

# Covariant Fluid Dynamics: a Long Wave-Length Approximation

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**Abstract.** In this paper we consider the Long-Wavelength Approximation Scheme (LWAS) in the framework of the covariant fluid approach to general relativistic dynamics, specializing to the particular case of irrotational dust matter. We discuss the dynamics of these models during the approach to any spacelike singularity where a BKL-type evolution is expected, studying the validity of this approximation scheme and the role of the magnetic part of the Weyl tensor,  $H_{ab}$ . Our analytic results confirm a previous numerical analysis: it is  $H_{ab}$  that destroys the pure Kasner-like approach to the singularity and eventually produces the bounce to another Kasner phase. Expanding regions evolve as separate universes where inhomogeneities and anisotropies die away.

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## 1. Introduction

Approximations methods are and have been very important in general relativity and its applications to cosmology and relativistic astrophysics. In the case of cosmology linear perturbation theory [1, 2] is at the heart of the present developments, in particular, in the analysis of the observed anisotropies of the Cosmic Microwave Background (see, e.g., [3]). Another approximation technique widely used in cosmology is the so-called Long Wave-Length Approximation Scheme (LWAS). Broadly speaking, it consists in neglecting spatial gradients of the variables describing the cosmological models, assuming they are small in comparison with their time derivatives. Since the time-scale of variation in cosmology is given by the local Hubble expansion rate, the approximation consists in neglecting inhomogeneities varying over a scale smaller than the local Hubble horizon. One can then try to get information about scales smaller than the Hubble radius by computing their effects through a series expansion in spatial gradients, which constitutes the LWAS (sometimes also called the gradient expansion method or long wavelength iteration). This scheme is then suitable to study large-scale structure formation and issues related with it. One important advantage of using the LWAS is the fact that neglecting spatial gradients we are still taking into account the non-linear character of the dynamical equations. Actually, the dynamical evolution that comes out will be fully non-linear and essentially the same as the one for homogeneous cosmological models.

In the literature there have been several approaches to the LWAS: using this approximation Belinski, Khalatnikov, and Lifshitz (BKL) studied [4] (see also [5]) the general behaviour of cosmological models near the initial singularity. In [6, 7] it was used to study several important issues in structure formation, from the gravitational instability mechanism to the evolution of the primordial inhomogeneities. Higher-order expansions in the LWAS and their applications have been developed in [8] by using a Hamiltonian approach. A closely related iterative method has been worked out in [9, 10]. Another approximation scheme close to the LWAS was presented in [11].

In this paper we study a different formulation of the LWAS based on the covariant fluid dynamics approach to cosmology (see, e.g., [12, 13, 14]). In this sense our work is complementary to the perturbative gauge invariant formalism in [15] (see also [14] and references therein). We focus on the study of irrotational dust models (IDM) with a cosmological constant  $\Lambda > 0$ , and more specifically, on the dynamical aspects near the singularities. One of the main conclusions of this study is that the LWAS breaks down due to a gravitomagnetic effect. This points at what it should be expected in the light of the standard BKL picture: that in the generic case the approach to a singularity has a first phase in which the solution goes into a Kasner phase, after which there is a “jump” into a transient period in which the solution moves away from a Kasner solution, and then goes into a new Kasner phase. With our language, and within the LWAS, this jumping is caused by a gravitomagnetic effect that is necessarily associated with non degenerate cases (here degenerate will mean two equal shear eigenvalues). From the point of view of the evolution equations, the LWAS reduces the quasi-linear Partial Differential Equations (PDEs) system of Einstein’s equations, or equivalently the covariant fluid equations, to Ordinary Differential Equations (ODEs). The evolution in this phase proceeds in a “silent” fashion [16, 17] approaching a Kasner singularity, till the effects of the gravito-magnetic Weyl tensor  $H_{ab}$  are no longer negligible and a bounce to a new Kasner phase is produced.

The plan of this paper is the following. In section 2 we review the fluid dynamical approach to IDMs and choose the most suitable form of the equation for the application of the LWAS. In section 3 we show how to implement the LWAS within the chosen approach and we present the first-order solution. In section 4 we discuss the validity of the approximation from the point of view of the covariant fluid dynamics approach. We end with some remarks and conclusions in section 5.

Throughout this paper we will use units in which  $8\pi G = c = 1$ . Tensorial components will be expressed both in coordinate charts and in vector basis. Then, our conventions for indices are: spacetime coordinate indices are denoted by the lower-case Latin letters  $a, \dots, l = 0, 1, 2, 3$ , and spacetime indices associated with an arbitrary basis  $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  by the rest of lower-case Latin letters  $m, \dots, z = 0, 1, 2, 3$ . Indices with respect to an orthonormal triad of spacelike vectors  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  are denoted by lower-case Greek letters  $\alpha, \dots, \lambda = 1, 2, 3$  and spatial coordinate indices by the rest of lower-case Greek letters  $\mu, \dots, \omega = 1, 2, 3$ .

## 2. Irrotational Dust Models

IDMs are important both for the description of the late universe as well as for the study of the gravitational instability mechanism. Their energy-momentum distribution is completely described by the fluid velocity  $\mathbf{u}$ , a unit timelike vector field ( $u^a u_a = -1$ ), and its associated energy density  $\rho$ . The energy-momentum tensor is given by

$$T_{ab} = \rho u_a u_b, \quad (1)$$

Since there is no pressure, the energy-momentum conservation equations tell us that the fluid worldlines are geodesics, i.e.,  $u^b \nabla_b u^a = 0$ . Assuming the fluid flow to be irrotational ( $u_{[a} \nabla_b u_{c]} = 0$ ) implies that the fluid velocity is the normal to a foliation of spacelike hypersurfaces. Then, there exists a time function  $\tau(x^a)$  such that  $\mathbf{u}$  is given by

$$\vec{u} = \frac{\partial}{\partial \tau}, \quad \mathbf{u} = -d\tau.$$

That is,  $\tau$  is at the same time the proper time of the fluid elements and a label of the hypersurfaces orthogonal to  $\mathbf{u}$ , which we will denote by  $\Sigma(\tau_1) \equiv \{\tau(x^a) = \tau_1 : \text{constant}\}$ . A system of coordinates for these hypersurfaces can be formed from any three independent first integrals of  $\mathbf{u}$ ,  $\{y^\mu(x^a)\}$  with  $u^a \partial_a y^\mu = 0$ . Then,  $\{\tau, y^\mu\}$  constitutes a set of comoving synchronous geodesic normal coordinates, in which the line element takes the following form

$$ds^2 = -d\tau^2 + h_{\mu\nu}(\tau, y^\sigma) dy^\mu dy^\nu, \quad (2)$$

where  $h_{\mu\nu}$  are the nonzero components of the induced positive-definite metric (first fundamental form) on the hypersurfaces  $\Sigma(\tau)$ . This coordinate system is fixed up to the transformations:  $\tau \rightarrow \tau' = \tau + \text{constant}$ , and  $y^\mu \rightarrow y^{\mu'} = f^{\mu'}(y^\nu)$ , being  $f^{\mu'}$  arbitrary functions.

