

Two-parameter non-linear spacetime perturbations: gauge transformations and gauge invariance

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Abstract. An implicit fundamental assumption in relativistic perturbation theory is that there exists a parametric family of spacetimes that can be Taylor expanded around a background. The choice of the latter is crucial to obtain a manageable theory, so that it is sometime convenient to construct a perturbative formalism based on two (or more) parameters. The study of perturbations of rotating stars is a good example: in this case one can treat the stationary axisymmetric star using a slow rotation approximation (expansion in the angular velocity Ω), so that the background is spherical. Generic perturbations of the rotating star (say parametrized by λ) are then built on top of the axisymmetric perturbations in Ω . Clearly, any interesting physics requires non-linear perturbations, as at least terms $\lambda\Omega$ need to be considered. In this paper we analyse the gauge dependence of non-linear perturbations depending on two parameters, derive explicit higher order gauge transformation rules, and define gauge invariance. The formalism is completely general and can be used in different applications of general relativity or any other spacetime theory.

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1. Introduction

An implicit fundamental assumption in relativistic perturbation theory is that there exists a parametric family of spacetimes such that the perturbative formalism is built as a Taylor expansion of this family around a background. The perturbations are then defined as the derivative terms of this series, evaluated on this background [1]. In most cases of interest one deals with an expansion in a single parameter, which can either be a formal one, as in cosmology [2, 3, 4] or in the study of quasi-normal modes of stars and black holes [5, 6], or can have a specific physical meaning, as in the study of binary black hole mergers via the close limit approximation [7, 8], or in the study of quasi-normal mode excitation by a physical source (see [6, 9] and references therein). Typically the perturbative expansion stops at the first order, but recent interesting developments deal with second order perturbations [2, 3, 4, 10, 11].

In some physical applications it may be instead convenient to construct a perturbative formalism based on two (or more) parameters, because the choice of background is crucial in having a manageable theory. The study of perturbations of stationary axisymmetric rotating stars (see [12, 13, 14] and references therein) is a good example. In this case, an analytic stationary axisymmetric solution is not known, at least for reasonably interesting equations of state. A common procedure is to treat axisymmetric stars using the so-called slow rotation approximation, so that the background is a star with spherical symmetry [15, 16]. In this approach the first order in Ω discloses frame dragging effects, with the star actually remaining spherical; Ω^2 terms carry the effects of rotation on the fluid. This is intuitive from a Newtonian point of view, as rotational kinetic energy goes like Ω^2 . This approximation is valid for angular velocities Ω much smaller than the mass shedding limit $\Omega_K \equiv \sqrt{M/R_{star}^3}$, with typical values for neutron stars $\Omega_K \sim 10^3 Hz$. Therefore the slow rotation approximation, despite the name, can still be valid for large angular velocities. In practice, the perturbative approach up to Ω^2 is accurate for most astrophysical situations, with the exception of newly born neutron stars (see [17] and references therein).

Given that the differential operators appearing in the perturbative treatment of a problem are those defined on the background, the theory is considerably simplified when the latter is spherical. Generic time dependent perturbations of the rotating star (parametrized by a dummy parameter λ and describing oscillations) are then built on top of the stationary axisymmetric perturbations in Ω . Clearly, in this approach any interesting physics requires non-linear perturbations, as at least terms of order $\lambda\Omega$ need to be considered. A similar approach could be used to study perturbations of the slowly rotating collapse, even if in specific cases [18, 19, 20] the perturbative expansion depends by one parameter only.

Classical studies in the literature have not analysed in full the gauge dependence and gauge invariance of the non-linear perturbation theory. For example, in [18] the second order perturbations are treated in a gauge invariant fashion on top of the first order perturbation in a given specific gauge. The perturbation variables used are therefore non gauge invariant under a complete second order gauge transformation [2, 21], but only invariant under “first order transformations acting at second order” [18]. While this may be perfectly satisfactory from the point of view of obtaining physical results, one may wish to convert results in a given gauge to a different one [2, 3], to compare results obtained in two different gauges, or to construct a fully gauge invariant formalism. To this end one needs to know the gauge

transformation rules and the rules for gauge invariance, either *up to order n* [2] or *at order n only*, as in [18]. The situation is going to be more complicated in the case of two parameters, as we shall see.‡

In this paper we keep in mind the above practical examples, but we do not make any specific assumption on the background spacetime and the two-parameter family it belongs to. As in [2, 24, 25], we do not even need to assume that the background is a solution of Einstein's field equations: the formalism is completely general and can be applied to any spacetime theory. We analyse the gauge dependence of perturbations in the case when they depend on two parameters, λ and Ω , derive explicit gauge transformation rules up to fourth order, i.e. including any term $\lambda^k \Omega^{k'}$ with $k + k' \leq 4$, and define gauge invariance. This choice of keeping fixed the total perturbative order is due to the generality of our approach. In practical applications one would be guided by the physical characteristics of the problem in deciding where to truncate the perturbative expansion. For example, in the case of a rotating star one could consider first order oscillations, parametrized by λ , on top of a stationary axisymmetric background described up to Ω^2 , neglecting therefore $\lambda^2 \Omega$ terms. Or instead, one could decide that $\lambda^2 \Omega$ terms are more interesting than the $\lambda \Omega^2$ ones in certain cases.

From a practical point of view, our aim is to derive the effects of gauge transformations on tensor fields T up to order $k + k' = 4$. It is indeed reasonable to assume that in a practical example like that of rotating stars, at most one will want to consider second order oscillations $\sim \lambda^2$ on top of a slowly rotating background described up to $O(\Omega^2)$, in order to take into account large oscillations and fluid deformations due to rotation.

We will show that the coordinate form of a two-parameter gauge transformation can be represented by:

$$\begin{aligned} \tilde{x}^\mu &= x^\mu + \lambda \xi_{(1,0)}^\mu + \Omega \xi_{(0,1)}^\mu \\ &+ \frac{\lambda^2}{2} \left(\xi_{(2,0)}^\mu + \xi_{(1,0)}^\nu \xi_{(1,0),\nu}^\mu \right) + \frac{\Omega^2}{2} \left(\xi_{(0,2)}^\mu + \xi_{(0,1)}^\nu \xi_{(0,1),\nu}^\mu \right) \\ &+ \lambda \Omega \left(\xi_{(1,1)}^\mu + \epsilon_0 \xi_{(1,0)}^\nu \xi_{(0,1),\nu}^\mu + \epsilon_1 \xi_{(0,1)}^\nu \xi_{(1,0),\nu}^\mu \right) + O^3(\lambda, \Omega), \end{aligned} \quad (1)$$

where the full expression is given in Eq. (75). Here $\xi_{(1,0)}^\mu$, $\xi_{(0,1)}^\mu$, $\xi_{(2,0)}^\mu$, $\xi_{(1,1)}^\mu$, and $\xi_{(0,2)}^\mu$ are independent vector fields and (ϵ_0, ϵ_1) are any two real numbers satisfying $\epsilon_0 + \epsilon_1 = 1$. Coupling terms like the $\lambda \Omega$ in (1) are the expected new features of the two-parameter case, cf. [2, 24]. Our main results are the explicit transformation rules for the perturbations of a tensor field T and the conditions for the gauge invariance of these perturbations.

The paper is organized as follows: in Section 2 we develop the necessary mathematical tools, deriving Taylor expansion formulae for two-parameter groups of diffeomorphisms and for general two-parameter families of diffeomorphisms. In Section 3 we set up an appropriate geometrical description of the gauge dependence of perturbations in the specific case of two-parameter families of spacetimes. In Section 4 we apply the tools developed in Section 2 to the framework introduced in Section 3, in order to define gauge invariance and formulas for gauge transformations, up to

‡ The concept of perturbation theory with more than one parameter has already been introduced, for first-order perturbations, in [22], where the standard definition of spacetime perturbations [23] is extended by using a $(4+n)$ -dimensional flat space in which space-times are embedded. The main aim of these works is to re-examine the gauge invariance of the metric.

fourth order in the two-parameter perturbative expansion. Section 5 is devoted to the conclusions. We follow the notation used previously in [2, 24, 25] for the case of one parameter perturbations.

2. Taylor expansion of tensor fields

In order to consider the issues of gauge transformations and gauge invariance in two-parameter perturbation theory we need first to introduce some mathematical tools concerning the two-parameter Taylor expansion of tensor fields. Since Taylor expansions are aimed to provide the value of a quantity at some point in terms of its value, and the value of its derivatives, at another point, a Taylor expansion of tensorial quantities can only be defined through a mapping between tensors at different points of the manifold under consideration. In this section we consider the cases where such a mapping is given by a two-parameter family of diffeomorphisms of \mathcal{M} , starting from the simplest case in which such a family constitutes a group.

2.1. Two-parameter groups of diffeomorphisms

Given a differentiable manifold \mathcal{M} , a two-parameter group of diffeomorphisms ϕ of \mathcal{M} can be represented as follows

$$\begin{aligned} \phi : \mathcal{M} \times \mathbb{R}^2 &\longrightarrow \mathcal{M} \times \mathbb{R}^2 \\ (p, \lambda, \Omega) &\longmapsto (\phi_{\lambda, \Omega}(p), \lambda, \Omega). \end{aligned} \quad (2)$$

For the purpose of introducing two-parameter perturbation theory, where perturbing first with respect to the parameter λ and afterwards with respect to the parameter Ω should be equivalent to the converse operation, we will assume that $\phi_{\lambda, \Omega}$ is such that it satisfies the following property[§]

$$\phi_{\lambda_1, \Omega_1} \circ \phi_{\lambda_2, \Omega_2} = \phi_{\lambda_1 + \lambda_2, \Omega_1 + \Omega_2}, \quad \forall \lambda, \Omega \in \mathbb{R}. \quad (3)$$

By one hand, this property implies that the two-parameter group is Abelian. On the other hand, it allow us to make the following useful decomposition of $\phi_{\lambda, \Omega}$ into two one-parameter groups of diffeomorphisms (flows) that remain implicitly defined by the equalities

$$\phi_{\lambda, \Omega} = \phi_{\lambda, 0} \circ \phi_{0, \Omega} = \phi_{0, \Omega} \circ \phi_{\lambda, 0}. \quad (4)$$

The action of the flows $\phi_{\lambda, 0}$ and $\phi_{0, \Omega}$ is generated by two vector fields, η and ζ respectively, acting on the tangent space of $\mathcal{M} \times \mathbb{R}^2$. The Lie derivatives of a generic tensor T with respect to η and ζ are

$$\mathcal{L}_\eta T := \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (\phi_{\lambda, 0}^* T - T) = \left[\frac{d}{d\lambda} \phi_{\lambda, 0}^* T \right]_{\lambda=0} = \left[\frac{\partial}{\partial \lambda} \phi_{\lambda, \Omega}^* T \right]_{\lambda=\Omega=0}, \quad (5)$$

$$\mathcal{L}_\zeta T := \lim_{\Omega \rightarrow 0} \frac{1}{\Omega} (\phi_{0, \Omega}^* T - T) = \left[\frac{d}{d\Omega} \phi_{0, \Omega}^* T \right]_{\Omega=0} = \left[\frac{\partial}{\partial \Omega} \phi_{\lambda, \Omega}^* T \right]_{\lambda=\Omega=0}, \quad (6)$$

where the superscript * denotes the pull-back map associated with the corresponding diffeomorphism [2]. Because the group is Abelian the vector fields η and ζ must commute

$$[\eta, \zeta] = 0. \quad (7)$$

[§] We adopt the convention that the first label in ϕ corresponds to the flow generated by η and parametrized by λ , while the second label corresponds to the flow generated by ζ and parametrized by Ω . Therefore $\phi_{\lambda, \Omega} \neq \phi_{\Omega, \lambda}$.

