A GENERALIZED BENDERS DECOMPOSITION BASED ALGORITHM FOR AN INVENTORY LOCATION PROBLEM WITH STOCHASTIC INVENTORY CAPACITY CONSTRAINTS

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ABSTRACT

This paper deals with an inventory location problem with order quantity and stochastic inventory capacity constraints, which is of a nonlinear, nonconvex mixed integer programming nature. The problem integrates strategic supply chain networks design decisions (i.e., warehouse location and customer assignment) with tactical inventory control decision for each warehouse (i.e., order size and reorder point). A novel decomposition approach, that deals with the nonconvex nature of the problem formulation is proposed and implemented, based on the Generalized Benders Decomposition. The proposed decomposition yields a master problem that addresses warehouses location and customer assignment decisions, and a set of underlying subproblems that deal with warehouse inventory control decisions. Based on this decomposition, nonlinearity of the original problem is captured by the subproblems that are solved at optimality, while the master problem is a mixed integer linear programming problem. The master is solved using a commercial solver, the subproblems are solved analytically by inspection, and cuts to be added into the master problem are obtained based on dual Lagrangian information. Optimal solutions were found for 160 instances in competitive times.

Keywords: Location; Generalized Benders Decomposition; Mixed Integer Nonconvex-Nonlinear Programming; Capacitated Inventory Location Problems; Strategic Supply Chain Network Design.

1. INTRODUCTION

Optimization models have been widely developed and employed in order to support decision making processes belonging to each organizational level. Over the years, mathematical models have become a key element for different organizations or industries. Nevertheless, despite the growing in computer capacities, the use of efficient analytic or algorithmic tools for solving these problems in competitive times is mandatory. These solution tools can be generic (i.e., for a wider class of problems) or specialized (i.e., for a specific class of problems), and the performance of these tools are usually assessed considering the solution quality and the computational times. A great number of works on operational research, and particularly this work, are focused on improving these performance indicators for several relevant problems in literature.

Furthermore, optimization models have been traditionally developed to support decisions related to specific problems that consider only a partial branch of the organization, yielding a partial system optimization, as it can be expected. Accordingly, the integration of decisions began as a new trend to develop mathematical models. This integration normally is achieved by considering decisions from different organizational levels or decisions in the same organizational level but made separately. Optimization models that integrate decisions might reach better solutions than models that are addressed separately, where in latter local optimums at organizational level may be obtained. Unfortunately, the integration of decisions typically generates models with a greater complexity.

An interesting example of the previous issues is the research on Inventory Location Problems (ILPs), which is also the focus of this research. In the last two decades a variety of ILPs have been proposed and studied, which integrate strategic facility location decisions (long term decisions) and those decisions related to supply chain inventory managing and planning (medium term decisions). Thus, ILPs are standard approaches to address long term supply chain network optimization problems, similar to Facility Location Problems (FLPs), which are the base or foundation of all the existent ILPs. Accordingly, tactical and operational decision making have to be addressed given the SCN topology obtained by the strategic models (Bitran et al., 1981, 1982; Hax and Candrea, 1984; Mourtis and Evers, 1995; Bradley and Arntzen, 1999; Miranda and Garrido, 2004).

This integration, which is proposed in ILP literature, usually yields Mixed Integer Nonlinear Programming Problems that require efficient solution approaches to solve them. Particularly, Benders Decomposition has been successfully developed and applied for solving mixed-integer linear problems using decomposition, projection and dualization (Benders, 1962; Rahmaniani, et al., 2017). This approach decomposes a problem into a master problems and a subproblem by separating the decision variables in two groups, where one set of variables is addressed by the Master Problem (MP) and the second set of variables belongs to the Subproblem (SP). Some years after the Benders's publication a generalization to deal with nonlinear, convex problems was developed, named Generalized Benders Decomposition (Geoffrion, 1972).

In this research a Benders Decomposition based solution approach is proposed and implemented for solving an Inventory Location Problem with Stochastic Inventory Capacity Constraints. The proposed decomposition generates a mixed-integer MP that is solved using a commercial solver, and a set of nonlinear SPs, which are solved analytically by inspection. This decomposition deals successfully with the nonlinearity and nonconvexity of the original formulation, in spite of the decomposition presented by Geoffrion (1972), which is focused on convex problems. Furthermore, global optimality is ensured based on the global convergence for all problems involved (MP and SPs) with a zero-gap certificate.

This document is organized as follows. The literature review of related topics is presented in Section 2. In Section 3 the mathematical modeling of the Inventory Location problem considered in this work is presented. Section 4 presents the proposed algorithm based on Generalized Benders Decomposition applied to the model explained in Section 3. Computational experimentation and results are presented and discussed in Section 5. Finally, Section 6 presents the conclusions of this work and a future research discussion.

2. LITERATURE REVIEW

The problem of locating different types of facilities has generated great interest in Operation Research and Management Science communities. Traditionally, FLPs consider a set of spatially distributed customers and a set of potential facilities to fulfill the customers demand. A great number of the FLPs models deal with the location of different types of industrial facilities (Daskin, 1995; Owen and Daskin, 1998; Drezner and Hamacher, 2002; Melo, et al., 2009; Eiselt and Marianov 2011, 2015; Drezner, 2014). Facility location decisions tend to be costly and their impact spans a long term horizon, and the optimal location for today may not be optimal under future conditions (Coyle, et al., 2003; Snyder, 2006).

The fierce competitiveness of markets forces the organizations to focus on their Supply Chains (SC), as stated in Simchi-Levi, et al. (2003). Supply Chain Management (SCM) involves decisions about a set of key elements (i.e., activities, processes and resources) required to be made in an efficient and timely manner. It is difficult to conceive SCM without considering mathematical models to support the planning, implementing and controlling the operations efficiently (Simchi-Levi, et al., 2004). Decisions involved are traditionally classified into three hierarchical levels: strategic (long term), tactical (medium term) and operational (short term). Designing the SC network structure has a significant impact into the overall performance and competitiveness (Miranda and Garrido, 2004; Shen, 2007; Melo et al., 2009; Farahani, et al., 2014).

Traditionally decisions belonging to different decisional levels are treated separately (Shen, 2007). Most organizations make decisions in a hierarchical and sequential mode leading that may lead to global sub-optimums (Fahimia, et al., 2013). Naturally, if the different elements of Supply Chain Network (SCN) are optimized separately the overall optimality might be unwarranted (Pourhejazy and Kwon, 2016).

SCN design is considered a strategic problem, consisting of determining facility locations (plants or warehouses), in order to meet customers demand at a minimum cost (Daskin, 1995; Owen and Daskin, 1998; Drezner and Hamacher, 2002; Melo, et al., 2009; Coyle et al., 2009; Perez-Loaiza, et al., 2017). Inventory management and facility location represent two relevant issues that must be addressed to efficiently and effectively design the SCN (Diabat, et al., 2015). Accordingly, ILPs are aimed to integrate the optimization of the key decision variables of inventory control and the location of facilities to design the SCN (Pourhejazy and Kwon, 2016). The development of models that integrate location and inventory control decisions has grew in the last years (Ağrah, et al., 2012). Shen, Z.J. (2007) shows an interesting review on the integrated supply chain design models considering different assumptions and modeling approaches used to develop some of the most popular models in this area. Farahani, et al. (2015) gives a comprehensive literature review on ILPs considering their modeling considerations, solution approaches and the application in different real contexts.

Normally ILPs integrate strategic decisions with tactical/operational decisions of SC. The review of Farahani et al. (2015) shows that many ILPs consider simultaneously the facility location and the management of a predefined inventory policy. Jayaraman (1998) analyzes the relationships among transportation, facility location and inventory issues and present a Mixed Integer Programming Model that integrates these three concerns. Later, Erlebacher and Meller (2000) presents a Mixed Integer Nonlinear problem considering the facility location and inventory control policies. Daskin et al. (2002) and Miranda and Garrido (2004) include safety stock due to variability of the customers demand into the model. Shen et al. (2003) includes the risk pooling into the mixed integer nonlinear model, this model is also reformulated as a set-covering problem. Miranda and Garrido (2006) integrates stochastic capacity constraints (order quantity and inventory) using a chance constrains approach to formulate it. Oszen, et al. (2008) presents an intuitive approach to build the capacity constraints that can be derived from a chance constraint formulation. Miranda and Garrido (2008) introduces some valid inequalities into the solution approach. Oszen, et al. (2009) considers a centralized logistic system where retailers can be sourced by more than one warehouse. Miranda and Cabrera (2010) presents a novel problem with stochastic capacity constraints considering a periodic review policy for the inventories. Escalona et al. (2015) considers a differentiated service level considering two demand classes using a critical level policy. Finally, recent ILPs with novel logistics and transportation strategies (multi-sourcing and reverse logistic strategies) are presented in Amiri-Aref et al. (2017) and Ross et al. (2017).

