On the Stability of the Einstein Static Universe

John D. Barrow[1], George F.R. Ellis[2], Roy Maartens[2,3], Christos G. Tsagas[2]

[1] *DAMTP, Centre for Mathematical Sciences, Cambridge University, Cambridge CB3 0WA, UK*
[2] *Department of Mathematics & Applied Mathematics, University of Cape Town, Cape Town 7701, South Africa*
[3] *Institute of Cosmology & Gravitation, University of Portsmouth, Portsmouth PO1 2EG, UK*

We show using covariant techniques that the Einstein static universe containing a perfect fluid is always neutrally stable against small inhomogeneous vector and tensor perturbations and neutrally stable against adiabatic scalar density inhomogeneities so long as $c_{\rm s}^2 > \frac{1}{5}$, and unstable otherwise. We also show that the stability is not significantly changed by the presence of a self-interacting scalar field source, but we find that spatially homogeneous Bianchi type IX modes destabilise an Einstein static universe. The implications of these results for the initial state of the universe and its pre-inflationary evolution are also discussed.

I. INTRODUCTION

The possibility that the universe might have started out in an asymptotically Einstein static state has recently been considered in [1], thus reviving the Eddington-Lemaître cosmology, but this time within the inflationary universe context. It is therefore useful to investigate stability of the family of Einstein static universes. In 1930, Eddington [2] showed instability against spatially homogeneous and isotropic perturbations, and since then the Einstein static model has been widely considered to be unstable to gravitational collapse or expansion. Nevertheless, the later work of Harrison and Gibbons on the entropy and the stability of this universe reveals that the issue is not as clear-cut as Newtonian intuition suggests.

In 1967, Harrison [3] showed that all physical inhomogeneous modes are oscillatory in a radiation-filled Einstein static model. Later, Gibbons [4] showed stability of a fluid-filled Einstein static model against *conformal* metric perturbations, provided that the sound speed satisfies $c_{\rm s}^2 \equiv dp/d\rho > \frac{1}{5}$.

The compactness of the Einstein static universe, with an associated maximum wavelength, is at the root of this "non-Newtonian" stability: the Jean's length is a significant fraction of the maximum scale, and the maximum scale itself is greater than the largest physical wavelength. Here we generalize Gibbons' results to include general scalar, vector, and tensor perturbations and a self-interacting scalar field. We also show that the Einstein static model is unstable to spatially homogeneous gravitational wave perturbations within the Bianchi type IX class of spatially homogeneous universes.

Consider a Friedmann universe containing a scalar field ϕ with energy density and pressure given by

$$\rho_\phi = \frac{1}{2}\dot\phi^2 + V(\phi),\ \ p_\phi = \frac{1}{2}\dot\phi^2 - V(\phi),$$

and a perfect fluid with energy density ρ and pressure $p = w\rho$, where $-\frac{1}{3} < w \le 1$. The cosmological constant Λ is absorbed into the potential V. The fluid has a barotropic equation of state $p = p(\rho)$, with sound speed given by $c_{\rm s}^2 = dp/d\rho$. The total equation of state is $w_{\rm t} = p_{\rm t}/\rho_{\rm t} = (p+p_\phi)/(\rho+\rho_\phi)$.

Assuming no interactions between fluid and field, they separately obey the energy conservation and Klein-Gordon equations,

$$\dot\rho + 3(1+w)H\rho = 0, \tag{1}$$

$$\ddot\phi + 3H\dot\phi + V'(\phi) = 0. \tag{2}$$

The Raychaudhuri field equation

$$\frac{\ddot a}{a} = -\frac{8\pi G}{3}\left[\frac{1}{2}(1+3w)\rho + \dot\phi^2 - V(\phi)\right], \tag{3}$$

has the Friedmann equation as a first integral,

$$H^2 = \frac{8\pi G}{3}\left[\rho + \frac{1}{2}\dot\phi^2 + V(\phi)\right] - \frac{K}{a^2}, \tag{4}$$

