tag{29}$$

There is no solution for y_h if $w > -\frac{2}{3}$, whereas y_c exists for all $\rho > 0$ branes, and $y_c \to \infty$ as $r \to 0$. For $-1 < w < -2/3$ we have $y_c < y_h$, while for a de Sitter brane ($w = -1$), we have $y_c = y_h$.

A. Regulator brane

To simplify the problem, and make it numerically tractable, we will assume that there is a second regulator brane at a fixed position $y_2 = L < y_c$. Since $y_c(t)$ increases with the expansion of the universe, the second brane remains within the regular coordinate region. The horizon y_h only exists for $w < -\frac{2}{3}$, which is the condition for inflation in the high-energy limit [11].

The fixed position of the regulator brane constrains its energy density and pressure via [12]

$$\frac{\rho_L + \lambda_2}{\lambda} = \frac{\sinh \mu L - (1+r)\cosh \mu L}{\cosh \mu L - (1+r)\sinh \mu L}\,, \tag{30}$$

$$\frac{p_L - \lambda_2}{\lambda} = -\frac{2(\rho_L + \lambda_2)}{3\lambda} - \frac{1}{3}\left\{\frac{\sinh \mu L - [1-(2+3w)r]\cosh \mu L}{\cosh \mu L - [1-(2+3w)r]\sinh \mu L}\right\}. \tag{31}$$

These imply

$$(1+w_L)r_L = -(1+w)r\left[\cosh \mu L - (1+r)\sinh \mu L\right]^{-1} \times \left\{\cosh \mu L - [1-(2+3w)r]\sinh \mu L\right\}^{-1}\,, \tag{32}$$

where $r_L = \rho_L/\lambda$, $w_L = p_L/\rho_L$. It follows that if the physical brane is de Sitter ($w = -1$), then so is the regulator brane (with $w_L = -1$). For $w > -\frac{2}{3}$ we have the limiting cases,

$$r \gg 1: \; (1+w_L)r_L \approx \frac{(1+w)}{[(2+3w)\sinh^2 \mu L]r}\,, \tag{33}$$

$$r \ll 1: \; (1+w_L)r_L \approx (1+w)re^{\mu L}\,. \tag{34}$$

Thus the regulator brane requires $(1+w_L)r_L \to 0$ for both the very high energy regime on the physical brane $r \to \infty$, and the low energy regime on the physical brane $r \to 0$.

B. New coordinates

It is convenient to introduce a new dimensionless variable for the normal coordinate,

$$z \equiv e^{\mu y}\,, \; 1 \le z < z_c = \sqrt{1 + \frac{2}{r}}\,, \tag{35}$$

where the range of validity follows from Eq. (29). The metric then reads

$${}^{(5)}d\bar{s}^2 = -N^2(t,z)dt^2 + A^2(t,z)d\vec{x}^{\,2} + \frac{dz^2}{\mu^2 z^2}\,, \tag{36}$$

with Eqs. (20) and (21) being rewritten as

$$A(t,z) = \frac{a(t)}{z}\left\{1 - \frac{r(t)}{2}[z^2 - 1]\right\}\,, \tag{37}$$

$$N(t,z) = \frac{n(t)}{z}\left\{1 + [2+3w(t)]\frac{r(t)}{2}[z^2 - 1]\right\}. \tag{38}$$

The wave equation (5) becomes

$$\frac{1}{N^2}\left[\ddot{F} + \left(3\frac{\dot{A}}{A} - \frac{\dot{N}}{N}\right)\dot{F}\right] + \frac{k^2}{A^2}F = \mu^2\left[z^2 F'' + \left(z + 3z^2\frac{A'}{A} + z^2\frac{N'}{N}\right)F'\right]\,, \tag{39}$$

where the prime from now on denotes $\partial/\partial z$.