A great number of papers focused on ILPs have used the Economic Order Quantity model (EOQ) to define the replenishment decisions at the warehouses or distribution centers. EOQ model is an important tool to balance the involved costs (i.e., ordering and holding costs). This theory was developed by Harris (1913) but some years later become as a robust tool applied in many contexts. Many models have been developed modifying the basic formula or other approaches trying to reach more suitable solutions for real problems (Pereira and Costa, 2014). The basic models of inventory control policy based on EOQ theory are clearly developed in Coyle et al. (2009), Hillier and Lieberman (2005), Chase, et al. (2004), Ballou (1999) among many other documents.

Integrating decisions that traditionally are treated separately tends to generate models with a higher complexity. Thus, the development and application of efficient solution approaches to solve these integrated models is required. The most popular solution approaches developed to solve ILPs have been Lagrangian relaxation and greedy heuristic based algorithms (Ağrah, et al., 2012). Daskin et al. (2002), Miranda and Garrido (2004, 2006, 2008), Snyder, et al. (2007) and Oszen et al. (2008) present different Lagrangian relaxation algorithms for different ILPs. Erlebacher and Meller (2000) proposes a set of algorithms based on greedy heuristic approaches. Shen et al. (2003) reformulates the problem into a set-covering formulation and develops a column generation based algorithm to solve it. Diabat, et al. (2015) presents an improved Lagrangian relaxation-based heuristic considering a multi-echelon ILP. An algorithm based on BD is used by Wheatley et al. (2015) to solve an uncapacitated ILP with nonlinear service constraints, which are derived by considering demand fill rate. An algorithm based on Generalized Benders Decomposition is presented in Ağrah, et al. (2012) considering an uncapacitated ILP with a multi-sourcing approach where a hybrid algorithm based on outer approximation to solve the subproblem is used. A comprehensive literature review of the most used solution approaches in ILPs contexts is presented in Schuster and Tancrez (2017) and Diabat, et al. (2015).

This research presents a Generalized Benders Decomposition based algorithm to solve the studied nonconvex, nonlinear ILP at optimality. Generalized Benders Decomposition (GBD) was developed by Geoffrion (1972), as a generalization of Bender Decomposition (BD) presented by Benders (1962), to solve nonlinear, convex models. BD was developed for solving a class of linear and mixed integer linear programming models. BD is a classical solution approach based on the decomposition scheme and iterative constraints generation (Costa, 2005). One of the principles used for BD is that the set variables of the problem can be classified under two types, complicating and noncomplicating variables. It is considered that the problem is much easier to solve when the complicating variables are temporarily fixed. Considering a set of fixed feasible values for the complicating variables, it is possible to solve the problem for the non-complicating variables. The decomposition generates two different problems: The Master Problem and the Subproblem. The Master Problem (MP) includes only the complicating variables as decisions and Subproblem (SP) only considers the noncomplicating variables as decisions. The iterative process uses the dual optimal information of SP to generate cuts that are added into the MP. If a model has at least one nonconvex function (i.e., objective function or constraints) neither BD nor GBD cannot guarantee optimality convergence due to the loss of strong duality (Li, et al., 2011). Li et al., (2014) proposes the Nonconvex Benders Decomposition to deal nonconvex problems based on convexification of the problem and the use of the solution algorithm based on the algorithm proposed by Geoffrion (1972). As the SP generated by the decomposition proposed in this paper is nonlinear the dual problem is obtained using the Lagrangian Dual problem. The optimal value of the dual problem is obtained using the Karush-Khun-Tucker (KKT) conditions. The related theoretical foundations are deeply explained in Bazaraa, et al. (1993), Bertsekas (1999) and other seminar documents focused on nonlinear programming and nonlinear theory.

3. A CAPACITATED INVENTORY LOCATION PROBLEM

The main focus of this paper is to present a novel algorithm to solve a Capacitated ILP, which is described and presented in this Section.

3.1 PROBLEM DESCRIPTION AND ASSUMPTIONS

The studied ILP, previously proposed in Miranda and Garrido (2006, 2008), considers jointly decisions and costs of warehouses location, customer assignment and inventory control for each warehouse in a single-commodity case. It is assumed that a single plant, in a fixed and known location, serves the set of selected or located warehouses. End customers present high volume stochastic demands, which are represented by their means and variances. Notice that each customer is assumed to be an aggregation of a set of end customers within a specific zone (Current and Schilling, 1990; Francis et al., 2004; Emir-Farinas and Francis, 2005, Caniato et al., 2005).

The model aims to support a long term SCN design problem, focused on warehouse location decisions and demand zone assignments. Naturally, this model can be used both to design a new SCN or to periodically analyze and re-optimize the SCN. The problem is aimed to minimize a long-term estimation of system costs including warehouse settings, transportation and inventory costs. The focus is not to optimize the inventory levels in short term, but instead to minimize expected long term system costs, including inventory costs, which are strongly dependent on network topology (i.e., warehouse location and customer assignment), as it has been widely studied in inventory-location literature (see Section 2).

Given the presence of stochastic demands, each warehouse must hold a safety stock to ensure a given service level (modeled as a stock-out probability), in addition to cycling inventory levels in this case, following the well known EOQ model (Erlebacher and Meller, 2000; Daskin et al., 2002; Shen et al., 2003; Miranda and Garrido, 2004). According to high volume demands (Escalona et al., 2015), a Normal approximation is employed to represent the behavior of warehouse demands.

The model considers a continuous review-inventory control policy for each warehouse with a fixed lot size Q and a reorder point r, where both are decision variables of the model. A single steady-state period is considered were all parameters and variables are not time dependent. The model integrates two capacity constraints; the first one focused on the maximum inventory levels, which is a probabilistic constraint, while the second one is focused in order sizes for each warehouse.

Natural and necessary extensions to this model are multi-period and multi-commodity formulations, allowing to model more realistic cases, mainly focused on real world industrial application. However, these extensions remain as a future research that should be based on methodological contributions of this paper and previous ILP literature.

3.2 MATHEMATICAL FORMULATION

This Section presents the mathematical formulation of the studied problem, following Miranda and Garrido (2006, 2008). Subsequently, some additional constraints are integrated into the formulation, in order to make it more mathematically tractable within the proposed solution approach.

Model decision variables are:

X_i : Binary variable, takes the value 1 if a warehouse is allocated in site *i*, 0 otherwise.
Y_{ij} : Binary variable, takes the value 1 if the customer *j* is assigned to the warehouse *i*, 0 otherwise
D_i : Mean of the demand assigned to the warehouse *i*
V_i : Variance of the demand assigned to the warehouse *i*
Q_i : Order quantity of the warehouse *i*

Parameters and sets of the model are:

N : Set of potential warehouses
M : Set of customers
d_j : Mean of the demand of the customer *j*
v_j : Variance of the demand of the customer *j*
FC_i : Operational and setting fixed cost of warehouse on the location *i*
RC_i : Unitary transportation cost between the plant and the warehouse *i*
TC_{ij} : Fixed transportation cost between the warehouse *i* and the customer *j*
AC_{ij} : Assignment cost of customer *j* to warehouse *i*, $AC_{ij} = RC_i \cdot d_j + TC_{ij}$
OC_i : Ordering cost of the warehouse *i*
HC_i : Unitary holding inventory cost of the warehouse *i*
LT_i : Lead-time of the warehouse *i*
$Z_{1-\alpha}$: Standard normal distribution value that accumulate $1-\alpha$
$Z_{1-\beta}$: Standard normal distribution value that accumulate $1-\beta$
$Q_{\max}^i$: Maximum order capacity of the warehouse *i*
$ICap_i$: Maximum inventory capacity of the warehouse *i*

The original mathematical formulation is as follows:

$$Min \quad \sum_{i\in N} FC_i \cdot X_i + \sum_{i\in N}\sum_{j\in M} AC_{ij} \cdot Y_{ij} + \sum_{i\in N}\left[\frac{OC_i \cdot D_i}{Q_i} + \frac{HC_i \cdot Q_i}{2} + HC_i \cdot Z_{1-\alpha}^i \cdot \sqrt{LT_i} \cdot \sqrt{V_i}\right] \quad (1)$$

s.t.:

$$\sum_{i \in N} Y_{ij} = 1 \qquad \forall j \in M \qquad (2)$$

$$Y_{ij} \le X_i \qquad \forall i \in N, \forall j \in M \qquad (3)$$

$$D_i = \sum_{j \in M} Y_{ij} \cdot d_j \qquad \forall i \in N \qquad (4)$$

$$V_i = \sum_{j \in M} Y_{ij} \cdot v_j \qquad \forall i \in N \qquad (5)$$

$$Q_i \le Q_{\max}^i \qquad \forall i \in N \qquad (6)$$

$$Q_i + \left(Z_{1-\alpha}^i + Z_{1-\beta}^i\right) \cdot \sqrt{LT_i} \cdot \sqrt{V_i} \le ICap_i \cdot X_i \qquad \forall i \in N \qquad (7)$$

$$X_i \in \{0,1\} \qquad \forall i \in N \qquad (8)$$

$$Y_{ij} \in \{0,1\} \qquad \forall i \in N, \forall j \in M \qquad (9)$$

Expression (1) is the total costs function to be minimized. The first term represents the fixed setting and operating costs for all installed warehouses. The second term is the total assignment costs (unitary and fixed transportation costs). The third term represents the costs of the inventory policy (ordering costs and holding costs of cycle inventory and safety stock). Equations (2) ensure that each customer is served by a single warehouse. Constraints (3) ensure that the customers are assigned to an installed warehouses. Constraints (4) and (5) compute demand mean and variance for each warehouse. Set of constraints (6) represent the maximum values for order sizes. Equations (7) ensure that the maximum inventory levels for each ordering period observe the available inventory capacity at least with a probability 1-β. Constraints (8) and (9) state the binary domain of the decision variables (X and Y). Notice that safety stock costs in expression (1), and inventory capacity constraints in equation (7), are derived based on Chance Constraint Programming, given the existence of stochastic demands and inventory levels, and assuming Normal demand behavior for the warehouses.