A very convenient description of these models is provided by the covariant fluid approach introduced by Ehlers [12] (see [13, 14] for details). For the purposes of this paper it will be useful to write the equations using an orthonormal tetrad adapted to the fluid velocity:  $\{\mathbf{e}_0 = \mathbf{u}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  with  $\mathbf{e}_m \cdot \mathbf{e}_n = \eta_{mn} \equiv \text{diag}(-1, 1, 1, 1)$ . For convenience we fix partially the freedom in the choice of the triad  $\{\mathbf{e}_\alpha\}$  by requiring it to propagate parallelly to the fluid flow

$$\dot{e}_\alpha^a = u^b \nabla_b e_\alpha^a = 0.$$

This choice also makes the local angular velocity to vanish

$$\Omega^\alpha \equiv \frac{1}{2} \varepsilon^{\alpha\beta\delta} \mathbf{e}_\beta \cdot \dot{\mathbf{e}}_\delta = 0 \quad (\varepsilon_{\alpha\beta\delta} = \eta_{\alpha\beta\delta 0}),$$

where  $\eta_{abcd}$  is the spacetime volume element. In this framework, the variables that we need to describe IDMs are the following. (i) *Metric variables*. The components of the triad vectors in adapted coordinates  $\{\tau, y^\mu\}$ ,  $e_\alpha^\mu$ . (ii) *Connection variables*. The spatial commutators,  $\gamma^\alpha_{\beta\lambda}$ , defined by the commutation relations between the triad vectors:

$$[\mathbf{e}_\beta, \mathbf{e}_\lambda] = \gamma^\alpha_{\beta\lambda} \mathbf{e}_\alpha, \quad \gamma^\alpha_{[\beta\lambda]} = \gamma^\alpha_{\beta\lambda}. \quad (3)$$

We can instead use, the following variables introduced by Schücking, Kundt and Behr (see, e.g., [18]):

$$\gamma^\alpha{}_{\beta\lambda} = 2a_{[\beta}\delta^\alpha{}_{\lambda]} + \varepsilon_{\beta\lambda\delta}n^{\alpha\delta} \iff a_\alpha = \frac{1}{2}\gamma^\beta{}_{\alpha\beta}, \quad n^{\alpha\beta} = \frac{1}{2}\varepsilon^{\lambda\delta(\alpha}\gamma^{\beta)}{}_{\lambda\delta}, \quad (4)$$

which contain the same information as  $\gamma^\alpha{}_{\beta\lambda}$ . (iii) *Kinematical variables*. The expansion  $\Theta (\equiv \nabla_a u^a)$  and the shear tensor of the fluid worldlines. The shear is a symmetric and trace-free *spatial*‡ tensor

$$\sigma_{ab} \equiv h_{(a}{}^c h_b)^d \nabla_d u_c - (\Theta/3)h_{ab} = \nabla_a u_b - (\Theta/3)h_{ab}.$$

(iv) *Matter variables*. The only non-zero component of the energy-momentum tensor is the energy density  $\rho = T_{00}$ . The Ricci tensor is, through Einstein's equations, given by

$$R_{ab} = \rho u_a u_b + \frac{1}{2}(\rho + 4\Lambda)g_{ab}, \quad (5)$$

where  $\Lambda$  denotes the cosmological constant. (v) *Weyl tensor variables*. The Weyl tensor  $C_{abcd}$  describes the spacetime curvature not determined locally by matter fields. Its ten independent components can be divided into two spatial, symmetric and trace-free tensors

$$E_{ab} = C_{acbd}u^c u^d, \quad H_{ab} = *C_{acbd}u^c u^d \quad (*C_{abcd} \equiv \frac{1}{2}\eta_{ab}{}^{ef}C_{cdef}),$$

which are called the gravito-electric and gravito-magnetic fields respectively. Whereas the gravito-electric field produces tidal forces, having a Newtonian analogue (the trace-free part of the Hessian of the Newtonian potential), the gravito-magnetic field has no Newtonian analogue.

The equations governing the behaviour of these quantities come from the Ricci identities applied to  $\mathbf{u}$ , the second Bianchi identities, Einstein's equations (5) and the Gauss-Codazzi equations (see [13, 14] for details). To obtain them we have to split the covariant derivative  $\nabla_a$  into a time derivative along the fluid worldlines,  $\dot{A}^{a\dots b\dots} \equiv u^c \nabla_c A^{a\dots b\dots}$ , and the induced covariant derivative on the hypersurfaces  $\Sigma(\tau)$ ,  $D_c A^{a\dots b\dots} \equiv h^a{}_e \dots h^f{}_b h_c{}^d \nabla_d A^{e\dots f\dots}$ , where  $A^{a\dots b\dots}$  is any arbitrary tensor field. To simplify the equation that will appear in this paper it is convenient to introduce the spatial divergence and curl of an arbitrary 2-index symmetric tensor§  $A_{ab}$  (see, e.g., [19])

$$\text{div}(A)_a \equiv D^b A_{ab}, \quad \text{curl} A_{ab} \equiv \varepsilon_{cd(a} D^c A_{b)}{}^d,$$

where  $\varepsilon_{abc}$  ( $\varepsilon^{abc} \varepsilon_{def} = 3!h^{[a} h^b h^c]_f$ ) corresponds to the volume element of the hypersurfaces  $\Sigma(\tau)$ . The projection of these definitions onto a triad  $\{\mathbf{e}_\alpha\}$  gives||

$$\text{curl}(A)_{\alpha\beta} = \varepsilon^{\lambda\delta}{}_{<\alpha} (\partial_{|\lambda|} - a_{|\lambda|}) A_{\beta>\delta} + \frac{1}{2} n^\delta{}_\delta A_{\alpha\beta} - 3n_{<\alpha}{}^\delta A_{\beta>\delta},$$

$$\text{div}(A)_\alpha = (\partial_\delta - 3a_\delta) A^\delta{}_\alpha - \varepsilon_\alpha{}^{\beta\delta} n_\beta{}^\lambda A_{\lambda\delta},$$

‡ *Spatial* means orthogonal to the fluid velocity.

§ These definitions are analogous to those for vector fields:  $\text{div}(A) \equiv D_a A^a$  and  $\text{curl} A_a \equiv \varepsilon_{abc} D^b A^c$ .

|| Angled brackets on indices denote the spatially projected, symmetric and tracefree part:  $A_{(\alpha\beta)} = A_{(\alpha\beta)} - (A^\lambda{}_\lambda/3)\delta_{\alpha\beta}$ .

where  $\boldsymbol{\partial}_\alpha \equiv e_\alpha^\mu \partial_{y^\mu}$ . Finally, for two arbitrary spatial symmetric tensors,  $A_{ab}$  and  $B_{ab}$ , we define the commutator as

$$[A, B]_{ab} \equiv 2A_{[a}{}^c B_{b]c}, \quad [A, B]_a \equiv \frac{1}{2}\varepsilon_{abc}[A, B]^{bc} = \varepsilon_{abc}A^b{}_d B^{cd}.$$

The resulting set of covariant equations is usually divided into two groups: *Evolutions equations*, giving the rate of change of our quantities along the fluid world-lines, and *constraint equations*, relations containing spatial derivatives only. The explicit form of these equations has been given in many places (see, e.g., [12, 13, 14]). In the case of IDMs they can be found in [20, 21]. In order to set up a suitable framework for the application of the LWAS, here we will look at these equations from a different point of view than the usual one. We consider as *main* variables the set  $\{e_\alpha^\mu, \Theta, \sigma_{\alpha\beta}\}$ , which will be the dynamical variables. Their evolution equations can be written as follows:

$$\dot{e}_\alpha{}^\mu = -\left(\frac{1}{3}\Theta\delta_\alpha{}^\beta + \sigma_\alpha{}^\beta\right)e_\beta{}^\mu, \quad (6)$$

$$\dot{\Theta} = -\frac{1}{2}\Theta^2 - \frac{3}{4}\sigma^{\alpha\beta}\sigma_{\alpha\beta} + \frac{3}{2}\Lambda - \frac{1}{4}{}^3R, \quad (7)$$

$$\dot{\sigma}_{\alpha\beta} = -\Theta\sigma_{\alpha\beta} - {}^3S_{\alpha\beta}. \quad (8)$$

