METRIC CONSTRUCTIONS OF TOPOLOGICAL INVARIANTS

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Abstract. We present a general mechanism for obtaining topological invariants from metric constructs. In more detail, we describe a process which, under very mild conditions, produces topological invariants out of a construction on a metric space together with a choice of scale (a non-negative value at each point of the space). Through Flagg's metric formalism of topology the results are valid for all topological spaces, not just the metrizable ones. We phrase the result in much greater generality than required for the topological applications, using the language of fibrations. We show that ordinary topological connectedness arises metrically, and we obtain metrically defined theories of homology and of homotopy.

1. Introduction

The definition of metric space makes explicit use of $[0, \infty]$ as the codomain of the metric function. However, it is only a tiny fraction of the structure on $[0, \infty]$ that gives rise to the theory of metric spaces. It is very natural to ask whether one can usefully replace $[0, \infty]$ by other structures to obtain a pleasant theory of more general metric spaces. In other words, one is called upon to list the properties of $[0, \infty]$ that are truly at the heart of the usual analytical arguments and thus axiomatise the structures that can serve as codomains of metric spaces.

This call was addressed by Kopperman ([14]) where the list of axioms describes a value semigroup with positives, i.e., an ordered semigroup together with a specified subset satisfying certain properties, and by Flagg ([9]) where the axioms describe a value quantale, i.e., a complete lattice with a binary operation satisfying a large amount of distributivity laws, and an important technical condition that identifies a certain subset to act as the set of positives. Interestingly, each of these formalisms allows for the metrizablity of all topological spaces, and that raises a natural question: can topology be effectively practiced using these metric models?

A theoretically affirmative answer was given in [19] where it is shown that the category **Top** is equivalent to the category $\mathbf{Met_c}$ of all metric spaces, where values are allowed to be taken in all value quantales, with morphisms the continuous functions. The standard models of topology and the metric ones are thus different representations of the same abstract category, and thus the models are equally powerful. However, that result does not settle the question of whether the metric formalism is convenient. We mention [4, 5] - recent results utilising the metric models for the study of topology. Part of the aim of this work is to provide further evidence in support of the metric formalism for general topology.

Let us briefly recount Flagg's metric formalism. The reader is referred to [19] for a short yet detailed presentation. A *value quantale* is a complete lattice L, with $0 < \infty$, where 0 is the smallest element in L and ∞ the largest, together with a commutative binary operation $+$ satisfying $0 + a = a$, $a + \bigwedge S = \bigwedge a + S$,

$a = \bigwedge\{b \in L \mid b \succ a\}$, and so that $a \wedge b \succ 0$ whenever $a, b \succ 0$, for all $a, b \in L$ and $S \subseteq L$. Here $\bigwedge$ is the meet in L, $b + S$ is defined element-wise, and $b \succ a$ is the well above relation, i.e., for all $S \subseteq L$, if $a \geq \bigwedge S$, then $b \geq s$ for some $s \in S$.

An *L-valued metric* space is then a triple (X, L, d) where X is a set, L is a value quantale, and $d\colon X \times X \to L$ is a function satisfying $d(x, x) = 0$ and $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in L$. With every such metric space there is associated its open ball topology consisting of the sets $U \subseteq X$ for all $x \in U$ there exists $\varepsilon \succ 0$ with $B_\varepsilon(x) \subseteq U$. Here $B_\varepsilon(x) = \{y \in X \mid d(x, y) \prec \varepsilon\}$. Flagg then proves that for every topological space (X, τ) there exists a value quantale L and an L-valued metric space (X, L, d) whose open ball topology is τ. A categorical perspective is taken in [19] where this metrization result is slightly strengthened as follows. Let $\mathbf{Met_c}$ be the category whose objects are all L-valued metric spaces (where L is allowed to vary), and whose morphisms $f\colon (X, L_X, d_X) \to (Y, L_Y, d_Y)$ are the functions $f\colon X \to Y$ satisfying the familiar continuity condition: for all $x \in X$ and $\varepsilon \succ 0_{L_Y}$, there exists $\delta \succ 0_{L_X}$ such that $d_Y(fx, fy) \prec \varepsilon$ for all $y \in X$ with $d_X(x, y) \prec \delta$. The open ball topology construction is then a functor $\mathbf{Met_c} \to \mathbf{Top}$ which is in fact an equivalence of categories.