C. Causal propagation in the bulk

The regulator brane will eventually have an effect on the physical brane via gravitational wave propagation. Approximate and numerical results will only be reliable up to the time when perturbations from the regulator brane reach the physical brane. The fastest signals in the bulk are zero-momentum ($k = 0$) on the brane. They follow null world-lines with (taking $n = 1$)

$$\frac{dz}{dt} = \pm\mu z n = \pm\mu\left[1 + \left(1 + \frac{3w}{2}\right) r(z^2-1)\right], \quad (40)$$

where, by Eq. (35), $r(z^2 - 1) < 2$. Hence for a FRW brane with $w > -\frac{2}{3}$, we have

$$\mu \leq \left|\frac{dz}{dt}\right| < 3(1+w)\mu\,. \quad (41)$$

In particular there is a lower limit on the time for causal propagation from the regulator brane at $z = z_L = e^{\mu L}$ to our FRW brane at $z = 1$, given by

$$\Delta t_L > \frac{e^{\mu L} - 1}{3\mu(1+w)}\,. \quad (42)$$

IV. COSMOLOGICAL BRANE PERTURBATIONS

In general, the wave equation (5) for an FRW brane does not separate in a Gaussian normal coordinate system where the physical brane is fixed at $y = 0$. The wave equation is only separable if the background metric functions $A(t, z)$ and $N(t, z)$ are themselves, and it can be shown that this is only possible in the special case of constant brane surface density (which is the case for a Minkowski brane [2] or de Sitter branes [6, 13]).

Moreover, even if one could define a set of independent eigenmodes for the bulk spacetime, the regulator brane in general represents a time-dependent boundary condition. Thus the regulator brane, moving relative to the normal coordinate system of the physical brane, will also lead to mixing between modes. In what follows we assume that the regulator brane remains at a fixed normal distance to the physical brane. One would expect this to *underestimate* the mode-mixing obtained when the second brane is free to move.

The form of the bulk metric functions given in Eqs. (37) and (38) suggests that it should be possible to obtain an approximate separable solution in the low-energy limit, $r \to 0$, or close to the brane, $z \to 1$, where A and N take the limiting forms

$$A(t, z) \to \frac{a(t)}{z}\,, \quad (43)$$

$$N(t, z) \to \frac{n(t)}{z}\,. \quad (44)$$

In this limit we can study the limiting behaviour of F analytically. (A similar approximation was independently discussed in Ref. [14].)

Formally we define the low-energy/near-brane regime by the condition

$$r(z^2-1) \ll 1\,. \quad (45)$$

If the regulator brane is fixed at finite $z_L < z_c$, then at sufficiently late times (in an expanding cosmology with $w > -1$) we will have $r \ll 1/(z_L^2 - 1)$ and the asymptotic solution will become a good approximation throughout the (finite) bulk. Conversely, the low-energy/near-brane condition, Eq. (45), will inevitably breakdown near the regulator brane in the limit $z_L \to z_c$.

We can analytically estimate the effect of mode-mixing at finite (but still small) r by trying to build up a perturbative solution starting from the low-energy/near-brane solutions, presented in the next section, and then calculating the corrections $\mathcal{O}(r)$, which we go on to do afterwards.

A. Separable solution for low-energy/near-brane limit

Using the canonical variable,

$$\tilde{F} = a z^{-3/2} F\,, \quad (46)$$

and conformal time η (so that $n = a$), we can write the wave equation (39) for $r(z^2 - 1) \to 0$ as

$$\frac{1}{a^2}\mathcal{D}_\eta \tilde{F}^{(0)} + \frac{k^2}{a^2}\tilde{F}^{(0)} - \mu^2 \mathcal{D}_z \tilde{F}^{(0)} = 0\,, \quad (47)$$

where the second-order self-adjoint operators are

$$\mathcal{D}_\eta \equiv \frac{\partial^2}{\partial\eta^2} - \frac{\ddot{a}}{a}\,, \quad (48)$$

$$\mathcal{D}_z \equiv \frac{\partial^2}{\partial z^2} - \frac{15}{4z^2}\,. \quad (49)$$

Here and from now on, a dot denotes $\partial/\partial\eta$.