The Taylor expansion of the pull-backs $\phi_{\lambda,0}^*T, \phi_{0,\Omega}^*T$ is given by (see [2])

$$\phi_{\lambda,0}^*T = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left[\frac{d^k}{d\lambda^k} \phi_{\lambda,0}^*T \right]_{\lambda=0} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathcal{L}_{\eta}^k T, \quad (8)$$

$$\phi_{0,\Omega}^*T = \sum_{k=0}^{\infty} \frac{\Omega^k}{k!} \left[\frac{d^k}{d\Omega^k} \phi_{0,\Omega}^*T \right]_{\Omega=0} = \sum_{k=0}^{\infty} \frac{\Omega^k}{k!} \mathcal{L}_{\zeta}^k T. \quad (9)$$

From this, using (4), we can derive the Taylor expansion of the two-parameter group of pull-backs $\phi_{\lambda,\Omega}^*T$:

$$\phi_{\lambda,\Omega}^*T = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \left[\frac{\partial^{k+k'}}{\partial \lambda^k \partial \Omega^{k'}} \phi_{\lambda,\Omega}^*T \right]_{\lambda=\Omega=0} = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \mathcal{L}_{\eta}^k \mathcal{L}_{\zeta}^{k'} T. \quad (10)$$

2.2. Two-parameter families of diffeomorphisms

Let us now consider the general case of a two-parameter family of diffeomorphisms Φ :

$$\begin{aligned} \Phi : \mathcal{M} \times \mathbb{R}^2 &\longrightarrow \mathcal{M} \times \mathbb{R}^2 \\ (p, \lambda, \Omega) &\longmapsto (\Phi_{\lambda,\Omega}(p), \lambda, \Omega). \end{aligned} \quad (11)$$

In this generic case the diffeomorphisms $\Phi_{\lambda,\Omega}$ do not form in general a group. In particular,

$$\Phi_{\lambda_1,\Omega_1} \circ \Phi_{\lambda_2,\Omega_2} \neq \Phi_{\lambda_1+\lambda_2,\Omega_1+\Omega_2},$$

which means that we cannot decompose $\Phi_{\lambda,\Omega}$ as in the case of a two-parameter group of diffeomorphisms. The Taylor expansion of the pull-back of $\Phi_{\lambda,\Omega}$ is formally given by

$$\Phi_{\lambda,\Omega}^*T = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \left[\frac{\partial^{k+k'}}{\partial \lambda^k \partial \Omega^{k'}} \Phi_{\lambda,\Omega}^*T \right]_{\lambda=\Omega=0}. \quad (12)$$

Since the diffeomorphisms $\Phi_{\lambda,\Omega}$ do not form a group we cannot write this expansion directly in terms of Lie derivatives as in the previous case, Eq. (10). Nevertheless, in order to study the characteristics and properties of the gauge transformations to be derived in Section 4, we would like to find an alternative way of expressing the expansion (12) in terms of suitable Lie derivatives, in a similar way as it was done in [2, 25] in the one-parameter case. To this end, new objects called *knight diffeomorphisms* were introduced in [2, 25]. Broadly speaking, they are a composition of one-parameter groups of diffeomorphisms that can reproduce the action of a given family of diffeomorphisms, in such a way that the more groups we compose the better the approximation is. When the number of composed groups tends to infinity the reproduction of the family is exact (see [2, 24, 25] for a detailed account on knight diffeomorphisms). The main aim for using these objects was to show [2, 25] that it is possible to expand a one-parameter family of diffeomorphism, at order n , using Lie derivatives with respect to a finite number n of vector fields. Knight diffeomorphisms constitute an elegant formulation of this kind of expansions. In order to apply these ideas to our case, we can think of translating the idea of knight diffeomorphisms from the one-parameter to the two-parameter case. We have studied this question and we have found that in the two-parameter case there are several ways of formulating knight diffeomorphisms, and they lead to different formal expansions of a two-parameter family of diffeomorphisms. Actually, we have found that some of these formulations

can be inconsistent in the sense that the differential operators that come out from them do not satisfy the Leibnitz rule. Then, we cannot associated to any of these operators a vector field whose Lie derivative will describe the action of that operator. Despite of these facts, we find that the main goal for introducing knight diffeomorphisms can be achieved in the two-parameter case. That is, we can still expand a two-parameter family of diffeomorphisms, at order n , using Lie derivatives with respect to a finite number of vector fields.

Then, here we are going to show how to approximate a given family of diffeomorphisms Φ , up to order $\lambda^k \Omega^{k'}$ with $k + k' = 4$, in terms of some differential operators that we will show later can be identified with Lie derivative operators with respect to some vector field. In other words, we are looking for an expansion of the following type:

$$\Phi_{\lambda, \Omega}^* T = \sum_{\substack{k, k' = 0 \\ k + k' = 4}}^4 \lambda^k \Omega^{k'} (\text{Combination of differential operators})_{k, k'} T. \quad (13)$$

We have performed this expansion term by term. For the sake of brevity we present here only the most relevant developments and results. We start by introducing the set of differential operators that we have used to build such an expansion by giving their actions on a general tensorial quantity T :

$$\mathcal{L}_{(1,0)} T := \left[\frac{\partial}{\partial \lambda} \Phi_{\lambda, \Omega}^* T \right]_{\lambda = \Omega = 0}, \quad (14)$$

$$\mathcal{L}_{(0,1)} T := \left[\frac{\partial}{\partial \Omega} \Phi_{\lambda, \Omega}^* T \right]_{\lambda = \Omega = 0}, \quad (15)$$

$$\mathcal{L}_{(2,0)} T := \left[\frac{\partial^2}{\partial \lambda^2} \Phi_{\lambda, \Omega}^* T \right]_{\lambda = \Omega = 0} - \mathcal{L}_{(1,0)}^2 T, \quad (16)$$

$$\mathcal{L}_{(1,1)} T := \left[\frac{\partial^2}{\partial \lambda \partial \Omega} \Phi_{\lambda, \Omega}^* T \right]_{\lambda = \Omega = 0} - (\epsilon_0 \mathcal{L}_{(1,0)} \mathcal{L}_{(0,1)} + \epsilon_1 \mathcal{L}_{(0,1)} \mathcal{L}_{(1,0)}) T, \quad (17)$$

$$\mathcal{L}_{(0,2)} T := \left[\frac{\partial^2}{\partial \Omega^2} \Phi_{\lambda, \Omega}^* T \right]_{\lambda = \Omega = 0} - \mathcal{L}_{(0,1)}^2 T, \quad (18)$$

$$\mathcal{L}_{(3,0)} T := \left[\frac{\partial^3}{\partial \lambda^3} \Phi_{\lambda, \Omega}^* T \right]_{\lambda = \Omega = 0} - 3 \mathcal{L}_{(1,0)} \mathcal{L}_{(2,0)} T - \mathcal{L}_{(1,0)}^3 T, \quad (19)$$

$$\begin{aligned} \mathcal{L}_{(2,1)} T := & \left[\frac{\partial^3}{\partial \lambda^2 \partial \Omega} \Phi_{\lambda, \Omega}^* T \right]_{\lambda = \Omega = 0} \\ & - 2 \mathcal{L}_{(1,0)} \mathcal{L}_{(1,1)} T - \mathcal{L}_{(0,1)} \mathcal{L}_{(2,0)} T - 2 \epsilon_2 \mathcal{L}_{(1,0)} \mathcal{L}_{(0,1)} \mathcal{L}_{(1,0)} T \\ & - (\epsilon_1 - \epsilon_2) \mathcal{L}_{(0,1)} \mathcal{L}_{(1,0)}^2 T - (\epsilon_0 - \epsilon_2) \mathcal{L}_{(1,0)}^2 \mathcal{L}_{(0,1)} T, \end{aligned} \quad (20)$$

$$\begin{aligned} \mathcal{L}_{(1,2)} T := & \left[\frac{\partial^3}{\partial \lambda \partial \Omega^2} \Phi_{\lambda, \Omega}^* T \right]_{\lambda = \Omega = 0} \\ & - 2 \mathcal{L}_{(0,1)} \mathcal{L}_{(1,1)} T - \mathcal{L}_{(1,0)} \mathcal{L}_{(0,2)} T - 2 \epsilon_3 \mathcal{L}_{(0,1)} \mathcal{L}_{(1,0)} \mathcal{L}_{(0,1)} T \\ & - (\epsilon_0 - \epsilon_3) \mathcal{L}_{(1,0)} \mathcal{L}_{(0,1)}^2 T - (\epsilon_1 - \epsilon_3) \mathcal{L}_{(0,1)}^2 \mathcal{L}_{(1,0)} T, \end{aligned} \quad (21)$$

|| The subscripts (p, q) denote the lowest order in the expansion (13) at which these operators will appear for the first time. See equation (30) below.

$$\mathcal{L}_{(0,3)}T := \left[\frac{\partial^3}{\partial\Omega^3} \Phi_{\lambda,\Omega}^* T \right]_{\lambda=\Omega=0} - 3\mathcal{L}_{(0,1)}\mathcal{L}_{(0,2)}T - \mathcal{L}_{(0,1)}^3 T, \quad (22)$$

$$\begin{aligned} \mathcal{L}_{(4,0)}T &:= \left[\frac{\partial^4}{\partial\lambda^4} \Phi_{\lambda,\Omega}^* T \right]_{\lambda=\Omega=0} \\ &- 4\mathcal{L}_{(1,0)}\mathcal{L}_{(3,0)}T - 3\mathcal{L}_{(2,0)}^2 T - 6\mathcal{L}_{(1,0)}^2\mathcal{L}_{(2,0)}T - \mathcal{L}_{(1,0)}^4 T, \end{aligned} \quad (23)$$

$$\begin{aligned} \mathcal{L}_{(3,1)}T &:= \left[\frac{\partial^4}{\partial\lambda^3\partial\Omega} \Phi_{\lambda,\Omega}^* T \right]_{\lambda=\Omega=0} \\ &- 3\mathcal{L}_{(1,0)}\mathcal{L}_{(2,1)}T - \mathcal{L}_{(0,1)}\mathcal{L}_{(3,0)}T - 3\epsilon_4\mathcal{L}_{(2,0)}\mathcal{L}_{(1,1)}T - 3\epsilon_5\mathcal{L}_{(1,1)}\mathcal{L}_{(2,0)}T \\ &- 3\mathcal{L}_{(1,0)}^2\mathcal{L}_{(1,1)}T - 3(\epsilon_0\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T + \epsilon_1\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)})\mathcal{L}_{(2,0)}T \\ &- (\epsilon_1 - \epsilon_2 - \epsilon_6)\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}^3 T - 3\epsilon_6\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}^2 T \\ &- 3(\epsilon_2 - \epsilon_6)\mathcal{L}_{(1,0)}^2\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}T - (\epsilon_0 - 2\epsilon_2 + \epsilon_6)\mathcal{L}_{(1,0)}^3\mathcal{L}_{(0,1)}T, \end{aligned} \quad (24)$$

$$\begin{aligned} \mathcal{L}_{(2,2)}T &:= \left[\frac{\partial^4}{\partial\lambda^2\partial\Omega^2} \Phi_{\lambda,\Omega}^* T \right]_{\lambda=\Omega=0} \\ &- 2\mathcal{L}_{(1,0)}\mathcal{L}_{(1,2)}T + 2\mathcal{L}_{(0,1)}\mathcal{L}_{(2,1)}T - 2\mathcal{L}_{(1,1)}^2 T \\ &- \epsilon_7\mathcal{L}_{(2,0)}\mathcal{L}_{(0,2)}T - \epsilon_8\mathcal{L}_{(0,2)}\mathcal{L}_{(2,0)}T - \mathcal{L}_{(1,0)}^2\mathcal{L}_{(0,2)}T - \mathcal{L}_{(0,1)}^2\mathcal{L}_{(2,0)}T \\ &- 4(\epsilon_0\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T + \epsilon_1\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)})\mathcal{L}_{(1,1)}T \\ &+ (\epsilon_3 + \epsilon_2 - \epsilon_1 + \epsilon_9)\mathcal{L}_{(0,1)}^2\mathcal{L}_{(1,0)}^2 T + (\epsilon_3 + \epsilon_2 - \epsilon_0 - \epsilon_9)\mathcal{L}_{(1,0)}^2\mathcal{L}_{(0,1)}^2 T \\ &- 2(\epsilon_3 + \epsilon_2 - \epsilon_0\epsilon_1 - \epsilon_9)\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T \\ &- 2(\epsilon_3 + \epsilon_2 - \epsilon_0\epsilon_1 + \epsilon_9)\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}T \\ &+ 2(\epsilon_3 - \epsilon_0\epsilon_1)\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}^2\mathcal{L}_{(1,0)}T + 2(\epsilon_2 - \epsilon_0\epsilon_1)\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}^2\mathcal{L}_{(0,1)}T, \end{aligned} \quad (25)$$

$$\begin{aligned} \mathcal{L}_{(1,3)}T &:= \left[\frac{\partial^4}{\partial\lambda\partial\Omega^3} \Phi_{\lambda,\Omega}^* T \right]_{\lambda=\Omega=0} \\ &- 3\mathcal{L}_{(0,1)}\mathcal{L}_{(1,2)}T - \mathcal{L}_{(1,0)}\mathcal{L}_{(0,3)}T - 3\epsilon_{10}\mathcal{L}_{(0,2)}\mathcal{L}_{(1,1)}T - 3\epsilon_{11}\mathcal{L}_{(1,1)}\mathcal{L}_{(0,2)}T \\ &- 3\mathcal{L}_{(0,1)}^2\mathcal{L}_{(1,1)}T - 3(\epsilon_0\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T + \epsilon_1\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)})\mathcal{L}_{(0,2)}T \\ &- (\epsilon_0 - \epsilon_3 - \epsilon_{12})\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}^3 T - 3\epsilon_{12}\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}^2 T \\ &- 3(\epsilon_3 - \epsilon_{12})\mathcal{L}_{(0,1)}^2\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T - (\epsilon_1 - 2\epsilon_3 + \epsilon_{12})\mathcal{L}_{(0,1)}^3\mathcal{L}_{(1,0)}T, \end{aligned} \quad (26)$$

$$\begin{aligned} \mathcal{L}_{(0,4)}T &:= \left[\frac{\partial^4}{\partial\Omega^4} \Phi_{\lambda,\Omega}^* T \right]_{\lambda=\Omega=0} \\ &- 4\mathcal{L}_{(0,1)}\mathcal{L}_{(0,3)}T - 3\mathcal{L}_{(0,2)}^2 T - 6\mathcal{L}_{(0,1)}^2\mathcal{L}_{(0,2)}T - \mathcal{L}_{(0,1)}^4 T, \end{aligned} \quad (27)$$

where the quantities ϵ_A ($A = 0, \dots, 12$) are real numbers which must satisfy the following conditions

$$\epsilon_0 + \epsilon_1 = 1, \quad \epsilon_4 + \epsilon_5 = 1, \quad \epsilon_7 + \epsilon_8 = 1, \quad \epsilon_{10} + \epsilon_{11} = 1. \quad (28)$$

These constants reflect the fact that, as already mentioned above, there is not a unique way of constructing an expansion of the type (13), actually the number of possibilities is infinite. In this paper we have restricted ourselves to expansions in which every single term has the form $\mathcal{L}_{(p_1,q_1)} \cdots \mathcal{L}_{(p_n,q_n)}T$ with $p_1 + q_1 \leq \cdots \leq p_n + q_n$. Then, the constants ϵ_A express the freedom that we have in constructing the expansion (13) with this criteria. This freedom and the non-uniqueness of the construction above is not a problem: the operators $\mathcal{L}_{(p,q)}$ and the corresponding vectors (see below) that we

are going to define should be thought of as a “basis” for the construction in terms of Lie derivative of each of the terms in the expansion (12). This “basis” is not unique, but the result at each order is.

