BRANE-WORLD COSMOLOGICAL PERTURBATIONS
A covariant approach

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The standard cosmological model, based on general relativity with an inflationary era, is very effective in accounting for a broad range of observed features of the universe. However, the ongoing puzzles about the nature of dark matter and dark energy, together with the problem of a fundamental theoretical framework for inflation, indicate that cosmology may be probing the limits of validity of general relativity. The early universe provides a testing ground for theories of gravity, since gravitational dynamics can lead to characteristic imprints on the CMB and other cosmological observations. Precision cosmology is in principle a means to constrain and possibly falsify candidate quantum gravity theories like M theory. Generalized Randall-Sundrum brane-worlds provide a phenomenological means to test aspects of M theory. I outline the 1+3-covariant approach to cosmological perturbations in these brane-worlds, and its application to CMB anisotropies.

I. INTRODUCTION

M theory, the 11-dimensional theory that encompasses the known superstring theories, is only partially understood, but is widely considered to be a promising potential route to quantum gravity [1], and is therefore an important candidate for cosmological testing. Currently there are not realistic M theory cosmological solutions, so that it is reasonable to use simplified phenomenological models that share some of the key features of M theory, especially branes. In brane cosmology, the observable universe is a 1+3-dimensional "brane" surface moving in a higher-dimensional "bulk" spacetime. Fields and particles in the non-gravitational sector are confined to the brane, while gravity propagates in the bulk. The simplest, and yet sufficiently general, phenomenological brane-world models are the cosmological generalizations [2–4] of the Randall-Sundrum II model [5]. In the RSII brane-world, the bulk is 5-dimensional anti-de Sitter spacetime, so that the extra dimension is infinite. The generalized brane-worlds (see [6,7] for recent reviews) also have non-compact extra dimension.[1] The other 6 extra spatial dimensions of M theory may be assumed to be stabilized and compactified on a very small scale, so that they do not affect the dynamics over the range of validity of the brane-world model, i.e. for energies sufficiently below the string scale, for which the brane may be treated as infinitely thin. The RSII models have the additional advantage that they provide a framework for investigating aspects of holography and the AdS/CFT correspondence.

What prevents gravity from 'leaking' into the infinite extra dimension at low energies is the negative bulk cosmological constant,

$$\Lambda_5 = -\frac{6}{\ell^2}, \tag{1}$$

where ℓ is the curvature radius if the bulk is AdS_5. Corrections to Newton's law in the weak-field static limit are $O(\ell^2/r^2)$ [5]:

$$\phi(r) = \frac{GM}{r}\left(1 + \frac{2\ell^2}{3r^2}\right) + \cdots \tag{2}$$

Experiments currently impose an upper bound $\ell \lesssim 1$ mm. On the brane, the negative Λ_5 is offset by the positive brane tension λ. The effective cosmological constant on the brane is

[1] Various extensions of the generalized RSII brane-worlds are not discussed here. The simplest extension is to introduce a second brane, so that the extra dimension is compact (but much larger than Planck scale), or a scalar field (gravitational sector) in the bulk, or both. See [7] for further discussion and references (including the "ekpyrotic" and cyclic models where the potential of the bulk scalar causes branes to collide, which may initiate a big bang and provide an alternative to inflation [8]). Other extensions involve corrections to the action, including 4-dimensional "induced gravity" and 5-dimensional Gauss-Bonnet corrections; see, e.g. [9].

$$\Lambda = \frac{1}{2}(\Lambda_5 + \kappa^2\lambda)\,, \quad \kappa^2 = 8\pi G = \frac{8\pi}{M_4^2}\,, \tag{3}$$

where $M_4 \sim 10^{19}$ GeV is the effective Planck scale on the brane. This is not the true fundamental gravity scale, which can be much lower, offering the possibility of a resolution of the hierarchy problem, as well as the exciting prospect that quantum gravity effects could be observable in particle accelerators and cosmic ray showers. The fundamental energy scale can be as low as $\sim$ TeV in some brane-world scenarios, but in generalized RSII models it is higher, $M_5 > 10^5$ TeV, and is related to M_4 via

$$M_5^3 = \frac{M_4^2}{\ell}\,. \tag{4}$$

The bound $\ell < 1$ mm implies that λ is above the electroweak scale, $\lambda > (100 \text{ GeV})^4$. At high energies ($\rho \gg \lambda$) in the early universe, gravity becomes 5-dimensional and there are significant corrections to standard cosmological dynamics. There are also corrections that can operate at low energies, mediated by bulk graviton or Kaluza-Klein (KK) modes. Both types of correction play an important role in cosmological perturbations. In particular, 5-dimensional gravitational-wave modes introduce nonlocal effects from the viewpoint of brane-bound observers [10,11].

Brane-world inflation

The unperturbed cosmological brane-world is a Friedmann brane in a Schwarzschild-AdS$_5$ bulk [2,12]. High-energy brane-world modifications to the dynamics of inflation on the brane have been investigated [4,13]. Essentially, the high-energy corrections provide increased Hubble damping,

$$V(\varphi) \gg \lambda \;\Rightarrow\; H \approx \frac{V}{M_4\sqrt{6\lambda}}\,, \tag{5}$$

thus making slow-roll inflation possible even for potentials $V(\varphi)$ that would be too steep in standard cosmology [4,14]. This can be seen clearly from the slow-roll parameters ($V \gg \lambda$)

$$\epsilon \approx \epsilon_{\rm gr}\left(\frac{4\lambda}{V}\right)\,,\; \eta \approx \eta_{\rm gr}\left(\frac{2\lambda}{V}\right), \tag{6}$$

where $\epsilon_{\rm gr}, \eta_{\rm gr}$ are the standard general relativity slow-roll parameters. Steep potentials can inflate at high energy and then naturally stop inflating when V drops below λ. These models can be constrained because they typically generate a blue spectrum of gravitational waves which can disturb nucleosynthesis [14]. They also raise the intriguing prospect that the inflaton could act as dark matter or quintessence at low energies [14,15].

Large-scale scalar perturbations generated by slow-roll inflation ($V \gg \lambda$) have an enhanced amplitude compared with the standard general relativity case [4]:

$$A_{\rm s}^2 \approx \left[\frac{64\pi}{75M_4^6}\frac{V^3}{V'^2}\right]\left(\frac{V}{\lambda}\right)^2. \tag{7}$$

This means that COBE-scale perturbations can be generated when the inflaton is well below M_4. For example,

$$\varphi_{\rm cobe} \approx \frac{300}{(M_4\ell)^{1/3}}\,M_4 \;\ll M_4\,, \tag{8}$$

for $V = \frac{1}{2}m^2\varphi^2$. The scalar spectral index is in general given by

$$n_{\rm s} = 1 - 6\epsilon + 2\eta\,, \tag{9}$$

and is driven closer to 1 (compared to general relativity) by high-energy effects.

High-energy inflation on the brane also generates a zero-mode (4-dimensional graviton mode) of tensor perturbations, and stretches it to super-Hubble scales. This zero-mode has the same qualitative features as in general relativity, remaining frozen at constant amplitude while beyond the Hubble horizon. Its amplitude is enhanced at high energies, although the enhancement is much less than for scalar perturbations [16]:

$$A_{\rm t}^2 \approx \left[\frac{8V}{25M_4^2}\right]\left(\frac{V}{\lambda}\right)^2, \tag{10}$$

$$\frac{A_{\rm t}^2}{A_{\rm s}^2} \approx \left[\frac{3M_4^2}{8\pi}\,\frac{V'^2}{V^2}\right]\left(\frac{\lambda}{V}\right). \tag{11}$$

Equation (11) means that brane-world effects suppress the large-scale tensor contribution to CMB anisotropies. The tensor spectral index has a smaller magnitude than in general relativity, but obeys the same consistency relation:

$$n_{\rm t} = -3\epsilon = -2\frac{A_{\rm t}^2}{A_{\rm s}^2}\,. \tag{12}$$

The massive KK modes (5-dimensional graviton modes which have an effective mass from a brane observer viewpoint) remain in the vacuum state during slow-roll inflation [16,17]. The evolution of the super-Hubble zero mode is the same as in general relativity, so that high-energy brane-world effects in the early universe serve only to rescale the amplitude. However, when the zero mode re-enters the Hubble horizon, massive KK modes can be excited. This may be a very small effect, but it remains to be properly quantified.