where $K = 0, \pm 1$, and together they imply

$$\dot H = -4\pi G\left[\dot\phi^2 + (1+w)\rho\right] + \frac{K}{a^2}. \tag{5}$$

II. THE EINSTEIN STATIC UNIVERSE

The Einstein static universe is characterized by $K = 1$, $\dot a = \ddot a = 0$. It is usually viewed as a fluid model with a cosmological constant that is given a priori as a fixed universal constant, and this is the view taken in previous stability investigations [2–4]. However, in the context of inflationary cosmology, where the Einstein static model may be seen as an initial state, we require a scalar field ϕ as well as a fluid, and Λ is not an a priori constant but determined by the vacuum energy of the scalar field. The vacuum energy is determined in turn by the potential $V(\phi)$ of the scalar field, which is given by some physical model. An initial Einstein static state of the universe arises if the field starts out in an equilibrium position:

$$V'(\phi_0) = 0\,, \quad \Lambda = 8\pi G\, V(\phi_0)\,. \tag{6}$$

We will first discuss the simple case where V is trivial, i.e., a flat potential. A general Einstein static model has $\dot{\rho} = 0 = V' = \ddot{\phi}$ by Eqs. (1)–(5), and satisfies

$$\frac{1}{2}(1+3w)\rho_0 + \dot{\phi}_0^2 = V_0, \tag{7}$$

$$(1+w)\rho_0 + \dot{\phi}_0^2 = \frac{1}{4\pi G a_0^2}. \tag{8}$$

If the field kinetic energy vanishes, that is if $\dot{\phi}_0 = 0$, then by Eq. (8), $(1+w)\rho_0 > 0$, so that there must be fluid in order to keep the universe static; but if the static universe has only a scalar field, that is if $\rho_0 = 0$, then the field must have nonzero (but constant) kinetic energy [5,1]. Qualitatively, this means that there must be effective total kinetic energy, i.e. $\rho_{\rm t} + p_{\rm t} > 0$, in order to balance the curvature energy a^{-2} in the absence of expansion or contraction, as shown by Eq. (5).

Equations (7) and (8) imply

$$w_{\rm t} = -\frac{1}{3}\,, \tag{9}$$

$$w_\phi = -\frac{1}{3}\left[\frac{V_0 + (1+3w)\rho_0/2}{V_0 - (1+3w)\rho_0/6}\right]. \tag{10}$$

It follows that $w_\phi \leq -\frac{1}{3}$ since $w > -\frac{1}{3}$ and $\rho_0 \geq 0$. In other words, the presence of matter with (non-inflationary) pressure drives the equation of state of the scalar field below $-\frac{1}{3}$. If the field has no kinetic energy, then by Eqs. (7) and (8), its equation of state is that of a cosmological constant,

$$\dot{\phi}_0 = 0 \;\Rightarrow\; w_\phi = -1\,. \tag{11}$$

If there is no fluid ($\rho_0 = 0$), then the field has maximal kinetic energy,

$$\rho_0 = 0 \;\Rightarrow\; w_\phi = -\frac{1}{3} = w_{\rm t}\,. \tag{12}$$

In this case, the field rolls at constant speed along a flat potential, $V(\phi) = V_0$. Dynamically, this pure scalar field case is equivalent to the case of $w = 1$ pure fluid with cosmological constant. This can be seen as follows: equating the radii a_0 of the pure-field and fluid-plus-Λ cases, using Eq. (8), implies $V_0 = (1+w)\rho_0$, by Eq. (7). This then leads to $w_\phi = -(3+5w)/(5+3w)$, by Eq. (10). However, we know that $w_\phi = -1$ for the fluid-plus-Λ case, and equating the two forms of w_ϕ leads to $w = 1$.

Thus there is a simple one-component type of Einstein static model, which may be realised by perfect fluid universes with cosmological constant (the pure scalar field case is equivalent to the $w = 1$ fluid model). This is the case considered by Eddington, Harrison and Gibbons. The general case, of most relevance to cosmology, involves a scalar field with nontrivial potential $V(\phi)$, so that the Einstein static model is an initial state, corresponding to an equilibrium position ($V_0' = 0$). This general case is a two-component model, since in addition to the scalar field, a fluid is necessary to provide kinetic energy and keep the initial model static.