On the other hand, it is straightforward to check that these operators are linear and satisfy the Leibnitz rule and hence they are derivatives. We can also check that they satisfy the rest of conditions of the theorem stated in Appendix A. Therefore, for each of them there is a vector field $\xi_{(p,q)}$ such that

$$\mathcal{L}_{\xi_{(p,q)}} T := \mathcal{L}_{(p,q)} T \quad (p, q \in \mathbb{N}). \quad (29)$$

In the particular case when Φ is a group of diffeomorphisms we recover the previous case (subsection 2.1), and $\mathcal{L}_{(p,q)} = 0$ if $p + q > 1$.

Using the differential operators we have just introduced we can express the Taylor expansion (12) of $\Phi_{\lambda,\Omega}$, up to fourth order in λ and Ω , in terms of the Lie derivatives associated with the vector fields $\xi_{(p,q)}$ (29):

$$\begin{aligned} \Phi_{\lambda,\Omega}^* T &= T + \lambda \mathcal{L}_{\xi_{(1,0)}} T + \Omega \mathcal{L}_{\xi_{(0,1)}} T \\ &+ \frac{\lambda^2}{2} \left\{ \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}_{\xi_{(1,0)}}^2 \right\} T + \frac{\Omega^2}{2} \left\{ \mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}_{\xi_{(0,1)}}^2 \right\} T \\ &+ \lambda \Omega \left\{ \mathcal{L}_{\xi_{(1,1)}} + \epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \right\} T \\ &+ \frac{\lambda^3}{6} \left\{ \mathcal{L}_{\xi_{(3,0)}} + 3 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}_{\xi_{(1,0)}}^3 \right\} T \\ &+ \frac{\lambda^2 \Omega}{2} \left\{ \mathcal{L}_{\xi_{(2,1)}} + 2 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(1,1)}} + \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(2,0)}} + 2 \epsilon_2 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \right. \\ &\left. + (\epsilon_1 - \epsilon_2) \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}^2 + (\epsilon_0 - \epsilon_2) \mathcal{L}_{\xi_{(1,0)}}^2 \mathcal{L}_{\xi_{(0,1)}} \right\} T \\ &+ \frac{\lambda \Omega^2}{2} \left\{ \mathcal{L}_{\xi_{(1,2)}} + 2 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,1)}} + \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,2)}} + 2 \epsilon_3 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \right. \\ &\left. + (\epsilon_0 - \epsilon_3) \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}}^2 + (\epsilon_1 - \epsilon_3) \mathcal{L}_{\xi_{(0,1)}}^2 \mathcal{L}_{\xi_{(1,0)}} \right\} T \\ &+ \frac{\Omega^3}{6} \left\{ \mathcal{L}_{\xi_{(0,3)}} + 3 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}_{\xi_{(0,1)}}^3 \right\} T \\ &+ \frac{\lambda^4}{24} \left\{ \mathcal{L}_{\xi_{(4,0)}} + 4 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(3,0)}} + 3 \mathcal{L}_{\xi_{(2,0)}}^2 + 6 \mathcal{L}_{\xi_{(1,0)}}^2 \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}_{\xi_{(1,0)}}^4 \right\} T \\ &+ \frac{\lambda^3 \Omega}{6} \left\{ \mathcal{L}_{\xi_{(3,1)}} + 3 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(2,1)}} + \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(3,0)}} + 3 \epsilon_4 \mathcal{L}_{\xi_{(2,0)}} \mathcal{L}_{\xi_{(1,1)}} \right. \\ &+ 3 \epsilon_5 \mathcal{L}_{\xi_{(1,1)}} \mathcal{L}_{\xi_{(2,0)}} + 3 \mathcal{L}_{\xi_{(1,0)}}^2 \mathcal{L}_{\xi_{(1,1)}} + 3 (\epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}) \mathcal{L}_{\xi_{(2,0)}} \\ &+ (\epsilon_1 - \epsilon_2 - \epsilon_6) \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}^3 + 3 \epsilon_6 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}^2 \\ &\left. + 3 (\epsilon_2 - \epsilon_6) \mathcal{L}_{\xi_{(1,0)}}^2 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} + (\epsilon_0 - 2 \epsilon_2 + \epsilon_6) \mathcal{L}_{\xi_{(1,0)}}^3 \mathcal{L}_{\xi_{(0,1)}} \right\} T \\ &+ \frac{\lambda^2 \Omega^2}{4} \left\{ \mathcal{L}_{\xi_{(2,2)}} + 2 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(1,2)}} + 2 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(2,1)}} + 2 \mathcal{L}_{\xi_{(1,1)}}^2 \right. \\ &+ \epsilon_7 \mathcal{L}_{\xi_{(2,0)}} \mathcal{L}_{\xi_{(0,2)}} + \epsilon_8 \mathcal{L}_{\xi_{(0,2)}} \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}_{\xi_{(1,0)}}^2 \mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}_{\xi_{(0,1)}}^2 \mathcal{L}_{\xi_{(2,0)}} \\ &+ 4 (\epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}) \mathcal{L}_{\xi_{(1,1)}} \\ &- (\epsilon_3 + \epsilon_2 - \epsilon_1 + \epsilon_9) \mathcal{L}_{\xi_{(0,1)}}^2 \mathcal{L}_{\xi_{(1,0)}}^2 - (\epsilon_3 + \epsilon_2 - \epsilon_0 - \epsilon_9) \mathcal{L}_{\xi_{(1,0)}}^2 \mathcal{L}_{\xi_{(0,1)}}^2 \\ &+ 2 (\epsilon_3 + \epsilon_2 - \epsilon_0 \epsilon_1 - \epsilon_9) \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \\ &\left. + 2 (\epsilon_3 + \epsilon_2 - \epsilon_0 \epsilon_1 + \epsilon_9) \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \right\} T \end{aligned}$$

$$\begin{aligned}
& -2(\epsilon_3 - \epsilon_0\epsilon_1)\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)}^2\mathcal{L}_{\xi(1,0)} - 2(\epsilon_2 - \epsilon_0\epsilon_1)\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)}^2\mathcal{L}_{\xi(0,1)} \Big\} T \\
& + \frac{\lambda\Omega^3}{6} \left\{ \mathcal{L}_{\xi(1,3)} + 3\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,2)} + \mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,3)} + 3\epsilon_{10}\mathcal{L}_{\xi(0,2)}\mathcal{L}_{\xi(1,1)} \right. \\
& + 3\epsilon_{11}\mathcal{L}_{\xi(1,1)}\mathcal{L}_{\xi(0,2)} + 3\mathcal{L}_{\xi(0,1)}^2\mathcal{L}_{\xi(1,1)} + 3(\epsilon_0\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)} + \epsilon_1\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)})\mathcal{L}_{\xi(0,2)} \\
& + (\epsilon_0 - \epsilon_3 - \epsilon_{12})\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)}^3 + 3\epsilon_{12}\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)}^2 \\
& \left. + 3(\epsilon_3 - \epsilon_{12})\mathcal{L}_{\xi(0,1)}^2\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)} + (\epsilon_1 - 2\epsilon_3 + \epsilon_{12})\mathcal{L}_{\xi(0,1)}^3\mathcal{L}_{\xi(1,0)} \right\} T \\
& + \frac{\Omega^4}{24} \left\{ \mathcal{L}_{\xi(0,4)} + 4\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(0,3)} + 3\mathcal{L}_{\xi(0,2)}^2 + 6\mathcal{L}_{\xi(0,1)}^2\mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^4 \right\} T \\
& + O^5(\lambda, \Omega). \tag{30}
\end{aligned}$$

In this expression we can see the way in which the parameters $\{\epsilon_A\}$ [real constants subject to the conditions (28)] describe the arbitrariness we have in the *reconstruction* of the Taylor expansion of a two-parameter family of diffeomorphisms in terms of Lie derivative operators.

To finish this section we will show how to recover the one-parameter case from the two-parameter case. The case when one of λ or Ω vanishes is trivial and it can be recovered from the above expressions just setting either $\lambda = 0$ or $\Omega = 0$. Let then consider the only other case, which arises when the two parameters λ and Ω are not longer independent, e.g. $\Omega = \Omega(\lambda)$. Then the specific way of recovering the single parameter (λ) case will depend on the specific function $\Omega(\lambda)$. Here we illustrate the simplest case of a linear relation $\Omega = a\lambda$ ($a \neq 0$). To arrive to the one-parameter expansion (see [2, 24, 25]), we need to study the consequences of the dependence between the two parameters. We can do that by looking at the definitions of the operators $\mathcal{L}_{(p,q)}$ in equations (14-27), and at their association with Lie derivative operators, described by equation (29). The result is a set of relations between the vector fields $\xi_{(p,q)}$ that can be summarized in the following relation:

$$\xi_{(p,q)} = \frac{1}{a^q}\xi_{(p+q,0)}. \tag{31}$$

Then, rescaling the vector fields $\xi_{(p,0)}$ in the following way: $\xi_{(p,0)} \rightarrow (p+1)^{-1}\xi_{(p,0)}$, we arrive at exactly the same expansions as in the one-parameter case [2, 24, 25]. In the case in which both parameters λ and Ω have a specific physical meaning, it will be the physics that will impose the functional dependence $\Omega(\lambda)$ in particular sub-cases.

3. Gauges in perturbation theory and the two-parameter case

Let us consider for the moment a spacetime $\{g^b, \mathcal{M}_0\}$ which we call the background, and a physical spacetime $\{g, \mathcal{M}\}$ which we attempt to describe as a perturbation of $\{g^b, \mathcal{M}_0\}$.¶ In relativistic perturbation theory we are used to write expressions of the form

$$g_{\mu\nu}(x) = g_{\mu\nu}^b(x) + \delta g_{\mu\nu}(x), \tag{32}$$

relating a perturbed tensor field such as the metric with the background value of the same field and with the perturbation. In doing this, we are implicitly assigning a correspondence between points of the perturbed and the background spacetimes [26]. Indeed through (32), which is a relation between the images of the fields in \mathbb{R}^m

¶ As manifolds \mathcal{M}_0 and \mathcal{M} are the same; for generality we assume that they are m -dimensional.