The main purpose of this work is to develop general machinery producing, among others, the following result. Consider a metric space (X, L, d) together with a specification of a non-negative value at each point, thought of as a scale on the space. Consider the diagram

$$\begin{tikzcd}
\mathbf{ScMet} \arrow[dr, "F"'] & & \mathbf{Met_C} \arrow[dl, "\hat{F}"] \\
& \mathcal{A} &
\end{tikzcd}$$

where $\mathbf{ScMet}$ is a suitably constructed category of scaled metric spaces. We show that any functor F gives rise to a canonical functor $\hat{F}$, provided $\mathcal{A}$ is small complete. Due to the equivalence $\mathbf{Met_c} \simeq \mathbf{Top}$ one obtains a general method for producing topological invariants. One may replace $\mathbf{Met_c}$ by $\mathbf{Met_u}$, the subcategory of $\mathbf{Met_c}$ consisting of the uniformly continuous functions, namely those f satisfying the uniform version of the continuity condition from above: for all $\varepsilon \succ 0_{L_Y}$ there exists $\delta \succ 0_{L_X}$ such that $d(fx, fy) \prec \varepsilon$ for all $x, y \in X$ with $d(x, y) \prec \delta$. The resulting invariants are than uniform invariants. The result is exemplified by considering three particular metric constructions which give rise to topological (resp. uniform) invariants related to connectedness, homology, and homotopy.

Rather than taking a geodesic path toward the result, the presentation takes the scenic route, framing the construction in the language of Grothendieck fibrations. The plan of the paper is as follows. Section 2 introduces scales on spaces (X, L, d); roughly the metric analogues of open coverings. Section 3 then presents, in a rather expository fashion, multivalued fibrations as a convenient framework for the relationship between metric spaces and scales on them. Section 4 is concerned with stating and establishing the main construction - a machine for generating invariants on the codomain of a multivalued fibration from a suitable functor. Finally, Section 5 specialises the general result to obtain three topological invariants: connectedness, a variant of the fundamental groupoid, and a homology theory.

2. Scales

In this section we introduce the concept of scale on a metric space, and the accompanying notions of open sets and continuity induced by a choice of a scale system.

Definition 2.1. A *scale* on a space (X, L, d) is a function $R\colon X \to L$ with $R(x) \succ 0$ for all $x \in X$. The quadruple (X, L, d, R) is then called a *scaled space* and R is a *scale on* X. A morphism $f\colon (X, L_X, d_X, R_X) \to (Y, L_Y, d_Y, R_Y)$ is a function $f\colon X \to Y$ which is *tolerant* of the scales, in the sense that $d_X(x, x') \prec R_X(x)$ implies $d_Y(fx, fx') \prec R_Y(fx)$, for all $x, x' \in X$.

It is immediate that all scaled spaces and their tolerant morphisms form a category, denoted by **ScMet**.

Definition 2.2. A *scale system* $\mathbf{\Sigma}$ is a full subcategory of **ScMet** such that for every space (X, L, d) there exists at least one scale R for which (X, L, d, R) is an object of $\mathbf{\Sigma}$, and so that if R_1, R_2 are such that $(X, L, d, R_i) \in \mathbf{\Sigma}$, $i = 1, 2$, then $(X, L, d, R) \in \mathbf{\Sigma}$ where $R\colon X \to L$ is given by $Rx = R_1x \wedge R_2x$.

In other words, a scale system is a choice of at least one scale for every possible space, closed under point-wise meets over the same space. We shall primarily be interested in the scale systems $\mathbf{\Sigma_c} = \mathbf{ScMet}$ and $\mathbf{\Sigma_u}$, the full subcategory spanned by the scaled spaces whose scale function is constant. Obviously, the hierarchy of scale systems is immense (see [18] for more examples); we consider just one more example, the scale system $\mathbf{\Sigma_g}$. Let $\mathbf{g}$ be a choice, for each value quantale L, of an element $g_L \succ 0$ in L. Then the scale system $\mathbf{\Sigma_g}$ consists, for each space (X, L, d) of just one scale: $x \mapsto g_L$. The choice g_L is to be thought of as what is considered negligibly small distances in L. Generally, we write $\mathbf{\Sigma_1} \le \mathbf{\Sigma_2}$ when $\mathbf{\Sigma_1}$ is a subcategory of $\mathbf{\Sigma_2}$ (e.g., $\mathbf{\Sigma_g} \le \mathbf{\Sigma_u} \le \mathbf{\Sigma_c}$). For the rest of this work let $\mathbf{\Sigma}$ denote a fixed scale system, and we'll say "$R \in \mathbf{\Sigma}$ on X" as a slightly abusive shorthand for "with the obvious space $X = (X, L, d)$, the scaled space (X, L, d, R) is an object of $\mathbf{\Sigma}$".

Definition 2.3. Let (X, L_X, d_X) and (Y, L_Y, d_Y) be spaces. A function $f\colon X \to Y$ is Σ-*continuous* if for every $R_Y \in \mathbf{\Sigma}$ on Y there exists $R_X \in \mathbf{\Sigma}$ on X such that $f\colon (X, L_X, d_X, R_X) \to (Y, L_Y, d_Y, R_Y)$ is tolerant.