Vector perturbations in the bulk metric can support vector metric perturbations on the brane, even in the absence of matter perturbations. However, there is no normalizable zero mode, and the massive KK modes stay in the vacuum state during brane-world inflation [18]. Therefore, as in general relativity, we can neglect vector perturbations in inflationary cosmology.

Perturbation evolution

The background dynamics of brane-world cosmology is known exactly. Large-scale cosmological perturbations on the brane are well understood [4,11,16,19,20]. However, without a solution for small-scale perturbations, we remain unable to predict the CMB anisotropies in brane-world cosmology, and the CMB provides the key means to test the scenario. The problem is that the 5-dimensional bulk perturbation equations must be solved in order to solve for perturbations on the brane. The 5-dimensional equations are partial differential equations for the 3-dimensional Fourier modes, with complicated boundary conditions. In fact, even the Sachs-Wolfe effect requires information from the 5-dimensional solutions; although the large-scale density perturbations can be determined without knowing the 5-dimensional solutions [11,19], the Sachs-Wolfe effect requires the large-scale metric perturbations, and these are related to the density perturbations in a way that involves the KK modes [20].

The theory of gauge-invariant perturbations in brane-world cosmology has been extensively investigated and developed [4,10,11,14,16–24] and is qualitatively well understood. The key remaining task is integration of the coupled brane-bulk perturbation equations; up to now, only special cases have been solved, where these equations effectively decouple. In general, and for the crucial case of calculating CMB anisotropies [20,22–24], the coupled system must be solved. From the brane viewpoint, the bulk effects, i.e. the high-energy corrections and the KK modes, act as source terms for the brane perturbation equations. At the same time, perturbations of matter on the brane can generate KK modes (i.e., emit 5-dimensional gravitons into the bulk) which propagate in the bulk and can interact with the brane. This nonlocal interaction amongst the perturbations is at the core of the complexity of the problem. It can be elegantly expressed via integro-differential equations [10], which take the form

$$A_k(t) = \int dt'\,\mathcal{G}(t,t')B_k(t')\,, \tag{13}$$

where $\mathcal{G}$ is the bulk retarded Green's function evaluated on the brane, and A_k, B_k are made up of brane metric and matter perturbations and their (brane) derivatives, and include high-energy corrections to the background dynamics.

II. COVARIANT DYNAMICS AND PERTURBATIONS

The 5D field equations are

$$^{(5)}G_{AB} = -\Lambda_5\, {}^{(5)}g_{AB} + \delta(y)\,\frac{8\pi}{M_5^3}\left[-\lambda g_{AB} + T_{AB}\right], \tag{14}$$

where y is a Gaussian normal coordinate orthogonal to the brane, which is at $y=0$, the induced metric on $\{y=\text{ const}\}$ is $g_{AB} = {}^{(5)}g_{AB} - n_A n_B$ with n^A the unit normal, and T_{AB} is the energy-momentum tensor of particles and fields confined to the brane (with $T_{AB}n^B = 0$). The effective field equations on the brane are derived from the Gauss-Codazzi equations and the Darmois-Israel junction conditions (using Z_2-symmetry) [3]:

$$G_{ab} = -\Lambda g_{ab} + \kappa^2 T_{ab} + 6\frac{\kappa^2}{\lambda}\mathcal{S}_{ab} - \mathcal{E}_{ab}\,, \tag{15}$$

where $\mathcal{S}_{ab} \sim (T_{ab})^2$ is the high-energy correction term, which is negligible for $\rho \ll \lambda$, while $\mathcal{E}_{ab}$ is the projection of the bulk Weyl tensor on the brane. This term encodes corrections from KK or 5D graviton effects. From the brane-observer viewpoint, the energy-momentum corrections in $\mathcal{S}_{ab}$ are local, whereas the KK corrections in $\mathcal{E}_{ab}$ are nonlocal, since they incorporate 5D gravity wave modes, as discussed above. These nonlocal corrections cannot be determined purely from data on the brane, and so the effective field equations are not a closed system. One needs to supplement them by 5D equations governing $\mathcal{E}_{ab}$, which are obtained from the 5D Einstein and Bianchi equations [3].

The trace free $\mathcal{E}_{ab}$ contributes an effective energy density ρ^*, pressure $\rho^*/3$, momentum density q^*_a and anisotropic stress π^*_{ab} on the brane. In a 1+3-covariant decomposition,

$$-\frac{1}{\kappa^2}\mathcal{E}_{ab} = \rho^*\left(u_a u_b + \frac{1}{3}h_{ab}\right) + q^*_a u_b + q^*_b u_a + \pi^*_{ab}\,, \tag{16}$$

where u^a is a physically determined 4-velocity on the brane and $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space at each event. The KK anisotropic stress π^*_{ab} incorporates the spin-0 ("Coulomb"), spin-1 (gravimagnetic) and spin-2 (gravitational wave) 4D modes of the 5D graviton. The KK momentum density q^*_a incorporates spin-0 and spin-1 modes, and the KK energy density ρ^* (the "dark radiation") incorporates the spin-0 mode. The brane "feels" the bulk gravitational field through these terms. In the background, $q^*_a = 0 = \pi^*_{ab}$, since only the "dark radiation" term is compatible with Friedmann symmetry.

The brane-world corrections can conveniently be consolidated into an effective total energy density, pressure, momentum density and anisotropic stress. Linearizing the general nonlinear expressions [11] we obtain

$$\rho^{\rm eff} = \rho\left(1 + \frac{\rho}{2\lambda} + \frac{\rho^*}{\rho}\right)\,, \tag{17}$$

$$p^{\rm eff} = p + \frac{\rho}{2\lambda}(2p+\rho) + \frac{\rho^*}{3}\,, \tag{18}$$

$$q^{\rm eff}_a = q_a\left(1 + \frac{\rho}{\lambda}\right) + q^*_a\,, \tag{19}$$

$$\pi^{\rm eff}_{ab} = \pi_{ab}\left(1 - \frac{\rho+3p}{2\lambda}\right) + \pi^*_{ab}\,, \tag{20}$$

where $\rho = \sum_i \rho^{(i)}$ and $p = \sum_i p^{(i)}$ are the total matter-radiation density and pressure, $q_a = \sum_i q^{(i)}_a$ is the total matter-radiation momentum density, and π_{ab} is the photon anisotropic stress (neglecting that of neutrinos, baryons and CDM).

Energy-momentum conservation,

$$\nabla^b T_{ab} = 0\,, \tag{21}$$

together with the 4D Bianchi identity, lead to

$$\nabla^a \mathcal{E}_{ab} = \frac{6\kappa^2}{\lambda}\nabla^a \mathcal{S}_{ab}\,, \tag{22}$$

which shows qualitatively how 1+3 spacetime variations in the matter-radiation on the brane can source KK modes. The 1+3-covariant decomposition of Eq. (21) leads to the standard energy and (linearized) momentum conservation equations,

$$\dot\rho + \Theta(\rho+p) + {\rm D}^a q_a = 0\,, \tag{23}$$

$$\dot q_a + 4Hq_a + {\rm D}_a p + (\rho+p)A_a + {\rm D}^b\pi_{ab} = 0\,, \tag{24}$$

where Θ is the volume expansion rate, which reduces to $3H$ in the background (H is the background Hubble rate), A_a is the 4-acceleration, and ${\rm D}_a$ is the covariant derivative in the rest space (i.e. ${\rm D}_a F^{b\cdots}{}_{\cdots c} = h_a{}^d h^b{}_e \cdots h_c{}^f \nabla_d F^{e\cdots}{}_{\cdots f}$). The absence of bulk source terms in the conservation equations is a consequence of having Λ_5 as the only 5D source in the bulk. If there is a bulk scalar field, then there is energy-momentum exchange between the brane and bulk (in addition to the gravitational interaction) [25].