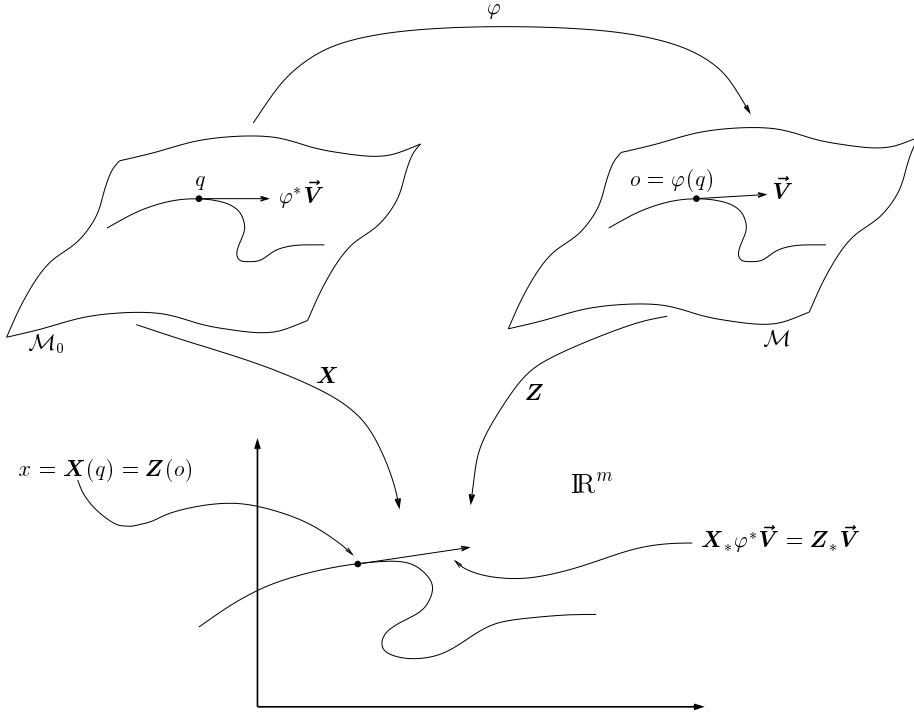


Figure 1. By choosing the coordinates on \mathcal{M}_0 and \mathcal{M} in such a way that $(Z \circ \varphi)^\mu = x^\mu$, a curve in \mathcal{M}_0 , and its φ -transformed in \mathcal{M} have the same representation in \mathbb{R}^m . Therefore, the components of the tangent vectors V and $\varphi^* V$ at the points $\varphi(p)$ and p are the same: $(\varphi^* V)^\mu(x) = (\varphi^* V)^\mu|_q = V(\mathbf{X} \circ \varphi^{-1})^\mu|_{\varphi(q)} = V z^\mu|_{\varphi(q)} = V^\mu(x)$.

rather than between the fields themselves on the respective manifolds \mathcal{M} and \mathcal{M}_0 , we are saying that there is a unique point x in \mathbb{R}^m that is at the same time the image of *two* points: one (say q) in \mathcal{M}_0 and one (o) in \mathcal{M} . This correspondence is what is usually called a gauge choice in the context of perturbation theory. Clearly, this is more than the usual assignment of coordinate labels to points of a single spacetime [27]. Furthermore, the correspondence established by relations such as (32) is not *per se* unique, but rather (32) typically defines a set of gauges, unless certain specific restrictions are satisfied by the fields involved (e.g., some metric components vanish). Leaving this problem aside, i.e. supposing that the gauge has been somehow completely fixed, let us look more precisely at the implications of (32), adopting the geometrical description illustrated in Figure 1.

If we call \mathbf{X} the chart on \mathcal{M}_0 and \mathbf{Z} the chart on \mathcal{M} we see that if we choose $z(o) = x(q)$, i.e. the correspondence between points of $\{g^b, \mathcal{M}_0\}$ and $\{g, \mathcal{M}\}$ implicit in (32), we are implicitly defining a map φ between \mathcal{M}_0 and \mathcal{M} , such that $\varphi = \mathbf{Z}^{-1} \circ \mathbf{X}$. Thus from the geometrical point of view a gauge choice is an identification of points of \mathcal{M}_0 and \mathcal{M} . Therefore, we could as well start directly assigning the point identification map φ first, calling φ itself a gauge, and defining coordinates adapted to it later.

This turns out to be a simpler way of proceeding in order to derive the gauge transformations in the following section.

Let us follow this idea in the specific case of two parameters introducing, in the spirit of [23, 2, 24, 25], an $(m+2)$ -dimensional manifold \mathcal{N} , foliated by m -dimensional submanifolds diffeomorphic to \mathcal{M} , so that $\mathcal{N} = \mathcal{M} \times \mathbb{R}^2$. We shall label each copy of \mathcal{M} by the corresponding value of the parameters λ, Ω . The manifold \mathcal{N} has a natural differentiable structure which is the direct product of those of \mathcal{M} and \mathbb{R}^2 . We can then choose charts on \mathcal{N} in which x^μ ($\mu = 0, 1, \dots, m-1$) are coordinates of each leaf $\mathcal{M}_{\lambda, \Omega}$ and $x^m = \lambda$, $x^{m+1} = \Omega$.

In this construction we are assuming that the perturbed spacetimes have the same manifold as the background one. With this we are not allowing the possibility of addressing questions like how can perturbations affect the differential structure of the background spacetime. These issues would require the use of a much more complicated mathematical apparatus, in particular the notion of limits of spacetimes introduced by Geroch [28] in order to define the background manifold as a limit $\lambda, \Omega \rightarrow 0$ of a family of *perturbed* manifolds $\mathcal{M}_{\lambda, \Omega}$. Instead, we consider perturbations as fields *living* on the background (as in [23, 2, 24, 25]), a standard approach in which these issues do not appear. More sophisticated structures of the extended manifold \mathcal{N} have been considered in [22], in an attempt to give the background metric the status of a gauge invariant quantity.

Coming back to our construction, if a tensor $T_{\lambda, \Omega}$ is given on each $\mathcal{M}_{\lambda, \Omega}$, we have that a tensor field T is automatically defined on \mathcal{N} by the relation $T(p, \lambda, \Omega) := T_{\lambda, \Omega}(p)$, with $p \in \mathcal{M}_{\lambda, \Omega}$.⁺ In particular, on each $\mathcal{M}_{\lambda, \Omega}$ one has a metric $g_{\lambda, \Omega}$ and a set of matter fields $\tau_{\lambda, \Omega}$, satisfying the set of field equations

$$\mathcal{E}[g_{\lambda, \Omega}, \tau_{\lambda, \Omega}] = 0. \quad (33)$$

Correspondingly, the fields g and τ are defined on \mathcal{N} .

We now want to define the perturbation in any tensor T , therefore we must find a way to compare $T_{\lambda, \Omega}$ with T_0 : this requires a prescription for identifying points of $\mathcal{M}_{\lambda, \Omega}$ with those of \mathcal{M}_0 . This is easily accomplished by assigning a diffeomorphism $\varphi_{\lambda, \Omega} : \mathcal{N} \rightarrow \mathcal{N}$ such that $\varphi_{\lambda, \Omega}|_{\mathcal{M}_0} : \mathcal{M}_0 \rightarrow \mathcal{M}_{\lambda, \Omega}$. Clearly, $\varphi_{\lambda, \Omega}$ can be regarded as the member of a two-parameter group of diffeomorphisms φ on \mathcal{N} , corresponding to the values of λ, Ω of the group parameter. Therefore, we could equally well give the vector fields ${}^\varphi\eta, {}^\varphi\zeta$ that generate φ . In the chart introduced above, ${}^\varphi\eta^m = 1$, ${}^\varphi\eta^{m+1} = 0$, ${}^\varphi\zeta^m = 0$, ${}^\varphi\zeta^{m+1} = 1$ but, except for these conditions, ${}^\varphi\eta, {}^\varphi\zeta$ remain arbitrary. For convenience, we shall also refer to such a pair of vector fields as a gauge. It is always possible to take the chart above defined such that ${}^\varphi\eta^\mu = {}^\varphi\zeta^\mu = 0$. So, in this chart, point of different submanifolds $\mathcal{M}_{\lambda, \Omega}$ connected by the diffeomorphism φ have the same \mathcal{M} -coordinates x^0, \dots, x^{m-1} , and differ only by the value of the coordinates λ, Ω . We call such a chart “adapted to the gauge φ ”: this is what is always used in practice.

The perturbation in T can now be defined simply as

$$\Delta_0^\varphi T_{\lambda, \Omega}^\varphi := \varphi_{\lambda, \Omega}^* T|_{\mathcal{M}_0} - T_0. \quad (34)$$

⁺ Tensor fields on \mathcal{N} constructed in this way are “tangent” to \mathcal{M} , in the sense that their components m and $m+1$ in the charts we have defined vanish identically.

The first term on the right-hand side of (34) can be Taylor-expanded using (10) to get

$$\Delta_0^\varphi T_{\lambda,\Omega}^\varphi = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \delta_\varphi^{(k,k')} T - T_0, \quad (35)$$

where

$$\delta_\varphi^{(k,k')} T := \left[\frac{\partial^{k+k'}}{\partial \lambda^k \partial \Omega^{k'}} \varphi_{\lambda,\Omega}^* T \right]_{\lambda=0, \Omega=0, \mathcal{M}_0} = \mathcal{L}_{\varphi\eta}^k \mathcal{L}_{\varphi\zeta}^{k'} T \Big|_{\mathcal{M}_0}, \quad (36)$$

which defines the perturbation of order (k, k') of T (notice that $\delta_\varphi^{(0,0)} T = T_0$). It is worth noticing $\Delta_0 T_{\lambda,\Omega}^\varphi$ and $\delta_\varphi^{(k,k')} T$ are defined on \mathcal{M}_0 ; this formalizes the statement one commonly finds in the literature that ‘‘perturbations are fields living in the background’’. It is important to appreciate that the parameters λ, Ω labelling the various spacetime models also serve to perform the expansion (35), and therefore determine what one means by ‘‘perturbations of order (k, k') ’’.

4. Gauge invariance and gauge transformations

Let us now suppose that two gauges φ and ψ , described by pairs of vector fields $(\varphi\eta, \varphi\zeta)$ and $(\psi\eta, \psi\zeta)$ respectively, are defined on \mathcal{N} , such that in the chart discussed above*

$$\begin{aligned} \varphi\eta^m &= \psi\eta^m &= 1, \\ \varphi\zeta^m &= \psi\zeta^m &= 0, \\ \varphi\eta^{m+1} &= \psi\eta^{m+1} &= 0, \\ \varphi\zeta^{m+1} &= \psi\zeta^{m+1} &= 1. \end{aligned} \quad (37)$$

Correspondingly, the integral curves of $(\varphi\eta, \varphi\zeta)$ and $(\psi\eta, \psi\zeta)$ define two two-parameter groups of diffeomorphisms φ and ψ on \mathcal{N} , that connect any two leaves of the foliation. Thus, $(\varphi\eta, \varphi\zeta)$ and $(\psi\eta, \psi\zeta)$ are everywhere transverse to $\mathcal{M}_{\lambda,\Omega}$ and points lying on the same integral surface of either of the two are to be regarded *as the same point* within the respective gauge: φ and ψ are both point identification maps, i.e. two different gauge choices.

The pairs of vector fields $(\varphi\eta, \varphi\zeta)$ and $(\psi\eta, \psi\zeta)$ can both be used to pull back a generic tensor T and therefore to construct two other tensor fields $\varphi_{\lambda,\Omega}^* T$ and $\psi_{\lambda,\Omega}^* T$, for any given value of (λ, Ω) . In particular, on \mathcal{M}_0 we now have three tensor fields, i.e. T_0 and

$$T_{\lambda,\Omega}^\varphi := \varphi_{\lambda,\Omega}^* T \Big|_{\mathcal{M}_0}, \quad T_{\lambda,\Omega}^\psi := \psi_{\lambda,\Omega}^* T \Big|_{\mathcal{M}_0}. \quad (38)$$

Since φ and ψ represent gauge choices for mapping a perturbed manifold $\mathcal{M}_{\lambda,\Omega}$ into the unperturbed one \mathcal{M}_0 , $T_{\lambda,\Omega}^\varphi$ and $T_{\lambda,\Omega}^\psi$ are the representations, in \mathcal{M}_0 , of the perturbed tensor according to the two gauges. We can write, using (34)–(36) and the expansion (10),

$$T_{\lambda,\Omega}^\varphi = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \delta_\varphi^{(k,k')} T = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \mathcal{L}_{\varphi\eta}^k \mathcal{L}_{\varphi\zeta}^{k'} T = T_0 + \Delta_0^\varphi T_{\lambda,\Omega}, \quad (39)$$

$$T_{\lambda,\Omega}^\psi = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \delta_\psi^{(k,k')} T = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \mathcal{L}_{\psi\eta}^k \mathcal{L}_{\psi\zeta}^{k'} T = T_0 + \Delta_0^\psi T_{\lambda,\Omega} \quad (40)$$

* In general, if the chart is adapted to the gauge φ , i.e. $\varphi\eta^\mu = \varphi\zeta^\mu = 0$, it is not adapted to the gauge ψ , so $\psi\eta^\mu \neq 0, \psi\zeta^\mu \neq 0$.

where $\delta_\varphi^{(k,k')}T$, $\delta_\psi^{(k,k')}T$ are the perturbations (36) in the gauges φ and ψ respectively, i.e.

$$\delta_\varphi^{(k,k')}T = \mathcal{L}_{\varphi_\eta}^k \mathcal{L}_{\varphi_\zeta}^{k'} T \Big|_{\mathcal{M}_0}, \quad (41)$$

$$\delta_\psi^{(k,k')}T = \mathcal{L}_{\psi_\eta}^k \mathcal{L}_{\psi_\zeta}^{k'} T \Big|_{\mathcal{M}_0}. \quad (42)$$

4.1. Gauge invariance

If $T_{\lambda,\Omega}^\varphi = T_{\lambda,\Omega}^\psi$, for any pair of gauges φ and ψ , we say that T is *totally gauge invariant*. This is a very strong condition, because then (39) and (40) imply that $\delta_\varphi^{(k,k')}T = \delta_\psi^{(k,k')}T$, for all gauges φ and ψ and for any (k, k') . In any practical case, however, one is interested in perturbations up to a fixed order. It is thus convenient to weaken the definition above, saying that T is *gauge invariant up to order (n, n')* iff for any two gauges φ and ψ

$$\delta_\varphi^{(k,k')}T = \delta_\psi^{(k,k')}T \quad \forall (k, k') \quad \text{with } k \leq n, k' \leq n'. \quad (43)$$

We have that a tensor field T is gauge invariant to order (n, n') iff in a given gauge φ we have that $\mathcal{L}_\xi \delta_\varphi^{(k,k')}T = 0$, for any vector field ξ defined on \mathcal{M} and for any $(k, k') < (n, n')$.