It is straightforward that all spaces together with all $\mathbf{\Sigma}$-continuous functions form a category, denoted by $\mathbf{Met_\Sigma}$. It is nothing but a play on words that $\mathbf{\Sigma_c}$-continuity coincides with the usual notion of continuity, while $\mathbf{\Sigma_u}$-continuity coincides with uniform continuity. In other words, $\mathbf{Met_{\Sigma_c}} = \mathbf{Met_c}$ and $\mathbf{Met_{\Sigma_u}} = \mathbf{Met_u}$. $\mathbf{\Sigma_g}$-continuous functions need not be continuous as their graphs may have small (as determined by $\mathbf{g}$) gaps.

Given a scaled space (X, L, d, R) we write $B_R(x) = \{y \in X \mid d(x, y) \prec R(x)\}$, and extend the notation to subsets $S \subseteq X$ by means of $B_R(S) = \bigcup_{x \in S} B_R(x)$.

Definition 2.4. Let (X, L, d) be a space. A subset $U \subseteq X$ is $\mathbf{\Sigma}$-*open* if there exists $R \in \mathbf{\Sigma}$ on X such that $U = B_R(U)$. $\mathbf{\Sigma}$-*closed* subsets are the complements of $\mathbf{\Sigma}$-open sets, and $\mathbf{\Sigma}$-*clopen* subsets are those that are both $\mathbf{\Sigma}$-open and $\mathbf{\Sigma}$-closed.

Evidently, $\mathbf{\Sigma_c}$-open (resp. $\mathbf{\Sigma_c}$-closed, $\mathbf{\Sigma_c}$-clopen) sets are the usual open (resp. closed, clopen) sets in the induced open ball topology, while $\mathbf{\Sigma_u}$-open (resp. $\mathbf{\Sigma_u}$-closed, $\mathbf{\Sigma_u}$-clopen) sets yields the notion of uniformly open (resp. uniformly closed, uniformly clopen) sets (and if d is symmetric, then these three notions coincide). For a set to qualify as being $\mathbf{\Sigma_g}$-open, every point in it is required to be surrounded by a ball in the set whose radius, as far as $\mathbf{g}$ is concerned, is not too small.

Proposition 2.5. *For a $\mathbf{\Sigma}$-continuous function $f\colon (X, L_X, d_X) \to (Y, L_Y, d_Y)$, the inverse image $W = f^{\leftarrow}(U)$ of every $\mathbf{\Sigma}$-open set $U \subseteq Y$ is $\mathbf{\Sigma}$-open in X. The converse need not hold.*

Proof. Let $R_Y \in \mathbf{\Sigma}$ on Y be a witness for U being $\mathbf{\Sigma}$-open, i.e., $B_{R_Y}(U) = U$. Since f is $\mathbf{\Sigma}$-continuous, there exists $R_X \in \mathbf{\Sigma}$ on X such that f is tolerant with respect to these scales. It follows at once then that $B_{R_X}(W) = W$, thus W is $\mathbf{\Sigma}$-open. That the converse generally fails may be seen, e.g., by considering $\mathbb{Q}$ with its usual metric. Its $\mathbf{\Sigma_u}$-open sets are just $\emptyset$ and $\mathbb{Q}$, and thus the inverse image of every function $f\colon \mathbb{Q} \to \mathbb{Q}$ preserves $\mathbf{\Sigma}$-open sets, but, obviously, not all such functions are $\mathbf{\Sigma_u}$-continuous (i.e., uniformly continuous). $\square$

We end this section with the following simple observations. Denote by $\tau_{\mathbf{\Sigma}}(X, L, d)$ the collection of all $\mathbf{\Sigma}$-open subsets of X. Then, for all spaces $X = (X, L, d)$

- $\emptyset, X \in \tau_{\mathbf{\Sigma}}(X)$.
- $\tau_{\mathbf{\Sigma}}(X)$ is closed under finite intersections and finite unions (this would not necessarily be true without the requirement that a scale system is closed under point-wise meets). In other words, $\tau_{\mathbf{\Sigma}}(X)$ is a lattice.
- If $\mathbf{\Sigma_1} \leq \mathbf{\Sigma_2}$, then $\tau_{\mathbf{\Sigma_1}}(X) \subseteq \tau_{\mathbf{\Sigma_2}}(X)$. In particular, the usual open ball topology $\tau(X)$, which coincides with $\tau_{\mathbf{\Sigma_c}}(X)$, is the ambient complete lattice in which all other lattices $\tau_{\mathbf{\Sigma}}(X)$ reside as sublattices.
- A scale system $\mathbf{\Sigma}$ is *saturated* if for all $R, S\colon X \to L$ with $Rx \geq Sx$, if $S \in \mathbf{\Sigma}$ on X, then $R \in \mathbf{\Sigma}$ on X. Clearly, every scale system $\mathbf{\Sigma}$ defines a unique saturated scale system $\bar{\mathbf{\Sigma}}$ containing it, and it holds that $\tau_{\bar{\mathbf{\Sigma}}}(X) = \tau_{\mathbf{\Sigma}}(X)$.
- A function $f\colon X \to Y$ is $\mathbf{\Sigma}$-continuous if, and only if, it is $\bar{\mathbf{\Sigma}}$-continuous.