Equation (22) may be thought of as the "nonlocal conservation equation". Linearizing the general 1+3-covariant decomposition [11], we obtain

$$\dot{\rho}^* + \frac{4}{3}\Theta\rho^* + \mathrm{D}^a q^*_a = 0\,, \tag{25}$$

$$\dot{q}^*_a + 4Hq^*_a + \frac{1}{3}\mathrm{D}_a\rho^* + \frac{4}{3}\rho^* A_a + \mathrm{D}^b\pi^*_{ab} = \frac{(\rho+p)}{\lambda}\left[-\mathrm{D}_a\rho + 3Hq_a + \frac{3}{2}\mathrm{D}^b\pi_{ab}\right]. \tag{26}$$

At linear order, spatial inhomogeneity ($\mathrm{D}_a\rho$), peculiar motions ($q_a = \rho v_a$) and anisotropic stresses (π_{ab}) in the matter-radiation on the brane are seen to be sources for KK modes (or 5D graviton emission into the bulk). Qualitatively and geometrically this can be understood as follows: the non-uniform 5D gravitational field generated by inhomogeneous and anisotropic 4D matter-radiation contributes to the 5D Weyl tensor, which nonlocally "backreacts" on the brane via its projection $\mathcal{E}_{ab}$. Note also that the source terms are suppressed at low energies, and during quasi-de Sitter inflation on the brane.

Equations (25) and (26) are propagation equations for ρ^* and q^*_a. There is no propagation equation on the brane for π^*_{ab}; if there were such an equation, then one could determine the KK modes purely from data on the brane, which would violate causality for 5D gravitational waves.

In the background, the modified Friedmann equations are

$$H^2 = \frac{\kappa^2}{3}\rho^{\rm eff} + \frac{1}{3}\Lambda + \frac{K}{a^2}\,, \tag{27}$$

$$\dot{H} = -\frac{\kappa^2}{2}(\rho^{\rm eff} + p^{\rm eff}) + \frac{K}{a^2}\,. \tag{28}$$

By Eq. (25), the KK energy density behaves like dark radiation:

$$\rho^* \propto \frac{1}{a^4}\,. \tag{29}$$

The source of the dark radiation is the tidal (Coulomb) effect of a 5D black hole in the bulk. When the black hole mass vanishes, the bulk geometry reduces to AdS_5 and $\rho^* = 0$. In order to avoid a naked singularity, we assume that the black hole mass is non-negative, so that $\rho^* \geq 0$. This additional effective relativistic degree of freedom is constrained by nucleosynthesis and CMB observations to be no more than ∼3% of the radiation energy density [20,22,26]:

$$\left.\frac{\rho^*}{\rho_{\rm rad}}\right|_{\rm nuc} \lesssim 0.03 \tag{30}$$

If $\rho^* = 0$ and $K = 0 = \Lambda$, then the exact solution of the Friedmann equations is [2]

$$a = \text{ const}\,[t(t+t_\lambda)]^{1/3(w+1)}\,, \quad t_\lambda = \frac{M_4}{\sqrt{\pi\lambda}} < 10^{-9}\,\text{sec}\,, \tag{31}$$

where $w = p/\rho$ is assumed constant. If $\rho^* \neq 0$ (but $K = 0 = \Lambda$), then the solution for the radiation era ($w = \frac{1}{3}$) is [22]

$$a = \text{ const}\,[t(t+t_\lambda)]^{1/4}\,, \quad t_\lambda = \frac{\sqrt{3}\,M_4}{4\sqrt{\pi\lambda}\,(1+\rho^*/\rho)}\,. \tag{32}$$

For $t \gg t_\lambda$ we recover from Eqs. (31) and (32) the standard behaviour, $a \propto t^{2/3(w+1)}$, whereas for $t \ll t_\lambda$, we have the very different behaviour, $a \propto t^{1/3(w+1)}$.

In the 1+3-covariant description of perturbations [11], we isolate the KK anisotropic stress π^*_{ab} as the term that must be determined from 5D equations. Once π^*_{ab} is determined in this way, the 1+3 perturbation equations on the brane form a closed system. The KK terms act as source terms modifying the standard general relativity perturbation equations, together with the high-energy corrections. For example, the propagation equation for the shear is [11]

$$\dot{\sigma}_{ab} + 2H\sigma_{ab} + E_{ab} - \frac{\kappa^2}{2}\pi_{ab} - \mathrm{D}_{\langle a}A_{b\rangle} = \frac{\kappa^2}{2}\pi^*_{ab} - \frac{\kappa^2}{4}(1+3w)\frac{\rho}{\lambda}\,\pi_{ab}\,, \tag{33}$$

where E_{ab} is the electric part of the 4D brane Weyl tensor (not to be confused with $\mathcal{E}_{ab}$). In general relativity, the right hand side is zero. In the brane-world, the first source term on the right is the KK term, the second term is the high-energy modification. The other modification is a straightforward high-energy correction of the background quantities H and ρ via the modified Friedmann equations.

In the 1+3-covariant approach, perturbative quantities are projected vectors ($V_a u^a = 0$) and projected symmetric tracefree tensors,

$$W_{ab} = W_{\langle ab \rangle} \equiv \left[h_a{}^c h_b{}^d - \frac{1}{3} h_{ab} h^{cd} \right] W_{cd} \,. \tag{34}$$

These are decomposed into (3D) scalar, vector and tensor modes as [6]

$$V_a = \mathrm{D}_a V + \bar{V}_a \,, \tag{35}$$

$$W_{ab} = \mathrm{D}_{\langle a} \mathrm{D}_{b \rangle} W + \mathrm{D}_{\langle a} \bar{W}_{b \rangle} + \bar{W}_{ab} \,, \tag{36}$$

where an overbar denotes a (3D) transverse quantity ($\mathrm{D}^a \bar{V}_a = 0 = \mathrm{D}^b \bar{W}_{ab}$). Purely scalar perturbations are characterized by

$$\bar{V}_a = \bar{W}_a = \bar{W}_{ab} = 0 \,, \tag{37}$$

and scalar quantities are formed via the (3D) Laplacian: $\mathcal{V} = \mathrm{D}^a \mathrm{D}_a V \equiv \mathrm{D}^2 V$. Purely vector perturbations are characterized by

$$V_a = \bar{V}_a \,, \; W_{ab} = \mathrm{D}_{\langle a} \bar{W}_{b \rangle} \,, \; \mathrm{curl}\, \mathrm{D}_a f = -2 \dot{f} \omega_a \,, \tag{38}$$

where ω_a is the vorticity, and purely tensor by

$$\mathrm{D}_a f = 0 = V_a \,, \; W_{ab} = \bar{W}_{ab} \,. \tag{39}$$

The KK energy density produces a scalar mode $\mathrm{D}_a \rho^*$ (which is present even if $\rho^* = 0$ in the background). The KK momentum density carries scalar and vector modes, and the KK anisotropic stress carries scalar, vector and tensor modes:

$$q^*_a = \mathrm{D}_a q^* + \bar{q}^*_a \,, \tag{40}$$

$$\pi^*_{ab} = \mathrm{D}_{\langle a} \mathrm{D}_{b \rangle} \pi^* + \mathrm{D}_{\langle a} \bar{\pi}^*_{b \rangle} + \bar{\pi}^*_{ab} \,. \tag{41}$$

Density perturbations

We define matter density and expansion (velocity) perturbation scalars, as in general relativity,

$$\Delta = \frac{a^2}{\rho} \mathrm{D}^2 \rho \,, \; Z = a^2 \mathrm{D}^2 \Theta \,, \tag{42}$$

and then define dimensionless KK scalars [11],

$$U = \frac{a^2}{\rho} \mathrm{D}^2 \rho^* \,, \; Q = \frac{a}{\rho} \mathrm{D}^2 q^* \,, \; \Pi = \frac{1}{\rho} \mathrm{D}^2 \pi^* \,. \tag{43}$$

A non-adiabatic isocurvature mode is associated with the KK fluctuations. We can see this from the 1+3-covariant expression which determines the non-adiabatic perturbations in the matter plus KK "fluid" [19]:

$$\dot{\rho}^{\,\mathrm{eff}}\, \mathrm{D}^2 p^{\mathrm{eff}} - \dot{p}^{\,\mathrm{eff}}\, \mathrm{D}^2 \rho^{\mathrm{eff}} = \frac{H\rho}{9a^2} \left[c_\mathrm{s}^2 - \frac{1}{3} + \left(\frac{2}{3} + w + c_\mathrm{s}^2 \right) \frac{\rho}{\lambda} \right] (3\rho U - 4\rho^* \Delta) \,, \tag{44}$$

where $c_\mathrm{s}^2 = \dot{p}/\dot{\rho}$, and the matter perturbations are assumed adiabatic. This mode is in general present, in particular when $\rho^* = 0$ in the background. However the mode will in general decay and be suppressed at low energies.

