Evolution of second-order cosmological perturbations

Karim A. Malik[1] and David Wands[2]
[1]*GRECO, Institut d'Astrophysique de Paris,*
C.N.R.S., 98bis Boulevard Arago,
75014 Paris, France
[2]*Institute of Cosmology and Gravitation,*
University of Portsmouth,
Portsmouth PO1 2EG, United Kingdom
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We present a method for constructing gauge-invariant cosmological perturbations which are gauge-invariant up to second order. As an example we give the gauge-invariant definition of the second order curvature perturbation on uniform density hypersurfaces. Using only the energy conservation equation we show that this curvature perturbation is conserved at second order on large scales for adiabatic perturbations.

I. INTRODUCTION

In this letter we report new results for the study of second-order perturbations about a Friedmann-Robertson-Walker (FRW) spacetime. The study of second-order perturbations to date has been limited by two main problems: firstly the difficulty in defining truly gauge-invariant perturbations at second order, and secondly the complexity of the resulting Einstein equations [1, 2].

We address the first of these challenges by extending an approach to the construction of gauge-invariant variables previously advocated for first-order perturbations [3]. We make an unambiguous physical definition of the perturbation, and by building this into the mathematical description of the physical perturbation, construct a gauge-invariant quantity. As an example, we give a gauge-invariant definition of the curvature perturbation on uniform density hypersurfaces.

We avoid much of the complexity of the second-order field equations by considering only the local energy conservation equation [4, 5] in the large-scale limit where we neglect all spatial derivatives. This gives us a simple result establishing the constancy of the large-scale curvature perturbation on uniform density hypersurfaces, up to and including second order, for adiabatic perturbations.

To use, for example, the observed Gaussian distribution of cosmic microwave background (CMB) anisotropies [6] to test models for the origin of structure in the very early universe, there is an implicit assumption that there is negligible growth of second order perturbations on large scales. Simple models of inflation produce an almost Gaussian distribution of density perturbations [7], and recent studies have shown that the comoving curvature perturbation does not evolve at second order on large scales during slow-roll inflation [8, 9, 10]. But non-linear evolution from the end of inflation until the time of last-scattering of the CMB could produce deviations from a Gaussian distribution. The existence of a constant curvature perturbation at second order shows that the observed distribution of CMB anisotropies can be used to directly constrain the distribution of perturbations produced in the very early universe in simple inflation models that predict adiabatic density perturbations after inflation.

II. SECOND-ORDER PERTURBATIONS

Observations on scales close to the Hubble scale seem to be consistent with an almost homogeneous and isotropic spacetime that can be described by small perturbations about a Friedmann-Robertson-Walker metric.

Any tensorial quantities can be split into a homogeneous background and inhomogeneous perturbation

$$\mathbf{T}(\eta, x^i) = \mathbf{T}_0(\eta) + \delta\mathbf{T}_1(\eta, x^i) + \frac{1}{2}\delta\mathbf{T}_2(\eta, x^i) + \dots \quad (2.1)$$

where we use subscripts 1 and 2 to denote first and second order perturbations.

We will consider perturbations about a spatially flat FRW background metric

$$ds^2 = a^2 \left[-d\eta^2 + \delta_{ij}dx^i dx^j \right] . \quad (2.2)$$

where η is conformal time and $a = a(\eta)$ is the scale factor. The metric tensor including second-order perturbations can be written as

$$g_{00} = -a^2 \left(1 + 2\phi_1 + \phi_2 \right) , \quad (2.3)$$

$$g_{0i} = a^2 \left(B_{1i} + \frac{1}{2}B_{2i} \right) , \quad (2.4)$$

$$g_{ij} = a^2 \left[(1 - 2\psi_1 - \psi_2) \delta_{ij} + 2C_{1ij} + C_{2ij} \right] . \quad (2.5)$$

We will refer to ψ as the curvature perturbation as it describes the intrinsic scalar curvature of constant-η hypersurfaces on large scales.

Perturbations can be split into scalar, vector, and tensor modes, according to their transformation behaviour

on spatial 3-hypersurfaces [11]. For instance we can write

$$2C_{ij} = 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}, \qquad (2.6)$$

where E is a scalar perturbation, F_i is a divergence-free vector, and h_{ij} a transverse, trace-free tensor perturbation.