In these equations,  ${}^3R$  and  ${}^3S_{\alpha\beta}$  denote the scalar curvature and the trace-free part of the Ricci tensor of the hypersurfaces  $\Sigma(\tau)$  respectively. They must be understood as given in terms of  $\gamma_{\alpha\beta}^\delta$  and their derivatives, through the expression

$${}^3R_{\alpha\beta} = \boldsymbol{\partial}_\lambda(\Gamma_{\alpha\beta}^\lambda) - \boldsymbol{\partial}_\beta(\Gamma_{\alpha\lambda}^\lambda) + \Gamma_{\alpha\beta}^\epsilon\Gamma_{\epsilon\lambda}^\lambda - \Gamma_{\alpha\lambda}^\epsilon\Gamma_{\epsilon\beta}^\lambda + \Gamma_{\alpha\epsilon}^\lambda\gamma_{\beta\lambda}^\epsilon,$$

where  $\Gamma_{\beta\delta}^\alpha \equiv \mathbf{e}^\alpha \cdot (\nabla_{\mathbf{e}_\delta} \mathbf{e}_\beta)$  are the Ricci rotation coefficients associated with the triad  $\{\mathbf{e}_\alpha\}$ , related to  $\gamma_{\beta\delta}^\alpha$  by  $\delta_{\alpha\epsilon}\Gamma_{\beta\lambda}^\epsilon = \delta_{\alpha\epsilon}\gamma_{\lambda\beta}^\epsilon + \delta_{\beta\epsilon}\gamma_{\alpha\lambda}^\epsilon + \delta_{\lambda\epsilon}\gamma_{\alpha\beta}^\epsilon$ . The remaining variables,  $\{\gamma_{\alpha\beta}^\delta, \rho, E_{\alpha\beta}, H_{\alpha\beta}\}$ , which will be called *auxiliary* variables, can be found in terms of the main ones, either from their definitions or from the constraints. The expressions are:

$$\gamma_{\alpha\beta}^\delta \boldsymbol{\partial}_\delta y^\mu = [\boldsymbol{\partial}_\alpha, \boldsymbol{\partial}_\beta] y^\mu, \quad (9)$$

$$\rho = \frac{1}{3}\Theta^2 - \frac{1}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta} - \Lambda + \frac{1}{2}{}^3R, \quad (10)$$

$$E_{\alpha\beta} = \frac{1}{3}\Theta\sigma_{\alpha\beta} - \sigma_{<\alpha}{}^\delta\sigma_{\beta>\delta} + {}^3S_{\alpha\beta}. \quad (11)$$

$$H_{\alpha\beta} = \text{curl}(\sigma)_{\alpha\beta}. \quad (12)$$

These quantities are subject to the following constraint:

$$\text{div}(\sigma)_\alpha - \frac{2}{3}\boldsymbol{\partial}_\alpha\Theta = 0. \quad (13)$$

The remaining equations involved in this formalism can be derived from those here presented. It should be noticed that Einstein's equations enter the scheme above only through equation (10), through the particular way  $\rho$  is related to the geometrical and kinematical variables.

### 3. Formulation of the Long Wave-Length Approximation Scheme: the first-order solution

As we have already mentioned the LWAS consists, roughly speaking, in neglecting spatial gradients with respect to time derivatives. One can find in the literature several ways of implementing the LWAS [6, 8, 9]. In the case of IDMs, the spatial gradients correspond to derivatives tangent to an intrinsically characterized foliation, namely, the one formed by hypersurfaces orthogonal to the fluid velocity. In this sense, this approximation scheme is physically well-motivated for the case of IDMs. Based on this fact, here we will adopt a new point of view based on the covariant fluid approach introduced in the previous section, which will allow us to clarify several geometrical and physical aspects of the LWAS. Although the formalism we develop can be applied in different contexts, here we focus in studying the approach to spacelike singularities, complementing in this way the work done by Langlois and Deruelle [10].

Let us consider the dynamical equations governing the behaviour of IDMs shown in the previous section. In order to apply the LWAS ideas it is very important to realise that in the evolution equations for the main quantities, Eqs. (6)-(8), only second-order spatial gradients appear, and they are encoded in the spatial curvature tensor, or more specifically, in  ${}^3R$  and  ${}^3S_{\alpha\beta}$ . Therefore, the zero- and first-order solutions can be found in one iteration (we will separate them later), which means that we can look directly for the first order solution just by neglecting in these equations terms of order greater than one in the spatial gradients. In our case, this means to neglect the spatial curvature:

$${}^3R \approx 0, \quad {}^3S_{ab} \approx 0, \quad (14)$$

which implies, through Eq. (12), that

$$D_a H_{bc} \approx 0 \implies \text{div}(H)_a \approx 0 \text{ and } \text{curl}H_{ab} \approx 0. \quad (15)$$

The first important consequence of (14) is that from the evolution equations (6)-(8) we obtain a closed system of equations for  $\Theta$  and  $\sigma_{\alpha\beta}$ .

$$\dot{\Theta} = -\frac{1}{2}\Theta^2 - \frac{3}{4}\sigma^{\alpha\beta}\sigma_{\alpha\beta} + \frac{3}{2}\Lambda, \quad (16)$$

$$\dot{\sigma}_{\alpha\beta} = -\Theta\sigma_{\alpha\beta}. \quad (17)$$

Second, the expressions for  $\rho$  and  $E_{\alpha\beta}$  become purely local

$$\rho = \frac{1}{3}\Theta^2 - \frac{1}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta} - \Lambda, \quad (18)$$

$$E_{\alpha\beta} = \frac{1}{3}\Theta\sigma_{\alpha\beta} - \sigma_{<\alpha}{}^\lambda\sigma_{\beta>\lambda}. \quad (19)$$

In order to integrate equations (16),(17) we will further specialize the triad  $\{\mathbf{e}_\alpha\}$  by taking into account the special structure of the first-order equations and the freedom:  $\mathbf{e}_\alpha \rightarrow \mathbf{e}_{\alpha'} = \mathcal{A}_{\alpha'}{}^\alpha \mathbf{e}_\alpha$  with  $\dot{\mathcal{A}}_{\alpha'}{}^\alpha = 0$ . We use this freedom to have an eigenbasis of the shear. That is,

$$\sigma_{\alpha\beta} = 0 \quad \text{for } \alpha \neq \beta, \quad (20)$$

This is possible because at the first order we have  ${}^3S_{\alpha\beta} = 0$ . In [22] (see also [23]), it was shown that a sufficient condition for this is  $H_{ab} = 0$ , and in [24] this result was extended to the case in which  $H_{ab}$  is transverse, i.e.  $\text{div}(H)_a = 0$ . We have already seen that this last condition holds at first order [see Eq. (15)]. As a consequence of (20) and (19),  $E_{\alpha\beta}$  is also diagonal. Hence  $[\sigma, E]_{ab} = 0$ .

We can now solve the equations for the first-order solution of the LWAS. The procedure that we will follow is very close to the one used to find the complete class of IDMs with flat spatial geometry [20]. We begin by solving equations (16),(17) for the expansion  $\Theta$  and the only two independent components of the shear, whose information can be encoded in the following two quantities (see, e.g., [17] and references therein)

$$\sigma_+ \equiv -\frac{3}{2}\sigma_{11}, \quad \sigma_- \equiv \frac{\sqrt{3}}{2}(\sigma_{22} - \sigma_{33}). \quad (21)$$

The equations for  $(\Theta, \sigma_+, \sigma_-)$  then are

$$\dot{\Theta} = -\frac{1}{2}(\Theta^2 + \sigma_+^2 + \sigma_-^2 - 3\Lambda), \quad (22)$$

$$\dot{\sigma}_{\pm} = -\Theta\sigma_{\pm}, \quad (23)$$

and the solution can then be expressed as

$$\Theta = \frac{\dot{V}}{V}, \quad \sigma_+ = \frac{\Sigma_+}{V}, \quad \sigma_- = \frac{\Sigma_-}{V}, \quad (24)$$

where  $V = V(\tau, y^\mu)$  denotes the proper volume associated with the fluid flow of  $\mathbf{u}$  and  $\Sigma_{\pm}$  are scalars independent of  $\tau$ . From (22,23) the equation for  $V$  is

$$2V\ddot{V} - \dot{V}^2 - 3\Lambda V^2 + 3\Sigma^2 = 0, \quad \Sigma^2 \equiv \frac{1}{3}\Sigma_+^2 + \Sigma_-^2, \quad (25)$$

and  $\Sigma^2$  is a first integral of the system (22,23). In order to write the solution of this equation one has to consider separately the following cases: (i)  $\Lambda > 0$ . Here, we have to consider also different cases according to a criterion that involves  $\Lambda$ ,  $\Sigma$ , and an integration function that we will call  $M$ . Then, when  $M^2 - 4\Lambda\Sigma^2 > 0$ , the solution of (25) can be written as

$$V(t, y^\mu) = \frac{1}{\Lambda} \sqrt{M^2 - 4\Lambda\Sigma^2} \sinh \left[ \frac{\sqrt{3\Lambda}}{2} (\tau - \mathcal{T}) \right] \sinh \left[ \frac{\sqrt{3\Lambda}}{2} (\tau - \mathcal{T} + \delta\mathcal{T}) \right], \quad (26)$$

where  $\mathcal{T}$  is an arbitrary function of the comoving coordinates  $y^\mu$ . When  $M^2 - 4\Lambda\Sigma^2 = 0$  we have

$$V(t, y^\mu) = \frac{M}{2\Lambda} \left( e^{\sqrt{2\Lambda}(\tau - \mathcal{T})} - 1 \right), \quad (27)$$

and finally, when  $M^2 - 4\Lambda\Sigma^2 < 0$ , the solution is:

$$V(t, y^\mu) = \frac{1}{\Lambda} \sqrt{4\Lambda\Sigma^2 - M^2} \sinh \left[ \frac{\sqrt{3\Lambda}}{2} (\tau - \mathcal{T}) \right] \cosh \left[ \frac{\sqrt{3\Lambda}}{2} (\tau - \mathcal{T} + \delta\mathcal{T}) \right]. \quad (28)$$