To prove this statement, let us first show that it is true for $(n, n') = (1, 0)$. In fact, if $\delta_\varphi^{(1,0)}T = \delta_\psi^{(1,0)}T$ for two arbitrary gauges φ, ψ , we have $\mathcal{L}_{\varphi_\eta - \psi_\eta} T|_{\mathcal{M}_0} = 0$. But since φ and ψ are arbitrary gauges, it follows that $\varphi_\eta - \psi_\eta$ is an arbitrary field ξ , and $\xi^m = \xi^{m+1} = 0$ because $\varphi_\eta^m = \psi_\eta^m = 1$, $\varphi_\eta^{m+1} = \psi_\eta^{m+1} = 0$, so ξ is tangent to \mathcal{M} . In the same way one proves the statement for $(n, n') = (0, 1)$. Now let us suppose that the statement is true for some (n, n') . Then, if one also has $\delta_\varphi^{(n+1, n')}T|_{\mathcal{M}_0} = \delta_\psi^{(n+1, n')}T|_{\mathcal{M}_0}$, it follows that $\mathcal{L}_{\varphi_\eta - \psi_\eta} \delta_\varphi^{(n, n')}T = 0$, while if $\delta_\varphi^{(n, n'+1)}T|_{\mathcal{M}_0} = \delta_\psi^{(n, n'+1)}T|_{\mathcal{M}_0}$, it follows that $\mathcal{L}_{\varphi_\zeta - \psi_\zeta} \delta_\varphi^{(n, n')}T = 0$, and we establish the result by induction over (n, n') .

As a consequence, T is gauge invariant to order (n, n') iff T_0 and all its perturbations of order lower than (n, n') are, in any gauge, either vanishing or constant scalars, or a combination of Kronecker deltas with constant coefficients. Thus, this generalizes to an arbitrary order (n, n') and to the two-parameter case the results of [2, 27, 23]. Further, it then follows that T is totally gauge invariant iff it is a combination of Kronecker deltas with coefficients depending only on λ, Ω .

4.2. Gauge transformations

If a tensor T is not gauge invariant, it is important to know how its representation on \mathcal{M}_0 changes under a gauge transformation. To this purpose it is natural to introduce, for each value of $(\lambda, \Omega) \in \mathbb{R}^2$, the diffeomorphism $\Phi_{\lambda, \Omega} : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ defined by

$$\Phi_{\lambda, \Omega} := \varphi_{-\lambda, -\Omega} \circ \psi_{\lambda, \Omega}. \quad (44)$$

Given that from the geometrical point of view adopted here φ and ψ are two gauges, Φ represents the gauge transformation. The action of $\Phi_{\lambda, \Omega}$ is illustrated in Fig. 2. We must stress that $\Phi : \mathcal{M}_0 \times \mathbb{R}^2 \rightarrow \mathcal{M}_0$ thus defined, *is not* a two-parameter group of diffeomorphisms in \mathcal{M}_0 . In fact, $\Phi_{\lambda_1, \Omega_1} \circ \Phi_{\lambda_2, \Omega_2} \neq \Phi_{\lambda_1 + \lambda_2, \Omega_1 + \Omega_2}$, essentially because the fields $(\varphi_\eta, \varphi_\zeta)$ and (ψ_η, ψ_ζ) have, in general, a non-vanishing commutator. However, it can be Taylor expanded, using the results of section (2.2).

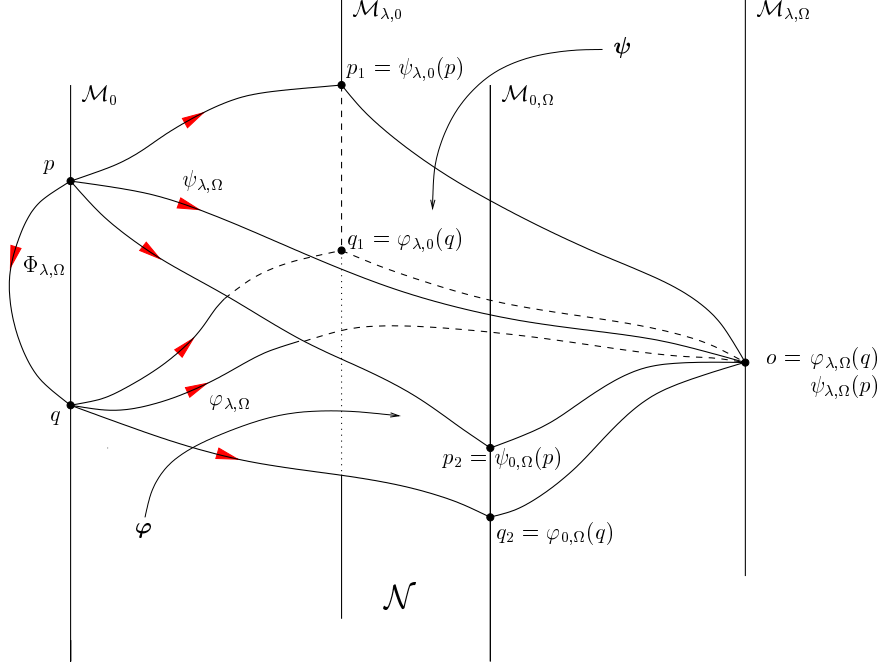


Figure 2. The action of a gauge transformation $\Phi_{\lambda, \Omega}$, represented on the background spacetime \mathcal{M}_0 . The gauge ψ is a two-dimensional submanifold embedded in $\mathcal{N} = \mathcal{M} \times \mathbb{R}^2$, represented by the surface connecting the points p , p_1 , p_2 , and o . Similarly, the gauge φ is represented by the surface connecting the points q , q_1 , q_2 , and o .

The tensor fields $T_{\lambda, \Omega}^\varphi$ and $T_{\lambda, \Omega}^\psi$, defined on \mathcal{M}_0 by the gauges φ and ψ , are connected by the linear map $\Phi_{\lambda, \Omega}^*$:

$$\begin{aligned} T_{\lambda, \Omega}^\psi &= \psi_{\lambda, \Omega}^* T|_{\mathcal{M}_0} = (\psi_{\lambda, \Omega}^* \varphi_{-\lambda, -\Omega}^* \varphi_{\lambda, \Omega}^* T)|_{\mathcal{M}_0} \\ &= \Phi_{\lambda, \Omega}^* (\varphi_{\lambda, \Omega}^* T)|_{\mathcal{M}_0} = \Phi_{\lambda, \Omega}^* T_{\lambda, \Omega}^\varphi. \end{aligned} \quad (45)$$

Thus, the gauge transformation to an arbitrary order (n, n') is given by the Taylor expansion of the pull-back $\Phi_{\lambda, \Omega}^* T$, whose terms are explicitly given in section 2.2. Up to fourth order, we have explicitly from (30)

$$\begin{aligned} T_{\lambda, \Omega}^\psi &= T_{\lambda, \Omega}^\varphi + \lambda \mathcal{L}_{\xi(1,0)} T_{\lambda, \Omega}^\varphi + \Omega \mathcal{L}_{\xi(0,1)} T_{\lambda, \Omega}^\varphi \\ &+ \frac{\lambda^2}{2} \left\{ \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^2 \right\} T_{\lambda, \Omega}^\varphi + \frac{\Omega^2}{2} \left\{ \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^2 \right\} T_{\lambda, \Omega}^\varphi \\ &+ \lambda \Omega \left\{ \mathcal{L}_{\xi(1,1)} + \epsilon_0 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + \epsilon_1 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \right\} T_{\lambda, \Omega}^\varphi \\ &+ \frac{\lambda^3}{6} \left\{ \mathcal{L}_{\xi(3,0)} + 3 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^3 \right\} T_{\lambda, \Omega}^\varphi \\ &+ \frac{\lambda^2 \Omega}{2} \left\{ \mathcal{L}_{\xi(2,1)} + 2 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(1,1)} + \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(2,0)} + 2 \epsilon_2 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \right\} T_{\lambda, \Omega}^\varphi \end{aligned}$$

$$\begin{aligned}
& + (\epsilon_1 - \epsilon_2) \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}^2 + (\epsilon_0 - \epsilon_2) \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(0,1)} \Big\} T_{\lambda,\Omega}^\varphi \\
& + \frac{\lambda \Omega^2}{2} \left\{ \mathcal{L}_{\xi(1,2)} + 2 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,1)} + \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,2)} + 2 \epsilon_3 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \right. \\
& + (\epsilon_0 - \epsilon_3) \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)}^2 + (\epsilon_1 - \epsilon_3) \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,0)} \Big\} T_{\lambda,\Omega}^\varphi \\
& + \frac{\Omega^3}{6} \left\{ \mathcal{L}_{\xi(0,3)} + 3 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^3 \right\} T_{\lambda,\Omega}^\varphi \\
& + \frac{\lambda^4}{24} \left\{ \mathcal{L}_{\xi(4,0)} + 4 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(3,0)} + 3 \mathcal{L}_{\xi(2,0)}^2 + 6 \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^4 \right\} T_{\lambda,\Omega}^\varphi \\
& + \frac{\lambda^3 \Omega}{6} \left\{ \mathcal{L}_{\xi(3,1)} + 3 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(2,1)} + \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(3,0)} + 3 \epsilon_4 \mathcal{L}_{\xi(2,0)} \mathcal{L}_{\xi(1,1)} \right. \\
& + 3 \epsilon_5 \mathcal{L}_{\xi(1,1)} \mathcal{L}_{\xi(2,0)} + 3 \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(1,1)} + 3 (\epsilon_0 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + \epsilon_1 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}) \mathcal{L}_{\xi(2,0)} \\
& + (\epsilon_1 - \epsilon_2 - \epsilon_6) \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}^3 + 3 \epsilon_6 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}^2 \\
& + 3 (\epsilon_2 - \epsilon_6) \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} + (\epsilon_0 - 2 \epsilon_2 + \epsilon_6) \mathcal{L}_{\xi(1,0)}^3 \mathcal{L}_{\xi(0,1)} \Big\} T_{\lambda,\Omega}^\varphi \\
& + \frac{\lambda^2 \Omega^2}{4} \left\{ \mathcal{L}_{\xi(2,2)} + 2 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(1,2)} + 2 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(2,1)} + 2 \mathcal{L}_{\xi(1,1)}^2 \right. \\
& + \epsilon_7 \mathcal{L}_{\xi(2,0)} \mathcal{L}_{\xi(0,2)} + \epsilon_8 \mathcal{L}_{\xi(0,2)} \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(2,0)} \\
& + 4 (\epsilon_0 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + \epsilon_1 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}) \mathcal{L}_{\xi(1,1)} \\
& - (\epsilon_3 + \epsilon_2 - \epsilon_1 + \epsilon_9) \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,0)}^2 - (\epsilon_3 + \epsilon_2 - \epsilon_0 - \epsilon_9) \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(0,1)}^2 \\
& + 2 (\epsilon_3 + \epsilon_2 - \epsilon_0 \epsilon_1 - \epsilon_9) \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \\
& + 2 (\epsilon_3 + \epsilon_2 - \epsilon_0 \epsilon_1 + \epsilon_9) \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \\
& - 2 (\epsilon_3 - \epsilon_0 \epsilon_1) \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,0)} - 2 (\epsilon_2 - \epsilon_0 \epsilon_1) \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(0,1)} \Big\} T_{\lambda,\Omega}^\varphi \\
& + \frac{\lambda \Omega^3}{6} \left\{ \mathcal{L}_{\xi(1,3)} + 3 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,2)} + \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,3)} + 3 \epsilon_{10} \mathcal{L}_{\xi(0,2)} \mathcal{L}_{\xi(1,1)} \right. \\
& + 3 \epsilon_{11} \mathcal{L}_{\xi(1,1)} \mathcal{L}_{\xi(0,2)} + 3 \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,1)} + 3 (\epsilon_0 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + \epsilon_1 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}) \mathcal{L}_{\xi(0,2)} \\
& + (\epsilon_0 - \epsilon_3 - \epsilon_{12}) \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)}^3 + 3 \epsilon_{12} \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)}^2 \\
& + 3 (\epsilon_3 - \epsilon_{12}) \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + (\epsilon_1 - 2 \epsilon_3 + \epsilon_{12}) \mathcal{L}_{\xi(0,1)}^3 \mathcal{L}_{\xi(1,0)} \Big\} T_{\lambda,\Omega}^\varphi \\
& + \frac{\Omega^4}{24} \left\{ \mathcal{L}_{\xi(0,4)} + 4 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(0,3)} + 3 \mathcal{L}_{\xi(0,2)}^2 + 6 \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^4 \right\} T_{\lambda,\Omega}^\varphi \\
& + O^5(\lambda, \Omega), \tag{46}
\end{aligned}$$

where the $\xi_{(p,q)}$ are now the generators of the gauge transformation $\Phi_{\lambda,\Omega}$.