The wave equation (47) is separable and hence its solution can be expressed in terms of KK eigenmodes,

$$\tilde{F}^{(0)}(\eta, z) = \sum_m v_m^{(0)}(\eta)\psi_m(z)\,, \quad (50)$$

where the mode functions are related to those in Eq. (9) by $v_m^{(0)} = a\varphi_m$ and $\psi_m = \mathcal{N}_m z^{-3/2} F_m$, with $\mathcal{N}_m$ a normalization constant. They obey the equations

$$\frac{1}{a^2}\mathcal{D}_\eta v_m^{(0)} + \frac{k^2}{a^2} v_m^{(0)} = -m^2 v_m^{(0)}\,, \quad (51)$$

$$\mathcal{D}_z \psi_m = -\frac{m^2}{\mu^2}\psi_m\,. \quad (52)$$

The bulk eigenmodes obey the same equation as in the case of a Minkowski brane, Eq. (11), and hence we have (for $m > 0$)

$$\psi_m = \mathcal{N}_m \sqrt{z} B_2\left(\frac{m}{\mu} z\right)\,, \quad (53)$$

where the coefficient of normalization ensures that the ψ_m constitute an orthonormal basis,

$$\int_1^{z_L} dz\, \psi_m(z)\psi_{m'}(z) = \delta_{mm'}\,. \quad (54)$$

In the radiation era, $w = \frac{1}{3}$, Eq. (28) shows that in the low-energy/ late-time regime,

$$a(\eta) = a_1 \eta\,, \quad a_1 \equiv \sqrt{2} a_*^2 \mu\,, \tag{55}$$

and Eq. (51) becomes

$$\ddot{v}_m^{(0)} + \left(k^2 + m^2 a_1^2 \eta^2\right) v_m^{(0)} = 0\,. \tag{56}$$

On large scales, or at late times, we can neglect the k-term, and the solutions are:

$$v_0^{(0)} = c_1 \eta + c_2\,, \tag{57}$$

$$v_m^{(0)} = \eta^{1/2} B_{1/4}\left(\frac{m a_1}{2}\eta^2\right)\,. \tag{58}$$

It follows that the massive modes $m \neq 0$ decay on super-horizon scales, unlike the massless mode $m = 0$. For $\eta \gg 1$ and k negligible, the massive modes behave as

$$v_m^{(0)} \approx \eta^{-1/2}\left[c_3 \cos\left(\frac{m a_1}{2}\eta^2\right) + c_4 \sin\left(\frac{m a_1}{2}\eta^2\right)\right]. \tag{59}$$

B. Perturbative mode-mixing

Using the bulk solution given in Eqs. (37) and (38), we can now write the wave equation (39) to first order in r as

$$\frac{1}{a^2}\mathcal{D}_\eta \tilde{F} + \frac{k^2}{a^2}\tilde{F} - \mu^2 \mathcal{D}_z \tilde{F} = r S[\tilde{F}]\,, \tag{60}$$

$$S[\tilde{F}] = \frac{(z^2-1)}{a^2}\Big[(2+3w)\mathcal{D}_\eta \tilde{F} - \frac{3}{2}(1+w)(5+3c_s^2)\mathcal{H}(\tilde{F}^{\cdot} - \mathcal{H}\tilde{F}) - k^2 \tilde{F}\Big] + (1-3w)\left[z\tilde{F}' + \frac{3}{2}\tilde{F}\right]\,, \tag{61}$$

where $\mathcal{H} = \dot{a}/a = \partial_\eta a/a$ and $c_s^2 = \dot{p}/\dot{\rho}$. The full solution is a series expansion,

$$\tilde{F} = \tilde{F}^{(0)} + \tilde{F}^{(1)} + \dots\,, \tag{62}$$

where the zero-order solution $\tilde{F}^{(0)}$ is given by the solution of Eq. (47) and successive terms in $\tilde{F}$ correspond to successively higher-order terms in r in the wave equation (39). Higher-order corrections can themselves be given as a sum over the zero-order bulk eigenmodes:

$$\tilde{F}^{(i)}(\eta, z) = \sum_m v_m^{(i)}(\eta)\psi_m(z)\,. \tag{63}$$

Being eigenmodes of the self-adjoint operator $\mathcal{D}_z$ in Eq. (49), the bulk modes ψ_m form an orthonormal basis for any function.