Where the function  $\delta\mathcal{T}$ , in cases (26,28), is given by

$$\delta\mathcal{T} = \frac{1}{\sqrt{3\Lambda}} \ln \left| \frac{M + 2\sqrt{\Lambda}\Sigma}{M - 2\sqrt{\Lambda}\Sigma} \right|. \quad (29)$$

(ii)  $\Lambda < 0$ . In this case the solution for the comoving volume is

$$V(t, y^\mu) = -\frac{1}{\Lambda} \sqrt{M^2 - 4\Lambda\Sigma^2} \sin \left[ \frac{\sqrt{-3\Lambda}}{2} (\tau - \mathcal{T}) \right] \cos \left[ \frac{\sqrt{-3\Lambda}}{2} (\tau - \mathcal{T} + \delta\mathcal{T}) \right],$$

where

$$\delta\mathcal{T} = -\frac{2}{\sqrt{-3\Lambda}} \arcsin \left[ \frac{M}{\sqrt{M^2 - 4\Lambda\Sigma^2}} \right]. \quad (30)$$

(iii)  $\Lambda = 0$ . The solution for this case is:

$$V(t, y^\mu) = \frac{3}{4} M (\tau - \mathcal{T}) (\tau - \mathcal{T} + \delta\mathcal{T}), \quad (31)$$

where

$$\delta\mathcal{T} \equiv \frac{4\Sigma}{\sqrt{3}M}. \quad (32)$$

In all these expressions for the comoving volume  $V(t, y^\mu)$ ,  $M$  is a time-independent scalar related to the energy density through equation (10):

$$\rho = \frac{M}{V}, \quad (33)$$

and a first integral of the energy-momentum conservation equation  $\dot{\rho} + \Theta\rho = 0$ . Taking into account that the proper volume (31) is defined up to a multiplicative time-independent scalar [see Eq. (24)], we naturally choose  $M$  to be a positive constant, and in fact we could even choose its value. On the other hand, the proper volume is related to the local scale factor  $a(\tau, y^\mu)$  by the expression

$$\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} \equiv 3H, \quad a^2 \equiv [\det(h_{\alpha\beta})]^{1/3},$$

where  $H$  is the local Hubble parameter.

Since we will be mainly interested in the behaviour near the singularities we can neglect the effect of the cosmological constant and consider only the case (iii) given by equations (31,32). This well-known fact can be checked from the expressions we obtain for  $V(t, y^\mu)$ , from where one can see that in the limit  $\tau \rightarrow \mathcal{T}$  all the cases behave like the case (iii), i.e.,  $V \sim \tau - \mathcal{T}$ . In any case, the procedure we will follow here to solve the different equations can be applied from the beginning to the end to all the cases with a non-zero cosmological constant.

Then, as we can see from (31,32), and assuming without loss of generality  $\Sigma \geq 0$  ( $\Rightarrow \delta\mathcal{T} > 0$ ), for  $\tau > \mathcal{T}$  we have an expanding fluid (note that this depends on which fluid element we are looking at, i.e. on  $y^\mu$ ), whereas for  $\tau < \mathcal{T} - \delta\mathcal{T}$  we have a collapsing one. The region  $\tau \in (\mathcal{T} - \delta\mathcal{T}, \mathcal{T})$  is excluded by the requirement of having a positive energy density. Then, for an expanding IDM,  $\mathcal{T}(y^\mu)$  is the Bang time corresponding to a fluid element with comoving coordinates  $y^\mu$ . In the same way,  $\mathcal{T} - \delta\mathcal{T}$  is the Crunch time for a contracting IDM.

The next step is to solve the equations for the triad vectors  $\{e_\alpha\}$  [Eq. (6)], which become

$$\dot{e}_1^\mu = -\frac{1}{3}(\Theta - 2\sigma_+)e_1^\mu,$$



$$\begin{aligned}\dot{e}_2^\mu &= -\frac{1}{3}(\Theta + \sigma_+ + \sqrt{3}\sigma_-)e_2^\mu, \\ \dot{e}_3^\mu &= -\frac{1}{3}(\Theta + \sigma_+ - \sqrt{3}\sigma_-)e_3^\mu.\end{aligned}$$

The result, in the case  $\Lambda = 0$ , is

$$e_\alpha^\mu = b_\alpha^\mu (\tau - \mathcal{T})^{-p_\alpha} (\tau - \mathcal{T} + \delta\mathcal{T})^{-q_\alpha}, \quad (34)$$

where the quantities  $b_\alpha^\mu$  in (34) are the time-independent components of a triad defining a Riemannian 3-dimensional geometry whose metric tensor is given by  $q_{\mu\nu} = \delta_{\alpha\beta} b_\alpha^\mu b_\beta^\nu$ , being  $b^\alpha_\mu$  the inverse matrix of  $b_\alpha^\mu$ . Moreover,  $q_\alpha \equiv 2/3 - p_\alpha$  and

$$p_1 \equiv \frac{\sqrt{3}\Sigma - 2\Sigma_+}{3\sqrt{3}\Sigma}, \quad p_2 \equiv \frac{\Sigma_+ + \sqrt{3}(\Sigma + \Sigma_-)}{3\sqrt{3}\Sigma}, \quad p_3 \equiv \frac{\Sigma_+ + \sqrt{3}(\Sigma - \Sigma_-)}{3\sqrt{3}\Sigma}, \quad (35)$$

are defined in the analogous way as they are defined for Bianchi I IDMs. Indeed, we can check that  $p_\alpha$  and  $q_\alpha$  satisfy the Kasner relations

$$\sum_{\alpha=1}^3 p_\alpha = \sum_{\alpha=1}^3 p_\alpha^2 = 1 \iff \sum_{\alpha=1}^3 q_\alpha = \sum_{\alpha=1}^3 q_\alpha^2 = 1,$$

so we will call them *local Kasner coefficients*. It is convenient, for practical purposes, to consider the fact that one can assume, without loss of generality, that in an open domain of a given fluid element located at  $y^\mu$ , the Kasner coefficients can be chosen to lie in the following intervals:

$$p_1 \in \left(-\frac{1}{3}, 0\right), \quad p_2 \in \left(0, \frac{2}{3}\right), \quad p_3 \in \left(\frac{2}{3}, 1\right). \quad (36)$$

Then, looking at (34) it is clear that the first-order solution has an initial singularity (in general not simultaneous) of the Kasner type, also known as a velocity-dominated singularity [25].