This can be related to the curvature perturbation $\mathcal{R}$ on uniform density surfaces, which is defined in the metric-based perturbation theory. The associated gauge-invariant quantity

$$\xi = \mathcal{R} + \frac{\delta\rho}{3(\rho+p)}\,, \tag{45}$$

may be defined for matter on the brane ($\xi^{\rm m}$), for the KK "fluid" (ξ^*) if $\rho^* \neq 0$, and for the total, effective fluid ($\xi^{\rm eff}$). If $\rho^* \neq 0$ in the background,

$$\xi^{\rm eff} = \xi^{\rm m} + \left[\frac{4\rho^*}{3(\rho+p)(1+\rho/\lambda)+4\rho^*}\right](\xi^* - \xi^{\rm m})\,. \tag{46}$$

In the case where $\rho^* = 0$ in the background, the evolution of the total curvature perturbation on large scales is [20]:

$$\dot{\xi}^{\,\rm eff} = \dot{\xi}^{\,\rm m} + H\left[c_{\rm s}^2 - \frac{1}{3} + \left(\frac{\rho+p}{\rho+\lambda}\right)\right]\frac{\delta\rho^*}{(\rho+p)(1+\rho/\lambda)}\,. \tag{47}$$

For adiabatic matter perturbations, $\dot{\xi}^{\,\rm m} = 0$, independent of brane-world modifications to the field equations, since this result depends on energy conservation only [27]. However, $\dot{\xi}^{\,\rm eff} \neq 0$ even for adiabatic matter perturbations. The KK effects on the brane contribute a non-adiabatic mode, although $\dot{\xi}^{\,\rm eff} \to 0$ at low energies.

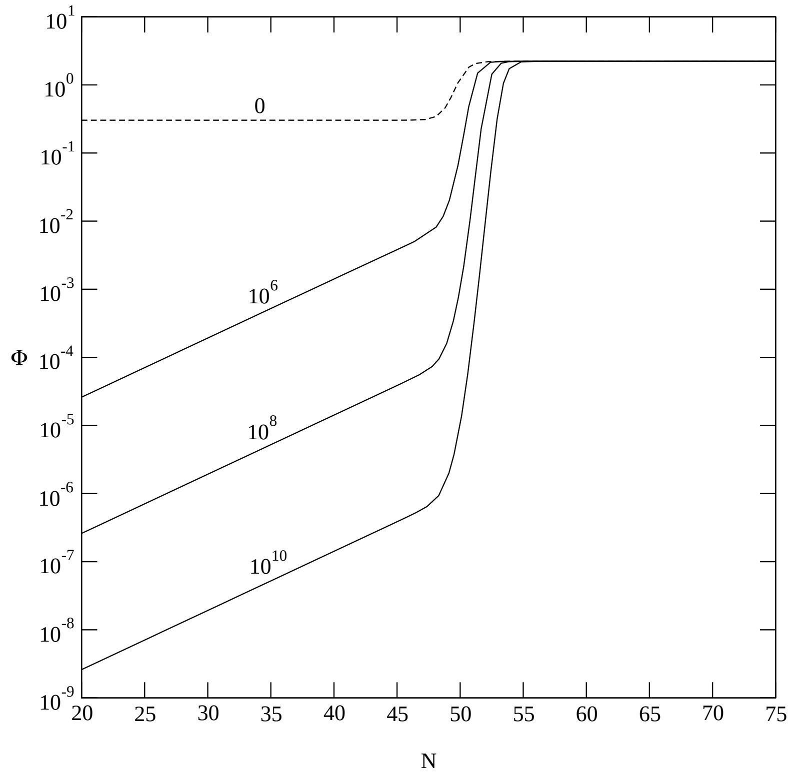

FIG. 1. The evolution of Φ along a fundamental world-line for a mode that is well beyond the Hubble horizon at $N \equiv \ln(a/a_0) = 0$, about 50 e-folds before inflation ends, and remains super-Hubble through the radiation era. A smooth transition from inflation to radiation is modelled by $w = \frac{1}{3}[(2-\alpha)\tanh(N-50)-(1-\alpha)]$, where α is a small positive parameter (chosen as $\alpha = 0.1$ in the plot). Labels on the curves indicate the value of ρ_0/λ, so that the general relativistic solution is the dashed curve ($\rho_0/\lambda = 0$). For $\rho_0/\lambda \gg 1$, inflation ends at $N = 50 - 2\ln[(1-2\alpha)/3] \approx 47.4$, and at $N = 50$ in general relativity. Only the lowest curve still has $\rho/\lambda \gg 1$ at the start of radiation-domination (N greater than about 53), and one can see that Φ is still growing.

The covariant density perturbation equations on the brane reduce to [19]

$$\dot{\Delta} = 3wH\Delta - (1+w)Z\,, \tag{48}$$

$$\dot{Z} = -2HZ - \left(\frac{c_{\rm s}^2}{1+w}\right){\rm D}^2\Delta - \kappa^2\rho U - \frac{1}{2}\kappa^2\rho\left[1 + (4+3w)\frac{\rho}{\lambda} - \left(\frac{4c_{\rm s}^2}{1+w}\right)\frac{\rho^*}{\rho}\right]\Delta\,, \tag{49}$$

$$\dot{U} = (3w-1)HU + \left(\frac{4c_{\rm s}^2}{1+w}\right)\left(\frac{\rho^*}{\rho}\right)H\Delta - \left(\frac{4\rho^*}{3\rho}\right)Z - a{\rm D}^2Q\,, \tag{50}$$

$$\dot{Q} = (3-1w)HQ - \frac{1}{3a}U - \frac{2}{3}a{\rm D}^2\Pi + \frac{1}{3a}\left[\left(\frac{4c_{\rm s}^2}{1+w}\right)\frac{\rho^*}{\rho} - 3(1+w)\frac{\rho}{\lambda}\right]\Delta\,. \tag{51}$$

It follows that the system closes on super-Hubble scales, since the KK anisotropic stress term Π occurs only via its Laplacian [11]. KK effects introduce two new isocurvature modes on large scales (associated with U and Q), as well as modifying the evolution of the adiabatic modes [19,23]. A simple illustration of this modification is shown in Fig. 1 (from [19]). The variable

$$\Phi = \kappa^2 a^2 \rho\Delta \tag{52}$$

is a covariant analogue of the Bardeen metric perturbation variable Φ_H. The large-scale behaviour of Φ through the inflationary and radiation eras is compared for different values of the inflationary energy scale relative to the brane tension. In general relativity, Φ is constant in both eras, whereas Φ grows during the high-energy regime on the brane.

Although the density perturbations can be found on super-Hubble scales, the Sachs-Wolfe effect requires Π in order to translate from density to metric perturbations. In the longitudinal gauge of the metric perturbation formalism, the gauge-invariant metric perturbations at last scattering are related by

$$\Phi_A - \Phi_H = -\kappa^2 a^2 \delta\pi^*\,, \tag{53}$$

where the radiation anisotropic stress on large scales is neglected, as in general relativity, and $\delta\pi^*$ is equivalent to the covariant quantity Π. In general relativity, the right hand side is zero. The brane-world corrections to the general relativistic direct Sachs-Wolfe effect are given by [20]

$$\frac{\delta T}{T} = \left(\frac{\delta T}{T}\right)_{\rm gr} - \frac{8}{3}\left(\frac{\rho_{\rm rad}}{\rho_{\rm cdm}}\right)S^* - \kappa^2 a^2 \delta\pi^* + \frac{2\kappa^2}{a^{5/2}}\int da\; a^{7/2}\,\delta\pi^*\,, \tag{54}$$

where S^* is the KK entropy perturbation (determined by $\delta\rho^*$). The KK term $\delta\pi^*$ cannot be determined by the 4D brane equations, so that $\delta T/T$ cannot be evaluated on large scales without solving the 5D equations.