3. Multivalued fibrations

In this section we introduce the concept of multivalued fibration as a framework for the main construction given in the next section. For the convenience of the reader we include a self-contained discussion of Grothendieck fibrations, as we notice some similarities with the above formalism of $\mathbf{\Sigma}$-continuity.

As motivation, notice that the condition of $\mathbf{\Sigma}$-continuity can be framed as follows. Let $\mathbf{Met_{all}}$ be the category whose objects are all spaces (X, L, d) and whose morphisms $f\colon (X, L_X, d_X) \to (Y, L_Y, d_Y)$ are all functions (continuous or not) $f\colon X \to Y$ between the underlying sets. There is then a forgetful functor $q : \mathbf{\Sigma} \to \mathbf{Met_{all}}$ which forgets the scale, mapping every tolerant function to itself. The category $\mathbf{Met_{\Sigma}}$ consisting of the $\mathbf{\Sigma}$-continuous functions can be described as the subcategory of $\mathbf{Met_{all}}$ containing all of the objects but only the morphisms $f\colon (X, L_X, d_X) \to (Y, L_Y, d_Y)$ satisfying the lifting condition: for all $e_Y \in \mathbf{\Sigma}$ with $qe_Y = (Y, L_Y, d_Y)$, there is a morphism $e_X \xrightarrow{e_f} e_Y$ with $qe_f = f$. In other words $\mathbf{\Sigma}$ carves out of $\mathbf{Met_{all}}$ the $\mathbf{\Sigma}$-continuous functions as those morphisms that can be lifted along q, provided their codomain can be lifted.

Let us briefly recall the basics of the theory of Grothendieck fibrations (see [2] for more details with terminology very close to ours, or [13, 3] for deeper treatments and different perspectives). Fix a category $\mathcal{B}$. For a functor $F\colon \mathcal{B}^{op} \to \mathbf{Set}$ one constructs the category $\int F$ whose objects are all pairs (B, x) with $B \in \mathrm{ob}(\mathcal{B})$ and $x \in FB$. The morphisms $(B, x) \to (B', x')$ are given as follows. For any morphism $b \colon B \to B'$ in $\mathcal{B}$ such that $x = (Fb)(x')$, there corresponds a morphism $b_{x,x'} \colon (B, x) \to (B', x')$. Identities and compositions are given as in $\mathcal{B}$, which implies that mapping (B, x) to B and $b_{x,x'}$ to b yields a functor $p \colon \int F \to \mathcal{B}$. Moreover, denoting by $[\mathcal{B}^{op}, \mathbf{Set}]$ the category of functors $F \colon \mathcal{B}^{op} \to \mathbf{Set}$, and by $\mathbf{Cat}/\mathcal{B}$ the category of categories over $\mathcal{B}$, namely the category of functors $p \colon \mathcal{E} \to \mathcal{B}$ with morphisms corresponding to commuting triangles, the Grothendieck construction $F \mapsto (\int F \to \mathcal{B})$ is functorial.

Identifying the essential image of $\int \colon [\mathcal{B}^{op}, \mathbf{Set}] \to \mathbf{Cat}/\mathcal{B}$ is achieved through the notion of discrete Grothendieck fibration. A functor $p \colon \mathcal{E} \to \mathcal{B}$ is a *discrete Grothendieck fibration* if for every $b \colon B \to B'$ in $\mathcal{B}$ and E' in $\mathcal{E}$ with $pE' = B'$, there exists a unique $e \colon E \to E'$ with $pe = f$. Let $\mathrm{dFib}(\mathcal{B})$ be the category of discrete Grothendieck fibrations $p \colon \mathcal{E} \to \mathcal{B}$, with the obvious notion of morphism of fibrations. There is then a functor $-_{\mathcal{B}} \colon \mathrm{dFib}(\mathcal{B}) \to [\mathcal{B}^{op}, \mathbf{Set}]$ given as follows. For a discrete Grothendieck fibration $p \colon \mathcal{E} \to \mathcal{B}$, and an object B in $\mathcal{B}$, let $\mathcal{E}_B = \{E \in \mathrm{ob}(\mathcal{E}) \mid pE = B\}$, the *fiber* over the object B. Given a morphism $b \colon B \to B'$, the definition of discrete Grothendieck fibration yields at once a function $\mathcal{E}_b \colon \mathcal{E}_{B'} \to \mathcal{E}_B$. The assignments $B \mapsto \mathcal{E}_B$ and $b \mapsto \mathcal{E}_b$ are functorial, thus resulting in a functor $p_{\mathcal{B}} \colon \mathcal{B}^{op} \to \mathbf{Set}$. The functoriality of the entire construction is routinely verified. The above constructions fit together in the following theorem.