III. INHOMOGENEOUS PERTURBATIONS OF THE FLUID MODEL

We consider first the effect of inhomogeneous density perturbations on the simple one-component fluid models. Density perturbations of a general Friedmann universe are described in the 1+3-covariant gauge-invariant approach by $\Delta = a^2 {\rm D}^2\rho/\rho$, where ${\rm D}^2$ is the covariant spatial Laplacian. The evolution of Δ is given by [6]

$$\ddot{\Delta} + \left(2 - 6w + 3c_{\rm s}^2\right) H\dot{\Delta} + \Big[12\left(w - c_{\rm s}^2\right)\frac{K}{a^2} + 4\pi G\left(3w^2 + 6c_{\rm s}^2 - 8w - 1\right)\rho + (3c_{\rm s}^2 - 5w)\Lambda\Big]\Delta - c_{\rm s}^2 {\rm D}^2\Delta - w\left({\rm D}^2 + 3\frac{K}{a^2}\right)\mathcal{E} = 0\,, \tag{13}$$

where $\mathcal{E}$ is the entropy perturbation, defined for a one-component source by

$$p\mathcal{E} = a^2 {\rm D}^2 p - \rho c_{\rm s}^2 \Delta\,. \tag{14}$$

For a simple (one-component perfect-fluid) Einstein static background model, $\mathcal{E} = 0$ and Eq. (13) reduces to

$$\ddot{\Delta}_k = 4\pi G(1+w)\left[1 + (3-k^2)c_{\rm s}^2\right]\rho_0\Delta_k\,, \tag{15}$$

where we have decomposed into Fourier modes with comoving index k (so that ${\rm D}^2 \to -k^2/a_0^2$). It follows that Δ_k is oscillating and non-growing, i.e. the Einstein static universe is stable against gravitational collapse, if and only if

$$(k^2 - 3)c_{\rm s}^2 > 1\,. \tag{16}$$

In spatially closed universes the spectrum of modes is discrete, $k^2 = n(n+2)$, where the comoving wavenumber n takes the values $n = 1, 2, 3\ldots$ in order that the harmonics be single-valued on the sphere [7,3]. Formally, $n = 0$ gives a spatially homogeneous mode ($k = 0$), corresponding to a change in the background, and the violation of the stability condition in Eq. (16) is consistent with the known result [2] that the Einstein static models collapse or expand under such perturbations. This follows from the Raychaudhuri equation (3).

For the first inhomogeneous mode ($n = 1$), the stability condition is also violated, so that the universe is unstable on the corresponding scale if there is initial data satisfying the constraint equations. However this is a gauge mode which is ruled out by the constraint equations ($G_{0i} = 8\pi G T_{0i}$). This mode reflects a freedom to change the 4-velocity of fundamental observers. (For multiple fluids with different velocities, one can in principle have isocurvature modes with $n = 1$ that are not forced to zero by the Einstein constraint equations.) The physical modes thus have $n \geq 2$. Stability of all these modes is guaranteed if

$$c_s^2 > \frac{1}{5}\,. \tag{17}$$

This condition was also found by Gibbons [4] in the restricted case of conformal metric perturbations; we have generalized the result to arbitrary adiabatic density perturbations. Harrison [3] demonstrated stability for a radiation-filled model and instability of the dust-filled model, which are included in the above result.

Thus *an Einstein static universe with a fluid that satisfies Eq. (17) (and with no scalar field), is neutrally stable against adiabatic density perturbations of the fluid for all allowed inhomogeneous modes.* The case $w = 1$ covers the pure scalar field (no accompanying fluid) models. When $c_{\rm s}^2 > 0$ the stability of the model is guaranteed by Eq. (16) for all but a finite number of modes. The universe becomes increasingly unstable as the fluid pressure drops. Clearly, a dust Einstein static universe ($c_{\rm s}^2 = 0$) is always unstable. In that case, Eq. (15) reads $\ddot{\Delta}_k = 4\pi G\rho_0\Delta_k$, implying exponential growth of Δ_k (the Jeans instability; see [3]).