This work considers two additional constraints, in order to avoid solutions that yield pitfalls arisen in a previous preliminary implementation of the proposed decomposition.

$$\sum_{j=1}^{M} Y_{ij} \cdot v_j \le V_{\max}^i \qquad \forall i \in N \qquad (10)$$

$$X_i \le \sum_{j=1}^{M} Y_{ij} \qquad \forall i \in N \qquad (11)$$

where:

$$V_{\max}^i = \left(\frac{ICap_i}{\left(Z_{1-\alpha} + Z_{1-\beta}\right) \cdot \sqrt{LT_i}} \right)^2 \qquad \forall i = 1, \ldots N \qquad (12)$$

Constraints (10) ensure subproblem feasibility within the proposed decomposition (as described in next section), where $V_{\max}^{i}$ is defined by expression (12). The right side of this expression is obtained through a mathematical manipulation of constraints (7) and represents a maximum feasible value for a warehouse demand variance based on inventory capacity constraint.

The set of constraints (11) avoid solutions that generate pitfalls in the iterations of the proposed algorithm. Particularly, these constraints avoid solutions in which some warehouses are selected ($X_i = 1$) and no customer are assigned to it. Otherwise, related dual variables cannot be computed properly. Notice that these constraints are not actually valid inequalities, indeed avoid feasible solutions that are not reasonable in practical terms, and also they are not optimal: for a solution that has a selected warehouse with no customers, it is always preferable to close it, thus yielding a system costs reduction (assuming $CF_i > 0$, $\forall i = 1,\ldots,N$).

4. BENDERS DECOMPOSITION BASED SOLUTION APPROACH

The aim of the proposed GBD based approach is to decompose the original problem in such way that the MP retains the NP hardness related to MILP structure of the problem, the SPs absorb the nonlinearity of the problem, and thus ensuring a zero duality gap based on solving SP at optimality. The last property relies on GBD ensures optimality (zero duality gap) if and only if the SPs presents strong duality and the master problem is solved exactly.

4.1 GENERAL ALGORITHM

The proposed algorithm based on GBD is as follows:

Step 1 (Initializing): Temporarily fix warehouse location and customer assignment decisions, yielding a subproblem which is equivalent to the original problem but only considering inventory control decisions as variables:

- The MP is defined by considering only the set of variables previously fixed as decision variables (warehouse location and customer assignment), and only the set of constraints from the original problem that involve these variables. This MP must integrate a set of cuts or constraints that ensure feasibility and optimality for the original problem.
- Feasibility and optimality cuts or constraints to be added into the MP are iteratively built up and added, until feasibility and optimality conditions for the original problem may be guaranteed. Given that constraints (10) and (11) are integrated into the formulation, any feasible solution of the MP yield always a feasible solution of the SP, and then only optimality cuts are going to be integrated into the MP.

Step 2: Solve the SP in terms of inventory control decisions variables, thus obtaining the related optimal dual variables.

Step 3: Build a new cut or constraint to be added into the MP, based on the optimal SP solutions (i.e., primal and dual variable values).

Step 4: Solve the MP with all added constraints, obtaining a new set of values for warehouse location and customer assignment decision variables.

- If the new values of MP decision variables (warehouse location and customer assignment) are equal to the obtained values in the previous algorithm iteration, then feasibility and optimality properties for the original problem can be guaranteed, and the algorithm ends.
- Otherwise, the SP must be solved once again based on these new values of MP decision variables as fixed, in other words, go to Step 2.

The proposed decomposition ensures zero duality gap for the original problem by ensuring the convergence of the MP and considering that this solution, providing a lower bound of the original problem, presents a zero duality Gap, due to global optimization conditions for the SP for every algorithm iteration.

4.2 DERIVATION OF THE SUBPROBLEM (SP)

Following definitions in Benders (1962) and Geoffrion (1972), we consider the binary variables (X,Y) are considered as the "complicating variables" (i.e., decision variables of the MP); consequently, variables (D,V,Q) are embraced by the SP.

Let $(\bar{X},\bar{Y})$ be a vector of feasible values for the variables (X,Y) considering constraints (2), (3), (8), (9), (10) and (11). Then, the SP can be written as follows:

$$Min \quad \rho(\bar{X},\bar{Y})+\sum_{i\in N}\varphi_i(D_i,V_i,Q_i) \tag{13}$$

s.t.:

$$D_i=\hat{D}_i \qquad \forall i\in N \tag{14}$$

$$V_i=\hat{V}_i \qquad \forall i\in N \tag{15}$$

$$Q_i\le Q^i_{\max} \qquad \forall i\in N \tag{6}$$

$$Q_i\le \hat{Q}_i \qquad \forall i\in N \tag{16}$$

where:

$$\rho(X,Y)=\sum_{i\in N}FC_i\cdot X_i+\sum_{i\in N}\sum_{j\in M}AC_{ij}\cdot Y_{ij} \tag{17}$$

$$\varphi_i(D_i, V_i, Q_i) = \frac{OC_i \cdot D_i}{Q_i} + \frac{HC_i \cdot Q_i}{2} + HC_i \cdot Z_{1-\alpha} \cdot \sqrt{LT_i} \cdot \sqrt{V_i} \tag{18}$$

$$\hat{D}_i = \sum_{j \in M} \bar{Y}_{ij} \cdot d_j \tag{19}$$

$$\hat{V}_i = \sum_{j \in M} \bar{Y}_{ij} \cdot v_j \tag{20}$$

$$\hat{Q}_i = ICap_i \cdot \bar{X}_i - \left(Z_{1-\alpha} + Z_{1-\beta}\right) \cdot \sqrt{LT_i} \cdot \sqrt{\bar{V}_i} \tag{21}$$

According to equation (17), the first term in equation (13) represents the part of the total cost function in equation (1), associated to the variables (X, Y) and evaluated in $(\bar{X}, \bar{Y})$. For fixed values of variables (X, Y) this term becomes constant, and then the SP is solved without considering it. It is remarkable that this SP is nonlinear and convex, although the original problem is nonconvex.

Notice that $(\bar{X}, \bar{Y})$ may yield a feasible or an infeasible solution for the original problem (1)-(9). However, given that constraints (10) and (11) are integrated into the problem formulation and also into the MP, always a feasible solution can be found.

4.3 SOLVING THE SUBPROBLEM

Before to solve the SP, it is decoupled into a set of independent subproblems, one SP for each warehouse *i*, SP_i ($i = 1, \ldots, N$) as shown in (22). The same as the original SP, each SP_i is of nonlinear, convex nature. These subproblems are solved analytically by inspection (or equivalently following Theil-Van de Panne conditions, 1960), as explained bellow. Then optimal dual variables are determined based on a simple and direct application of the well know KKT conditions.

$$\begin{aligned} Min \quad & \varphi_i(D_i, V_i, Q_i) \\ s.t.: \quad & \\ & D_i = \hat{D}_i \\ & V_i = \hat{V}_i \\ & Q_i \le Q_{\max}^i \\ & Q_i \le \hat{Q}_i \end{aligned} \tag{22}$$

To solve each SP_i, let $(\bar{D}, \bar{V})$ be the optimal value for (D, V) in (22). Notice that although (D, V) are indeed SP decision variables, its values, $(\hat{D}, \hat{V})$, can be known in advance based on equations (19) y (20). Accordingly, $(\bar{D}, \bar{V})$ are in addition the optimal values of (D, V).

Subsequently, the optimal value of Q, $\bar{Q}$, is determined analytically by inspection based on the well known EOQ model but observing capacity constraints. Therefore, the optimal value of (D_i, V_i, Q_i) is computed as:

$$\bar{D}_i = \hat{D}_i \tag{23}$$

$$\bar{V}_i = \hat{V}_i \tag{24}$$

$$\bar{Q}_i = \min\left\{Q_i^*, \hat{Q}_i, Q_{\max}^i\right\} \tag{25}$$

where:

$$Q_i^* = \sqrt{\frac{2 \cdot OC_i \cdot \bar{D}_i}{HC_i}} \tag{26}$$

For each set of constraints in SP a vector of dual variables is defined, independent of the way in which SP is decoupled and solved. Let λ_1, λ_2, μ_1 and μ_2 be the dual variables assigned to constraints (14), (15), (6) and (16), respectively. These variables are used as dual multipliers to build a lagrangian dual problem. The domain of each variable depends on the nature of the associated constraint. Specifically, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\mu_1, \mu_2 \geq 0$.