Theorem 3.1. $\mathrm{dFib}(\mathcal{B}) \underset{\int}{\overset{-_{\mathcal{B}}}{\rightleftarrows}} [\mathcal{B}^{op}, \mathbf{Set}]$ *is an equivalence of categories.*

The discrete case of Grothendieck's theory of fibrations is somewhat degenerate, but it illustrates the point well; the theorem above provides a convenient translation mechanism between functors $\mathbf{Set}$ and discrete fibrations, where it is generally simpler to work with discrete fibrations. Grothendieck's original construction is an extension of the equivalence in the discrete case, where $[\mathcal{B}^{op}, \mathbf{Set}]$ is replaced by $[\mathcal{B}^{op}, \mathbf{Cat}]$ and $\mathrm{dFib}(\mathcal{B})$ is replaced by the category $\mathrm{Fib}(\mathcal{B})$ of Grothendieck fibrations.

In the context of the above discussion concerning $\mathbf{\Sigma}$-continuity, we are led to another extension of the equivalence in the discrete case, namely when lifts exist but are not unique, nor do different lifts of the same morphism necessarily have isomorphic domains.

Definition 3.2. A *multivalued fibration* is a functor $p \colon \mathcal{E} \to \mathcal{B}$ with the property that for all $b \colon B \to B'$ in $\mathcal{B}$, and E' in $\mathcal{E}$ with $pE' = B'$, there exists (a not necessarily unique) $e \colon E \to E'$ in $\mathcal{E}$ with $pe = b$. If, further, $pe = \mathrm{id}_B$ implies e is an identity morphism, and if no two lifts of a given morphism b have the same domain, then p is a *discrete multivalued fibration.*

If $p \colon \mathcal{E} \to \mathcal{B}$ is a discrete multivalued fibration, and $\mathcal{E}_B = \{E \in \mathrm{ob}(\mathcal{E}) \mid pE = B\}$ is the discrete fiber over B, then, given a morphism $b \colon B \to B'$, one immediately obtains a multivalued function $\mathcal{E}_b \colon \mathcal{E}_{B'} \to \mathcal{E}_B$, namely $E \in \mathcal{E}_b(E')$ if, and only if, there exists $e \colon E \to E'$ with $pe = b$. Let us thus be explicit about the formalism of multivalued functions. The category $\mathbf{Set_{MV}}$ is the Kleisli category associated to the covariant monad structure on $\mathcal{P}_* \colon \mathbf{Set} \to \mathbf{Set}$ given by unions, where $\mathcal{P}_*(T)$ is the set of all non-empty subsets of T. In more detail the objects of $\mathbf{Set_{MV}}$ are all sets, and morphisms $f \colon S \to T$ are functions $f \colon S \to \mathcal{P}_*(T)$, with composition

$(f \circ g)(x) = \bigcup_{y \in g(x)} f(y)$. It is easy to verify that a discrete multivalued fibration as above gives rise to a functor $\mathcal{B}^{op} \to \mathbf{Set_{MV}}$.

The Grothendieck construction $\int F$ easily adapts to operate on functors $F\colon \mathcal{B}^{op} \to \mathbf{Set_{MV}}$, as follows. The objects of $\int F$ are pairs (B, x) with $B \in \mathrm{ob}(\mathcal{B})$ and $x \in FB$ (just as in the classical construction). A morphism $(B, x) \to (B', x')$ is a $b_{x,x'}$ where $b\colon B \to B'$ satisfies $x \in (Fb)(x')$. Projecting onto the first coordinate gives a discrete multivalued fibration $\int F \to \mathcal{B}$. The details are very similar to the classical discrete Grothendieck fibration case (which is presented in great detail in [2]), including the following theorem, where $\mathrm{dFib}(\mathcal{B})_{\mathrm{MV}}$ is the category of discrete multivalued fibration over $\mathcal{B}$.

Theorem 3.3. $\mathrm{dFib}(\mathcal{B})_{\mathrm{MV}} \underset{\int}{\overset{-_{\mathcal{B}}}{\rightleftarrows}} [\mathcal{B}^{op}, \mathbf{Set_{MV}}]$ *is an equivalence of categories.*

Obviously there is a more general theory of multivalued fibrations, related to appropriate notions of multivalued functors. However, as this is not the aim of this work, and since the discussion above is sufficient justification for adopting Grothendieck's terminology for our purposes, we postpone the development of the multivalued theory in full to future work.