The first-order corrections are given by

$$\frac{1}{a^2}\mathcal{D}_\eta \tilde{F}^{(1)} + \frac{k^2}{a^2}\tilde{F}^{(1)} - \mu^2 \mathcal{D}_z \tilde{F}^{(1)} = r S[\tilde{F}^{(0)}]. \tag{64}$$

Substituting the decomposition (63) and projecting onto the basis of the functions ψ_m, leads to the following equation that each of the coefficients $v_m^{(1)}(\eta)$ must satisfy:

$$\frac{1}{a^2}\mathcal{D}_\eta v_m^{(1)} + \frac{k^2}{a^2} v_m^{(1)} + m^2 v_m^{(1)} = r\left\{\sum_n (I_{mn} - \delta_{mn})\Big[-(2+3w)n^2 v_n^{(0)} - \frac{3}{2a^2}(1+w)(5+3c_s^2)\mathcal{H}\left\{\dot{v}_n^{(0)} - \mathcal{H}v_n^{(0)}\right\} - 3(1+w)\frac{k^2}{a^2}v_n^{(0)}\Big] + (1-3w)\sum_n\left(J_{mn} - \frac{3}{2}\delta_{mn}\right)v_n^{(0)}\right\}\,. \tag{65}$$

Here the matrix coefficients I_{mn} and J_{mn} are given respectively by

$$I_{mn} = \int_1^{z_L} dz\, \psi_m(z) z^2 \psi_n(z)\,, \tag{66}$$

$$J_{mn} = \int_1^{z_L} dz\, \psi_m(z) z \frac{d}{dz}\psi_n(z)\,. \tag{67}$$

It is the presence of off-diagonal terms in the matrices I and J that lead to mode mixing for $r > 0$. Note that for $n = 0$ we have $\psi_0 \propto z^{-3/2}$ and hence $J_{m0} = -3/2\delta_{m0}$.

For a regulator brane at fixed z_L, we expect mode-mixing to occur mainly at early times (maximum r) and become small for $r \ll 1/(z_L^2 - 1)$. Both analytic and numerical approaches are thus limited to finite $z_L < z_c$. At a given time (fixed r) the mixing becomes largest at large z. However the coordinate singularity necessarily limits our analysis to $z_L < z_c = \sqrt{1 + 2/r}$, and hence $r(z_L^2 - 1) < 2$ in all cases, suggesting a perturbative analysis should not be too bad in most cases.

The zero-mode growing-mode solution for $k = 0$ given in Eq. (57), has $\dot{v}_0^{(0)} = \mathcal{H}v_0^{(0)}$ and hence the whole right-hand-side of Eq. (65) vanishes even at finite r. However an initial zero-mode configuration ($v_m = 0$ for all $m \neq 0$) with finite k evolves into a mixed mode solution because the zero-order $v_0^{(0)} \neq 0$ acts as a source term at first order for all $v_m^{(1)}$ with $I_{m0} \neq 0$.

We can see from Eq. (65) that the zero-mode with $n = 0$ evolves independently of the massive modes only when $r = 0$ (Minkowski brane) or $w = -1$ (de Sitter).

We have checked the approximation by numerical solution of the full wave equation (39), in the case where $w = \frac{1}{3}$ and only the lowest eigenmode ($m = 0$) solution is excited at lowest order, corresponding to an initial form of F that is constant in z, and with $\dot{F} = 0$. In this case all the massive modes vanish initially.