The physical explanation of this rather unexpected stability lies in the Jeans length associated with the model [4]. Although there are always unstable modes (i.e., with wavelength above the Jeans scale) in a flat space, in a closed universe there is an upper limit on wavelength. It turns out that, for sufficiently large speed of sound, all physical wavelengths fall below the Jeans length. By Eq. (8), the maximum wavelength $2\pi a_0$ depends on the equation of state and is given by

$$\lambda_{\rm max} = \sqrt{\frac{\pi}{G\rho_0(1+w)}}\,. \tag{18}$$

The Jeans length is $2\pi a_0/n_{\rm j}$, where $\ddot{\Delta}_{k_{\rm j}} = 0$ and $k_{\rm j}^2 = n_{\rm j}(n_{\rm j}+2)$. Hence, by Eq. (15), we have

$$\lambda_{\rm j} = \left(\frac{c_{\rm s}}{\sqrt{4c_{\rm s}^2+1} - c_{\rm s}}\right)\lambda_{\rm max}\,. \tag{19}$$

Stability means $\lambda < \lambda_{\rm j}$, which leads to Eq. (17). For dust, $\lambda_{\rm j} = 0$ and all the modes are clearly unstable. For radiation, $\lambda_{\rm j} = 0.61\lambda_{\rm max}$, and for a stiff fluid $\lambda_{\rm j} = 0.81\lambda_{\rm max}$. In both of these cases the Jeans length comprises a considerable portion of the size of the universe, and is greater than all allowed wavelengths. This would also place important restrictions on the evolution of non-linear density inhomogeneities by shock damping or black hole formation.

We also note that the stability condition entails neutral stability; the oscillations in Δ are not damped by expansion, since the background is static. In the cosmological context, where an initial Einstein static state begins to expand under a homogeneous perturbation, the expansion will damp the inhomogeneous perturbations. However, the Einstein static will not be an attractor; instead, the attractor will be de Sitter spacetime.

Vector perturbations of a fluid are governed by the comoving dimensionless vorticity $\varpi_a = a\omega_a$, whose modes satisfy the propagation equation

$$\dot{\varpi}_k = -\left(1 - 3c_{\rm s}^2\right)H\varpi_k\,. \tag{20}$$

For a fluid Einstein static background, this reduces to

$$\dot{\varpi}_k = 0\,, \tag{21}$$

so that any initial vector perturbations remain frozen. Thus there is *neutral stability against vector perturbations for all equations of state on all scales.*

Gravitational-wave perturbations of a perfect fluid may be described in the covariant approach [8] by the comoving dimensionless transverse-traceless shear $\Sigma_{ab} = a\sigma_{ab}$, whose modes satisfy

$$\ddot{\Sigma}_k + 3H\dot{\Sigma}_k + \left[\frac{k^2}{a^2} + 2\frac{K}{a^2} - \frac{8\pi G}{3}(1+3w)\rho + \frac{2}{3}\Lambda\right]\Sigma_k = 0\,. \tag{22}$$

For the Einstein static background, this becomes

$$\ddot{\Sigma}_k + 4\pi G\rho_0(k^2+2)(1+w)\Sigma_k = 0\,, \tag{23}$$

so that there is *neutral stability against tensor perturbations for all equations of state on all scales.* However this analysis does not cover spatially homogeneous modes. It turns out that there are various unstable spatially homogeneous anisotropic modes, for example a Bianchi type IX mode which we discuss below. The associated anisotropies can die away in the case where the instability results in expansion. If this is the case, despite the extra spatially homogeneous unstable modes, the expanding inflationary universe will be an attractor.