Summarizing, expressions (24), (31,32), and (34) solve the first-order evolution. From them we can find the time dependence of all the variables. With regard to the spatial dependence, which is encoded inside the functions  $\mathcal{T}$ ,  $\{\mathbf{b}_\alpha\}$ , and  $\Sigma_\pm$ , the only restriction comes from the constraint (13) (the *momentum constraint*) which is equivalent to the following three equations

$$\partial_1(\sigma_+ + \Theta) - 3a_1\sigma_+ - \sqrt{3}n_{23}\sigma_- = 0, \quad (37)$$

$$\partial_2(\sigma_+ + \sqrt{3}\sigma_- - 2\Theta) - 3a_2(\sigma_+ + \sqrt{3}\sigma_-) + \sqrt{3}n_{13}(\sqrt{3}\sigma_+ - \sigma_-) = 0, \quad (38)$$

$$\partial_3(\sigma_+ - \sqrt{3}\sigma_- - 2\Theta) - 3a_3(\sigma_+ - \sqrt{3}\sigma_-) - \sqrt{3}n_{12}(\sqrt{3}\sigma_+ + \sigma_-) = 0, \quad (39)$$

To see which kind of restrictions these equations impose we need to compute the quantities  $a_\alpha$  and  $n_{\alpha\beta}$ , or equivalently the commutators  $\gamma^\alpha_{\beta\lambda}$ . Using the expressions (3) we get

$$\gamma^\alpha_{\beta\lambda} = \left\{ \gamma'^\alpha_{\beta\lambda} + 2\delta^\alpha_{[\beta} \left[ (\partial'_{\lambda]} p_\alpha) \ln \frac{P}{Q} + \frac{\partial'_{\lambda]} P}{P} p_\alpha + \frac{\partial'_{\lambda]} Q}{Q} q_\alpha \right] \right\} P^{p_\lambda - p_\alpha - p_\beta} Q^{q_\lambda - q_\alpha - q_\beta}, \quad (40)$$

where  $\gamma_{\beta\delta}^{\alpha}$  are the (time-independent) commutator functions associated with the triad  $\{\mathbf{b}_\alpha\}$ , underlined indices do not follow the usual index summation convention, and we have used the following definitions:

$$\partial'_{\alpha} \equiv b_{\alpha}^{\mu} \partial_{y^{\mu}}, \quad P \equiv \tau - \mathcal{T}, \quad Q \equiv \tau - \mathcal{T} + \delta\mathcal{T}.$$

Then, from (4,40) we have

$$a_{\alpha} = \left\{ a'_{\alpha} + \frac{1}{2} \left[ (\partial'_{\alpha} p_{\alpha}) \ln \frac{P}{Q} + (p_{\alpha} - 1) \frac{\partial'_{\alpha} P}{P} + (q_{\alpha} - 1) \frac{\partial'_{\alpha} Q}{Q} \right] \right\} P^{-p_{\alpha}} Q^{-q_{\alpha}},$$

$$n^{\alpha\beta} = \begin{cases} n'^{\alpha\alpha} P^{2p_{\alpha}-1} Q^{2q_{\alpha}-1} & \text{if } \alpha = \beta, \\ \left\{ n'^{\alpha\beta} - \frac{1}{2} \epsilon^{\alpha\beta\delta} \partial'_{\delta} \left[ (p_{\alpha} - p_{\beta}) \ln \frac{P}{Q} \right] \right\} P^{-p_{\alpha}} Q^{-q_{\alpha}} & (\alpha \neq \delta \neq \beta) \text{ if } \alpha \neq \beta. \end{cases}$$

Introducing all these expressions into the constraints (37-39), we can see that they are equivalent to the following three time-independent equations

$$\partial'_1 \left[ \Sigma_+ + \frac{3}{2} M(\mathcal{T} - \frac{1}{2} \delta\mathcal{T}) \right] = 3a'_1 \Sigma_+ + \sqrt{3} n'_{23} \Sigma_-, \quad (41)$$

$$\partial'_2 \left[ \Sigma_+ + \sqrt{3} \Sigma_- - 3M(\mathcal{T} - \frac{1}{2} \delta\mathcal{T}) \right] = 3a'_2 (\Sigma_+ + \sqrt{3} \Sigma_-) - \sqrt{3} n'_{13} (\sqrt{3} \Sigma_+ - \Sigma_-), \quad (42)$$

$$\partial'_3 \left[ \Sigma_+ - \sqrt{3} \Sigma_- - 3M(\mathcal{T} - \frac{1}{2} \delta\mathcal{T}) \right] = 3a'_3 (\Sigma_+ - \sqrt{3} \Sigma_-) + \sqrt{3} n'_{12} (\sqrt{3} \Sigma_+ + \Sigma_-). \quad (43)$$

This is not an obvious result since equations (37)-(39) are linear combinations of functions depending on  $\{\tau, y^{\mu}\}$  with coefficients depending only on  $\{y^{\mu}\}$  and, in principle, one would expect that from each of them we would get several equations. What we have seen is that we only get one for each of them. Equations (41)-(43) are then constraints on the quantities  $\mathcal{T}$ ,  $\{\mathbf{b}_\alpha\}$  and  $\Sigma_{\pm}$ .

At this point, the construction of the first-order solution is essentially finished. As we have seen, the time dependence is explicitly known whereas the spatial one is encoded in the quantities  $\mathcal{T}$ ,  $\{\mathbf{b}_\alpha\}$  and  $\Sigma_{\pm}$ . Now, we can split the zero and first order terms by taking into account that the zero order is characterized by the absence of any spatial gradient. Then, the splitting is given by the relations

$$b_{\alpha}^{\mu} = \delta_{\alpha}^{\mu} + {}_1 b_{\alpha}^{\mu}, \quad (44)$$

$$\Sigma_{\pm} = {}_o \Sigma_{\pm} + {}_1 \Sigma_{\pm}, \quad \mathcal{T} = {}_o \mathcal{T} + {}_1 \mathcal{T}, \quad (45)$$

where  ${}_o \Sigma_{\pm}$  and  ${}_o \mathcal{T}$  are constants. From (44) we deduce that the commutator functions  $\gamma_{\beta\delta}^{\alpha}$ , or equivalently,  $a'_{\alpha}$  and  $n'^{\alpha\beta}$  do not have a zero-order part. Moreover, it also gives a splitting of the spatial derivative  $\partial'_{\alpha}$

$$\partial'_{\alpha} = \partial_{y^{\mu}} + {}_1 b_{\alpha}^{\mu} \partial_{y^{\mu}}. \quad (46)$$

Then, it also follows that equations (41)-(43) are automatically satisfied at zero-order. At first order they are

$$\partial_1 [{}_1 \Sigma_+ + \Xi] = 3a'_{1o} \Sigma_+ + \sqrt{3} n'_{23o} \Sigma_-,$$

$$\partial_2 [{}_1 \Sigma_+ + \sqrt{3} {}_1 \Sigma_- - 2\Xi] = 3a'_{2o} ({}_o \Sigma_+ + \sqrt{3} {}_o \Sigma_-) - \sqrt{3} n'_{13o} (\sqrt{3} {}_o \Sigma_+ - {}_o \Sigma_-),$$

$$\partial_3 \left[ {}_1\Sigma_+ - \sqrt{3}{}_1\Sigma_- - 2\Xi \right] = 3a'_3({}_o\Sigma_+ - \sqrt{3}{}_o\Sigma_-) + \sqrt{3}n'_{12}(\sqrt{3}{}_o\Sigma_+ + {}_o\Sigma_-),$$

where

$$\Xi \equiv \frac{3}{2}M_1\mathcal{T} - \frac{1}{\sqrt{3}{}_o\Sigma}({}_o\Sigma_{+1}\Sigma_+ + {}_o\Sigma_{-1}\Sigma_-), \quad {}_o\Sigma^2 \equiv \frac{1}{3}({}_o\Sigma_+^2 + {}_o\Sigma_-^2).$$

It is worth noting that our zero order solution is exactly Bianchi I, and so is the first order part from the point of view of the evolution equations.

