Fuzzy Differential Equations for Modeling and Control of Fuzzy Systems

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Abstract. A survey of the methodologies associated with the modeling and control of uncertain nonlinear systems has been given due importance in this paper. The basic criteria that highlights the work is relied on the various patterns of techniques incorporated for the solutions of fuzzy differential equations (FDEs) that corresponds to fuzzy controllability subject. The solutions which are generated by these equations are considered to be the controllers. Currently, numerical techniques have come out as superior techniques in order to solve these types of problems. The implementation of neural networks technique is contributed in the complex way of dealing the appropriate solutions of the fuzzy systems.

Keywords: Modeling, Fuzzy Differential Equation, Fuzzy system.

1 Introduction

In recent days, many methods involving uncertainties have used fuzzy numbers [1-8], where the uncertainties of the system are represented by fuzzy coefficients. Fuzzy method is a highly favorable tool for uncertain nonlinear system modeling. The fuzzy models approximate uncertain nonlinear systems with several linear piecewise systems (Takagi-Sugeno method) [9]. Mamdani models use fuzzy rules to achieve a good level of approximation of uncertainties [10].

In comparison with the normal systems, FDEs are considered to be very noncomplex. It is feasible for them to apply directly for nonlinear control. Fuzzy control through FDEs requires solution of the FDEs. Several approaches are incorporated. Some numerical approaches, such as Nystrom method [11] and Runge-Kutta method [12] can also be implemented for resolving FDEs. Laplace transform has been utilized for second-order FDE in [13]. The results of feedback control in refer to the wave equation has been illustrated in [14], whereas the open loop control in concerned to the wave equation has been demonstrated in [15].

Neural networks can also be implemented for resolving FDEs. [16] proposed a static neural network in order to resolve FDE. [17] illustrated that the solution of ordinary differential equation (ODE) can be estimated with the help of neural network. [18] implemented neural approximations of ODEs to dynamic systems. [19] implemented dynamics neural networks for the approximation of the first-order ODE. In [20] a feed-forward neural network is suggested in order to resolve an elliptic PDE in 2D. In [21] by employing a feed forward neural network, controlled heat problem has been solved. In this paper, a survey on the numerical solutions of the PDEs and FDEs is given. The solutions which are generated by these equations are considered to be the controllers. Here, it has been presented that the roots of the mentioned equations can be extracted with different methods. The needs in mathematical modelling of efficient numerical algorithms as an alternative to classical methods of applied mathematics makes enormous progress for obtaining efficient numerical methods. In this paper, the advantages of the numerical methods in terms of accuracy is discussed. Studying of previous works by other researchers shows that no study has been done as a survey for the solutions of these equations, so that this survey will be a good beginning for those showing interest in the field of these kinds of equations.

2 Mathematical preliminaries

The following definitions are used in this paper.

Definition 1: If v is: 1) normal, there exists $\vartheta_0 \in \Re$ in such a manner $v(\vartheta_0) = 1$, 2) convex, $v(\gamma\vartheta + (1-\gamma)\vartheta) \geq \min\{v(\vartheta), v(\theta)\}, \forall\vartheta, \theta \in \Re, \forall\gamma \in [0,1]$ 3) upper semi-continuous on $\Re$, $v(\vartheta) \leq v(\vartheta_0) + \varepsilon, \forall\vartheta \in N(\vartheta_0), \forall\vartheta_0 \in \Re, \forall\varepsilon > 0, N(\vartheta_0)$ is a neighborhood, 4) $v^+ = \{\vartheta \in \Re, v(\vartheta) > 0\}$ is compact, then v is a fuzzy variable, $v \in E: \Re \to [0,1]$.

Definition 2: The fuzzy number v in association to the α -level is illustrated as

$$[v]^\alpha = \{\vartheta \in \Re, v(\vartheta) \geq \alpha\} \tag{1}$$

where $0 < \alpha \leq 1, v \in E$.