III. GAUGE TRANSFORMATIONS

Under a second-order coordinate transformation

$$\widetilde{x^\mu} = x^\mu + \xi_1^\mu + \frac{1}{2}\left(\xi_{1,\nu}^\mu \xi_1^{\ \nu} + \xi_2^\mu\right), \qquad (3.7)$$

any tensor $\mathbf{T}$ and its perturbations defined in Eq. (2.1) transform as [2]

$$
\begin{aligned}
\widetilde{\delta\mathbf{T}_1} &= \delta\mathbf{T}_1 + \mathrm{L}_{\xi_1}\mathbf{T}_0 \\
\widetilde{\delta\mathbf{T}_2} &= \delta\mathbf{T}_2 + \mathrm{L}_{\xi_2}\mathbf{T}_0 + \mathrm{L}_{\xi_1}^2\mathbf{T}_0 + 2\mathrm{L}_{\xi_1}\delta\mathbf{T}_1 .
\end{aligned} \quad (3.8)
$$

Thus under a first-order transformation $\xi_1^\mu = (\alpha_1, \beta_1^i)$, a scalar quantity such as the density, ρ, transforms at first-order as

$$\widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1, \qquad (3.9)$$

while at second order, writing $\xi_2^\mu = (\alpha_2, \beta_2^i)$, we have

$$
\begin{aligned}
\widetilde{\delta\rho_2} &= \delta\rho_2 + \rho_0'\alpha_2 + \alpha_1\left[\rho_0''\alpha_1 + \rho_0'\alpha_1{}' + 2\delta\rho_1{}'\right] \\
&\quad + (2\delta\rho_1 + \rho_0'\alpha_1)_{,i}\,\beta_1^i.
\end{aligned} \quad (3.10)
$$

For the first order curvature perturbation we have

$$\widetilde{\psi}_1 = \psi_1 - \mathcal{H}\alpha_1, \qquad (3.11)$$

where $\mathcal{H} \equiv a'/a$, while at second order we get from Eq. (3.8),

$$
\begin{aligned}
\widetilde{\psi}_2 &= \psi_2 - \alpha_1\left[\mathcal{H}\alpha_1{}' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\alpha_1 - 2\psi_1' - 4\mathcal{H}\psi_1\right] \\
&\quad - \mathcal{H}\alpha_2 - (\mathcal{H}\alpha_1 - 2\psi_1)_{,i}\,\beta_1^i.
\end{aligned} \quad (3.12)
$$

IV. GAUGE CHOICES AND GAUGE-INVARIANT VARIABLES

A gauge-invariant theory of linear perturbations about FRW metric was proposed by Bardeen [11] and subsequently developed by many authors (for example see Refs.[12, 13, 14, 15, 16]). No such gauge-invariant formalism has been developed for non-linear cosmological perturbations [24]. According to the Stewart-Walker lemma [18] any truly gauge-independent perturbation must be constant in the background spacetime. This apparently limits ones ability to make a gauge-invariant study of quantities that evolve in the background spacetime, e.g., density perturbations in an expanding cosmology.

In practice one can construct gauge-invariant definitions of unambiguous, that is physically defined, perturbations. These are not unique gauge-independent perturbations, but are gauge-invariant in the sense commonly used by cosmologists to define a physical perturbation. We draw a distinction here between quantities that are automatically gauge-independent, i.e., those that have no gauge dependence (such as perturbations about a constant scalar field), and quantities that are in general gauge-dependent (such as the curvature perturbation) but can have a gauge-invariant definition once their gauge-dependence is fixed (such as the curvature perturbation on uniform-density hypersurfaces). Although this approach has been widely used, at least implicitly, to construct gauge-invariant quantities at first-order [3, 13], it has not previously been used at higher-order. In this letter we show that it is possible to define gauge-invariant quantities at second-order corresponding to physical perturbations.

1. Uniform density hypersurfaces

The uniform density hypersurfaces are defined by setting $\widetilde{\delta\rho} = 0$ to the required order in perturbation theory. We find that for a specific spatial gauge this leads to a specific temporal gauge to the required order.