V. DISCUSSION

We have shown how in the low-energy/ near-brane limit we can decompose bulk metric perturbations into

RS modes which evolve independently in the low-energy limit ($r \to 0$). At finite $r > 0$ and away from the brane, $z^2 > 1$, the bulk metric is not separable in Gaussian normal coordinates defined with respect to a generic FRW brane and this leads to mode mixing. At late times/low energies, the bulk metric becomes separable for $r(z^2-1) \ll 1$.

Our analytic approximation shows explicitly how mode-mixing occurs when an initial massless mode, generated during inflation, re-enters the Hubble horizon. The key result is Eq. (65).

On the physical brane at $z = 1$, the 4D tensor metric perturbations, which are in principle constrained, e.g., by cosmic microwave background observations, have an amplitude

$$F|_{\rm brane} = \frac{1}{a(\eta)} \left[\tilde{F}^{(0)}(\eta, 1) + \tilde{F}^{(1)}(\eta, 1) \right] , \qquad (68)$$

to lowest order in our approximation. The massive modes contribute to the 4D tensor metric perturbations but their amplitude is suppressed at the brane due to the RS volcano-type potential for $0 < m^2 < 15\mu^2/4$ [2]. The massive modes contribute an anisotropic stress term in the "dark radiation" term [7]

$$\delta E^i_j = -\frac{1}{2} \left(h^i_j\right)'' - \frac{A'}{A} \left(h^i_j\right)' , \qquad (69)$$

where the derivatives here are with respect to y.

We note that any analytic or numerical analysis based on Gaussian normal coordinates faces some significant practical limitations that may mean that it is not particularly well-suited for more detailed calculations. In particular, the existence of a coordinate singularity at $z = z_c$, a finite proper distance from the brane, makes it impossible to treat an infinite bulk. Introducing a second brane at a fixed Gaussian normal distance $z = z_L < z_c$ may require an unphysical equation of state on the second brane. Thus unphysical effects may propagate to the "physical" (e.g., radiation-dominated) brane in a finite time.

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Note added:

In the final stages of this work, Ref. [16] appeared, dealing with the same topic.

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APPENDIX A: DE SITTER BRANE

The bulk background metric is given by

$$^{(5)}d\bar{s}^2 = N^2(y) \left[-dt^2 + e^{2H_0 t} \delta_{ij} dx^i dx^j \right] + dy^2 , \qquad (A1)$$

where $a = e^{H_0 t}$, H_0 is constant and $N(y)$ is given by

$$N(y) = \cosh \mu y - (1 + r) \sinh \mu y , \qquad (A2)$$

with r constant. The wave equation has the form

$$\ddot{F} + 3H_0 \dot{F} + \frac{k^2}{a^2} F = N^2 F'' + 4NN'F' , \qquad (A3)$$

and is separable, so that $F(t,y)$ can be written in the form of Eq. (9), and

$$\ddot{\varphi}_m + 3H_0\dot{\varphi}_m + \left(m^2 + \frac{k^2}{a^2}\right)\varphi_m = 0\,, \qquad (A4)$$

$$F_m'' + 4\frac{N'}{N}F_m' + \frac{m^2}{N^2}F_m = 0\,. \qquad (A5)$$

The general solutions are given by

$$\varphi_m(t) = \exp\left(-\frac{3}{2}H_0 t\right) B_\nu \left(\frac{ke^{-H_0 t}}{H_0}\right)\,, \qquad (A6)$$

with

$$\nu^2 = \frac{9}{4} - \frac{m^2}{H_0^2}\,, \qquad (A7)$$

and

$$F_m(y) = \frac{A^\nu_{3/2}\left(\sqrt{1+\mu^2 N^2/H_0^2}\right)}{N^{3/2}}\,, \qquad (A8)$$

where $A^\nu_{3/2}$ is a linear combination of associated Legendre functions. Equation (A7) indicates the existence of a mass gap [6, 9, 13] between the zero mode and the start of the massive KK tower at $m = \frac{3}{2}H_0$.