Given that every SP is a nonlinear problem, Geoffrion (1962) considers the lagrangian dual problem where all the constraints are added into the lagrangian function.

Following Geoffrion (1962), Bazaara (1993) and Wolsey and Nemhauser (1999), for the general case shown in (27), the associated lagrangian dual problem is presented in (28).

$$\begin{aligned} & Min \quad f(x) \\ & s.t: \\ & \qquad g(x) \leq 0 \\ & \qquad h(x) = 0 \\ & \qquad x \in X \end{aligned} \tag{27}$$

$$\underset{\mu \geq 0, \lambda \in \mathbb{R}}{Max} \quad \inf_{x \in X} \left\{ f(x) + \mu^T \cdot g(x) + \lambda^T \cdot h(x) \right\} \tag{28}$$

Accordingly, the lagrangian dual problem associated with the SP can be written as:

$$\underset{\mu \geq 0, \lambda \in \mathbb{R}}{Max} \quad \inf_{D,V,Q} \left\{ \sum_{i \in N} \varphi_i\left(D_i, V_i, Q_i\right) + \mu^T \cdot g\left(D, V, Q\right) + \lambda^T \cdot h\left(D, V, Q\right) \right\} \tag{29}$$

where:

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \tag{30}$$

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{31}$$

$$g(D,V,Q) = \begin{pmatrix} Q - Q_{\max} \\ Q_i - \hat{Q} \end{pmatrix} \leq \vec{0} \tag{32}$$

$$h(D,V,Q) = \begin{pmatrix} \hat{D} - D \\ \hat{V} - V \end{pmatrix} = \vec{0} \tag{33}$$

Beside these definitions, and according to the characterization made on (18), the first term in (29) is the summation of the objective function of each SP_i. For a general problem as is shown in (34) the necessary conditions of KKT can be expressed as is shown in (35) (Bazarra, 1993).

$$\underset{x \in X}{Min} \left\{ f(x) \,/\, g_{i=1,\dots,l}(x) \leq 0, h_{j=1,\dots,k}(x) = 0 \right\} \tag{34}$$

$$\begin{aligned} & \nabla f(x) + \sum_{i=1}^{l} \mu_i \cdot \nabla g(x) + \sum_{j=1}^{k} \lambda_j \cdot \nabla h(x) = 0 \\ & \mu_i \cdot g(x) = 0 \qquad \forall i = 1,\dots,l \\ & \mu_i \geq 0 \,,\; \lambda_j \in \mathbb{R} \qquad \forall i = 1,\dots,l, \forall j = 1,\dots,k \end{aligned} \tag{35}$$

Applying these conditions for each SP_i and considering the optimal values $\left(\bar{D},\bar{V},\bar{Q}\right)$, yields the equation system shown in (36). Solving this equation system allows to obtain the optimal values for the dual variables of every SP_i, $\left(\bar{\lambda},\bar{\mu}\right)$.

$$\begin{pmatrix} \dfrac{OC_i}{\bar{Q}_i} \\ \dfrac{HC_i \cdot Z_{1-\alpha} \cdot \sqrt{LT_i}}{2 \cdot \sqrt{\bar{V}_i}} \\ -\dfrac{OC_i \cdot \bar{D}_i}{\bar{Q}_i^2} + \dfrac{HC_i}{2} \end{pmatrix} + \mu_{1i} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu_{2i} \cdot \begin{pmatrix} 0 \\ \dfrac{\left(Z_{1-\alpha} + Z_{1-\beta}\right) \cdot \sqrt{LT_i}}{2 \cdot \sqrt{\bar{V}_i}} \\ 1 \end{pmatrix} + \lambda_{1i} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_{2i} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = 0 \tag{36-a}$$

$$\mu_{1i} \cdot \left(\bar{Q}_i - Q_{\max}^i \right) = 0 \tag{36-b}$$

$$\mu_{2i} \cdot \left(\bar{Q}_i + \left(Z_{1-\alpha} + Z_{1-\beta} \right) \cdot \sqrt{LT_i} \cdot \sqrt{\bar{V}_i} - ICap_i \cdot \bar{X}_i \right) = 0 \tag{36-c}$$

$$\mu_{1i}, \mu_{2i} \geq 0 \;,\; \lambda_{1i}, \lambda_{2i} \in \mathbb{R} \tag{36-d}$$

In the general BD or GBD algorithm, when MP decision variables are fixed the SP may be feasible or not:

- If the SP is feasible then there are two possible cases. The first case is when the SP has at least one optimal and bounded solution, in which an optimality cut must be added into the MP. The second case occurs when the SP is unbounded, case in which the algorithm ends due to the original problem is unbounded too.

- If the SP is infeasible then a feasibility cut must be added into the MP.

However, by adding constraints (10) and (11) to the MP the feasibility of each SP_i is assured, and moreover each SP is bounded. Thus, only optimality cuts are required to be added into the MP.

4.4 OPTIMALITY CUTS

Once the optimal primal and dual variables values $\left(\bar{D},\bar{V},\bar{Q},\bar{\lambda},\bar{\mu}\right)$ are obtained, as is shown in Section 4.3, it is possible to generate an optimality cut to be added into the MP, as shown in (36), where Z is the objective function of the MP.

$$\begin{aligned} Z \geq \rho\left(X,Y\right)+\sum_{i=1}^{N}\varphi\left(\bar{D}_i^p,\bar{V}_i^p,\bar{Q}_i^p\right)+\sum_{i=1}^{N}\bar{\mu}_{1,i}^p\cdot\left(\bar{Q}_i^p-Q_{\max}^i\right)+\sum_{i=1}^{N}\bar{\mu}_{2,i}^p\cdot\left(\bar{Q}_i^p+\delta_i\cdot\sqrt{\bar{V}_i^p}-ICap_i\cdot X_i\right) \\ +\sum_{i=1}^{N}\bar{\lambda}_{1,i}^p\cdot\left(\sum_{j\in M}Y_{ij}\cdot d_j-\bar{D}_i^p\right)+\sum_{i=1}^{N}\bar{\lambda}_{2,i}^p\cdot\left(\sum_{j\in M}Y_{ij}\cdot v_j-\bar{V}_i^p\right) \end{aligned} \tag{36}$$

Accordingly, the MP at each iteration k can be written as follow:

$$Min \quad Z \tag{37}$$

s.t.:

$$\sum_{i\in N}Y_{ij}=1 \qquad \forall j\in M \tag{2}$$

$$Y_{ij}\leq X_i \qquad \forall i\in N,\forall j\in M \tag{3}$$

$$\begin{aligned} Z \geq \rho\left(X,Y\right)+\sum_{i=1}^{N}\left[\varphi\left(\bar{D}_i^p,\bar{V}_i^p,\bar{Q}_i^p\right)+\bar{\lambda}_{1,i}^p\cdot\left(\sum_{j\in M}Y_{ij}\cdot d_j-\bar{D}_i^p\right)\right] \\ +\sum_{i=1}^{N}\left[\bar{\lambda}_{2,i}^p\cdot\left(\sum_{j\in M}Y_{ij}\cdot v_j-\bar{V}_i^p\right)+\bar{\mu}_{1,i}^p\cdot\left(\bar{Q}_i^p-Q_{\max}^i\right)\right] \qquad \forall p\in P^k \\ +\sum_{i=1}^{N}\left[\bar{\mu}_{2,i}^p\cdot\left(\bar{Q}_i^p+\delta_i\cdot\sqrt{\bar{V}_i^p}-ICap_i\cdot X_i\right)\right] \end{aligned} \tag{38}$$

$$X_i\in\{0,1\} \qquad \forall i\in N \tag{8}$$

$$Y_{ij}\in\{0,1\} \qquad \forall i\in N,\forall j\in M \tag{9}$$

The set P^k in (38) represents the set of cuts obtained and added into the MP after k algorithm iterations. For the initial iteration $P^{k=0}=\varnothing$, and the MP is unbounded $\left(Z=-\infty\right)$. Thus, an auxiliary optimization problem is built and solved to generate an initial MP feasible solution and to start the algorithm, as described in the following section.

5. COMPUTATIONAL IMPLEMENTATION AND RESULTS

The computational application of the proposed approach was made considering 160 instances. These instances were created from 5 base instances. From each one of these 5 base instances, 32 instances with different sizes were created. Base instances were generated from a random distribution in a square area of 2000[km] of side. Every base instance considers 20 potential sites to install a warehouse and the location of 40 customers. The instances were named using the following notation *N_M_I*, where: *N* represents the number of potential warehouses, *M* represents the number of customers and *I* the number of the base instance. The parameter *N* takes values in {5, 10, 15, 20}, *M* takes values in {5, 10, 15, 20, 25, 30, 35, 40} and finally *I* takes values in {1, 2, 3, 4, 5}.