From Eq. (3.9) we see that setting $\widetilde{\delta\rho_1} = 0$ to first-order requires a gauge shift from an arbitrary gauge

$$\widetilde{\alpha}_1 = -\frac{\delta\rho_1}{\rho_0'}. \qquad (4.13)$$

Leaving for the moment the spatial gauge dependence, we see from Eq. (3.10), and using Eqs. (4.13) to fix the first-order temporal gauge shift, that to second order we require

$$\widetilde{\alpha}_2 = -\frac{1}{\rho_0'}\left[\delta\rho_2 - \frac{1}{\rho_0'}\delta\rho_1{}'\delta\rho_1 + \delta\rho_{1,i}\widetilde{\beta}_1^i\right]. \qquad (4.14)$$

To completely fix the second-order temporal gauge shift (4.14) picking out uniform density hypersurfaces we must also specify the first-order spatial gauge shift $\widetilde{\beta}_1^i$. For example, a natural choice is to pick worldlines comoving with the fluid. The fluid 3-velocity transforms as

$$\widetilde{v}^i = v^i - \beta^{i\prime}. \qquad (4.15)$$

Thus from an arbitrary spatial gauge we can transform to the comoving gauge by the spatial gauge transformation

$$\widetilde{\beta}^i = \int v^i d\eta. \qquad (4.16)$$

In this case there is a constant of integration corresponding to the arbitrary choice of spatial coordinates labelling the worldlines on an initial time-slice.

We are now able to construct gauge-invariant definitions for any metric or matter perturbations for comoving observers on uniform density hypersurfaces. As an example we give the curvature perturbation. To first-order, using Eqs. (3.11) and (4.13), we recover the well-known expression [4, 19]

$$-\zeta_1 \equiv \left.\widetilde{\psi_1}\right|_\rho = \psi_1 + \mathcal{H}\frac{\delta\rho_1}{\rho_0'} \,. \qquad (4.17)$$

For the second order curvature perturbation on uniform density hypersurfaces, along comoving worldlines, we use Eq. (3.12) with Eqs. (4.13), (4.14) and (4.16), to give

$$
\begin{aligned}
-\zeta_2 &\equiv \left.\widetilde{\psi_2}\right|_\rho \\
&= \psi_2 + \frac{\mathcal{H}}{\rho_0'}\delta\rho_2 - 2\frac{\mathcal{H}}{\rho_0'^2}\delta\rho_1'\delta\rho_1 - 2\frac{\delta\rho_1}{\rho_0'}\left(\psi_1' + 2\mathcal{H}\psi_1\right) \\
&\quad + \frac{\delta\rho_1^2}{\rho_0'^2}\left(\mathcal{H}\frac{\rho_0''}{\rho_0'} - \mathcal{H}' - 2\mathcal{H}^2\right) \\
&\quad + 2\left(\psi_1 + \mathcal{H}\frac{\delta\rho_1}{\rho_0'}\right)_{,i}\widetilde{\beta}_1^i \,. \qquad (4.18)
\end{aligned}
$$

We can also give gauge-invariant definitions for scalar quantities on uniform density hypersurfaces with comoving worldlines. As an example we write the pressure perturbation on uniform-density hypersurfaces at first order

$$\left.\widetilde{\delta P_1}\right|_\rho = \delta P_1 - \frac{P_0'}{\rho_0'}\delta\rho_1 \,. \qquad (4.19)$$

We can identify this as the usual gauge-invariant definition of the non-adiabatic part of the pressure perturbation. At second order we have

$$
\begin{aligned}
\left.\widetilde{\delta P_2}\right|_\rho &= \delta P_2 - \frac{P_0'}{\rho_0'}\delta\rho_2 + 2\left(\delta P_1 - \frac{P_0'}{\rho_0'}\delta\rho_1\right)_{,i}\widetilde{\beta}_1^i \\
&\quad + P_0'\left\{\left(\frac{\delta\rho_1'}{\rho_0'} - \frac{\delta P_1'}{P_0'}\right)\frac{2\delta\rho_1}{\rho_0'} + \left(\frac{P_0''}{P_0'} - \frac{\rho_0''}{\rho_0'}\right)\frac{\delta\rho_1^2}{\rho_0'^2}\right\} \,.
\end{aligned}
$$
$$(4.20)$$

These results are readily extended to systems involving scalar fields, as scalar fields obey the same transformation rules as the energy density, given in Eqs. (3.9) and (3.10). Hence one can write down the comoving curvature perturbation (i.e., the curvature perturbation on uniform scalar field hypersurfaces) or the relative entropy perturbation between two fields.

For adiabatic perturbations the local pressure is a unique function of the local density, $P = P(\rho)$. Hence we can identify the non-adiabatic part of the pressure perturbation, to first and second orders, as

$$
\begin{aligned}
\delta P_{1\,\mathrm{nad}} &= \delta P_1 - c_s^2\delta\rho_1 \,, \\
\delta P_{2\,\mathrm{nad}} &= \delta P_2 - c_s^2\delta\rho_2 - \frac{dc_s^2}{d\rho_0}\delta\rho_1^2 \,, \qquad (4.21)
\end{aligned}
$$

where the extra term on the right-hand-side of the expression for the second-order non-adiabatic pressure perturbation arises from the local variation of the adiabatic sound speed, $c_s^2 \equiv dP/d\rho$. Note that the non-adiabatic pressure perturbation is automatically gauge-invariant at first order, but the second order non-adiabatic perturbation is only gauge-invariant if the first order non-adiabatic pressure perturbation vanishes [2].