Therefore $[v]^\alpha = v^+ = \{\vartheta \in \Re, v(\vartheta) > 0\}$ Since $\alpha \in [0,1]$, $[v]^\alpha$is a bounded mentioned as $\underline{v}^\alpha \leq [v]^\alpha \leq \overline{v}^\alpha$ The α -level of v in midst of $\underline{v}^\alpha$ and $\overline{v}^\alpha$ is explained as

$$[v]^\alpha = (\underline{v}^\alpha, \overline{v}^\alpha) \tag{2}$$

$\underline{v}^\alpha$ and $\overline{v}^\alpha$ signify the function of α. We state $\underline{v}^\alpha = d_A(\alpha), \overline{v}^\alpha = d_B(\alpha), \alpha \in [0,1]$

If $v_1, v_2 \in E$, the fuzzy operations are as follows

Sum,

$$[v_1 \oplus v_2]^\alpha = [v_1]^\alpha + [v_2]^\alpha = [\underline{v_1}^\alpha + \underline{v_2}^\alpha, \overline{v_1}^\alpha + \overline{v_2}^\alpha] \tag{3}$$

subtract,

$$[v_1 \ominus v_2]^\alpha = [v_1]^\alpha - [v_2]^\alpha = [\underline{v_1}^\alpha - \underline{v_2}^\alpha, \overline{v_1}^\alpha - \overline{v_2}^\alpha] \tag{4}$$

or multiply,

$$\underline{\omega}^\alpha \leq [v_1 \odot v_2]^\alpha \leq \overline{\omega}^\alpha \; or \; [v_1 \odot v_2]^\alpha = (\underline{\omega}^\alpha, \overline{\omega}^\alpha) \tag{5}$$

where, $\overline{\omega}^\alpha = \overline{v_1}^\alpha \overline{v_2}^1 + \overline{v_1}^1 \overline{v_2}^\alpha - \overline{v_1}^1 \overline{v_2}^1$,$\underline{\omega}^\alpha = \underline{v_1}^\alpha \underline{v_2}^1 + \underline{v_1}^1 \underline{v_2}^\alpha - \underline{v_1}^1 \underline{v_2}^1$ and $\alpha \in [0,1]$.

Definition 3: The second-order singular nonlinear PDE can be illustrated by utilizing the equation mentioned below

$$\frac{\partial^2\xi(x,t)}{\partial t^2}+\frac{2}{t}\frac{\partial\xi(x,t)}{\partial t}=F(x,\xi(x,t),\frac{\partial\xi(x,t)}{\partial x},\frac{\partial^2\xi(x,t)}{\partial x^2}) \quad (6)$$

in which t and x are independent variables, ξ is the dependent variable, F is a nonlinear function of $x, \xi,\ \xi_x$ and ξ_{xx} , also the initial conditions for the PDE (6) are illustrated as below

$$\xi(x,0)=f(x),\ \xi_t(x,0)=g(x)$$

Definition 4: Consider the following controlled unknown nonlinear system

$$\dot{x}=f_1(x_1,u,t) \quad (7)$$

where $f_1(x_1,u)$ is unknown vector function, $x_1\in\Re^n$ is an internal state vector, and $x\in\Re^m$ is the input vector.

The following FDE can be used to model the uncertain nonlinear system (7),

$$\frac{d}{dt}x=f(x,u) \quad (8)$$

where $x\in\Re^n$ is the fuzzy variable that corresponds to the state x_1 in (7), $f(t,x)$ is a fuzzy vector function that relates to $f_1(x_1,u)$, and $\frac{d}{dt}x$ is the fuzzy derivative.

3 Numerical methods for solving partial and fuzzy differential equations

3.1 Predictor-corrector method

The Predictor-corrector methodology is broadly utilized in order to resolve initial value problems. In [22], three numerical methodologies for resolving fuzzy ODEs are proposed. These methodologies are Adams-Bashforth, Adams-Moulton and predictor-corrector. Predictor-corrector is extracted by blending Adams-Bashforth and Adams-Moulton methodologies. Convergence and stability of the suggested methodologies are proved. Considering the convergence order of the Euler methodology which is $O(h)$ (as given in [23]), a higher order of convergency is achievable by utilizing the suggested methodologies in [22], to be mentioned that a predictor-corrector methodology of convergence order $O(h^m)$ is utilized where the Adams-Bashforth m-step methodology and Adams-Moulton $(m-1)$ -step methodology are taken to be as predictor and corrector, respectively. By going with the ideas of [24], the suggested methodologies in [22] can resolve the stiff problems.

In [25] a numerical solution in concerned with hybrid FDE is researched. The improved predictor–corrector methodology is selected and altered in order to resolve the hybrid FDEs on the basis of the Hukuhara derivative. The symbolic systems associated with the computer to be mentioned as Maple and Mathematica are employed to carry out complex computations of algorithm. It is displayed that the solutions extracted using predictor-corrector methodology are more precise and well matched with the exact solutions.