The initial solution for the algorithm is obtained using a basic Facility Location Problem, where two sets of constraints are integrated to ensure SPs feasibility. The model uses the MP variables $\left(X,Y\right)$ and a subset of parameters from the original model. The mathematical formulation is as follows:

$$\underset{(X,Y)}{Min} \quad \rho\left(X,Y\right) \qquad (17)$$

s.t.:

$$\sum_{i\in N} Y_{ij} = 1 \qquad \forall j \in M \qquad (2)$$

$$Y_{ij} \le X_i \qquad \forall i \in N, \forall j \in M \qquad (3)$$

$$\sum_{j=1}^{M} Y_{ij} \cdot v_j \le V_{\max}^{i} \qquad \forall i \in N \qquad (10)$$

$$X_i \le \sum_{j=1}^{M} Y_{ij} \qquad \forall i \in N \qquad (11)$$

$$X_i \in \left\{0,1\right\} \qquad \forall i \in N \qquad (8)$$

$$Y_{ij} \in \left\{0,1\right\} \qquad \forall i \in N, \forall j \in M \qquad (9)$$

The objective function (17) is sum of warehouses settings and assignment costs. Sets of constraints (2), (3), (8) and (9) are the same as in the original model. Constraints (10) and (11) are derived from the original constraints (7) and the definition made on (12). Constraints are valid inequalities to ensure that a warehouse is open only if at least one customer is assigned to it.

The proposed algorithm is implemented in Microsoft Visual C++ 2010, and master problem is solved using Cplex 12.5, both using a computer with a processor Intel Core I7 of 3.4 GHz and 8 GB of RAM in a 64-bit Operating System.

The notation of the results is showed in the Table 1.

Table 1-Notation used in tables of results

NOTATION	
N	Number of potential warehouses
M	Number of customers
OF	Optimal Objective Function
T	Computing time [s]
NCA	Number of cuts added

The following tables show the results for each base instance. Tables 2-6 show the results obtained for each base instance, the optimal solution was reached for each and every of these 160 instances.

Table 2-Results of base instance 1

N	**5**	**5**	**5**	**5**	**5**	**5**	**5**	**5**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
OF	282,336.24	554,434.37	770,046.77	957,136.84	1,242,964.02	1,578,384.49	1,583,497.47	1,836,125.71
T	0.144	0.088	0.089	0.046	0.212	0.142	0.048	0.052
NCA	7	6	5	3	5	6	3	3
N	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
OF	268,123.24	466,713.86	657,455.23	843,767.27	1,006,153.99	1,227,083.68	1,363,669.12	1,575,606.95
T	0.455	0.288	1.457	3.163	1.126	2.149	4.646	5.569
NCA	13	9	22	31	16	16	28	29
N	**15**	**15**	**15**	**15**	**15**	**15**	**15**	**15**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
OF	268,123.24	466,713.86	571,136.42	785,754.97	951,789.76	1,154,815.38	1,316,919.51	1,536,874.27
T	1.075	6.631	0.859	46.581	57.811	96.095	394.99	780.43
NCA	26	47	13	91	90	97	156	203
N	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
OF	268,123.24	449,567.32	558,464.41	748,307.26	951,789.76	1,118,771.67	1,240,319.96	1,396,095.90
T	4.271	12.715	5.374	48.736	716.95	1164.6	1810.9	392.59
NCA	52	52	27	80	181	211	246	115

Table 3-Results of base instance 2

N	**5**	**5**	**5**	**5**	**5**	**5**	**5**	**5**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	273,509.31	488,596.29	728,109.15	971,747.72	1,202,002.78	1,352,426.42	1,650,183.90	1,847,962.81
T	0.065	0.127	0.24	0.099	0.128	0.113	0.12	0.109
NCA	3	4	5	3	3	3	3	3
N	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	249,114.18	440,739.38	698,268.74	816,057.92	1,058,996.82	1,259,315.81	1,382,592.91	1,545,264.18
T	1.252	1.708	50.601	2.143	17.514	13.621	8.593	21.432
NCA	18	20	81	15	39	36	22	44
N	**15**	**15**	**15**	**15**	**15**	**15**	**15**	**15**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	234,623.95	440,739.38	633,728.35	777,430.84	957,822.21	1,152,265.98	1,314,181.59	1,527,322.99
T	2.227	76.867	224.59	79.18	178.72	225.44	267.04	5984.9
NCA	29	88	127	69	99	96	96	304
N	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	217,304.71	412,105.94	633,294.85	777,430.84	937,273.62	1,152,265.98	1,283,385.74	1,459,502.27
T	7.109	221.75	12201	16428	1939.5	67309	38639	95420
NCA	32	124	470	580	283	1103	805	1101

Table 4-Results of base instance 3

N	**5**	**5**	**5**	**5**	**5**	**5**	**5**	**5**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	309,773.67	663,268.42	857,475.38	1,134,696.25	1,324,234.81	1,696,998.56	1,792,011.57	1,933,157.61
T	0.082	0.63	0.172	0.312	0.156	0.313	0.243	0.125
NCA	4	12	4	6	4	5	5	4
N	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	267,418.10	496,809.74	700,722.82	891,762.16	1,087,393.04	1,353,302.51	1,404,540.69	1,574,895.32
T	0.784	4.483	1.504	1.941	10.119	5.058	4.099	6.685
NCA	15	31	16	12	26	20	17	19
N	**15**	**15**	**15**	**15**	**15**	**15**	**15**	**15**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	244,661.41	492,831.52	613,697.72	841,690.07	1,014,224.90	1,242,187.28	1,415,749.10	1,481,751.09
T	1.31	92.139	52.678	60.781	300.86	196.6	7006.6	473.58
NCA	20	135	78	61	96	106	404	139
N	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	223,911.16	459,349.51	613,697.72	776,053.35	971,087.48	1,206,745.15	1,291,004.46	1,426,955.46
T	2.829	763.36	1966	6857.7	8617.6	25254	6300.3	9579.7
NCA	17	220	270	450	367	492	244	508

Table 5-Results of base instance 4

N	**5**	**5**	**5**	**5**	**5**	**5**	**5**	**5**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	343,840.59	693,779.62	892,452.76	1,048,419.31	1,377,994.71	1,630,786.16	1,774,022.88	2,021,612.05
T	0.072	0.151	0.143	0.105	0.196	0.072	0.238	0.075
NCA	7	7	7	6	8	4	4	4
N	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	269,752.30	521,275.87	714,412.91	863,829.27	1,041,778.56	1,302,147.92	1,417,713.59	1,623,970.44
T	0.442	1.069	10.883	5.025	5.244	8.314	4.964	1.595
NCA	18	24	58	38	30	36	25	13
N	**15**	**15**	**15**	**15**	**15**	**15**	**15**	**15**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	245,527.63	474,834.38	663,728.94	789,644.75	1,041,778.56	1,239,589.13	1,403,783.93	1,503,614.61
T	0.64	5.771	26.729	20.907	862.57	266.43	461.11	483.64
NCA	18	40	79	55	238	158	186	151
N	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	238,971.87	459,965.31	639,091.51	755,769.46	990,445.43	1,162,732.61	1,324,579.24	1,441,995.04
T	2.822	208.05	4485	442.23	7963.3	9564.9	21154	2364.4
NCA	43	208	534	172	468	495	701	240

Table 6-Results of base instance 5

N	**5**	**5**	**5**	**5**	**5**	**5**	**5**	**5**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	335,246.32	604,107.67	865,731.80	1,066,347.32	1,237,721.23	1,502,105.47	1,679,858.10	1,873,356.59
T	0.064	0.083	0.129	0.08	0.051	0.084	0.059	0.152
NCA	5	6	8	3	3	4	3	4
N	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	318,176.43	479,904.04	786,961.71	966,229.42	1,170,330.56	1,383,688.76	1,539,986.14	1,553,323.32
T	0.406	0.22	14.621	5.5056	2.593	2.255	8.274	0.703
NCA	17	8	67	33	20	17	37	8
N	**15**	**15**	**15**	**15**	**15**	**15**	**15**	**15**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	317,046.57	479,904.04	647,006.72	912,726.76	1,037,050.31	1,211,723.26	1,362,499.71	1,553,323.32
T	1.762	9.828	4.901	39.816	9.05	15.727	49.544	689.93
NCA	35	51	40	62	29	39	63	192
N	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
FO	255,818.67	435,024.37	647,006.72	806,329.66	986,809.08	1,141,832.15	1,271,783.91	1,397,482.35
T	1.62	7.715	4619.3	148.37	7262.5	5718.8	11776	5909
NCA	31	43	515	116	535	458	474	388

Analyzing the Tables 2-6 it is possible to get insights about the behavior of the solutions. In a specific column the number of customers is fixed but going down 5 more potential warehouses are added in each instance, from 5 to 20. Thus, each feasible solution of an instance is also feasible in all the instances bellow for the same column. Having in mind that the optimal solution is found for all the instances, this value is in fact an upper-bound for the optimal value for all the instances bellow in the same column. Moreover, in some cases the optimal solution of an instance is also optimal for some of the instances bellow (e.g. instances 10_5_1, 15_5_1 and 20_5_1).