Conveniently, every functor $q\colon \mathcal{D} \to \mathcal{C}$ has a canonical multivalued fibration p, as in the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook] \arrow[d, two heads, "p"] & \mathcal{D} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, hook] & \mathcal{C} \end{tikzcd}$$

associated with it (the horizontal arrows are inclusions), as we now (easily) establish. Given a functor $q\colon \mathcal{D} \to \mathcal{C}$, declare a morphism $c\colon C \to C'$ in $\mathcal{C}$ to be *light* (with respect to q) if it can be lifted to $\mathcal{D}$ along q whenever its codomain can be so lifted (explicitly, for all D' with $qD' = C'$ there exists $d\colon D \to D'$ with $qd = c$).

Lemma 3.4. *Let $q\colon \mathcal{D} \to \mathcal{C}$ be a functor. Then restricting $\mathcal{C}$ to the light morphisms results in a subcategory $\mathcal{B}$, restricting $\mathcal{D}$ to those morphisms d such that qd is light results in a subcategory $\mathcal{E}$, and the restriction of q to $\mathcal{E}$ yields a multivalued fibration $p\colon \mathcal{E} \to \mathcal{B}$.*

Proof. Clearly, if the codomain of $c = \mathrm{id_C}$ lifts to D, then c lifts to $\mathrm{id_D}$. Given a composition $c = c' \circ c''$ with c', c'' light, if the codomain of c lifts to D, then lifting first c' and then c'' results in a lift of c. Thus $\mathcal{B}$ is a category, and it follows at once that so is $\mathcal{E}$, and obviously $p = q|_{\mathcal{E}}$ is a multivalued fibration. $\square$

The motivating discussion at the beginning of this section can now be rephrased as the claim that the forgetful functor $\mathbf{c\Sigma} \to \mathbf{Met_\Sigma}$, where $\mathbf{c\Sigma}$ is the subcategory of $\mathbf{\Sigma}$ specified by the $\mathbf{\Sigma}$-continuous functions, is the multivalued fibration associated to the forgetful functor $q\colon \mathbf{\Sigma} \to \mathbf{Met_{all}}$.

Remark 3.5. It is interesting to note that the proof that the composition of two light morphisms is light in the case of $\mathbf{\Sigma_c}$ (resp. $\mathbf{\Sigma_u}$) is a rather unorthodox proof that the composition of two continuous (resp. uniformly continuous) functions is continuous (resp. uniformly continuous), revealing how formal the result is.

We conclude by noting that much of the above is familiar in the context of continuity or uniform continuity, though is rarely explicitly given in terms of multivaluedness, namely, in the familiar $\varepsilon - \delta$ definition of continuity, the dependence of δ on ε is multivalued.

4. The main construction

Suppose that $\mathcal{B}$ is a category whose objects one wishes to study and that $\mathcal{A}$ is a category whose objects one wishes to use as invariants in the study of $\mathcal{B}$. In other words, we seek a functor $\mathcal{B} \to \mathcal{A}$. We give conditions under which such a functor is canonically obtained from a functor $\mathcal{D} \to \mathcal{A}$ for an auxiliary category $\mathcal{D}$. We phrase the theorem with applications for $\mathbf{Sigma} \to \mathbf{Met_{all}}$ in mind, given in the next section.

Generally, given a functor $p\colon \mathcal{E} \to \mathcal{B}$, the *fiber* $\mathcal{E}_B$ over B (under p) is the subcategory of $\mathcal{E}$ consisting of those objects that project to B, and those morphisms that project to id_B. The fiber is *small* if it is small as a category.

Theorem 4.1. *Consider the diagram*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook] \arrow[d, two heads, "p"] & \mathcal{D} \arrow[d, "q"] \arrow[ddr, "F"] & \\
\mathcal{B} \arrow[r, hook] \arrow[drr, "\hat{F}"'] & \mathcal{C} \arrow[d, phantom, "\Uparrow\pi"] & \\
 & & \mathcal{A}
\end{tikzcd}$$

in which q and F are given functors, p is the multivalued fibration associated to q (cf. Lemma 3.4), and the horizontal arrows are inclusions. Suppose $\mathcal{A}$ is small complete, p has small fibers, and for all $b\colon B \to B'$ and morphisms $e_1\colon E_1 \to E'$ and $e_2\colon E_2 \to E'$ with $pe_1 = b = pe_2$, there exist morphisms $e_1'\colon E \to E_1$ and $e_2'\colon E \to E_2$, each projecting to id_B, with $e_1 \circ e_1' = e_2 \circ e_2'$. Then there exist a functor $\hat{F}\colon \mathcal{B} \to \mathcal{A}$ and a natural transformation