In [26] an improved predictor-corrector method is presented in order to resolve FDE under generalized differentiability. The generalized characterization theorem is used for converting a FDE into two ODE systems. The significance of transforming a FDE to a system of ODEs is that any numerical technique which is suitable for ODEs can be applied. The improved predictor-corrector three-step methodology can be generated to improved predictor-corrector m-step methodologies of convergence order $O(h^m)$.

The predictor-corrector technique is efficient since it utilizes information from previous steps. The drawback of predictor-corrector technique is that the number of iterations so long as it approaches is unknown. Furthermore, this technique is very difficult to program. As long as the solutions for sufficient points are defined, another technique such as the Adomian decomposition technique must be utilized.

3.2 Adomian decomposition method

In [27] the Adomian decomposition method is used for finding the fuzzy solution of homogeneous fuzzy PDEs with specific fuzzy boundary and initial conditions. Seikkala derivative is utilized for resolving fuzzy heat equation with specific fuzzy boundary and initial conditions. The crisp form of heat equation is resolved by utilizing Adomian Decomposition method. After that the solution is extended in fuzzy form as a Seikkala solution.

In [28] the Adomian decomposition method is implemented for finding the numerical solution of of hybrid FDEs. This methodology considers the approximate solution of a nonlinear equation as an infinite series which generally converges to the accurate solution. The comparison between the approximation solutions and the exact solutions shows that the convergency is quite close.

The highly advantage of the Adomian decomposition technique is related to its application for all types of integral equations, linear or non-linear, homogeneous or non-homogeneous having constant coefficients or having variable coefficients. The drawback of this technique is that even though the series can be quickly convergent in a so much minute region, it has extremely slow convergence rate in the broader region, as well as the truncated series solution is an imprecise solution in that region. There are other numerical techniques for solving FDEs such as Euler technique, which is usually the next method investigated after the Adomian decomposition method. The Euler method is clear, and simple to understand.

3.3 Euler method

In [23], the FDE is substituted by its parametric form. The classical Euler technique is implemented for resolving the novel system that contains two classical ODEs with initial conditions. The capability of technique is demonstrated by resolving several linear as well as nonlinear first-order FDEs.

In [29] two improvised Euler type methodologies to be mentioned as Max-Min improved Euler methodology and average improved Euler methodology are suggested for extracting numerical solution of linear as well as nonlinear ODEs at par with fuzzy initial condition. In this paper all the possible blends of lower as well as upper bounds in concerned with the variable are considered and then resolved by the suggested methodologies. Also, an exact method is laid down.

In [30] the numerical solution associated with linear, non-linear as well as system of ODEs with fuzzy initial condition is researched. Two Euler type methodologies namely Max-Min Euler methodology and average Euler methodology are laid down for extracting numerical solution related to the FDEs. Several investigators in their works have considered the left and right bounds of the variables in the differential equations. In this paper, the investigators constructed the methodologies by taking into account all possible combinations of lower as well as upper bounds of the variable. The solution extracted by Max-Min Euler methodology very closely matches with the outcomes extracted by [23] and exact solution.

For many higher order systems, it is very difficult to make the Euler approximation effective. Euler methodology is not very accurate and stable. Neural network is comparatively simple as well as computationally rapid. Due to the superior estimation abilities of neural networks, the estimated solution for FDE is extremely near to the exact solution.

3.4 Neural network method

In [31] a technique in order to resolve both ODEs and PDEs is presented and is dependent on the function approximation abilities of feedforward neural networks. This technique results in the development of solution presented in a differentiable and closed analytic form. This form applies a feedforward neural network as the basic estimation element that its parameters (weights and biases) are adjusted to diminish a suitable error function. In order to train the network, optimization methodologies have implemented, that need the calculation of the gradient error considering the network parameters. In the suggested methodology the model function is presented as the sum of two terms. The first term suffices the initial/boundary conditions, also does not include adjustable parameters. The second term includes a feedforward neural network to be trained in order to suffice the differential equation. The implementation of a neural architecture sums up several attractive features to the technique:

1- The implementation of neural networks supplies a solution with highly superior generalized attributes. Compared results with the finite element methodology which are depicted in this work describe this point vividly.

2- The technique is simple and can be implemented to ODEs, systems of ODEs and also to PDEs stated on orthogonal box boundaries. Furthermore, the process is in advancement to deal with the case of irregular (arbitrarily shaped) boundaries.

3- The technique can be tested in hardware, utilizing neuro processors, and also it proposes the chance to handle real-time complex differential equation problems that occur in several engineering applications.

4- The technique can also be effectively imposed on parallel architectures.