Thus we can write the gauge-invariant pressure perturbation on uniform density hypersurfaces as

$$\left.\widetilde{\delta P_1}\right|_\rho = \delta P_{1\,\mathrm{nad}} \,, \qquad (4.22)$$

$$\left.\widetilde{\delta P_2}\right|_\rho = \delta P_{2\,\mathrm{nad}} - 2\frac{\delta\rho_1}{\rho_0'}\delta P_{1\,\mathrm{nad}}' \,. \qquad (4.23)$$

We see that the pressure perturbation will vanish on hypersurfaces of uniform density for adiabatic perturbations.

2. Uniform curvature hypersurfaces

Instead of defining quantities on uniform density hypersurfaces we can choose to work with uniform curvature slices. In some scenarios this can have the advantage of staying non-singular even when the uniform density hypersurfaces become ill-defined [20, 21].

We can define uniform curvature hypersurfaces by $\widetilde{\psi_1} = 0$ and $\widetilde{\psi_2} = 0$, which fixes the temporal gauge shift and $\widetilde{E}_1 = 0$ and $\widetilde{F}_1^i = 0$ to fix the spatial gauge shift to first-order. This implies for the first order temporal gauge shift, using Eq. (3.11),

$$\bar{\alpha}_1 = \frac{\psi_1}{\mathcal{H}} \,, \qquad (4.24)$$

and at second order, using Eq. (3.12),

$$\bar{\alpha}_2 = \frac{1}{\mathcal{H}}\left[\psi_2 + 2\psi_1^2 + \frac{1}{\mathcal{H}}\psi_1'\psi_1 + \psi_{1,k}\bar{\beta}_1^k\right] \,, \qquad (4.25)$$

where we used Eq. (4.24) and fix the spatial gauge shift [12, 13]

$$\bar{\beta}_1^i = E_{1,}{}^i + F_1^i \,. \qquad (4.26)$$

The density perturbation on uniform curvature hypersurfaces is then, at first order, using Eqs. (3.9),

$$\left.\widetilde{\delta\rho_1}\right|_\psi \equiv \delta\rho_1 + \frac{\rho_0'}{\mathcal{H}}\psi_1 \,. \qquad (4.27)$$

At first order the curvature perturbation defined on uniform density hypersurfaces and the density perturbation on uniform curvature hypersurfaces are simply related by

$$\left.\widetilde{\delta\rho_1}\right|_\psi = \frac{\rho_0'}{\mathcal{H}}\left.\widetilde{\psi_1}\right|_\rho \,. \qquad (4.28)$$

From Eq. (3.10) we get the definition of the second order density perturbation on uniform curvature hypersurfaces

$$\widetilde{\delta\rho_2}\Big|_{\psi} \equiv \delta\rho_2 + \frac{\rho_0'}{\mathcal{H}}\psi_2 + \frac{1}{\mathcal{H}}\left(2\rho_0' + \frac{\rho_0''}{\mathcal{H}} - \frac{\rho_0'\mathcal{H}'}{\mathcal{H}^2}\right)\psi_1^2$$
$$+2\frac{\rho_0'}{\mathcal{H}^2}\psi_1'\psi_1 + \frac{2}{\mathcal{H}}\psi_1\delta\rho_1{}'$$
$$+2\left(\delta\rho_1 + \frac{\rho'}{\mathcal{H}}\psi_1\right)_{,i}\bar{\beta}_1^i. \quad (4.29)$$

Again, these results are readily extended to scalar fields which obey the same gauge transformations rules as the energy density.

V. EVOLUTION OF CURVATURE ON LARGE SCALES

Having defined gauge-invariant second-order variables, we now turn to finding evolution equations for these quantities.

A fundamental question in cosmology is how density perturbations evolve in the large-scale regime where gravitational perturbations cannot be neglected. In particular it is important to establish whether non-linear evolution could introduce significant non-Gaussianity, e.g., in CMB anisotropies, even in inflationary models where the large scale structure of the universe is supposed to arise from purely Gaussian fluctuations at very early times.