A simple approximation to $\delta\pi^*$ on large scales is discussed in [22] and the Sachs-Wolfe effect is estimated as

$$\frac{\delta T}{T} \sim \left(\frac{\delta\pi^*}{\rho}\right)_{\rm in}\left(\frac{t_{\rm eq}}{t_{\rm ls}}\right)^{2/3}\left[\frac{\ln(t_{\rm in}/t_4)}{\ln(t_{\rm eq}/t_4)}\right], \tag{55}$$

where t_4 is the 4D Planck time and $t_{\rm in}$ is the time when the KK anisotropic stress is induced on the brane, which is expected to be of the order of the 5D Planck time.

Vector perturbations

The vorticity propagation equation on the brane is the same as in general relativity,

$$\dot{\omega}_a + 2H\omega_a = -\frac{1}{2}{\rm curl}\,A_a\,. \tag{56}$$

Taking the curl of the conservation equation (24) (for the case of a perfect fluid, $q_a = 0 = \pi_{ab}$), and using the identity in Eq. (38), one obtains

$${\rm curl}\,A_a = -6Hc_{\rm s}^2\omega_a\,, \tag{57}$$

as in general relativity, so that Eq. (56) becomes

$$\dot{\omega}_a + [2-3c_{\rm s}^2]H\omega_a = 0\,, \tag{58}$$

which expresses the conservation of angular momentum. In general relativity, vector perturbations vanish when the vorticity is zero. By contrast, in brane-world cosmology, bulk effects can source vector perturbations even in the absence of vorticity [6]. This can be seen via the divergence equation for the magnetic part H_{ab} of the 4D Weyl tensor on the brane:

$$\mathrm{D}^2\bar{H}_a = 2\kappa^2(\rho+p)\left[1+\frac{\rho}{\lambda}\right]\omega_a + \frac{4}{3}\kappa^2\rho^*\omega_a - \frac{1}{2}\kappa^2\mathrm{curl}\,\bar{q}^*_a\,, \tag{59}$$

where $H_{ab} = \mathrm{D}_{\langle a}\bar{H}_{b\rangle}$. Even when $\omega_a = 0$, there is a source for gravimagnetic terms on the brane from the KK quantity $\mathrm{curl}\,\bar{q}^*_a$.

We define covariant dimensionless vector perturbation quantities for the vorticity and the KK gravimagnetic term:

$$\bar{\alpha}_a = a\,\omega_a\,,\ \bar{\beta}_a = \frac{a}{\rho}\mathrm{curl}\,\bar{q}^*_a\,. \tag{60}$$

On large scales, we can find a closed system for these vector perturbations on the brane [6]:

$$\dot{\bar{\alpha}}_a + \left(1-3c_{\rm s}^2\right)H\bar{\alpha}_a = 0\,, \tag{61}$$

$$\dot{\bar{\beta}}_a + (1-3w)H\bar{\beta}_a = \frac{2}{3}H\left[4\left(3c_{\rm s}^2-1\right)\frac{\rho^*}{\rho} - 9(1+w)^2\frac{\rho}{\lambda}\right]\bar{\alpha}_a\,. \tag{62}$$

Thus we can solve for $\bar{\alpha}_a$ and $\bar{\beta}_a$ on super-Hubble scales, as for density perturbations. Vorticity in the brane matter is a source for the KK vector perturbation $\bar{\beta}_a$ on large scales. Vorticity decays unless the matter is ultra-relativistic or stiffer ($w \geq \frac{1}{3}$), and this source term typically provides a decaying mode. There is another pure KK mode, independent of vorticity, but this mode decays like vorticity. For $w \equiv p/\rho = \mathrm{const}$, the solutions are

$$\bar{\alpha}_a = b_a\left(\frac{a}{a_0}\right)^{3w-1}\,, \tag{63}$$

$$\bar{\beta}_a = c_a\left(\frac{a}{a_0}\right)^{3w-1} + b_a\left[\epsilon_w\frac{8\rho_0^*}{3\rho_0}\left(\frac{a}{a_0}\right)^{2(3w-1)} + 2(1+w)\frac{\rho_0}{\lambda}\left(\frac{a}{a_0}\right)^{-4}\right]\,, \tag{64}$$

where $\dot{b}_a = 0 = \dot{c}_a$ and $\epsilon_w = (1,0)$ for $(w \neq \frac{1}{3}, w = \frac{1}{3})$.

Inflation will redshift away the vorticity and the KK mode, which is consistent with the analysis in [18] of vector perturbations generated during inflation.

Tensor perturbations

The covariant description of tensor modes on the brane is via the shear, which satisfies the wave equation [6]

$$\mathrm{D}^2\bar{\sigma}_{ab} - \ddot{\bar{\sigma}}_{ab} - 5H\dot{\bar{\sigma}}_{ab} - \left[2\Lambda + \frac{1}{2}\kappa^2\left\{\rho - 3p - (\rho+3p)\frac{\rho}{\lambda}\right\}\right]\bar{\sigma}_{ab}$$
$$= -\kappa^2\left(\dot{\bar{\pi}}^*_{ab} + 2H\bar{\pi}^*_{ab}\right)\,. \tag{65}$$

Unlike the density and vector perturbations, there is no closed system on the brane for large scales. The KK anisotropic stress $\bar{\pi}^*_{ab}$ is an unavoidable source for tensor modes on the brane. These modes and their effect on the CMB are discussed in the following section.

III. CMB ANISOTROPIES IN THE BRANE-WORLD

The perturbation equations in the previous section form the basis for an analysis of scalar and tensor CMB anisotropies in the brane-world. The full system of equations on the brane, including the Boltzmann equation for photons, has been given for scalar [23] and tensor [24] perturbations. But the systems are not closed, as discussed above, because of the presence of the KK anisotropic stress π^*_{ab}, which acts a source term. For example, in the tight-coupling radiation era, the scalar perturbation equations may be decoupled to give an equation for the gravitational potential Φ, defined by

$$E_{ab} = \mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\Phi\,. \tag{66}$$

In general relativity, this equation in Φ has no source term, but in the brane-world there is a source term made up of π^*_{ab} and its time-derivatives. At low energies ($\rho \ll \lambda$), and for a flat background ($K = 0$), the equation is [23]

$$3x\Phi_k'' + 12\Phi_k' + x\Phi_k$$
$$= \frac{\text{const}}{\lambda}\left[\pi_k^{*\prime\prime} - \frac{1}{x}\,\pi_k^{*\prime} + \left(\frac{2}{x^3} - \frac{3}{x^2} + \frac{1}{x}\right)\pi_k^*\right], \tag{67}$$

where $x = k/(aH)$, a prime denotes d/dx, and Φ_k, π_k^* are the Fourier modes of Φ and π_{ab}^*. In general relativity the right hand side is zero, so that the equation may be solved for Φ_k, and then for the remaining perturbative variables, which gives the basis for initializing CMB numerical integrations. At high energies, earlier in the radiation era, the decoupled equation is fourth order [23]:

$$729x^2\Phi_k'''' + 3888x\Phi_k''' + (1782 + 54x^2)\Phi_k'' + 144x\Phi_k' + (90 + x^2)\Phi_k = \text{const}\left[243\left(\frac{\pi_k^*}{\rho}\right)'''' + \right.$$
$$\left. -\frac{810}{x}\left(\frac{\pi_k^*}{\rho}\right)''' + \frac{18(135 + 2x^2)}{x^2}\left(\frac{\pi_k^*}{\rho}\right)'' - \frac{30(162 + x^2)}{x^3}\left(\frac{\pi_k^*}{\rho}\right)' + \frac{x^4 + 30(162 + x^2)}{x^4}\left(\frac{\pi_k^*}{\rho}\right)\right]. \tag{68}$$

The formalism and machinery are ready to compute the temperature and polarization anisotropies in brane-world cosmology, once a solution, or at least an approximation, is given for π_{ab}^*. The resulting power spectra will reveal the nature of the brane-world imprint on CMB anisotropies, and would in principle provide a means of constraining or possibly falsifying the brane-world models. Once this is achieved, the implications for the fundamental underlying theory, i.e. M theory, would need to be explored.