To complete the construction of the first-order solution, we give the expression for the remaining quantities, that is, the gravito-electric and -magnetic tensor fields  $E_{\alpha\beta}$  and  $H_{\alpha\beta}$ . As we said before, equation (19) tells us that  $E_{\alpha\beta}$  is diagonal since  $\sigma_{\alpha\beta}$  is diagonal. Then, we only need to compute the quantities  $E_+$  and  $E_-$  defined as in (21). The result is

$$E_+ = \frac{1}{3}\Theta\sigma_+ + \frac{1}{3}(\sigma_+^2 - \sigma_-^2) = \frac{1}{3V^2} \left\{ \frac{3}{2}M(\tau - \mathcal{T} + \frac{1}{2}\delta\mathcal{T})\Sigma_+ + \Sigma_+^2 - \Sigma_-^2 \right\},$$

$$E_- = \frac{1}{3}(\Theta - 2\sigma_+)\sigma_- = \frac{1}{3V^2} \left\{ \frac{3}{2}M(\tau - \mathcal{T} + \frac{1}{2}\delta\mathcal{T}) - 2\Sigma_+ \right\} \Sigma_-.$$

The gravito-magnetic field  $H_{\alpha\beta}$  has five independent components, which can be given in terms of the quantities  $H_{\pm}$  [defined as in Eq. (21)], and  $H_1 \equiv \sqrt{3}H_{23}$ ,  $H_2 \equiv \sqrt{3}H_{13}$ , and  $H_3 \equiv \sqrt{3}H_{12}$ . Their form is given by

$$H_+ = -\frac{3}{2}n_{11}\sigma_+ - \frac{\sqrt{3}}{2}(n_{22} - n_{33})\sigma_-$$

$$= -\frac{2\Sigma_+}{M}n'_{11}P^{2(p_1-1)}Q^{2(q_1-1)} - \frac{2\Sigma_-}{\sqrt{3}M} \left\{ n'_{22}P^{2(p_2-1)}Q^{2(q_2-1)} - n'_{33}P^{2(p_3-1)}Q^{2(q_3-1)} \right\},$$

$$H_- = -\frac{\sqrt{3}}{2}(n_{22} - n_{33})\sigma_+ + \frac{1}{2}(n_{11} - 2n_{22} - 2n_{33})\sigma_-$$

$$= -\frac{2\Sigma_+}{\sqrt{3}M} \left\{ n'_{22}P^{2(p_2-1)}Q^{2(q_2-1)} - n'_{33}P^{2(p_3-1)}Q^{2(q_3-1)} \right\}$$

$$+ \frac{2\Sigma_-}{3M} \left\{ n'_{11}P^{2(p_1-1)}Q^{2(q_1-1)} - 2n'_{22}P^{2(p_2-1)}Q^{2(q_2-1)} - 2n'_{33}P^{2(p_3-1)}Q^{2(q_3-1)} \right\},$$

$$H_1 = -\sqrt{3}n_{23}\sigma_+ + (\boldsymbol{\partial}_1 - a_1)\sigma_-$$

$$= \frac{4}{3M} \left\{ \partial_1({}_1\Sigma_-) - {}_o\Sigma_-a'_1 - \sqrt{3}{}_o\Sigma_+n'_{23} - {}_o\Sigma_- \frac{\partial_1 V}{V} + \frac{\sqrt{3}}{2}{}_o\Sigma_+\partial_1 \left[ (p_2 - p_3) \ln \frac{P}{Q} \right] \right.$$

$$\left. - \frac{1}{2}{}_o\Sigma_- \left[ (\partial_1 p_1) \ln \frac{P}{Q} + (p_1 - 1) \frac{\partial_1 P}{P} + (q_1 - 1) \frac{\partial_1 Q}{Q} \right] \right\} P^{-p_1-1}Q^{-q_1-1},$$

$$H_2 = \frac{3}{2}n_{13}(\sigma_+ + \sqrt{3}\sigma_-) + \frac{1}{2}(\boldsymbol{\partial}_2 - a_2)(\sqrt{3}\sigma_+ - \sigma_-)$$

$$= \frac{2}{3M} \left\{ \partial_2(\sqrt{3}{}_1\Sigma_+ - {}_1\Sigma_-) - (\sqrt{3}{}_o\Sigma_+ - {}_o\Sigma_-)a'_2 + 3({}_o\Sigma_+ + \sqrt{3}{}_o\Sigma_-)n'_{13} \right.$$

$$\left. - (\sqrt{3}{}_o\Sigma_+ - {}_o\Sigma_-) \frac{\partial_2 V}{V} + \frac{3}{2}({}_o\Sigma_+ + \sqrt{3}{}_o\Sigma_-) \partial_2 \left[ (p_1 - p_3) \ln \frac{P}{Q} \right] \right.$$

$$\left. - \frac{1}{2}(\sqrt{3}{}_o\Sigma_+ - {}_o\Sigma_-) \left[ (\partial_2 p_2) \ln \frac{P}{Q} + (p_2 - 1) \frac{\partial_2 P}{P} + (q_2 - 1) \frac{\partial_2 Q}{Q} \right] \right\} P^{-p_2-1}Q^{-q_2-1},$$

$$H_3 = \frac{3}{2}n_{12}(\sigma_+ - \sqrt{3}\sigma_-) - \frac{1}{2}(\boldsymbol{\partial}_3 - a_3)(\sqrt{3}\sigma_+ + \sigma_-)$$

$$= \frac{2}{3M} \left\{ -\partial_3(\sqrt{3}{}_1\Sigma_+ + {}_1\Sigma_-) + (\sqrt{3}{}_o\Sigma_+ + {}_o\Sigma_-)a'_3 + 3({}_o\Sigma_+ - \sqrt{3}{}_o\Sigma_-)n'_{12} \right.$$

$$\left. + (\sqrt{3}{}_o\Sigma_+ + {}_o\Sigma_-) \frac{\partial_3 V}{V} - \frac{3}{2}({}_o\Sigma_+ - \sqrt{3}{}_o\Sigma_-) \partial_3 \left[ (p_1 - p_2) \ln \frac{P}{Q} \right] \right.$$

$$\left. + \frac{1}{2}(\sqrt{3}{}_o\Sigma_+ + {}_o\Sigma_-) \left[ (\partial_3 p_3) \ln \frac{P}{Q} + (p_3 - 1) \frac{\partial_3 P}{P} + (q_3 - 1) \frac{\partial_3 Q}{Q} \right] \right\} P^{-p_3-1}Q^{-q_3-1}.$$

**Table 1.** Asymptotic behaviour of the first-order solution.

| Quantity                 | Behaviour as $\tau \rightarrow \mathcal{T}$   | Behaviour as $\tau \rightarrow \infty$         |
|--------------------------|---|--|
| $\Theta$                 | $(\tau - \mathcal{T})^{-1}$   | $2\tau^{-1}$                                   |
| $\sigma_{\pm}$           | $\frac{4\Sigma_{\pm}}{3M\delta\mathcal{T}}(\tau - \mathcal{T})^{-1}$  | $\frac{4\Sigma_{\pm}}{3M}\tau^{-2}$            |
| $\frac{\sigma}{\Theta}$  | $\frac{4\Sigma}{3M\delta\mathcal{T}}$   | $\frac{2\Sigma}{3M}\tau^{-1}$                  |
| $\frac{\rho}{\Theta^2}$  | $\frac{4\Sigma_{\pm}}{3M\delta\mathcal{T}}(\tau - \mathcal{T})$   | $\frac{1}{3}$                                  |
| $\frac{E_{+}}{\Theta^2}$ | $\frac{1}{3} \left\{ \Sigma_{+} + \frac{4(\Sigma_{+}^2 - \Sigma_{-}^2)}{3M\delta\tau} \right\} \frac{4}{3M\delta\mathcal{T}}$ | $\frac{2\Sigma_{+}}{9M}\tau^{-1}$              |
| $\frac{E_{-}}{\Theta^2}$ | $\frac{1}{3} \left\{ 1 - \frac{8\Sigma_{+}}{3M\delta\tau} \right\} \frac{4\Sigma_{-}}{3M\delta\mathcal{T}}$                   | $\frac{2\Sigma_{-}}{9M}\tau^{-1}$              |
| $\frac{H_{+}}{\Theta^2}$ | $\mathcal{H}_{+}^{BB}(y^{\mu})(\tau - \mathcal{T})^{2p_1}$  | $\mathcal{H}_{+}^{\infty}(y^{\mu})\tau^{-2/3}$ |
| $\frac{H_{-}}{\Theta^2}$ | $\mathcal{H}_{-}^{BB}(y^{\mu})(\tau - \mathcal{T})^{2p_1}$  | $\mathcal{H}_{-}^{\infty}(y^{\mu})\tau^{-2/3}$ |
| $\frac{H_1}{\Theta^2}$   | $\mathcal{H}_1^{BB}(y^{\mu})(\tau - \mathcal{T})^{-p_1}$  | $\mathcal{H}_1^{\infty}(y^{\mu})\tau^{-2/3}$   |
| $\frac{H_2}{\Theta^2}$   | $\mathcal{H}_2^{BB}(y^{\mu})(\tau - \mathcal{T})^{-p_2}$  | $\mathcal{H}_2^{\infty}(y^{\mu})\tau^{-2/3}$   |
| $\frac{H_3}{\Theta^2}$   | $\mathcal{H}_3^{BB}(y^{\mu})(\tau - \mathcal{T})^{-p_3}$  | $\mathcal{H}_3^{\infty}(y^{\mu})\tau^{-2/3}$   |