We can now relate the perturbations in the two gauges. To order (n, n') with $n + n' \leq 4$, these relations can be derived by substituting (39), (40) in (46):

$$\delta_\psi^{(1,0)} T - \delta_\varphi^{(1,0)} T = \mathcal{L}_{\xi(1,0)} T_0, \tag{47}$$

$$\delta_\psi^{(0,1)} T - \delta_\varphi^{(0,1)} T = \mathcal{L}_{\xi(0,1)} T_0, \tag{48}$$

$$\delta_\psi^{(2,0)} T - \delta_\varphi^{(2,0)} T = 2 \mathcal{L}_{\xi(1,0)} \delta_\varphi^{(1,0)} T + \left\{ \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^2 \right\} T_0, \tag{49}$$

$$\begin{aligned} \delta_\psi^{(1,1)} T - \delta_\varphi^{(1,1)} T &= \mathcal{L}_{\xi(1,0)} \delta_\varphi^{(0,1)} T + \mathcal{L}_{\xi(0,1)} \delta_\varphi^{(1,0)} T \\ &+ \left\{ \mathcal{L}_{\xi(1,1)} + \epsilon_0 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + \epsilon_1 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \right\} T_0, \end{aligned} \tag{50}$$

$$\delta_\psi^{(0,2)}T - \delta_\varphi^{(0,2)}T = 2\mathcal{L}_{\xi(0,1)}\delta_\varphi^{(0,1)}T + \left\{ \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^2 \right\} T_0, \quad (51)$$

$$\begin{aligned} \delta_\psi^{(3,0)}T - \delta_\varphi^{(3,0)}T &= 3\mathcal{L}_{\xi(1,0)}\delta_\varphi^{(2,0)}T + 3\left\{ \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^2 \right\} \delta_\varphi^{(1,0)}T \\ &+ \left\{ \mathcal{L}_{\xi(3,0)} + 3\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^3 \right\} T_0, \end{aligned} \quad (52)$$

$$\begin{aligned} \delta_\psi^{(2,1)}T - \delta_\varphi^{(2,1)}T &= 2\mathcal{L}_{\xi(1,0)}\delta_\varphi^{(1,1)}T + \mathcal{L}_{\xi(0,1)}\delta_\varphi^{(2,0)}T + \left\{ \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^2 \right\} \delta_\varphi^{(0,1)}T \\ &+ 2\left\{ \mathcal{L}_{\xi(1,1)} + \epsilon_0\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)} + \epsilon_1\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)} \right\} \delta_\varphi^{(1,0)}T \\ &+ \left\{ \mathcal{L}_{\xi(2,1)} + 2\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(1,1)} + \mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(2,0)} + 2\epsilon_2\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)} \right. \\ &\left. + (\epsilon_1 - \epsilon_2)\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)}^2 + (\epsilon_0 - \epsilon_2)\mathcal{L}_{\xi(1,0)}^2\mathcal{L}_{\xi(0,1)} \right\} T_0, \end{aligned} \quad (53)$$

$$\begin{aligned} \delta_\psi^{(1,2)}T - \delta_\varphi^{(1,2)}T &= 2\mathcal{L}_{\xi(0,1)}\delta_\varphi^{(1,1)}T + \mathcal{L}_{\xi(1,0)}\delta_\varphi^{(0,2)}T + \left\{ \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^2 \right\} \delta_\varphi^{(1,0)}T \\ &+ 2\left\{ \mathcal{L}_{\xi(1,1)} + \epsilon_0\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)} + \epsilon_1\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)} \right\} \delta_\varphi^{(0,1)}T \\ &+ \left\{ \mathcal{L}_{\xi(1,2)} + 2\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,1)} + \mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,2)} + 2\epsilon_3\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)} \right. \\ &\left. + (\epsilon_0 - \epsilon_3)\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)}^2 + (\epsilon_1 - \epsilon_3)\mathcal{L}_{\xi(0,1)}^2\mathcal{L}_{\xi(1,0)} \right\} T_0, \end{aligned} \quad (54)$$

$$\begin{aligned} \delta_\psi^{(0,3)}T - \delta_\varphi^{(0,3)}T &= 3\mathcal{L}_{\xi(0,1)}\delta_\varphi^{(0,2)}T + 3\left\{ \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^2 \right\} \delta_\varphi^{(0,1)}T \\ &+ \left\{ \mathcal{L}_{\xi(0,3)} + 3\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^3 \right\} T_0, \end{aligned} \quad (55)$$

$$\begin{aligned} \delta_\psi^{(4,0)}T - \delta_\varphi^{(4,0)}T &= 4\mathcal{L}_{\xi(1,0)}\delta_\varphi^{(3,0)}T + 6\left\{ \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^2 \right\} \delta_\varphi^{(2,0)}T \\ &+ 4\left\{ \mathcal{L}_{\xi(3,0)} + 3\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^3 \right\} \delta_\varphi^{(1,0)}T \\ &+ \left\{ \mathcal{L}_{\xi(4,0)} + 4\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(3,0)} + 3\mathcal{L}_{\xi(2,0)}^2 + 6\mathcal{L}_{\xi(1,0)}^2\mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^4 \right\} T_0, \end{aligned} \quad (56)$$

$$\begin{aligned} \delta_\psi^{(3,1)}T - \delta_\varphi^{(3,1)}T &= 3\mathcal{L}_{\xi(1,0)}\delta_\varphi^{(2,1)}T + \mathcal{L}_{\xi(0,1)}\delta_\varphi^{(3,0)}T + 3\left\{ \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^2 \right\} \delta_\varphi^{(1,1)}T \\ &+ 3\left\{ \mathcal{L}_{\xi(1,1)} + \epsilon_0\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)} + \epsilon_1\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)} \right\} \delta_\varphi^{(2,0)}T \\ &+ \left\{ \mathcal{L}_{\xi(3,0)} + 3\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^3 \right\} \delta_\varphi^{(0,1)}T \\ &+ 3\left\{ \mathcal{L}_{\xi(2,1)} + 2\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(1,1)} + \mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(2,0)} + 2\epsilon_2\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)} \right. \\ &\left. + (\epsilon_1 - \epsilon_2)\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)}^2 + (\epsilon_0 - \epsilon_2)\mathcal{L}_{\xi(1,0)}^2\mathcal{L}_{\xi(0,1)} \right\} \delta_\varphi^{(1,0)}T \\ &+ \left\{ \mathcal{L}_{\xi(3,1)} + 3\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(2,1)} + \mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(3,0)} + 3\epsilon_4\mathcal{L}_{\xi(2,0)}\mathcal{L}_{\xi(1,1)} + 3\epsilon_5\mathcal{L}_{\xi(1,1)}\mathcal{L}_{\xi(2,0)} \right. \\ &\left. + 3\mathcal{L}_{\xi(1,0)}^2\mathcal{L}_{\xi(1,1)} + 3(\epsilon_0\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)} + \epsilon_1\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)})\mathcal{L}_{\xi(2,0)} \right. \\ &\left. + (\epsilon_1 - \epsilon_2 - \epsilon_6)\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)}^3 + 3\epsilon_6\mathcal{L}_{\xi(1,0)}\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)}^2 \right. \\ &\left. + 3(\epsilon_2 - \epsilon_6)\mathcal{L}_{\xi(1,0)}^2\mathcal{L}_{\xi(0,1)}\mathcal{L}_{\xi(1,0)} + (\epsilon_0 - 2\epsilon_2 + \epsilon_6)\mathcal{L}_{\xi(1,0)}^3\mathcal{L}_{\xi(0,1)} \right\} T_0, \end{aligned} \quad (57)$$

$$\begin{aligned} \delta_\psi^{(2,2)}T - \delta_\varphi^{(2,2)}T &= 2\mathcal{L}_{\xi(1,0)}\delta_\varphi^{(1,2)}T + 2\mathcal{L}_{\xi(0,1)}\delta_\varphi^{(2,1)}T \\ &+ \left\{ \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^2 \right\} \delta_\varphi^{(0,2)}T + \left\{ \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^2 \right\} \delta_\varphi^{(2,0)}T \end{aligned}$$

$$\begin{aligned}
& + 4 \left\{ \mathcal{L}_{\xi(1,1)} + \epsilon_0 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + \epsilon_1 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \right\} \delta_\varphi^{(1,1)} T \\
& + 2 \left\{ \mathcal{L}_{\xi(2,1)} + 2 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(1,1)} + \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(2,0)} + 2 \epsilon_2 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \right. \\
& \left. + (\epsilon_1 - \epsilon_2) \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}^2 + (\epsilon_0 - \epsilon_2) \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(0,1)} \right\} \delta_\varphi^{(0,1)} T \\
& + 2 \left\{ \mathcal{L}_{\xi(1,2)} + 2 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,1)} + \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,2)} + 2 \epsilon_3 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \right. \\
& \left. + (\epsilon_0 - \epsilon_3) \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)}^2 + (\epsilon_1 - \epsilon_3) \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,0)} \right\} \delta_\varphi^{(1,0)} T \\
& + \left\{ \mathcal{L}_{\xi(2,2)} + 2 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(1,2)} + 2 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(2,1)} + 2 \mathcal{L}_{\xi(1,1)}^2 \right. \\
& + \epsilon_7 \mathcal{L}_{\xi(2,0)} \mathcal{L}_{\xi(0,2)} + \epsilon_8 \mathcal{L}_{\xi(0,2)} \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(2,0)} \\
& + 4 (\epsilon_0 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + \epsilon_1 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}) \mathcal{L}_{\xi(1,1)} \\
& - (\epsilon_3 + \epsilon_2 - \epsilon_1 + \epsilon_9) \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,0)}^2 - (\epsilon_3 + \epsilon_2 - \epsilon_0 - \epsilon_9) \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(0,1)}^2 \\
& + 2 (\epsilon_3 + \epsilon_2 - \epsilon_0 \epsilon_1 - \epsilon_9) \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \\
& + 2 (\epsilon_3 + \epsilon_2 - \epsilon_0 \epsilon_1 + \epsilon_9) \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \\
& \left. - 2 (\epsilon_3 - \epsilon_0 \epsilon_1) \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,0)} - 2 (\epsilon_2 - \epsilon_0 \epsilon_1) \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}^2 \mathcal{L}_{\xi(0,1)} \right\} T_0, \quad (58)
\end{aligned}$$

$$\begin{aligned}
\delta_\psi^{(1,3)} T - \delta_\varphi^{(1,3)} T & = 3 \mathcal{L}_{\xi(0,1)} \delta_\varphi^{(1,2)} T + \mathcal{L}_{\xi(1,0)} \delta_\varphi^{(0,3)} T + 3 \left\{ \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^2 \right\} \delta_\varphi^{(1,1)} T \\
& + 3 \left\{ \mathcal{L}_{\xi(1,1)} + \epsilon_0 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + \epsilon_1 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \right\} \delta_\varphi^{(0,2)} T \\
& + \left\{ \mathcal{L}_{\xi(0,3)} + 3 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^3 \right\} \delta_\varphi^{(1,0)} T \\
& + 3 \left\{ \mathcal{L}_{\xi(1,2)} + 2 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,1)} + \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,2)} + 2 \epsilon_3 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} \right. \\
& \left. + (\epsilon_0 - \epsilon_3) \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)}^2 + (\epsilon_1 - \epsilon_3) \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,0)} \right\} \delta_\varphi^{(0,1)} T \\
& + \left\{ \mathcal{L}_{\xi(1,3)} + 3 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,2)} + \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,3)} + 3 \epsilon_{10} \mathcal{L}_{\xi(0,2)} \mathcal{L}_{\xi(1,1)} + 3 \epsilon_{11} \mathcal{L}_{\xi(1,1)} \mathcal{L}_{\xi(0,2)} \right. \\
& + 3 \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,1)} + 3 (\epsilon_0 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + \epsilon_1 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)}) \mathcal{L}_{\xi(0,2)} \\
& + (\epsilon_0 - \epsilon_3 - \epsilon_{12}) \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)}^3 + 3 \epsilon_{12} \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)}^2 \\
& \left. + 3 (\epsilon_3 - \epsilon_{12}) \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + (\epsilon_1 - 2 \epsilon_3 + \epsilon_{12}) \mathcal{L}_{\xi(0,1)}^3 \mathcal{L}_{\xi(1,0)} \right\} T_0, \quad (59)
\end{aligned}$$

$$\begin{aligned}
\delta_\psi^{(0,4)} T - \delta_\varphi^{(0,4)} T & = 4 \mathcal{L}_{\xi(0,1)} \delta_\varphi^{(0,3)} T + 6 \left\{ \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^2 \right\} \delta_\varphi^{(0,2)} T \\
& + 4 \left\{ \mathcal{L}_{\xi(0,3)} + 3 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^3 \right\} \delta_\varphi^{(0,1)} T \\
& + \left\{ \mathcal{L}_{\xi(0,4)} + 4 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(0,3)} + 3 \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(0,2)} + 6 \mathcal{L}_{\xi(0,1)}^2 \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^4 \right\} T_0. \quad (60)
\end{aligned}$$

This result is, of course, consistent with the characterization of gauge invariance given in subsection 4.1. Equations (47) and (48) imply that $T_{\lambda,\Omega}$ is gauge invariant to the order (1,0) or (0,1) iff $\mathcal{L}_\xi T_0 = 0$, for any vector field on \mathcal{M}_0 . Equation (49) implies that $T_{\lambda,\Omega}$ is gauge invariant to the order (2,0) iff $\mathcal{L}_\xi T_0 = 0$ and $\mathcal{L}_\xi \delta_\varphi^{(1,0)} T = 0$, for any vector field on \mathcal{M}_0 , and so on for all the orders.