$$\begin{tikzcd}
 & \mathcal{D} \arrow[dr, "F"] & \\
\mathcal{E} \arrow[ur, hook] \arrow[rr, bend left] \arrow[rr, bend right] \arrow[dr, two heads, "p"'] \arrow[rr, phantom, "\pi\Uparrow"] & & \mathcal{A} \\
 & \mathcal{B} \arrow[ur, "\hat{F}"'] &
\end{tikzcd}$$

satisfying the universal property depicted in the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook] \arrow[d, two heads, "p"] & \mathcal{D} \arrow[d, "q"] \arrow[ddr, "F"] & \\
\mathcal{B} \arrow[r, hook] \arrow[drr, "\hat{F}"] \arrow[drr, bend right, "\forall H"'] & \mathcal{C} \arrow[d, phantom, "\forall\alpha \quad \exists!\beta \quad \Uparrow\pi"] & \\
 & & \mathcal{A}
\end{tikzcd}$$

which, in more detail, reads as: for all functors $H\colon \mathcal{B} \to \mathcal{A}$ together with a natural transformation from $\mathcal{E} \xrightarrow{p} \mathcal{B} \xrightarrow{H} \mathcal{A}$ to $\mathcal{E} \to \mathcal{D} \xrightarrow{F} \mathcal{A}$ there exists a unique natural *transformation* $\beta\colon H \to \hat{F}$ *such that* $\alpha = \pi \circ (\beta \bullet p)$ *(where* $\circ$ *is vertical composition and* $\bullet$ *is horizontal composition of natural transformations).*

Proof. Given an object B in $\mathcal{B}$ let $\mathcal{E}_B$ be the fiber over B and let $F_B\colon \mathcal{E}_B \to \mathcal{A}$ be the restriction of the functor F. By the assumption of small fibers, the limit $\hat{F}B = \lim F_B$ exists. For every object E in $\mathcal{E}_B$, let $\pi_E\colon \hat{F}B \to FE$ be the canonical projection. Let now $b\colon B \to B'$ be a morphism in $\mathcal{B}$, and we must construct a morphism $\hat{F}b\colon \hat{F}B \to \hat{F}B'$. Such a morphism amounts to constructing a cone

$$\begin{tikzcd} FE' \arrow[rr, "Fe'"] & & FE'' \\ & \hat{F}B \arrow[ul] \arrow[ur] & \end{tikzcd}$$

from $\hat{F}B$ to the diagram given by $F_{B'}$, namely E', E'' project to B' and e' projects to $\mathrm{id}_{B'}$. Given any E' with $pE' = B'$ we obtain the morphism $\hat{F}B \to FE'$ as follows. Firstly, by the lifting property, there exists $e\colon E \to E'$ with $pe = b$, and we may thus consider

$$\begin{tikzcd} FE' & \\ FE \arrow[u, "Fe"] & \hat{F}B \arrow[l, "\pi_E"] \arrow[ul] \end{tikzcd}$$

obtaining a morphisms from $\hat{F}B$ to each object of the diagram $F_{B'}$. To verify independence of the choice of e, suppose that e_1, e_2 are two morphisms with $pe_i = b$. Then, by the assumed condition on p in the statement of the theorem, we obtain the commuting diagram

$$\begin{tikzcd} E_1 \arrow[dr, "e_1"'] & E \arrow[l] \arrow[r] \arrow[d, "e"] & E_2 \arrow[dl, "e_2"] \\ & E' & \end{tikzcd}$$

and we must show that $Fe_1 \circ \pi_{E_1} = Fe_2 \circ \pi_{E_2}$. Notice that applying F to the top part of this diagram yields a portion of the diagram F_B. We thus have

$$\begin{tikzcd} & \hat{F}B \arrow[dl, "\pi_{E_1}"'] \arrow[d, "\pi_E"'] \arrow[dr, "\pi_{E_2}"] & \\ FE_1 \arrow[dr, "Fe_1"'] & FE \arrow[l] \arrow[r] \arrow[d, "Fe"'] & FE_2 \arrow[dl, "Fe_2"] \\ & FE' & \end{tikzcd}$$

in which all triangles commute, and thus the entire diagram commutes, yielding $Fe_1 \circ \pi_{E_1} = Fe \circ \pi_E = Fe_2 \circ \pi_{E_2}$, as required.