This technique is simple and can be employed to both ODEs as well as PDEs by developing the suitable form of the trial solution. The technique displays superior generalization performance as the deviation at the test points is in no case major than the maximum deviation at the training points. This is in contrast with the finite element technique in the case that the deviation at the testing points is extremely higher in comparison with the deviation at the training points.

In [32] a modified technique is proposed in order to obtain the numerical solutions of fuzzy PDEs by utilizing fuzzy artificial neural networks. Utilizing modified fuzzy neural network ensures that the training points get selected over an open interval without training the network in the range of first and end points. This novel technique is on the basis of substituting each x in the training set (where $\in [a,b]$) by the polynomial $Q(x) = \epsilon(x+1)$ in such a manner that $Q(x) \in (a,b)$, by selecting an appropriate $\epsilon \in (0,1)$. Also, it can be suggested that the proposed methodology can deal efficiently with all types of fuzzy PDEs as well as to generate precise estimated solution entirely for all domain and not only at the training set.

4 Comparison of numerical methods

In this section, an example of application has been laid down in order to compare the efficiency of the numerical methods to approximate the solution of FDEs. **Example 1.** A tank with a heating system is shown in Figure 1, where $R = 0.5$ and the thermal capacitance is $C = 2$. The temperature is x. The model is [33],

$$\frac{d}{dt}x(t) = -\frac{1}{RC}x(t) \tag{11}$$

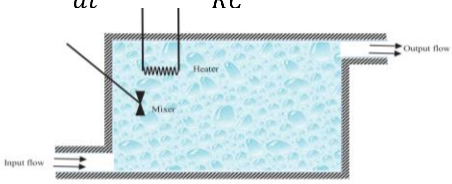

Fig. 1. Thermal system

where $t \in [0,1]$ and x is the amount of sinking in each moment. If the initial position is $x(0) = (\alpha-1, 1-\alpha)$ and $\alpha \in [0,1]$, then the exact solutions of (9) are

$$x(t,\alpha) = [(\alpha-1)e^t, (1-\alpha)e^t] \tag{10}$$

To approximate the solution (10), we use four popular methods: Predictor-corrector method, Adomian decomposition method, Euler method, and Neural network method. The errors of these methods are shown in Table 1. The lower and upper bounds of absolute errors are shown in Figure 2 and Figure 3 respectively. The approximation errors of the neural network method is smaller than the other methods.

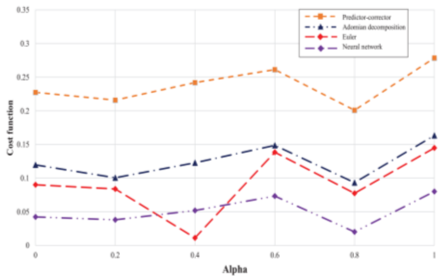

Fig. 2. The lower bounds of absolute errors

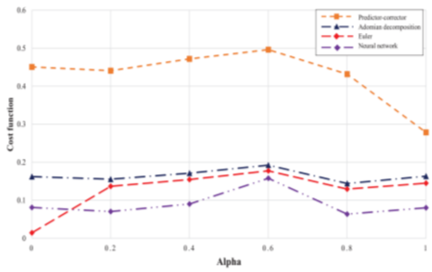

Fig. 3. The upper bounds of absolute errors

Table 1. Approximation errors

α	Predictor-corrector	Adomian decomposition	Euler	Neural network
0	[0.2275,0.4507]	[0.1196,0.1623]	[0.0902,0.1423]	[0.0423,0.0812]
0.2	[0.2159,0.4407]	[0.1006,0.1553]	[0.0839,0.1367]	[0.0381,0.0702]
0.4	[0.2419,0.4718]	[0.1228,0.1713]	[0.0112,0.1545]	[0.0519,0.0901]
0.6	[0.2613,0.4962]	[0.1486,0.1923]	[0.1385,0.1773]	[0.0734,0.1578]
0.8	[0.2009,0.4319]	[0.0933,0.1441]	[0.0774,0.1295]	[0.0201,0.0635]
1	[0.2785,0.2785]	[0.1633,0.1633]	[0.1448,0.1448]	[0.0801,0.0801]

5 Conclusions

In this paper, some of numerical methodologies are demonstrated as a solution of PDEs and FDEs. This survey illustrates that the roots of the differential equation can be extracted with different algorithms. However, in few cases there exist no roots in differential equation. For obtaining the roots of system in a case that there is no exact solution, iteration methodologies can be utilized for estimating the solution. This survey supplies an input for those showing interest in the field of differential equations.

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