It is also possible to find the explicit expressions for the generators $\xi_{(p,q)}$ of the gauge transformation Φ in terms of the gauge vector fields $(\varphi\eta, \varphi\zeta)$ and $(\psi\eta, \psi\zeta)$. We write here their expressions up to second order:

$$\xi_{(1,0)} = \psi\eta - \varphi\eta, \quad (61)$$

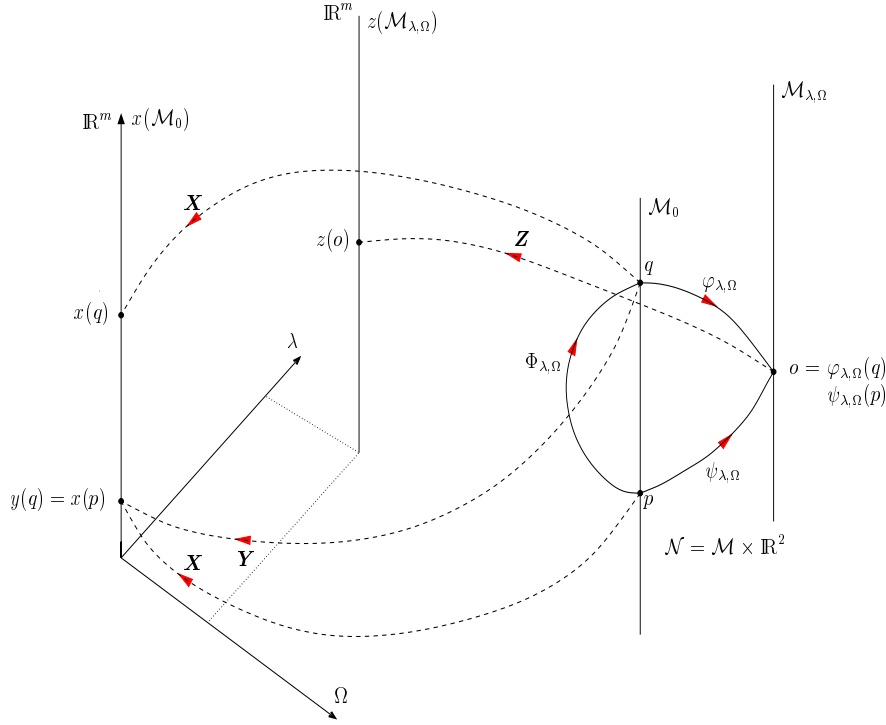


Figure 3. The picture shows how two different gauge choices, φ and ψ , map two different points in the background manifold \mathcal{M}_0 , namely q and p respectively, to the same point o in the physical spacetime $\mathcal{M}_{\lambda\Omega}$. The coordinate version of this fact is shown on the left part of picture. We can look at this either from the *active* point of view (the two points q and p are given coordinates in the same chart by the map \mathbf{X} , namely $x^\mu(q)$ and $x^\mu(p)$), or from the *passive* point of view (the point q is given coordinates in two different charts by maps \mathbf{X} and \mathbf{Y} , corresponding to two different points of \mathbb{R}^m , $x(q)$ and $y(q) = x(p)$). Finally, \mathbf{Z} assigns coordinates to the point o of the physical spacetime $\mathcal{M}_{\lambda\Omega}$.

$$\xi_{(0,1)} = \psi\zeta - \varphi\zeta, \quad (62)$$

$$\xi_{(2,0)} = [\varphi\eta, \psi\eta], \quad (63)$$

$$\xi_{(1,1)} = \epsilon_0[\varphi\eta, \psi\zeta] + \epsilon_1[\varphi\zeta, \psi\eta], \quad (64)$$

$$\xi_{(0,2)} = [\varphi\zeta, \psi\zeta]. \quad (65)$$

4.3. Coordinate transformations

Up to now, we have built a two-parameter formalism using a geometrical, coordinate-free language. However, in order to carry out explicit calculations in a practical case, one has to introduce systems of local coordinates. In this respect, all our expressions are immediately translated into components simply by using the expression of the components of the Lie derivative of a tensor. Nonetheless, much of the literature on the subject is written using coordinate systems, and gauge transformations are most often represented by the corresponding coordinate transformations. For this reason, we devote this subsection to describe how to establish the translation between the two languages, giving in particular the explicit transformation of coordinates (further details are in [3]).

Let us consider the situation described in Fig. 3. We have considered two gauge choices, represented by the groups of diffeomorphisms φ and ψ , under which the point o in the physical manifold $\mathcal{M}_{\lambda,\Omega}$ corresponds to two different points in the background manifold \mathcal{M}_0 , namely $q = \varphi_{\lambda,\Omega}^{-1}(o)$ and $p = \psi_{\lambda,\Omega}^{-1}(o)$. The transformation relating these two gauge choices is described by the two-parameter family of diffeomorphisms $\Phi_{\lambda,\Omega} = \varphi_{\lambda,\Omega}^{-1} \circ \psi_{\lambda,\Omega}$, so that $\Phi_{\lambda,\Omega}(p) = q$. This gauge transformation maps a tensor field T on $q \in \mathcal{M}_0$ to the tensor field $(\Phi^*T)(p) = \Phi^*(T(q))$ on $p \in \mathcal{M}_0$.

Now, let us consider a chart $(\mathcal{U}, \mathbf{X})$ on an open subset \mathcal{U} of \mathcal{M}_0 . The gauges $\varphi_{\lambda,\Omega}$ and $\psi_{\lambda,\Omega}$ define two maps from $\mathcal{M}_{\lambda,\Omega}$ to \mathbb{R}^m :

$$\begin{aligned} \mathbf{X} \circ \varphi_{\lambda,\Omega}^{-1} : \mathcal{M}_{\lambda,\Omega} &\longrightarrow \mathbb{R}^m & \mathbf{X} \circ \psi_{\lambda,\Omega}^{-1} : \mathcal{M}_{\lambda,\Omega} &\longrightarrow \mathbb{R}^m \\ o &\longmapsto x(q(o)), & o &\longmapsto x(p(o)). \end{aligned} \quad (66)$$

Then, we can look at the gauge transformation Φ in two different ways: from the *active* point of view or from the *passive* point of view. In the first case, one considers a diffeomorphism which changes the point on the background \mathcal{M}_0 . To these points one associates different values of the coordinates in the chart $(\mathcal{U}, \mathbf{X})$. So the coordinate change is given by

$$x^\mu(p) \longrightarrow x^\mu(q) \quad (67)$$

or, defining the pull-back of x as $\tilde{x}^\mu(p) := x^\mu(\Phi(p))$,

$$x^\mu(p) \longrightarrow \tilde{x}^\mu(p). \quad (68)$$

If we instead consider the passive point of view, we need to introduce a new chart (\mathcal{U}', y) :

$$\mathbf{Y} := \mathbf{X} \circ \Phi_{\lambda,\Omega}^{-1}, \quad (69)$$

such that the two sets of coordinates are related by

$$y^\mu(q) = x^\mu(p), \quad (70)$$

so we can say that the gauge transformation does not change the point on \mathcal{M}_0 , but it changes the chart from $(\mathcal{U}, \mathbf{X})$ to $(\mathcal{U}', \mathbf{Y})$, i.e. the labels of the point of \mathcal{M}_0 . The coordinate transformation is then

$$x^\mu(q) \longrightarrow y^\mu(q). \quad (71)$$

Now, let us consider the transformation of a vector field V . From the active point of view, the components of V in the chart $(\mathcal{U}, \mathbf{X})$, V^μ , are related with the ones of the transformed vector field \tilde{V} , \tilde{V}^μ , by

$$\tilde{V}^\mu = (\mathbf{X}_* \tilde{V})^\mu = (\mathbf{X}_* \Phi_{\lambda,\Omega}^* V)^\mu. \quad (72)$$

From the passive point of view, we can use the properties relating the pull-back and push-forward maps associated with diffeomorphisms:

$$\mathbf{X}_* \Phi_{\lambda,\Omega}^* V = \mathbf{X}_* \Phi_{*\lambda,\Omega}^{-1} V = \mathbf{Y}_* V, \quad (73)$$

so we get the well known result that the components of the transformed vector \tilde{V} in the coordinate system \mathbf{X} are defined in terms of the components of the vector V in the new coordinate system \mathbf{Y} :

$$\tilde{V}^\mu(x(p)) = (\mathbf{Y}_* V(q))^\mu = V'^\mu(y(q)) = \left(\frac{\partial y^\mu}{\partial x^\nu} \right) \Big|_{x(q)} V^\nu(x(q)). \quad (74)$$

In order to write down explicit expressions, we will apply the expansion of the pull-back of Φ^* [See equation (30)] to the coordinate functions x^μ . Then, the *active* coordinate transformation is given by