Figures 1 - 5 show the behavior of the optimal objective function for each group of instances associated to each base instance. The optimal objective function value for a fixed number of potential warehouses performs a non-decreasing behavior when the number of customers is increased. In most cases the curve for an instance tends to show a linear growth. Nevertheless in some cases adding five customers generate a marginal increase between the optimal objective function values of the instances (e.g. optimal values of instances 10_35_5 and 10_40_5).

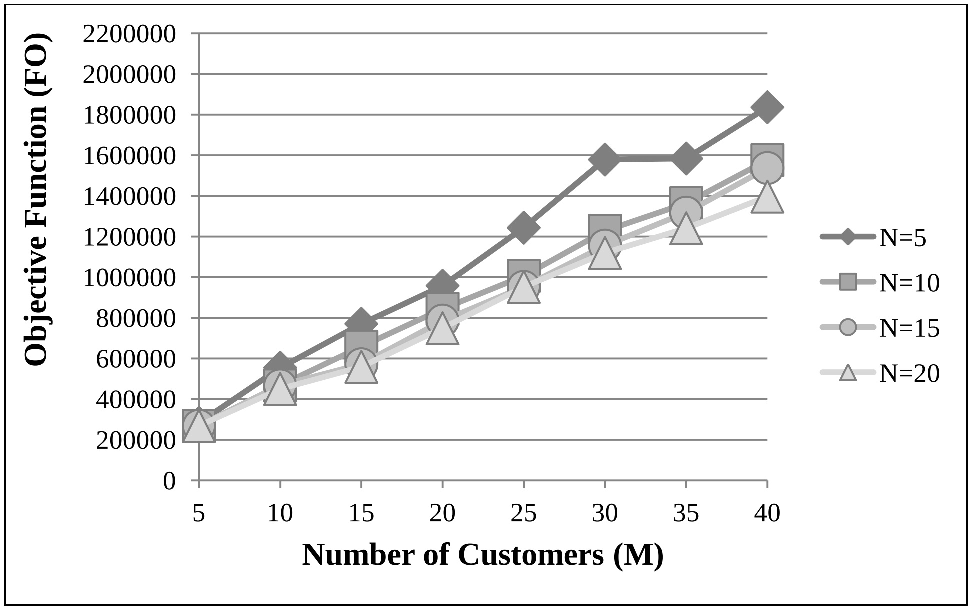

Figure 1-Optimal objective function values for base instance 1

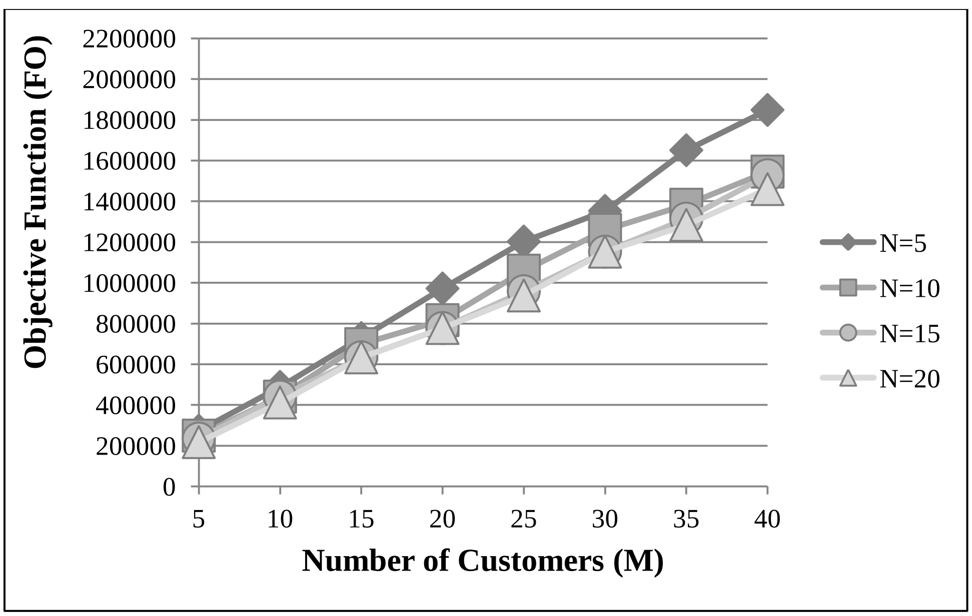

Figure 2-Optimal objective function values for base instance 2

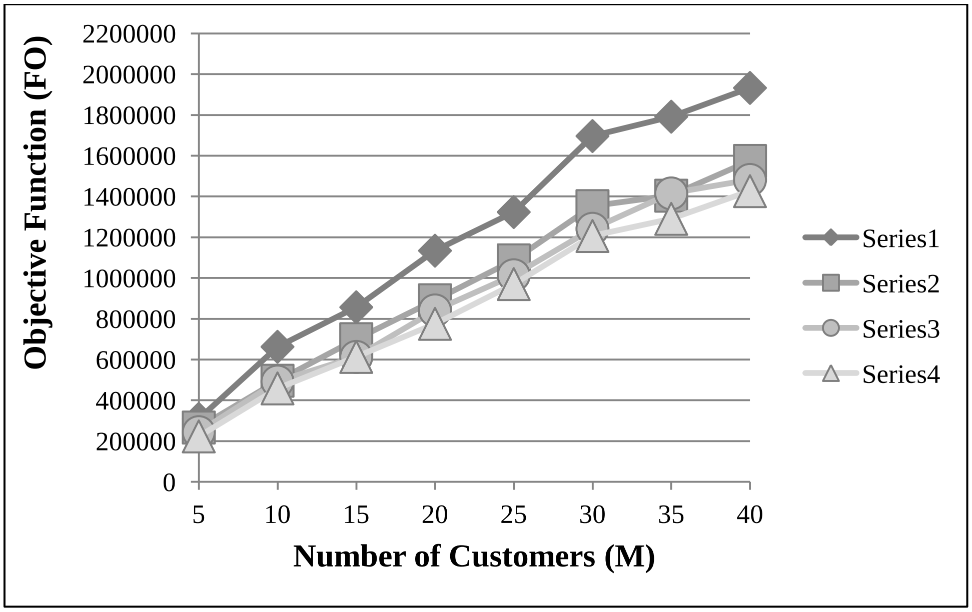

Figure 3-Optimal objective function values for base instance 3

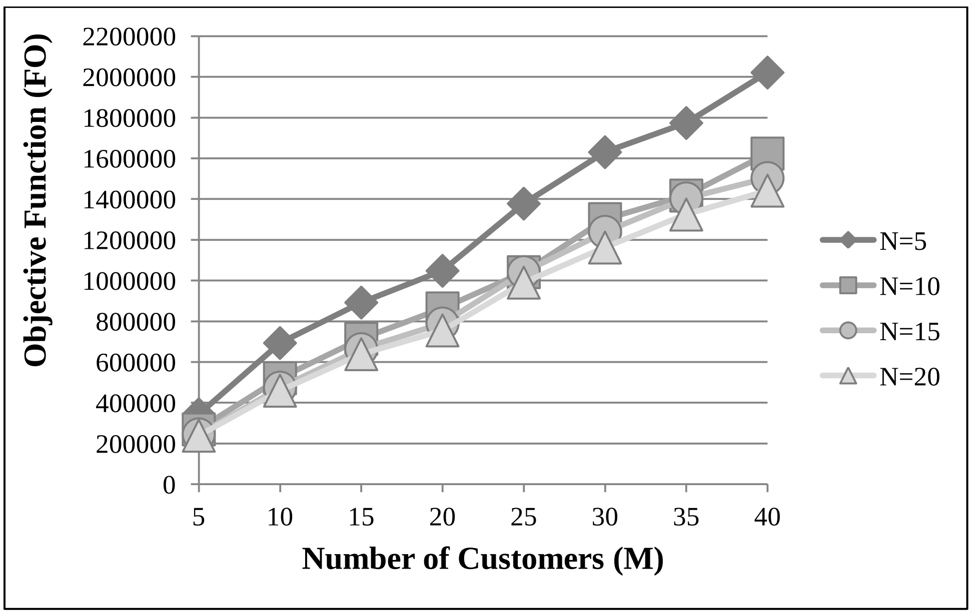

Figure 4-Optimal objective function values for base instance 4

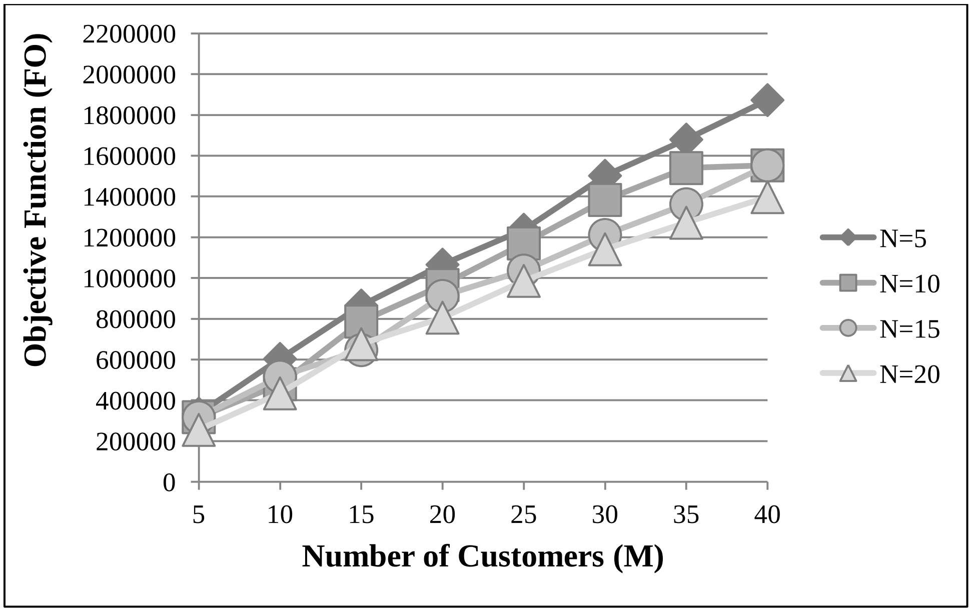

Figure 5-Optimal objective function values for base instance 5

Another important result is related to the computing time for solving the instances. Due to the nature of the solution approach, the MP tends to be more complex to solve with each iteration, due to the number of cuts incorporated increases. Figure 6 shows a histogram and the cumulative curve of the total times to solve the 160 instances.

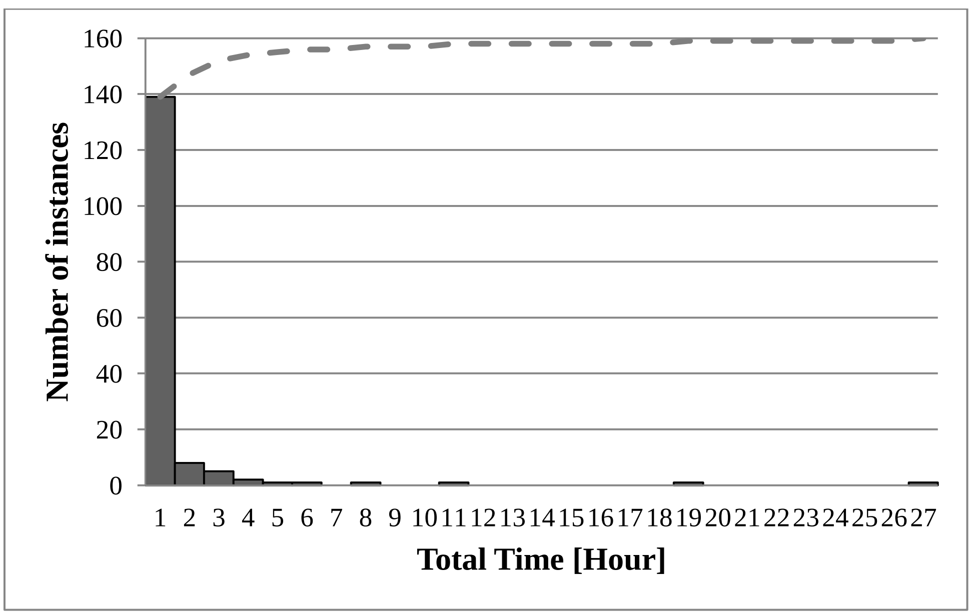

Figure 6-Histogram of computing times.

Analyzing the previous figure it is possible to notice that most of the instances are solved in less than an hour (86.875% of the instances). According to Table 7 it is possible to notice that an 80.6% of the instances are solved in less than ten minutes. Moreover the 67.5% of the instances need less than one minute to be solved.

Table 7-Computational times

Time [s]	Number of instances	Percentage [%]	Cumulative Number of instances	Cumulative Percentage [%]
$T \leq 1$	49	30.6%	49	30.6
$1 < T \leq 10$	42	26.3%	91	56.9
$10 < T \leq 60$	17	10.6%	108	67.5
$60 < T \leq 300$	14	8.8%	122	76.3
$300 < T \leq 600$	7	4.4%	129	80.6
$600 < T \leq 1,800$	6	3.8%	135	84.4
$1,800 < T \leq 3,600$	10	6.3%	139	86.9
$3,600 < T \leq 18,000$	16	10.0%	155	96.9
$18,000 < T \leq 36,000$	2	1.3%	157	98.1
$36,000 < T$	3	1.9%	160	100.0