Despite the complexity of the field equations at second order (see e.g. Ref. [22]) it is sufficient to use the local conservation of energy-momentum to establish the conservation of ζ if we neglect spatial gradients, which we expect to be valid on sufficiently large scales. This strategy was first employed in Ref. [4] to establish the conservation of ζ at first order. We will also neglect terms quadratic in first-order vector and tensor perturbations which can be calculated using the first-order equations of motion and can be shown to be small on large scales in an expanding universe. We defer a full treatment including the effect of these quadratic terms to future work.

We define the energy momentum tensor as

$$T^{\mu\nu} \equiv (\rho + P)\,u^\mu u^\nu + Pg^{\mu\nu} + \Pi^{\mu\nu}, \quad (5.30)$$

where $\Pi^{\mu\nu}$ is the trace-free anisotropic stress tensor, and u^μ is the fluid 4-velocity.

Energy conservation is given by $u_\nu \nabla_\mu T^{\mu\nu} = 0$. In the homogeneous background we have

$$\rho_0' + 3\frac{a'}{a}\left(\rho_0 + P_0\right) = 0, \quad (5.31)$$

while dropping spatial gradient terms we obtain

$$\delta\rho_1{}' + 3\mathcal{H}\left(\delta\rho_1 + \delta P_1\right) - 3\left(\rho_0 + P_0\right)\psi_1' \simeq 0, (5.32)$$
$$\delta\rho_2{}' + 3\mathcal{H}\left(\delta\rho_2 + \delta P_2\right) - 3\left(\rho_0 + P_0\right)\psi_2'$$
$$- 6\psi_1'\left[\delta\rho_1 + \delta P_1 + 2\left(\rho_0 + P_0\right)\psi_1\right] \simeq 0, (5.33)$$

to first and second order respectively. These can be written in terms of the gauge-invariant curvature perturbation, ζ defined in Eqs. (4.17) and (4.18), giving

$$\zeta_1' \simeq -\frac{\mathcal{H}}{(\rho + P)}\widetilde{\delta P_1}|_\rho, \quad (5.34)$$

and

$$\zeta_2' \simeq -\frac{\mathcal{H}}{(\rho + P)}\widetilde{\delta P_2}|_\rho$$
$$-\frac{2}{\rho_0 + P_0}\left[\widetilde{\delta P_1}|_\rho - 2(\rho_0 + P_0)\zeta_1\right]\zeta_1', (5.35)$$

where the gauge-invariant pressure perturbation on uniform density hypersurfaces is given in terms of the non-adiabatic pressure perturbation in Eqs. (4.22) and (4.23). Thus the curvature perturbation on uniform density hypersurfaces is constant at first and second order for adiabatic perturbations on large scales if we can neglect spatial gradients [25].

Hence we expect an initially Gaussian distribution of adiabatic curvature perturbations in the very early universe will remain Gaussian in the large scale limit. By the same token, primordial non-Gaussianity of the perturbations may be indicative of non-adiabatic evolution in the early universe.

VI. CONCLUSION

In summary, we have given a procedure for defining gauge-invariant cosmological perturbations at first and second order. As an example we have given a gauge-invariant definition of the curvature perturbation on uniform density hypersurfaces. We expect that this prescription could be extended to higher orders if desired.

We were then able to show, using only the local energy conservation equation, that the curvature perturbation remains constant on large scales for adiabatic perturbations where we neglect spatial gradients. As shown in Ref. [5], a conserved perturbation exists of any quantity that obeys an autonomous local conservation equation. Thus we can construct conserved perturbations of the local energy density of any fluid with a barotropic equation of state.

Note added: While writing up this letter, a non-linear result for the constancy of ζ in a long-wavelength approximation [1] was reported by Rigopoulos and Shellard [23].

Society. Algebraic computations of tensor components were performed using the GRTensorII package for Maple.

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[24] After completing this work, our attention was drawn to a recent paper by Nakamura [17] where a general procedure is proposed for constructing gauge-invariant perturbations upto third order of a generic spacetime.

[25] An alternative argument for the existence of a conserved quantity at second order was given recently in Ref. [5] defined in terms of the density perturbation on uniform expansion hypersurfaces. One can verify that for adiabatic perturbations the second-order part of the conserved quantity defined in Eq. (11) of Ref. [5] coincides with $\zeta_2 - 2\zeta_1^2$, using the gauge-invariant defintions of ζ_1 and ζ_2 given in Eqs. (4.17) and (4.18), written in terms of the density perturbation evaluated on spatially flat hypersurfaces.