In summary, for the simplest models (one-component perturbations), we find neutral stability on all physical inhomogeneous scales against adiabatic density perturbations if $c_{\rm s}^2 > \frac{1}{5}$, and against vector and tensor perturbations for any $c_{\rm s}^2$ and w, thus generalizing previous results.

IV. SCALAR-FIELD PERTURBATIONS

We turn now to the case of a self-interacting scalar field, i.e., where the scalar field is governed by a non-flat potential $V(\phi)$, with initial Einstein static state at $\phi = \phi_0$, i.e., $V_0' = 0 = \dot{\phi}_0$. This is the general dynamical problem: the stability analysis of the Einstein static universe as an initial equilibrium position within a physically motivated (non-flat) potential. (Some realizations of this scenario are discussed in [1].) Since we are interested only in the behaviour close to the Einstein static solution, we can treat H and V' as small in the perturbation equations. The lowest-order solution for density perturbations, $\Delta_k^{(0)}$, thus corresponds to $H = 0 = V'$, and is given by the fluid perturbation solution, Eq. (15),

$$\Delta_k^{(0)}(t) = A_k \cos\omega_0 t + B_k \sin\omega_0 t \,, \tag{24}$$

$$\omega_0^2 = 4\pi G(1+w)\rho_0 \left[(k^2-3)c_{\rm s}^2 - 1\right] \,, \tag{25}$$

with A_k and B_k constants for each mode. The next order contains contributions from the scalar field perturbations,

$$\Delta_k = \Delta_k^{(0)} + \Delta_k^{(1)} + \cdots \,, \quad \Delta_k^{(1)} = \Delta_{\phi\, k} \,, \tag{26}$$

during the time when the background is close to the Einstein static equilibrium position.

The entropy term in Eq. (13) has no intrinsic fluid contribution since we assume the fluid is adiabatic. A scalar field generically has intrinsic entropy perturbations, which follow from Eq. (14) as [9]

$$\mathcal{E}_\phi = \frac{1-c_\phi^2}{w_\phi}\,\Delta_\phi \,. \tag{27}$$

(See [10] for the corresponding expression in the metric-based perturbation formalism.) These entropy perturbations have a stabilizing effect on density perturbations of the scalar field: the entropy term in Eq. (13) cancels the preceding Laplacian term $-c_\phi^2 {\rm D}^2\Delta_\phi$, which would produce instability when $c_\phi^2 < 0$; what remains is the term $-{\rm D}^2\Delta_\phi$, which contributes to stability.

For an initial Einstein static state, the entropy contribution from the scalar field to lowest order is,

$$[w_\phi \mathcal{E}_\phi]_k^{(1)} = 2\Delta_k^{(1)} \,, \tag{28}$$

where we used $[c_\phi^2]^{(0)} = -1$. There is also a relative entropy contribution, arising from the relative velocity between the field and the fluid. We expect this contribution to be negligible.

We now expand all the dynamical quantities in Eq. (13) to incorporate the lowest order effect of the self-interacting scalar field. In particular, $a = a_0 + a_{(1)}$ and $\rho = \rho_0 + \rho_{(1)}$, where $\rho_{(1)} = \frac{1}{2}\dot\phi_{(1)}^2 + V_{(1)}$, and similarly for the pressure. Then $w \to w + w_{(1)}$, where

$$w_{(1)} = \frac{p_{(1)} - w\rho_{(1)}}{\rho} \,, \tag{29}$$

and $c_{\rm s}^2 \to c_{\rm s}^2 + c_{(1)}^2$, where $c_{(1)}^2$ is determined by the form of the potential $V(\phi)$ near ϕ_0. It also follows from the background equations that

$$H_{(1)}^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot\phi_{(1)}^2 + V_{(1)}\right] + \frac{2}{a_0^3}\, a_{(1)} \,, \tag{30}$$

$$\ddot\phi_{(1)} = -V'_{(1)} \,. \tag{31}$$

Using these results, Eq. (13) gives the evolution equation for the contribution from scalar field perturbations:

$$\ddot\Delta_k^{(1)} + \tilde\omega_0^2 \Delta_k^{(1)} = F\dot\Delta_k^{(0)} + G\Delta_k^{(0)} \,, \tag{32}$$

where

$$\tilde\omega_0^2 = 4\pi G(1+w)\rho_0\left[(k^2-3)(c_{\rm s}^2+2) - 1\right] \tag{33}$$

$$F(t) = -(2 - 6w + 3c_{\rm s}^2)H_{(1)} \,, \tag{34}$$

$$\begin{aligned} G(t) = \frac{1}{a_0^2}\Bigg\{ & 12[c_{(1)}^2 - w_{(1)}] + 24[w - c_{\rm s}^2]\frac{a_{(1)}}{a_0} \\ & + 6ww_{(1)} + 6c_{(1)}^2 - 8w_{(1)} \\ & + [3w^2 - 8w + 6c_{\rm s}^2 - 1]\frac{\rho_{(1)}}{\rho_0} \\ & + k^2\left[c_{(1)}^2 - 2c_{\rm s}^2\frac{a_{(1)}}{a_0}\right]\Bigg\} \,. \end{aligned} \tag{35}$$

The change in the frequency, $\omega_0 \to \tilde\omega_0$, shows the stabilizing effect of scalar-field entropy perturbations, which effectively increase the adiabatic sound speed term in ω_0: $c_{\rm s}^2 \to c_{\rm s}^2 + 2$. The solution of Eq. (32) is given by Green's method in the form

$$\begin{aligned} \Delta_k^{(1)} = & \frac{\cos\tilde\omega_0 t}{\tilde\omega_0}\int \Delta_k^{(0)}\left[(F\sin\tilde\omega_0 t)^{\cdot} - G\sin\tilde\omega_0 t\right] dt \\ & - \frac{\sin\tilde\omega_0 t}{\tilde\omega_0}\int \Delta_k^{(0)}\left[(F\cos\tilde\omega_0 t)^{\cdot} - G\cos\tilde\omega_0 t\right] dt \,. \end{aligned} \tag{36}$$

Since $\tilde\omega_0^2 > 0$, for $k^2 \geq 3$, we see that stability is not changed by the introduction of scalar-field perturbations. We note that if general relativity is extended to include higher-order curvature corrections to the gravitational Lagrangian, then the existence and stability conditions for the Einstein static universe are changed in interesting ways [12] which are linked to the situation of general relativity plus a pure scalar field through the conformal equivalence of the two problems [13].

V. SPATIALLY HOMOGENEOUS TENSOR PERTURBATIONS

We now investigate the stability of the Einstein static universe to spatially homogeneous gravitational-wave perturbations of the Bianchi type IX kind. The anisotropy in these spatially homogeneous perturbations, which is absent within the Friedmann family, is what allows for tensor modes.

The Einstein static universe is a particular exact solution of the Bianchi type IX, or Mixmaster, universe containing a perfect fluid and a cosmological constant. The Mixmaster is a spatially homogeneous closed (compact space sections) universe of the most general type. It contains the closed isotropic Friedmann universes as particular subcases when a fluid is present. Physically, the Mixmaster universe arises from the addition of expansion anisotropy and 3-curvature anisotropy to the Friedmann universe. It displays chaotic behaviour on approach to the initial and final singularities if $w < 1$. This is closely linked to the fact that, despite being a closed universe, its spatial 3-curvature is negative except when it is close to isotropy.