However, the first step required is the solution or estimate of π_{ab}^*. This solution will be of the form, expressed in Fourier modes (and assuming no incoming 5D gravitational waves):

$$\pi_k^*(t) \propto \int dt'\ \mathcal{G}(t,t')F_k(t')\,, \tag{69}$$

where $\mathcal{G}$ is a retarded Green's function evaluated on the brane. The functional F_k will be determined by the covariant brane perturbation quantities and their derivatives. It is known in the case of a Minkowski background [28], but not in the cosmological case. Once $\mathcal{G}$ and F_k are determined or estimated, the numerical integration in Eq. (69) can in principle be incorporated into a modified version of a CMB numerical code.

In order to make some progress towards understanding brane-world signatures on CMB anisotropies, we need to consider approximations to the solution. The nonlocal nature of π_k^*, as reflected in Eq. (69), is fundamental, but is also the source of the great complexity of the problem. The lowest level approximation to π_k^* is local. Despite removing the key aspect of the KK anisotropic stress, we can get a feel for its influence on the CMB if we capture at least part of its qualitative properties. The key qualitative feature is that inhomogeneity and anisotropy on the brane are a source for KK modes in the bulk which "backreact" or "feed back" onto the brane.

The simplest case is that of tensor perturbations. The transverse traceless part of inhomogeneity and anisotropy on the brane is given by the transverse traceless anisotropic stresses in the geometry, i.e. by the photon anisotropic stress $\bar\pi_{ab}$ and the shear anisotropy $\bar\sigma_{ab}$. The photon anisotropic stress in turn is sourced by the shear to lowest order (neglecting the role of the octupole and higher Boltzmann moments), so that $F_k \approx F[\bar\sigma_k]$. The simplest local approximation which reflects the essential qualitative feature of the spin-2 KK modes is [24]

$$\kappa^2\bar\pi_{ab}^* = -\zeta H\bar\sigma_{ab}\,, \tag{70}$$

where ζ is a dimensionless KK parameter, which is assumed to be a comoving constant in a first approximation,

$$\dot\zeta = 0\,. \tag{71}$$

The limit $\zeta = 0$ corresponds to no KK effects on the brane, and $\zeta = 0 = \lambda^{-1}$ gives the general relativity limit.

For tensor perturbations, there is no freedom over the choice of frame (i.e. u^a), and thus there is no gauge ambiguity in Eq. (70). However, for scalar (or vector) perturbations, this relation could only hold in one frame, since π_{ab}^* is frame-invariant in linear theory while σ_{ab} is not:

$$u^a \to u^a + v^a \quad \Rightarrow \quad \pi_{ab}^* \to \pi_{ab}^*\,,\ \sigma_{ab} \to \sigma_{ab} + \mathrm{D}_{\langle a}v_{b\rangle}\,. \tag{72}$$

Thus for scalar perturbations, we would need an alternative, frame-invariant, local approximation.

The approximation in Eq. (70) has the qualitative form of a shear viscosity, which suggests that KK effects lead to a damping of tensor anisotropies. This is indeed consistent with the conversion of part of the zero-mode at Hubble re-entry into massive KK modes [16,17]. The conversion may be understood equivalently as the emission of KK gravitons into the bulk. This leads to a loss of energy in the 4D graviton modes on the brane, i.e. to an effective damping. The approximation in Eq. (70), although local, therefore also incorporates this key feature qualitatively.

The 1+3-covariant transverse traceless quantities are the electric (E_{ab}) and magnetic parts of the brane Weyl tensor, the shear, and the anisotropic stresses. They are expanded in electric ($Q^{(k)}_{ab}$) and magnetic ($\hat{Q}^{(k)}_{ab}$) parity tensor harmonics [29], with dimensionless coefficients:

$$\bar{E}_{ab} = \sum_k \left(\frac{k}{a}\right)^2 \left[E_k Q^{(k)}_{ab} + \hat{E}_k \hat{Q}^{(k)}_{ab}\right], \tag{73}$$

$$\bar{H}_{ab} = \sum_k \left(\frac{k}{a}\right)^2 \left[H_k Q^{(k)}_{ab} + \hat{H}_k \hat{Q}^{(k)}_{ab}\right], \tag{74}$$

$$\bar{\sigma}_{ab} = \sum_k \frac{k}{a} \left[\sigma_k Q^{(k)}_{ab} + \hat{\sigma}_k \hat{Q}^{(k)}_{ab}\right], \tag{75}$$

$$\bar{\pi}_{ab} = \rho \sum_k \left[\pi_k Q^{(k)}_{ab} + \hat{\pi}_k \hat{Q}^{(k)}_{ab}\right], \tag{76}$$

$$\bar{\pi}^*_{ab} = \rho \sum_k \left[\pi^*_k Q^{(k)}_{ab} + \hat{\pi}^*_k \hat{Q}^{(k)}_{ab}\right]. \tag{77}$$

Using $\bar{H}_{ab} = \mathrm{curl}\,\bar{\sigma}_{ab}$, we arrive at the coupled equations [24]

$$\frac{k}{a^2}\left(\sigma'_k + \mathcal{H}\sigma_k\right) + \frac{k^2}{a^2}E_k - \frac{\kappa^2}{2}\rho\pi_k = -\kappa^2(1+3w)\frac{\rho^2}{4\lambda}\pi_k + \frac{\kappa^2}{2}\rho\pi^*_k, \tag{78}$$

$$\begin{aligned} &\frac{k^2}{a^2}\left(E'_k + \mathcal{H}E_k\right) - k\left[\frac{k^2}{a^2} + \frac{3K}{a^2} - \frac{\kappa^2}{2}(1+w)\rho\right]\sigma_k + \frac{\kappa^2}{2}\rho\pi'_k - \frac{\kappa^2}{2}(3w+2)\mathcal{H}\rho\pi_k \\ &= -\frac{\kappa^2}{12\lambda}\left[6k(1+w)\rho^2\sigma_k - 3(\rho'+3p')\rho\pi_k - 3(3w+1)\rho\left(\rho\pi'_k + \rho'\pi_k\right) - 9(3w+1)\rho^2\mathcal{H}\pi_k\right] \\ &\quad - \frac{2}{3}k\kappa^2\rho^*\sigma_k - \frac{\kappa^2}{2}\left[\rho\pi^{*\prime}_k + \left(\rho' + \mathcal{H}\rho\right)\pi^*_k\right], \end{aligned} \tag{79}$$

where τ is conformal time, a prime denotes $d/d\tau$, $\mathcal{H} = a'/a$, and the equation-of-state parameter w is not assumed constant. Equations (78) and (79), with all brane-world terms on the right-hand sides, determine the tensor anisotropies in the CMB, once π_k and π^*_k are given. The former is determined by the Boltzmann equation in the usual way [29], since high-energy corrections are negligible at and after nucleosynthesis. The latter is given by the approximation Eq. (70), which gives

$$\kappa^2\rho\pi^*_k = -\zeta\mathcal{H}\frac{k}{a^2}\sigma_k. \tag{80}$$

We will also assume $K = 0 = \rho^*$ in the background. The KK parameter ζ (together with the brane tension λ) then controls brane-world effects on the tensor CMB anisotropies in this simple local approximation.

Note that the 4D metric perturbation variable, H_T, which characterizes the amplitude of 4D gravitational waves, is related in flat models to the covariant variables by

$$H_{Tk} = \frac{\sigma'_k}{k} + 2E_k. \tag{81}$$

If the photon anisotropic stress π_k can be neglected, Eq. (78) implies

$$kH_{Tk} = -\sigma'_k - (\zeta+2)\mathcal{H}\sigma_k. \tag{82}$$

IV. CMB TENSOR POWER SPECTRA

In the tight coupling regime we can neglect the photon anisotropic stress (i.e. $\pi_k = 0$), and the variable

$$u_k \equiv a^{1+\zeta/2}\sigma_k \tag{83}$$

satisfies the equation of motion

$$u_k'' + \left[k^2 + 2K - \frac{(a^{-1-\zeta/2})''}{a^{-1-\zeta/2}}\right] u_k = 0\,, \tag{84}$$

by Eq. (80). In flat models ($K = 0$) on large scales, the solution is

$$\sigma_k = A_k a^{-(2+\zeta)} + B_k a^{-(2+\zeta)} \int d\tau\, a(\tau)^{2+\zeta}\,, \tag{85}$$

where A_k (decaying mode) and B_k are constants of integration. If we let $\zeta \to 0$, we recover the general relativity solution.