In order to understand the dynamical behaviour of IDMs using the first-order approximation of the LWAS, we have computed the asymptotic behaviour in time, both near the Big-Bang singularity ( $\tau \rightarrow \mathcal{T}$ ) and for large future proper time ( $\tau \rightarrow \infty$ ) of the main physical quantities that are used in the covariant fluid approach. The results are summarized in Table 1. These results represent the generic behaviour of IDMs, there are particular cases, characterized by the vanishing of some of the connection quantities  $n_{\alpha\beta}$  or some the components of the gradient of  $\mathcal{T}$ , in which the behaviour indicated in Table 1 has to be revised. For the sake of brevity we have not included these particular cases here. Moreover, to obtain this asymptotic behaviour we have used the fact that in the neighbourhood of a fluid element one can always assume that the Kasner coefficients are distributed as in (36).

#### 4. On the validity of the Long Wavelength approximation scheme

The LWAS is based on the general idea that spatial gradients can be neglected with respect to time derivatives. This assumption puts restrictions to the validity of the expressions we have given above. This issue was already studied by Deruelle and Langlois in [10]. Their criterion to consider the approximation as valid is to compare the third-order terms arising from the first-order solution and to check whether or not they are smaller than the first-order ones. To that end, they computed the neglected terms from the first-order solution, essentially the spatial curvature terms  ${}^3R$  and  ${}^3S_{\mu\nu}$ . Then, they

realized that in general the effect of the local anisotropy makes some of spatial curvature components to blow up. Then, they argue that when this happens one shall recover the oscillatory BKL behaviour.

Our calculations agree with their results. First, for late times ( $\tau \rightarrow \infty$ ), the system evolves towards an Einstein-de Sitter model as expected (see Table 1), whereas for  $\Lambda > 0$  it would evolve toward the deSitter model. On the contrary, we have found that indeed some components of the spatial curvature blow up in general as one approaches the initial singularity ( $\tau - \mathcal{T} \rightarrow 0$ ). Nevertheless, here we want to take a complementary point of view to deal with this issue. In particular, we only need to analyze the quantities used in the covariant fluid approach constructed the first-order solution. In our scheme there is a set of main variables, namely  $\{e_\alpha^\mu, \Theta, \sigma_{\alpha\beta}\}$ , and a set of auxiliary variables,  $\{\gamma_{\alpha\beta}^\delta, \rho, E_{\alpha\beta}, H_{\alpha\beta}\}$ . The idea being that the secondary variables can be computed from the main ones. Of particular interest are the gravito-electric and -magnetic fields. Their first-order form has been given in Table 1. As we can see, their behaviour near the initial singularity is completely different. While the normalized<sup>¶</sup> gravito-electric field,  $\Theta^{-2}E_{\alpha\beta}$ , a dimensionless quantity, is finite there (but depending on the fluid element we consider), the normalized gravito-magnetic field,  $\Theta^{-2}H_{\alpha\beta}$ , has components that blow up. These divergences constitute an indication of the breakdown of the approximation scheme as we approach the initial singularity. Here again it is the local anisotropy that produces the blowing up of the gravito-magnetic field: this first-order quantity becomes dominant over the zero-order ones and the scheme breaks down. This can be seen in Table 1, where the dependence of the components of  $\Theta^{-2}H_{\alpha\beta}$  on the Kasner coefficients  $p_\alpha$  is explicit. In contrast, the gravito-electric field does not depend on them and its normalization is regular. Moreover, from Table 1 we also see the fact that the effect of matter, described by the normalized energy density  $\Theta^{-2}\rho$  (essentially, the cosmological density parameter  $\Omega$ ), can be neglected near the initial singularity, a well-known result.

Another interesting difference between the behaviour of the gravito-electric and -magnetic tensors is their local/non-local character. Whereas  $E_{\alpha\beta}$  is completely local at first-order, that is, for a fixed fluid element  $y^\alpha$  it only depends on the value of the main variables at that point [see Eq. (19)],  $H_{\alpha\beta}$  does depend on spatial gradients of the main quantities. More specifically, on the shear-rate spatial gradients [see Equation (12)]. In this sense, and in the spirit of the LWAS,  $H_{ab}$  is a purely first-order quantity and the spatial Ricci tensor is a purely second-order quantity. The blowing up of  $H_{\alpha\beta}$  (at first order) and  ${}^3R_{\alpha\beta}$  (as computed from the first-order solution) are not independent facts. To understand this we have to look at the evolution of the spatial Ricci tensor:

$${}^3\dot{R} = -\frac{2}{3}\Theta {}^3R - 2\sigma^{\alpha\beta} {}^3S_{\alpha\beta}, \quad (47)$$

$${}^3\dot{S}_{\alpha\beta} = \frac{2}{3}\Theta {}^3S_{\alpha\beta} + \sigma_{<\alpha}{}^\delta {}^3S_{\beta>\delta} - \frac{1}{6}\sigma_{\alpha\beta} {}^3R + \text{curl}(H)_{\alpha\beta}. \quad (48)$$

In IDMs the gravito-magnetic field has its origin in the anisotropy,  $H_{ab} = \text{curl}(\sigma)_{ab}$ , and it is the only source for the Ricci tensor as we can see from equations (47,48). First,

<sup>¶</sup> Note that we are normalizing using the expansion, which is related to the Hubble length, the natural scale of the system, by  $\Theta = 3H$ .

it can generate  ${}^3S_{ab}$  through equation (48), and  ${}^3S_{ab}$  is a source for the scalar spatial curvature  ${}^3R$  in equation (47).

On the other hand, the dynamical picture we have when approaching the spacelike singularity is essentially the BKL picture, in which the system has a Kasner-like evolution with some determined Kasner coefficients, say  $p'_\alpha$ , and at some point the system jumps to another Kasner phase characterized by other Kasner coefficients, say  $p''_\alpha$ . Within our approach, we can define the following time-dependent Kasner coefficients

$$p_1 \equiv \frac{\sqrt{3}\sigma - 2\sigma_+}{3\sqrt{3}\sigma}, \quad p_2 \equiv \frac{\sigma_+ + \sqrt{3}(\sigma + \sigma_-)}{3\sqrt{3}\sigma}, \quad p_3 \equiv \frac{\sigma_+ + \sqrt{3}(\sigma - \sigma_-)}{3\sqrt{3}\sigma}. \quad (49)$$

For the first-order solution, these become the time-independent ones defined in (35), because all the components of the shear have the same time dependence. To see how they change at higher order within the LWAS, it is enough to consider the evolution equation for the shear (8) with the trace-free part of the three-dimensional Ricci tensor,  ${}^3S_{\alpha\beta}$  (which is a second-order quantity) computed from the first-order solution. Now, in order to write down the evolution equations for the shear it is important to consider some important facts. First, that we are considering a basis in which the shear tensor diagonalizes, so it can be described only by the scalars  $(\sigma_+, \sigma_-)$ . Second, in the computation of  ${}^3S_{\alpha\beta}$  one can see that the diagonal terms dominate over the non-diagonal ones as one approaches the initial singularity at  $\tau \rightarrow \mathcal{T}$ . Therefore, for our purposes we can consider that  ${}^3S_{\alpha\beta}$  is diagonal and described by the two quantities

$${}^3S_+ \equiv -\frac{3}{2} {}^3S_{11}, \quad {}^3S_- \equiv \frac{\sqrt{3}}{2} ({}^3S_{22} - {}^3S_{33}),$$