Considering the nature of the proposed solution approach it may be relevant to analyze the relation between computing times and the number of cuts added into the MP. Figure 7 shows the behavior of computing times according to the number of cut added (NCA) for the 160 instances, putting aside the impact of the specific instance characteristics (e.g. number of warehouses, numbers of customers).

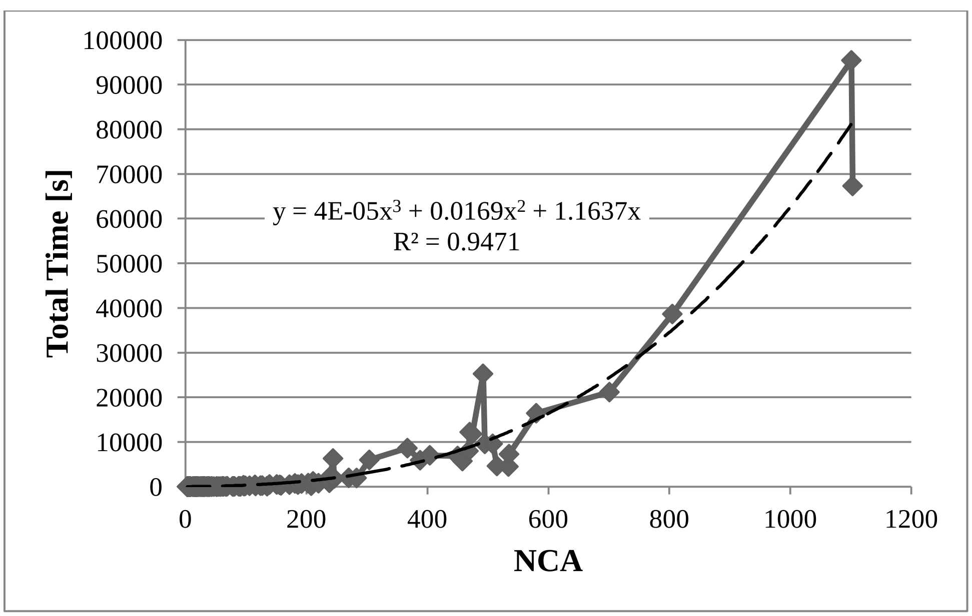

Figure 7-Relationship between NCA and total time

According to Figure 7 it is possible to identify a strong relationship that explains the computing time by the number of cuts added, with a more accentuated tendency than linear. Naturally, there is more characteristic that should be considered for a better understanding of this relationship (e.g. number of potential warehouses, number of customer, spatial distribution).

Table 8-Average results

N	**5**	**5**	**5**	**5**	**5**	**5**	**5**	**5**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
OF	308,941.23	600,837.27	822,763.17	1,035,669.49	1,276,983.51	1,552,140.22	1,695,914.78	1,902,442.95
T	0.0854	0.2158	0.1546	0.1284	0.1486	0.1448	0.1416	0.1026
NCA	5.2	7	5.8	4.2	4.6	4.4	3.6	3.6
N	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
OF	274,516.85	481,088.58	711,564.28	876,329.20	1,072,930.59	1,305,107.73	1,421,700.49	1,574,612.04
T	0.6678	1.5536	15.8132	3.55552	7.3192	6.2794	6.1152	7.1968
NCA	16.2	18.4	48.8	25.8	26.2	25	25.8	22.6
N	**15**	**15**	**15**	**15**	**15**	**15**	**15**	**15**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
OF	261,996.56	478,377.69	625,859.63	821,449.48	1,000,533.15	1,200,116.21	1,362,626.77	1,520,577.26
T	1.4028	38.2472	61.9514	49.453	281.8022	160.0584	1635.8568	1682.496
NCA	25.6	72.2	67.4	67.6	110.4	99.2	181	197.8
N	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**
M	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
OF	240,825.93	443,202.49	624,165.37	772,778.11	967,481.08	1,156,469.51	1,282,214.66	1,424,406.20
T	3.7302	242.718	4655.3348	4785.0072	5299.97	21802.26	15936.04	22733.138
NCA	35	129.4	363.2	279.6	366.8	551.8	494	470.4

Finally, Table 8 summarizes the previous results by averaging the results of the five base instances. In order to isolate the effect of the size of the instances the average is made considering the instances with the same number of potential warehouses and customers.