The diagonal type IX universe has three expansion scale factors $a_i(t)$, determined by the Einstein equations, which are [11]:

$$\frac{(\dot{a}_1 a_2 a_3)^{\cdot}}{a_1 a_2 a_3} = \frac{4[(a_2^2 - a_3^2)^2 - a_1^4]}{(a_1 a_2 a_3)^2} + \Lambda + 4\pi G(1-w)\rho \,, \tag{37}$$

and the two equations obtained by the cyclic interchanges $a_1 \to a_2 \to a_3 \to a_1$, together with the constraint

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} = \Lambda - 4\pi G(1+3w)\rho \,, \tag{38}$$

and the perfect fluid conservation equation (with w constant)

$$\rho(a_1 a_2 a_3)^{w+1} = \text{ const} \,. \tag{39}$$

The Einstein static model is the particular solution with all $a_i = a_0 = \text{const}$. We consider the stability of this static isotropic solution by linearizing the type IX equations about it: $a_i(t) = a_0 + \delta a_i(t)$, $\rho(t) = \rho_0 + \delta\rho(t)$, and $\delta p = w\delta\rho$.

The linearized field equations lead to

$$\ddot{\delta a}_i + \frac{12}{a_0^2}\,\delta a_i = \frac{8}{a_0^2}\,(\delta a_1 + \delta a_2 + \delta a_3) + 4\pi G(1-w)\delta\rho \,. \tag{40}$$

The conservation equation to linear order gives

$$\delta\rho = -4\left(\frac{\Lambda}{8\pi G} + \rho_0\right)\left(\frac{\delta a_1 + \delta a_2 + \delta a_3}{a_0}\right). \tag{41}$$

If we define an arithmetic-mean perturbed scale factor by

$$\delta A(t) \equiv \frac{\delta a_1 + \delta a_2 + \delta a_3}{a_0} \,, \tag{42}$$

then it obeys

$$\ddot{\delta A} = 3\delta A\left[\frac{4}{a_0^2} - (1-w)(\Lambda + 8\pi G\rho_0)\right] = \frac{3\delta A}{a_0^2}(1+3w) \,. \tag{43}$$

The first term on the right-hand side arises from the pure spatially homogeneous gravitational wave modes of Bianchi type IX (i.e., with $\delta\rho = 0 = \delta p$). The second term on the right-hand side arises from the matter perturbations. Thus we see that the perturbation grows and the Einstein static solution is unstable to spatially homogeneous pure gravitational-wave perturbations of Bianchi type IX,

$$\delta A \propto \exp\left(2t\sqrt{\frac{3}{a_0}}\right). \tag{44}$$

When the matter perturbations are included the instability remains unless $1 + 3w < 0$, which we have ruled out for a fluid. In this case, the perturbations oscillate. This condition corresponds to a violation of the strong energy condition and this ensures that a Mixmaster universe containing perfect fluid matter will expand forever and approach isotropy. Notice that the matter effects disappear at this order when $w = 1$. This is a familiar situation in anisotropic cosmologies where a $w = 1$ fluid behaves on average like a simple form of anisotropy "energy".

The instability to spatially homogeneous gravitational wave modes is not surprising. Mixmaster perturbations allow small distortions of the Einstein static solution to occur which conserve the volume but distort the shape. Some directions expand whilst others contract. Note that these $SO(3)$-invariant homogeneous anisotropy modes are different to the inhomogeneous modes considered in the perturbation analysis. Mixmaster oscillatory behaviour is not picked out by the eigenfunction expansions of perturbations of Friedmann models.

The relationship between the stability of the Einstein static to inhomogeneous and spatially homogeneous gravitational-wave perturbations can be considered in the light of the relationships between the Bianchi type modes and inhomogeneous modes. In open universes, Lukash [14] has pointed out the correspondence between Bianchi type VII anisotropy modes and the inhomogeneous gravitational wave perturbation spectrum that emerges when appropriate eignfunctions are chosen for solutions of the Helmholz equation on negatively curved spaces. These Bianchi type modes correspond to choosing complex wavenumbers [15,16]. In the case of a closed universe of Bianchi type IX a similar characterisation of the homogeneous tensor mode as arising by choice of an imaginary wave number would lead to the Mixmaster instability found above in the case of $k^2 < -2$. If such modes are admitted in the spectrum of perturbation modes for the closed geometry, then they will lead to instability. However, since superpositions of them lead to non self-adjointness of the Laplacian [17], there will be problems with quantum analogues of these modes and it is not clear that they are physically admissible in the real universe.