This also means that within this approximation the shear and the trace-free Ricci tensor commute, which in turn implies that the gravitomagnetic field is transverse:  $\text{div}(H)_a = 0$ . Taking into account all these considerations, the evolution equations for the Kasner coefficients (49) can be expressed in a compact way by introducing the following vector:  $\vec{p} = (p_1, p_2, p_3)$ . And the evolution of  $\vec{p}$  is given by

$$\dot{\vec{p}} = \Omega \mathcal{A} \vec{p}, \quad (50)$$

where

$$\Omega \equiv \frac{\sigma_+ {}^3S_- - \sigma_- {}^3S_+}{3\sqrt{3}\sigma^2}, \quad \mathcal{A} \equiv \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix},$$

where  ${}^3S_{\pm}$  are computed from the first-order solution, and  $\sigma_{\pm}$  from the zero-order one. Equation (50) can be solved for each fluid element and the solution can be expressed in the following form

$$\vec{p}(\tau, y^\mu) = \vec{p}_0(y^\mu) + \frac{1 - \cos[\sqrt{3}I(\tau, y^\mu)]}{3} \mathcal{A}^2 \vec{p}_0(y^\mu) + \frac{1}{\sqrt{3}} \sin[\sqrt{3}I(\tau, y^\mu)] \mathcal{A} \vec{p}_0(y^\mu), \quad (51)$$

where

$$I(\tau, y^\mu) \equiv \int_{\tau_0}^{\tau} \Omega(\tau', y^\mu) d\tau', \quad \mathcal{A}^2 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

From the first order solution we find that in the generic case, i.e. without extra assumptions, the quantity  $I(\tau, y^\mu)$  behaves, near the initial singularity, as

$$I(\tau, y^\mu) \xrightarrow{\tau \rightarrow \mathcal{T}} \mathcal{I}(y^\mu)(p_1 - p_2)(\tau - \mathcal{T})^{-2p_3}.$$

Then, equation (51) shows the change that the Kasner coefficients will suffer when the spatial curvature is taken into account. However, this equation will not be valid once the Kasner coefficients have changed their value significantly, since we are approximating the spatial curvature from the first-order solution, where the Kasner coefficients are *static*. This means that the behaviour of the spatial curvature along the different spatial directions does not change. In order to take into account this fact one should compute the spatial curvature from equations (47,48), which describe the evolution of  ${}^3R$  and  ${}^3S_{\alpha\beta}$ , mainly driven by a second-order quantity in the LWAS:  $\text{curl}(H)_{ab}$ . This is in agreement with the observation in numerical simulations of the homogeneous Bianchi IX models, which exhibit a BKL behaviour, that the gravito-magnetic tensor  $H_{ab}$  generates the transition between different Kasner epochs (see [17]).

Finally, we want to remark that the discussion presented here is based on the generic solution of the LWAS. One can restrict the free functions and parameters that appear in the first-order solution in order to make it well behaved near the initial singularity. More specifically, we can impose the following restrictions on the first-order solution:

$$\mathcal{T} = \mathcal{T}_0 : \text{constant}, \quad \text{and} \quad n'_{11} = 0. \quad (52)$$

The first condition means that the initial singularity must be simultaneous. Actually, without loss of generality we can choose  $\mathcal{T}_0 = 0$ . The second condition in (52) is a restriction on the triad  $\{\mathbf{b}_\alpha\}$ . Then, under conditions (52) one can check that

$$\frac{H_{\alpha\beta}}{\Theta^2} \xrightarrow{\tau \rightarrow \mathcal{T}} 0, \quad \frac{{}^3R_{\alpha\beta}}{\Theta^2} \xrightarrow{\tau \rightarrow \mathcal{T}} 0. \quad (53)$$

With respect to the results shown in Table 1 the only thing that changes is the behaviour of  $H_{ab}$  near the singularity: the precise behaviour can be found in Table 2. Although for the component  $\Theta^{-2}H_3$  the expression in Table 2 suggests a logarithmic singularity for the case  $p_3 = 1$ ,  $p_1 = p_2 = 0$ , this is not the case. For this very particular case, the coefficient  $\mathcal{H}_3^0(y^\mu)$  vanishes identically for that situation and hence the result (53) is general for the particular case (52).

## 5. Conclusions and Discussion

In this work we have considered for the first time the application of the LWAS in the context of the covariant fluid approach to cosmology [13, 14]. This produces a system of ODEs for our main variables  $\{e_\alpha{}^\mu, \Theta, \sigma_{\alpha\beta}\}$ , giving a “silent” evolution [16, 17] that, for a collapsing fluid, proceeds towards a Kasner singularity. However, our analytical results show that at a certain point the gravito-magnetic field  $H_{ab}$  becomes dominant, producing a bounce to a new Kasner phase and the breakdown of the scheme. These results are consistent with the BKL picture for the approach to a generic spacelike singularity. On the other hand, for an expanding fluid, our results show that anisotropies die away and

**Table 2.** Asymptotic behaviour of the first-order solution in the particular case characterized by the relations (52).

| Quantity               | Behaviour as $\tau \rightarrow 0$             |
|------------------------|---|
| $\frac{H_+}{\Theta^2}$ | $\mathcal{H}_+^0(y^\mu)\tau^{2p_2}$           |
| $\frac{H_-}{\Theta^2}$ | $\mathcal{H}_-^0(y^\mu)\tau^{2p_2}$           |
| $\frac{H_1}{\Theta^2}$ | $\mathcal{H}_1^0(y^\mu)\tau^{1-p_1} \ln \tau$ |
| $\frac{H_2}{\Theta^2}$ | $\mathcal{H}_2^0(y^\mu)\tau^{1-p_2} \ln \tau$ |
| $\frac{H_3}{\Theta^2}$ | $\mathcal{H}_3^0(y^\mu)\tau^{1-p_3} \ln \tau$ |

the local evolution is more and more Robertson-Walker-like. This confirms previous analyses [10, 16, 17] and is also in line with recent investigations on the evolution of non-linearities on superhorizon scales [26].

The LWAS is based on the assumption that spatial gradients can be neglected with respect to time derivatives, whose scale is given by the Hubble parameter. Although the approximation scheme is physically well-motivated it turns out that there is not a well-established mathematical formulation of the principles on which it is based. The difficulty in having such a formulation lies in the difficulty to establish a meaningful way of smoothing a clumpy cosmological models, which in turn is related to the averaging problem in cosmology, a problem not yet clarified.

Most of the times the Long-Wavelength Approximation has been directly applied to the metric components, and then the spatial gradients of the metric has been neglected, which of course is a coordinate dependent procedure. In this work we have applied it to the quantities of the covariant fluid approach, which are intrinsically defined once the fluid velocity is prescribed, but basically the way in which we have implemented it is equivalent to the previous one. The reason for this is that we have neglected completely the spatial curvature tensor  ${}^3R_{ab}$ , which means that we have not only neglected spatial derivatives of the connection components,  $\partial_\delta \Gamma_{\alpha\beta}^\lambda$ , but also the product of them,  $\Gamma_{\alpha\beta}^\delta \Gamma_{\lambda\gamma}^\epsilon$ . The second approximation basically means that we have neglected the spatial gradients of the metric.

However, in principle we can generalize this approximation scheme by applying the idea at the level of the connection instead of applying it at the level of the metric. That is, to neglect all the spatial gradients of the connection, which implies that we will have a non-vanishing spatial curvature for the first iteration of the scheme. As one can see, the equations for this case will have essentially the same form as those of the Bianchi models, which are homogeneous models with homogeneous spatial curvature. So we will be still inside the spirit of the approximation. The advantage that this generalization can have is that it would allow us to include the possibility of having recollapsing regions (regions in which  ${}^3R > 0$ ), and in this way one could be able to explore a more realistic



universe model by using the LWAS, for example studying the evolution, within this approach, of a model corresponding to a Robertson-Walker universe with initially small perturbations. Since the integration functions of the equations of the LWAS would be space dependent this means that we could have regions in which the spatial curvature would lead to recollapse, and other regions that would expand forever. This would give a picture closer to the idea we have of the universe from observations, where we have void regions that are in expansion and others that are accreting matter and collapsing to form cosmic structures. From a mathematical point of view we expect that the generalized approach outlined above should be related to the improved LWAS introduced in [27] and, for the covariant quantities, should produce a system of ODEs able to describe the evolution from expansion to recollapse, much like the equations for “silent” models.

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