The average values of optimal objective function, total time and the number of cuts added into the MP are presented in Figure 8, 9 and 10 respectively.

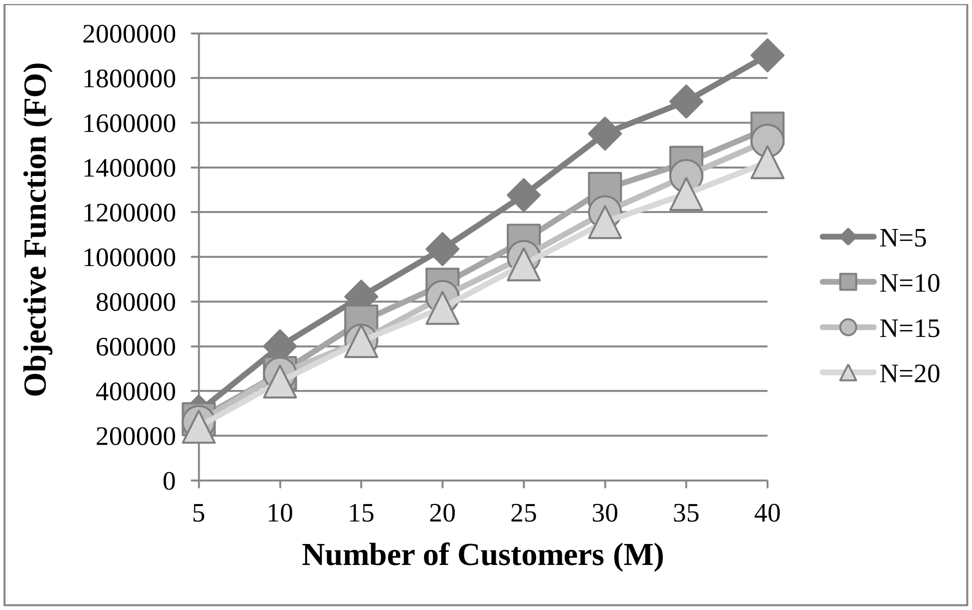

Figure 8-Optimal objective function values for average results

As expected, Figure 8 confirms the same behavior of the optimal values for each base instance.

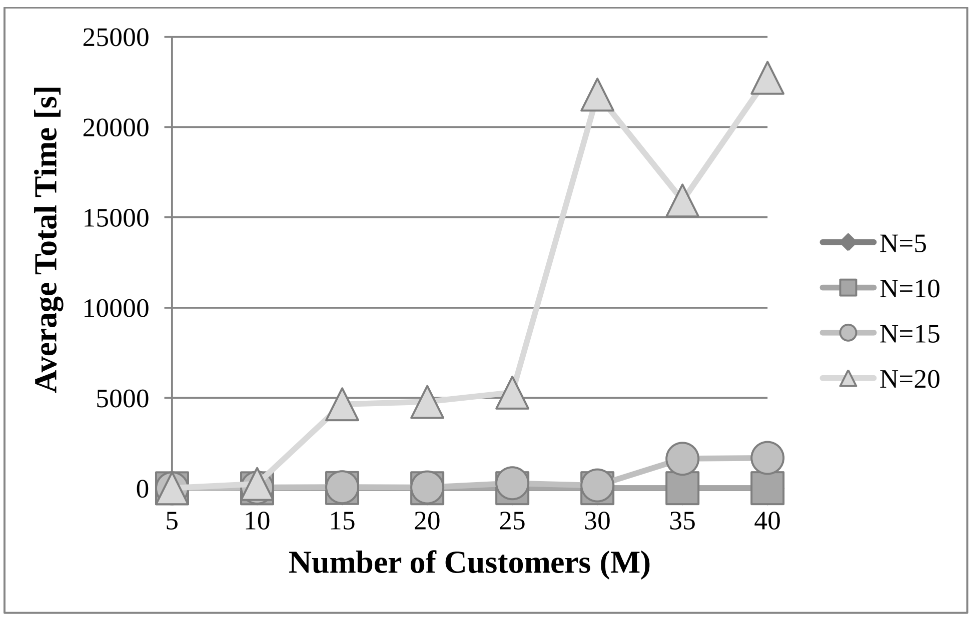

Figure 9-Total times for average results

Analyzing Figure 9 it is possible to visualize that the computing times for instances with 20 potential warehouses are notably greater that other instances with lower values of *N*. Moreover, solving times of the other instances tend to be relatively low, especially highlighting the global optimality ensured with the proposed solution approach.

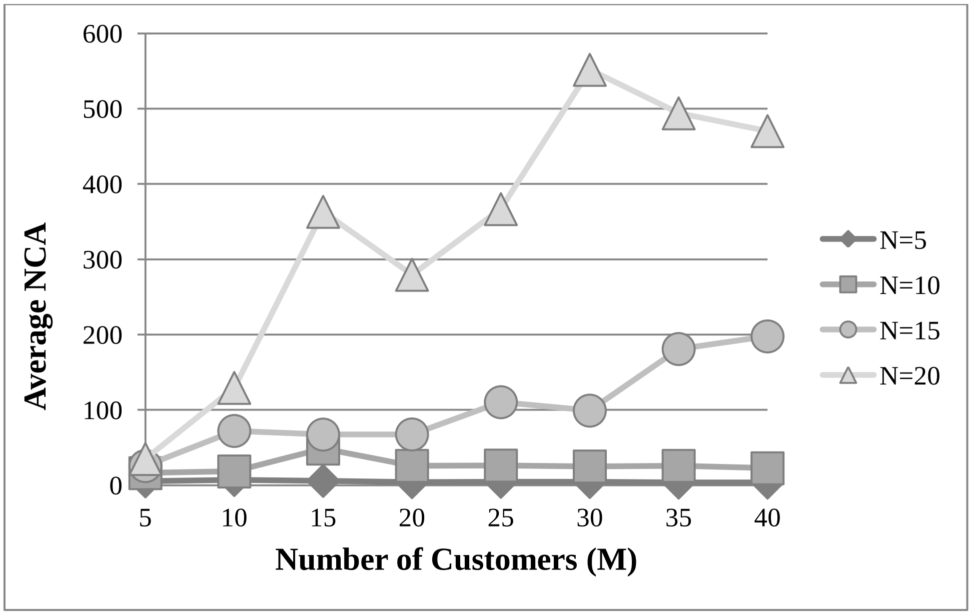

Figure 10-Number of cuts added for average results

In general terms, the behavior showed in Figure 10 it is similar to the performance of total times in Figure 9. It is clearly observed that the instances with more cuts are those with 20 potential warehouses. By contrasting the information obtained from Figure 9 and 10, it is possible to observe that the number of cuts added it is strongly related to the total time needed to solve all instance as previously suggested by Figure 7. This insight suggests further research focused on reducing the number of cuts.

6. CONCLUSIONS AND FUTURE WORK

This paper studied a joint Inventory Location Problem with Stochastic Inventory Capacity Constraints, which considers decisions related to both the structure of the supply chain network and the sizing of inventories at each allocated warehouse. As a consequence, the mathematical structure of the studied mixed integer nonlinear nonconvex programming problem requires efficient solution approaches for obtaining optimal solutions in competitive times. Accordingly, this paper proposes a novel Generalized Benders Decomposition based solution approach that ensures optimality. It is remarkable that, despite of nonconvex model structure, the proposed solution approach ensures global optimality.

Due to this study is focused on long term optimization models, whose usage is sporadic, computing times can be considered not as important as the quality of the solutions. In other words, computing times can be longer than for real time or short term optimization problem. However, the time for solving the problem is a relevant performance indicator to classify an algorithmic approach. It is remarkable that for the real world based medium sized instances considered in this study, 75% of the instances were solved in less than four minutes, especially considering the complexity of the model. The sizes of the employed instances can be considered as medium/small. However, these instances may represent real world sizes for specific industry or company cases.

The proposed solution approach introduces an interesting and novel strategy to decompose the problem based on the decomposition scheme of GBD. Setting the binary variables as the master problem decision variables yields a set of subproblems that can be analytically solved at optimality. As a consequence, the lagrangian dual information is obtained using closed mathematical expressions, and it is properly employed to build the cuts to be added iteratively into the master problem. Furthermore, the master problems can be solved at optimality using a standard commercial solver, given its mixed integer linear programming nature. Then the proposed strategy deals with the nonconvexity of the original problem and ensures global optimality.

In terms of future research, it worth to be mentioned the application of the proposed solution approach to other inventory location problems, considering other inventory control policies, more complex supply chain, or considering other type of constraints. Moreover, the model can be adapted to deal with unique features and requirements of specific industries and/or type of commodities (e.g. final products, raw materials, spare parts). The existence of more extended supply chain networks, where sub-networks are embedded into a common shared network, may lead to the use of nested decomposition approaches. Natural extensions are multi-period and multi-commodity applications. Further important issues are potential enhancements to the proposed algorithm in order to improve the general performance of the algorithm such as computational aspects and also algorithmic design issues (e.g. lazy constraints, convergence criteria and approaches for solving the master problem).

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