VI. CONCLUSIONS

There is considerable interest in the existence of preferred initial states for the universe and in the existence of stationary cosmological models. So far this interest has focussed almost entirely upon the de Sitter universe as a possible initial state, future attractor, or global stationary state for an eternal inflationary universe. Of the other two homogeneous spacetimes, the Einstein static provides an interesting candidate to explore whether it could play any role in the past evolution of our Universe. It is important to know whether it can provide a natural initial state for a past eternal universe, whether it allows the universe to evolve away from this state, and whether under any circumstances it can act as an attractor for the very early evolution of the universe. We might also ask whether it is not possible for it to provide the globally static background state for an inhomogeneous eternal universe in which local regions undergo expansion or contraction, manifesting an instability of the Einstein static universe. With these questions in mind we have investigated in detail the situations under which Einstein static universe is stable and unstable.

We have shown that the Einstein static universe is neutrally stable against inhomogeneous vector and tensor linear perturbations, and against scalar density perturbations if $c_s^2 > \frac{1}{5}$, extending earlier results of Gibbons for purely conformal density perturbations. However, we find that spatially homogeneous gravitational-wave perturbations of the most general type destabilise a static universe. We pointed out the link that can be forged between this homogeneous instability and the behaviour of the inhomogeneous gravitational wave spectrum by choosing modes with imaginary wave number. Our results show that if the universe is in a neighbourhood of the Einstein static solution, it stays in that neighbourhood, but the Einstein static is not an attractor (because the stability is neutral, with non-damped oscillations). Expansion away from the static state can be triggered by a fall in the pressure of the matter. Typically, expansion away from the static solution will lead to inflation. If inflation occurs, then perturbations about a Friedmann geometry will rapidly be driven to zero. The nonlinear effects (which will certainly be important in these models because of the initial infinite time scale envisaged) will be discussed in a further paper, as will other aspects of the spatially homogeneous anisotropic modes.

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[1] G.F.R. Ellis and R. Maartens, gr-qc/0211082.
[2] A.S. Eddington, MNRAS **90**, 668 (1930).
[3] E.R. Harrison, Rev. Mod. Phys. **39**, 862 (1967).
[4] G.W. Gibbons, Nucl. Phys. B**292**, 784 (1987); ibid., **310**, 636 (1988).
[5] A.B. Burd and J.D. Barrow, Nucl. Phys. B**308,** 929 (1988).
[6] M. Bruni, P.K.S. Dunsby, and G.F.R. Ellis, Ap. J. **395**, 34 (1992).
[7] E.M. Lifshitz, J. Phys. (USSR) **10**, 116 (1946).
[8] P.K.S. Dunsby, B.A. Bassett and G.F.R. Ellis, Class. Quantum Grav. **14**, 1215 (1997); A.D. Challinor, Class. Quantum Grav. **17**, 871 (2000); R. Maartens, C.G. Tsagas, and C. Ungarelli, Phys. Rev. D**63**, 123507 (2001).
[9] M. Bruni, G.F.R. Ellis, and P.K.S. Dunsby, Class. Quantum Grav. **9**, 921 (1992).
[10] B.A. Bassett, F. Tamburini, D. Kaiser, and R. Maartens, Nucl. Phys. B**561**, 188 (1999).
[11] L. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, Pergamon, Oxford (4th ed.) (1975).
[12] J.D. Barrow and A.C. Ottewill, J. Phys. A**16**, 2757 (1983).
[13] J.D. Barrow and S. Cotsakis, Phys. Lett. B**214**, 515 (1988).
[14] V. Lukash, Nuovo Cimento **B35**, 268 (1976).
[15] V. Lukash, see Y. Zeldovich and I.D. Novikov, *Relativistic Astrophysics*, vol.2 (U Chicago Press, Chicago, 1983), p. 306.
[16] J.D. Barrow, Canadian J. Phys. **64**, 152 (1986).
[17] U. Moschella and R. Schaeffer, Phys. Rev. D **57**, 2147 (1998).