$$\begin{aligned}
\tilde{x}^\mu(p) &= x^\mu(q) = (\Phi^* x^\mu)(p) \\
&= x^\mu(p) + \lambda \xi_{(1,0)}^\mu + \Omega \xi_{(0,1)}^\mu \\
&+ \frac{\lambda^2}{2} \left(\xi_{(2,0)}^\mu + \xi_{(1,0)}^\nu \xi_{(1,0),\nu}^\mu \right) + \frac{\Omega^2}{2} \left(\xi_{(0,2)}^\mu + \xi_{(0,1)}^\nu \xi_{(0,1),\nu}^\mu \right) \\
&+ \lambda \Omega \left(\xi_{(1,1)}^\mu + \epsilon_0 \xi_{(1,0)}^\nu \xi_{(0,1),\nu}^\mu + \epsilon_1 \xi_{(0,1)}^\nu \xi_{(1,0),\nu}^\mu \right) \\
&+ \frac{\lambda^3}{6} \left(\xi_{(3,0)}^\mu + 3 \xi_{(1,0)}^\nu \xi_{(2,0),\nu}^\mu + \xi_{(1,0)}^\rho \xi_{(1,0),\rho}^\nu \xi_{(1,0),\nu}^\mu \right) \\
&+ \frac{\lambda^2 \Omega}{2} \left(\xi_{(2,1)}^\mu + 2 \xi_{(1,0)}^\nu \xi_{(1,1),\nu}^\mu + \xi_{(0,1)}^\nu \xi_{(2,0),\nu}^\mu + 2 \epsilon_2 \xi_{(1,0)}^\rho \xi_{(0,1),\rho}^\nu \xi_{(1,0),\nu}^\mu \right. \\
&+ \left. (\epsilon_1 - \epsilon_2) \xi_{(0,1)}^\rho \xi_{(1,0),\rho}^\nu \xi_{(1,0),\nu}^\mu + (\epsilon_0 - \epsilon_2) \xi_{(1,0)}^\rho \xi_{(1,0),\rho}^\nu \xi_{(0,1),\nu}^\mu \right) \\
&+ \frac{\lambda \Omega^2}{2} \left(\xi_{(1,2)}^\mu + 2 \xi_{(0,1)}^\nu \xi_{(1,1),\nu}^\mu + \xi_{(1,0)}^\nu \xi_{(0,2),\nu}^\mu + 2 \epsilon_3 \xi_{(0,1)}^\rho \xi_{(1,0),\rho}^\nu \xi_{(0,1),\nu}^\mu \right. \\
&+ \left. (\epsilon_0 - \epsilon_3) \xi_{(1,0)}^\rho \xi_{(0,1),\rho}^\nu \xi_{(0,1),\nu}^\mu + (\epsilon_1 - \epsilon_3) \xi_{(0,1)}^\rho \xi_{(0,1),\rho}^\nu \xi_{(1,0),\nu}^\mu \right) \\
&+ \frac{\Omega^3}{6} \left(\xi_{(0,3)}^\mu + 3 \xi_{(0,1)}^\nu \xi_{(0,2),\nu}^\mu + \xi_{(0,1)}^\rho \xi_{(0,1),\rho}^\nu \xi_{(0,1),\nu}^\mu \right) \\
&+ \frac{\lambda^4}{24} \left(\xi_{(4,0)}^\mu + 4 \xi_{(1,0)}^\nu \xi_{(3,0),\nu}^\mu + 3 \xi_{(2,0)}^\nu \xi_{(2,0),\nu}^\mu \right. \\
&+ \left. 6 \xi_{(1,0)}^\rho \xi_{(1,0),\rho}^\nu \xi_{(2,0),\nu}^\mu + \xi_{(1,0)}^\sigma \xi_{(1,0),\sigma}^\rho \xi_{(1,0),\rho}^\nu \xi_{(1,0),\nu}^\mu \right) \\
&+ \frac{\lambda^3 \Omega}{6} \left(\xi_{(3,1)}^\mu + 3 \xi_{(1,0)}^\nu \xi_{(2,1),\nu}^\mu + \xi_{(0,1)}^\nu \xi_{(3,0),\nu}^\mu + 3 \epsilon_4 \xi_{(2,0)}^\nu \xi_{(1,1),\nu}^\mu \right. \\
&+ \left. 3 \epsilon_5 \xi_{(1,1)}^\nu \xi_{(2,0),\nu}^\mu + 3 \xi_{(1,0)}^\rho \xi_{(1,0),\rho}^\nu \xi_{(1,1),\nu}^\mu + 3 (\epsilon_0 \xi_{(1,0)}^\rho \xi_{(0,1),\rho}^\nu + \epsilon_1 \xi_{(0,1)}^\rho \xi_{(1,0),\rho}^\nu) \xi_{(2,0),\nu}^\mu \right. \\
&+ \left. (\epsilon_1 - \epsilon_2 - \epsilon_6) \xi_{(0,1)}^\sigma \xi_{(1,0),\sigma}^\rho \xi_{(1,0),\rho}^\nu \xi_{(1,0),\nu}^\mu + 3 \epsilon_6 \xi_{(1,0)}^\sigma \xi_{(0,1),\sigma}^\rho \xi_{(1,0),\rho}^\nu \xi_{(1,0),\nu}^\mu \right. \\
&+ \left. 3 (\epsilon_2 - \epsilon_6) \xi_{(1,0),\sigma}^\rho \xi_{(0,1),\rho}^\nu \xi_{(1,0),\nu}^\mu + (\epsilon_0 - 2 \epsilon_2 + \epsilon_6) \xi_{(1,0),\sigma}^\rho \xi_{(1,0),\sigma}^\nu \xi_{(1,0),\rho}^\mu \xi_{(0,1),\nu}^\mu \right) \\
&+ \frac{\lambda^2 \Omega^2}{4} \left(\xi_{(2,2)}^\mu + 2 \xi_{(1,0)}^\nu \xi_{(1,2),\nu}^\mu + 2 \xi_{(0,1)}^\nu \xi_{(2,1),\nu}^\mu + 2 \xi_{(1,1)}^\nu \xi_{(1,1),\nu}^\mu \right. \\
&+ \left. \epsilon_7 \xi_{(2,0)}^\nu \xi_{(0,2),\nu}^\mu + \epsilon_8 \xi_{(0,2)}^\nu \xi_{(2,0),\nu}^\mu + \xi_{(1,0)}^\rho \xi_{(1,0),\rho}^\nu \xi_{(0,2),\nu}^\mu + \xi_{(0,1)}^\rho \xi_{(0,1),\rho}^\nu \xi_{(2,0),\nu}^\mu \right. \\
&+ \left. 4 (\epsilon_0 \xi_{(1,0)}^\rho \xi_{(0,1),\rho}^\nu + \epsilon_1 \xi_{(0,1)}^\rho \xi_{(1,0),\rho}^\nu) \xi_{(1,1),\nu}^\mu \right. \\
&- (\epsilon_3 + \epsilon_2 - \epsilon_1 + \epsilon_9) \xi_{(0,1)}^\sigma \xi_{(0,1),\sigma}^\rho \xi_{(1,0),\rho}^\nu \xi_{(1,0),\nu}^\mu \\
&- (\epsilon_3 + \epsilon_2 - \epsilon_0 - \epsilon_9) \xi_{(1,0),\sigma}^\rho \xi_{(1,0),\sigma}^\nu \xi_{(0,1),\rho}^\mu \xi_{(0,1),\nu}^\mu \\
&+ 2 (\epsilon_3 + \epsilon_2 - \epsilon_0 \epsilon_1 - \epsilon_9) \xi_{(1,0),\sigma}^\rho \xi_{(0,1),\sigma}^\nu \xi_{(1,0),\rho}^\mu \xi_{(0,1),\nu}^\mu \\
&+ 2 (\epsilon_3 + \epsilon_2 - \epsilon_0 \epsilon_1 + \epsilon_9) \xi_{(0,1),\sigma}^\rho \xi_{(1,0),\sigma}^\nu \xi_{(0,1),\rho}^\mu \xi_{(1,0),\nu}^\mu \\
&- 2 (\epsilon_3 - \epsilon_0 \epsilon_1) \xi_{(1,0),\sigma}^\rho \xi_{(0,1),\sigma}^\nu \xi_{(0,1),\rho}^\mu \xi_{(1,0),\nu}^\mu - 2 (\epsilon_2 - \epsilon_0 \epsilon_1) \xi_{(0,1),\sigma}^\rho \xi_{(1,0),\sigma}^\nu \xi_{(1,0),\rho}^\mu \xi_{(0,1),\nu}^\mu \left. \right) \\
&+ \frac{\lambda \Omega^3}{6} \left(\xi_{(1,3)}^\mu + 3 \xi_{(0,1)}^\nu \xi_{(1,2),\nu}^\mu + \xi_{(1,0)}^\nu \xi_{(0,3),\nu}^\mu + 3 \epsilon_{10} \xi_{(0,2)}^\nu \xi_{(1,1),\nu}^\mu \right. \\
&+ \left. 3 \epsilon_{11} \xi_{(1,1)}^\nu \xi_{(0,2),\nu}^\mu + 3 \xi_{(0,1)}^\rho \xi_{(0,1),\rho}^\nu \xi_{(1,1),\nu}^\mu + 3 (\epsilon_0 \xi_{(1,0)}^\rho \xi_{(0,1),\rho}^\nu + \epsilon_1 \xi_{(0,1)}^\rho \xi_{(1,0),\rho}^\nu) \xi_{(0,2),\nu}^\mu \right)
\end{aligned}$$

$$\begin{aligned}
& + (\epsilon_0 - \epsilon_3 - \epsilon_{12})\xi_{(1,0)}^\sigma \xi_{(0,1)}^\rho \xi_{(0,1),\sigma}^\nu \xi_{(0,1),\nu}^\mu + 3\epsilon_{12}\xi_{(0,1)}^\sigma \xi_{(1,0),\sigma}^\rho \xi_{(0,1),\rho}^\nu \xi_{(0,1),\nu}^\mu \\
& + 3(\epsilon_3 - \epsilon_{12})\xi_{(0,1)}^\sigma \xi_{(0,1),\sigma}^\rho \xi_{(1,0),\rho}^\nu \xi_{(0,1),\nu}^\mu + (\epsilon_1 - 2\epsilon_3 + \epsilon_{12})\xi_{(0,1)}^\sigma \xi_{(0,1),\sigma}^\rho \xi_{(0,1),\rho}^\nu \xi_{(1,0),\nu}^\mu \\
& + \frac{\Omega^4}{24} \left(\xi_{(0,4)}^\mu + 4\xi_{(0,1)}^\nu \xi_{(0,3),\nu}^\mu + 3\xi_{(0,2)}^\nu \xi_{(0,2),\nu}^\mu \right. \\
& \left. + 6\xi_{(0,1)}^\rho \xi_{(0,1),\rho}^\nu \xi_{(0,2),\nu}^\mu + \xi_{(0,1)}^\sigma \xi_{(0,1),\sigma}^\rho \xi_{(0,1),\rho}^\nu \xi_{(0,1),\nu}^\mu \right), \tag{75}
\end{aligned}$$

where the vector fields $\xi_{(p,q)}^\mu$ and their derivatives are evaluated in $x(p)$. This expression gives the relation between the coordinates, in the chart $(\mathcal{U}, \mathbf{X})$, of the two points p and q of \mathcal{M}_0 .

On the other hand, the *passive* coordinate transformation is found by inverting (75):

$$y^\mu(q) := x^\mu(p) = x^\mu(q) - \lambda \xi_{(1,0)}^\mu(x(p)) - \Omega \xi_{(0,1)}^\mu(x(p)) + O^2(\lambda, \Omega), \tag{76}$$

and then by expanding $x(p)$ around $x(q)$. We obtain in this way an expression of the form

$$y^\mu(q) = x^\mu(q) - \lambda \xi_{(1,0)}^\mu(x(q)) - \Omega \xi_{(0,1)}^\mu(x(q)) + O^2(\lambda, \Omega), \tag{77}$$

which gives the relation between the coordinates of any arbitrary point $q \in \mathcal{M}_0$ in the two charts $(\mathcal{U}, \mathbf{X})$ and $(\mathcal{U}', \mathbf{Y})$. Such a relation is needed to find the transformation of the components of a tensor field, by using (74), as it is usually done in textbooks for first order gauge transformations [29, 30]. However, in order to determine these transformation rules it is much simpler to apply directly the expressions (47-60), computing explicitly the Lie derivatives of the tensor field.

5. Conclusions

Many astrophysical systems (in particular, oscillating relativistic rotating stars) can be well described by perturbation theory depending on two parameters. A well-founded description of two-parameter perturbations can be very useful for such applications, specially in order to handle properly perturbations at second order and beyond. For example, one may wish to compare results derived in different gauges.

In this paper we have studied the problem of gauge dependence of non-linear perturbations depending on two parameters, considering perturbations of arbitrary order in a geometrical perspective, and generalizing the results of the one-parameter case [2, 24] to the case of two parameters. We have constructed a geometrical framework in which a *gauge choice* is a two-parameter *group* of diffeomorphisms, while a *gauge transformation* is a two-parameter *family* of diffeomorphisms. We have shown that any two-parameter family of diffeomorphisms can be expanded in terms of Lie derivatives with respect to vectors $\xi_{(p,q)}^\mu$. In terms of this expansion, which can be deduced order by order, we have derived general expressions for transformations of coordinates and tensor perturbations, and the conditions for gauge invariance of tensor perturbations. We have computed these expressions up to fourth order in the perturbative expansion, i.e. up to terms $\lambda^k \Omega^{k'}$ with $k + k' = 4$.

The way in which the expansion of a two-parameter family of diffeomorphisms was derived in this paper is order by order, constructing derivative operators that can be rewritten as Lie derivatives with respect some vector fields. The development of an underlying geometrical structure, analogous to the knight diffeomorphisms introduced

in the one-parameter case [2], would be interesting for two reasons: first, in order to have a deeper mathematical understanding of the theory, and second, in order to derive a close formula, valid at all orders, for gauge transformations and gauge invariance conditions. The present paper has been devoted to the derivation of the useful formulae for practical applications. In particular, our expressions will be useful to compare results derived in different gauges, and can form the basis for the construction of a gauge invariant theory of two-parameter systems in the line of works done for the one-parameter case like as for example [31, 32, 33, 34, 35]. We leave the development of a more formal framework for future work.

Appendix A. Proof of the statement (29)

The aim of this Appendix is to give a proof of a theorem that allows us to make the statement contained in equation (29).

Theorem: “Let \mathcal{L} be a derivative operator acting on the set of all the tensor fields defined on a differentiable manifold \mathcal{M} and satisfying the following conditions: (i) It is linear and satisfies the Leibniz rule; (ii) it is tensor-type preserving; (iii) it commutes with every contraction of a tensor field; and (iv) it commutes with the exterior differentiation d . Then, there exists a vector field ξ such that \mathcal{L} is equivalent to the Lie derivative operator with respect to ξ , that is, \mathcal{L}_ξ .”

First of all, notice that the operators introduced in equations (14-27) satisfy the conditions of the theorem. In particular, properties (iii) and (iv) follow from the fact that Φ^* commutes with contractions and the exterior derivative (see [36]). For more details on this question see, e.g., [37].

The proof of the theorem is as follows: When acting on functions, \mathcal{L} defines a vector field ξ through the relation

$$\mathcal{L}f =: \xi(f), \quad \forall f \in \mathcal{F}(\mathcal{M}), \quad (\text{A.1})$$

where $\mathcal{F}(\mathcal{M})$ denotes the algebra of C^∞ functions on \mathcal{M} . What we want to prove is that on an arbitrary tensor field T ,

$$\mathcal{L}T = \mathcal{L}_\xi T. \quad (\text{A.2})$$

Clearly (A.2) holds for an arbitrary tensor field T iff it holds for an arbitrary vector field V . For the latter equation (A.2) is equivalent to the following expression

$$\mathcal{L}V = [\xi, V]. \quad (\text{A.3})$$

Applying this to any function f we obtain

$$(\mathcal{L}V)(f) = \xi[V(f)] - V[\xi(f)], \quad \forall f \in \mathcal{F}(\mathcal{M}). \quad (\text{A.4})$$

Therefore, to prove (A.2) is equivalent to prove (A.4). To this end, let us consider the action of the operator \mathcal{L} on the function $V(f)$. Using (A.1) we have

$$\mathcal{L}[V(f)] = \xi[V(f)]. \quad (\text{A.5})$$

On the other hand, using the properties (i)-(iv) of \mathcal{L} we have

$$\begin{aligned} \mathcal{L}[V(f)] &= \mathcal{L}(df(V)) = \mathcal{L}[\mathcal{C}(df \otimes V)] = \mathcal{C}[d(\mathcal{L}f) \otimes V + df \otimes \mathcal{L}V] \\ &= d(\mathcal{L}f)V + df(\mathcal{L}V) = V(\mathcal{L}f) + (\mathcal{L}V)(f). \end{aligned} \quad (\text{A.6})$$

Then, this in combination with (A.5), and using (A.1), leads to equation (A.4), which is what we wanted to prove.

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