We can solve Eq. (84) on all scales in the high-energy ($\rho \gg \lambda$ and $a \propto \tau^{1/3}$) and low-energy ($\rho \ll \lambda$ and $a \propto \tau$) radiation-dominated regimes, and during matter-domination ($a \propto \tau^2$). The solutions are

$$u_k(\tau) = \sqrt{k\tau}\left[c_1 J_{(5+\zeta)/6}(k\tau) + c_2 Y_{(5+\zeta)/6}(k\tau)\right] \quad \text{(high energy radiation)}, \tag{86}$$

$$u_k(\tau) = \sqrt{k\tau}\left[c_3 J_{(3+\zeta)/2}(k\tau) + c_4 Y_{(3+\zeta)/2}(k\tau)\right] \quad \text{(low energy radiation)}, \tag{87}$$

$$u_k(\tau) = \sqrt{k\tau}\left[c_5 J_{(5+2\zeta)/2}(k\tau) + c_6 Y_{(5+2\zeta)/2}(k\tau)\right] \quad \text{(matter domination)}, \tag{88}$$

where c_i are integration constants and J_n, Y_n are Bessel functions. The solutions for the electric part of the brane Weyl tensor can be found from Eq. (78). For modes of cosmological interest the wavelength is well outside the Hubble radius at the transition from the high energy regime to the low energy. It follows that the regular solution (labelled by c_1) in the high-energy regime will only excite the regular solution (c_3) in the low-energy, radiation-dominated era. Performing a series expansion, we arrive at the appropriate initial conditions for large-scale modes in the low-energy radiation era:

$$H_{Tk} = 1 - \frac{(k\tau)^2}{2(3+\zeta)} + \frac{(k\tau)^4}{8(3+\zeta)(5+\zeta)} + O[(k\tau)^6], \tag{89}$$

$$\sigma_k = -\frac{k\tau}{3+\zeta} + \frac{k^3\tau^3}{2(3+\zeta)(5+\zeta)} + O[(k\tau)^5], \tag{90}$$

$$E_k = \frac{(4+\zeta)}{2(3+\zeta)} - \frac{(k\tau)^2(8+\zeta)}{4(3+\zeta)(5+\zeta)} + O[(k\tau)^4]. \tag{91}$$

In the limit $\zeta \to 0$, we recover the general relativity results [29].

For modes that are super-Hubble at matter-radiation equality (i.e. $k\tau_{\rm eq} \ll 1$), the above solution joins smoothly onto the regular solution labelled by c_5 in Eq. (88). For $k\tau_{\rm eq} \gg 1$, the shear during matter domination takes the form

$$\sigma_k = -2^{(3+2\zeta)/2}\Gamma[(5+2\zeta)/2](k\tau)^{-(3+2\zeta)/2} J_{(5+2\zeta)/2}(k\tau). \tag{92}$$

In the opposite limit, the wavelength is well inside the Hubble radius at matter-radiation equality. The asymptotic form of the shear in matter domination is then

$$\sigma_k \sim \frac{\Gamma[(3+\zeta)/2]}{\sqrt{\pi}} \left(\frac{2\tau_{\rm eq}}{\tau}\right)^{1+\zeta/2} (k\tau)^{-(1+\zeta/2)} \sin\left(k\tau - \frac{\pi}{4}\zeta\right). \tag{93}$$

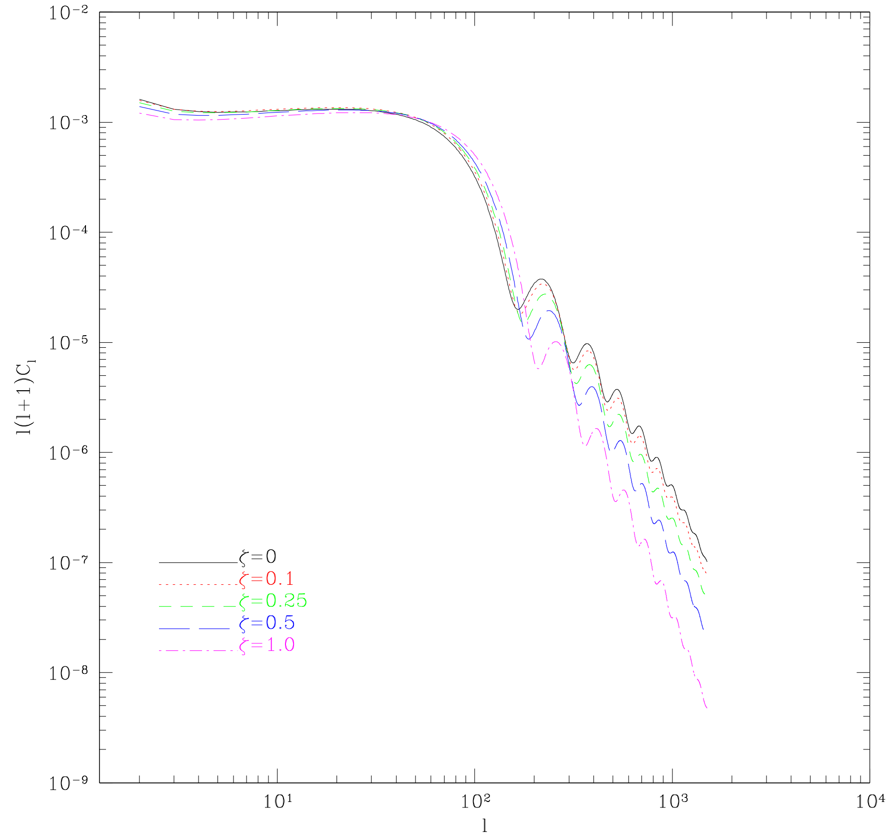

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FIG. 2. The temperature power spectrum for tensor perturbations in brane-world models, using the approximation in Eq. (70), with ζ the dimensionless KK parameter. Models are shown with $\zeta = 0.0$, 0.1, 0.25, 0.5 and 1.0. The initial tensor power spectrum is scale invariant. The background cosmology is the (concordance) spatially flat ΛCDM model with density parameters $\Omega_b = 0.035$, $\Omega_c = 0.315$, $\Omega_\Lambda = 0.65$, no massive neutrinos, and the Hubble constant $H_0 = 65\,\mathrm{km s^{-1} Mpc^{-1}}$.

The initial conditions Eqs. (89)–(91), are used in a modified version of the CAMB code [30] to obtain the tensor temperature and polarization power spectra [24]. The temperature and electric and magnetic polarization spectra are shown in Figs. 2–4 for a scale-invariant initial power spectrum. The normalization is set by the initial power in the gravity wave background. Within the local approximation to π^*_k, the power spectra are insensitive to high-energy effects: the $\zeta = 0$ curve in Fig. 2 is indistinguishable from that of the general relativity model. For the computations, the lowest value of the brane tension λ, consistent with the limit $\lambda > (100 \text{ GeV})^4$, is used, but the results are largely insensitive to the value of λ within the local approximation.

There are three notable effects arising from our approximation to the KK stress:
(1) the power on large scales reduces with increasing KK parameter ζ;
(2) features in the spectrum shift to smaller angular scales with increasing ζ;
(3) the power falls off more rapidly on small scales as ζ increases.

Neglecting scattering effects, the shear is the only source of linear tensor anisotropies [29]. For $1 \ll l < 60$ the dominant modes to contribute to the temperature C_ls are those whose wavelengths subtend an angle $\sim 1/l$ when the shear first peaks (around the time of Hubble crossing). The small suppression in the C_ls on large scales with increasing ζ arises from the reduction in the peak amplitude of the shear at Hubble entry [see Eq. (92)], qualitatively interpreted as the loss of energy in the 4D graviton modes to 5D KK modes.

Increasing ζ also has the effect of adding a small positive phase shift to the oscillations in the shear on sub-Hubble scales, as shown e.g. by Eq. (93). The delay in the time at which the shear first peaks leads to a small increase in the maximum l for which $l(l+1)C_l$ is approximately constant, as is apparent in Fig. 2. The phase shift of the subsequent peaks in the shear has the effect of shifting the peaks in the tensor C_ls to the right. For $l > 60$ the main contribution to the tensor anisotropies at a given scale is localized near last scattering and comes from modes with wavenumber $k \sim l/\tau_0$, where τ_0 is the present conformal time. On these scales the gravity waves have already entered the Hubble radius at last scattering. Such modes are undergoing adiabatic damping by the expansion and this results in the sharp decrease in the anisotropies on small scales. Increasing the KK parameter ζ effectively produces more damping and hence a sharper fall off of power. The transition to a slower fall off in the C_ls at $l \sim 200$ is due to the weaker dependence of the shear amplitude on wave-number at last scattering for modes that have entered the Hubble radius during radiation domination. The asymptotic expansion of Eq. (92) gives the shear amplitude $\propto k^{-(2+\zeta)}$ at fixed τ, whereas for modes that were sub-Hubble at matter-radiation equality Eq. (93) gives the amplitude $\propto k^{-(1+\zeta/2)}$.

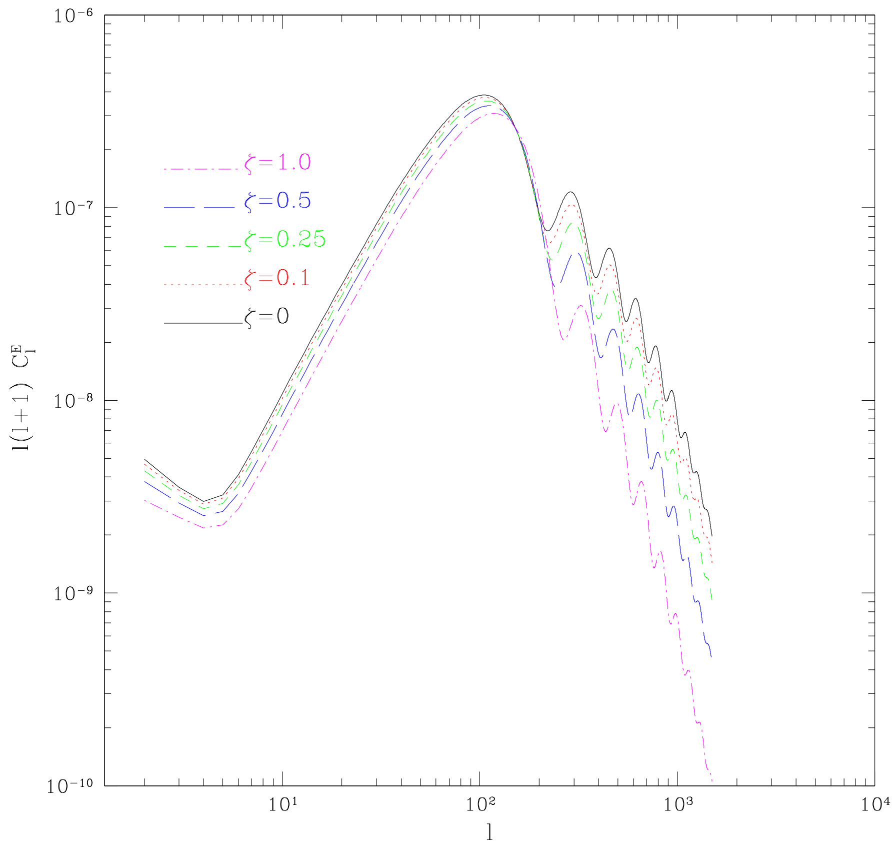

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FIG. 3. The electric polarization power spectrum for tensor perturbations for the same brane-world models as in Fig. 2.

Similar comments apply to the tensor polarization C_l^E and C_l^B, shown in Figs. 3 and 4. As with the temperature anisotropies, there is the same shifting of features to the right and increase in damping on small scales. Since polarization is only generated at last scattering (except for the feature at very low l that arises from scattering at reionization, optical depth 0.03) the large-scale polarization is suppressed, since the shear (and hence the temperature quadrupole at last scattering) is small for super-Hubble modes. In matter domination the large-scale shear is $\sigma_k = -k\tau/(5+2\zeta)$; the reduction in the magnitude of the shear with increasing KK parameter ζ is clearly visible in the large-angle polarization.

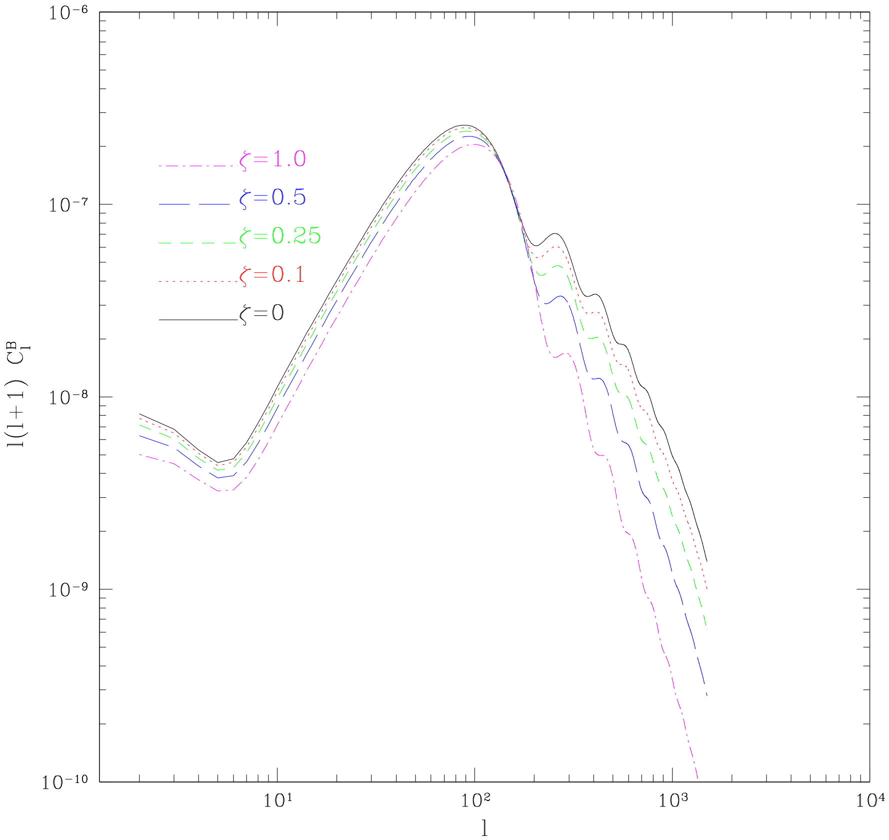

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FIG. 4. The magnetic polarization power spectrum for tensor perturbations for the same brane-world models as in Fig. 2.

V. CONCLUSION

In principle, observations can constrain the KK parameter ζ, which determines the brane-world effect on tensor anisotropies in the CMB by controlling the generation of 5D modes within a simplified local approximation, Eq. (70). The other brane-world parameter λ, the brane tension, is not constrained within this approximation. This may indicate that the approximation entails a hidden low-energy assumption, or it may simply be accidental.

The local approximation to π_k^* introduced in [24] is a first step towards the calculation of the brane-world imprint on CMB anisotropies on small scales, starting with the simplest case of tensor anisotropies. In practice, the tensor power spectra are not measured, and the prospect of useful data is still far off. What is more important is the theoretical task of improving on the simplified local approximation in Eq. (70). This approximation encodes aspects of the qualitative features of brane-world tensor anisotropies, primarily the loss of energy in 4D graviton modes via 5D graviton emission, which may be expected to survive in modified form within more realistic approximations. However, a proper understanding of brane-world effects must incorporate the nonlocal nature of the KK graviton modes, as reflected in the general form of Eq. (69).

Furthermore, it is the scalar anisotropies which dominate the measured power spectra, and it is therefore of even greater importance to develop the scalar analysis. In this case, even the first step of a local approximation has to confront the problem of the frame ambiguity in Eq. (70). An alternative frame-invariant local approximation is needed, as a first step. Nonlocal approximations, and ultimately numerical integration of the nonlocal equation for π_k^*, must be developed.

Once these more realistic nonlocal approximations are developed and the brane-world imprint on the CMB is computed, there are two key tasks:

- generalize the CMB brane-world computations to include a bulk scalar field and other extensions of the RSII-type models;

- determine the implications of the brane-world imprint for the fundamental theory, M theory, which underlies the key aspects of phenomenological brane-world cosmologies.

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