The morphisms $\hat{F}B \to FE'$ are thus well-defined, and we now show that they form a cone over $F_{B'}$, namely that the outer part of the diagram

$$\begin{tikzcd}
FE'_1 \arrow[rrrr, "Fe'"] & & & & FE'_2 \\
& FE_1 \arrow[ul, "Fe_1"] \arrow[rr, "Fe"] & & FE_2 \arrow[ur, "Fe_2"'] & \\
& & \hat{F}B \arrow[ul, "\pi_{E_1}"] \arrow[ur, "\pi_{E_2}"'] & &
\end{tikzcd}$$

commutes for all $e'\colon E'_1 \to E'_2$ with $pe' = \mathrm{id}_{B'}$. For the rest of the diagram the morphisms e_1 and e_2 may be chosen arbitrarily, as long as e_i has E'_i as codomain and $pe_i = b$. For a given morphism e' proceed as follows. Form the commutative diagram

$$\begin{tikzcd}
E_1 \arrow[r, "e_s"] \arrow[d, "e"'] & E_t \arrow[r, "e_t"] & E'_1 \arrow[d, "e'"] \\
E_2 \arrow[rr, "e_2"] & & E'_2
\end{tikzcd}$$

by choosing e_2 and e_t to satisfy $pe_2 = b = pe_t$, and then, again by the assumed condition on p in the statement of the theorem, add the morphisms e and e_s, which satisfy $pe = \mathrm{id}_B = pe_s$. We then take $e_1 = e_t \circ e_s$, and note that e_1 and e_2 may be used in the diagram above. Since $pe = \mathrm{id}_B$ it follows that the inner triangle in that diagram commutes, while the trapezoid commutes since it is simply F applied to the auxiliary commutative diagram. The commutativity of the entire diagram follows and with it the claim regarding the cone over $F_{B'}$.

We thus obtain $\hat{F}b\colon \hat{F}B \to \hat{F}B'$ for each $b\colon B \to B'$. The functoriality of the construction follows easily. Indeed, to verify that $\hat{F}\mathrm{id}_B\colon \hat{F}B \to \hat{F}B$ is the identity morphism all we need to do is show that $\pi_E \circ \hat{F}\mathrm{id}_B = \pi_E$, for all E with $pE = B$. Since the diagram

$$\begin{tikzcd}
\hat{F}B \arrow[r, "\hat{F}\mathrm{id}_B"] \arrow[d, "\pi_{E'}"'] & \hat{F}B \arrow[d, "\pi_E"] \\
FE' \arrow[r, "Fe"] & FE
\end{tikzcd}$$

commutes for all $e\colon E' \to E$ with $pe = \mathrm{id}_B$, choosing $e = \mathrm{id}_E$ gives the desired equality. Finally, given $\begin{tikzcd} B \arrow[r, "b"] \arrow[rr, bend left, "b''=b'\circ b"] & B' \arrow[r, "b'"] & B'' \end{tikzcd}$ then, again, to show that $\hat{F}b'' = \hat{F}b' \circ \hat{F}b$ one needs to examine the diagram

$$\begin{tikzcd}
\hat{F}B \arrow[r, "\hat{F}b"] \arrow[rr, bend left, "\hat{F}b''"] & \hat{F}B' \arrow[r, "\hat{F}b'"] & \hat{F}B'' \arrow[d, "\pi_{E''}"] \\
& & FE''
\end{tikzcd}$$

and argue for the equality of the two mophirms $\hat{F}B \to FE''$, for each E'' with $pE'' = B''$, which we now fix. By first lifting b' and then lifting b we may find

$\begin{tikzcd} E \arrow[r, "e"] \arrow[rr, bend left, "e''=e'\circ e"] & E' \arrow[r, "e'"] & E'' \end{tikzcd}$ projecting precisely to $\begin{tikzcd} B \arrow[r, "b"] \arrow[rr, bend left, "b''=b'\circ b"] & B' \arrow[r, "b'"] & B'' \end{tikzcd}$. Applying F, the diagram above is augmented to become

$$\begin{tikzcd} \hat{F}B \arrow[r, "\hat{F}b"] \arrow[rr, bend left, "\hat{F}b''"] \arrow[d, "\pi_E"] & \hat{F}B' \arrow[r, "\hat{F}b'"] & \hat{F}B'' \arrow[d, "\pi_{E''}"] \\ FE \arrow[r, "Fe"] \arrow[rr, bend right, "Fe''"'] & FE' \arrow[r, "Fe'"] & FE'' \end{tikzcd}$$

in which, by definition of $\hat{F}$, each of the squares commutes, as well as the outer part of the diagram (consisting of the two bent morphisms and the outer vertical morphisms). The required equality is now a simple diagram chase.

The canonical natural transformation π is given, for all objects E in $\mathcal{E}$, by the components $\pi_E \colon \hat{F}(pE) \to FE$, the canonical projection from the limit $\hat{F}(pE)$ onto FE. The naturality condition

$$\begin{tikzcd} \hat{F}(pE) \arrow[r, "\hat{F}(pe)"] \arrow[d, "\pi_E"] & \hat{F}(pE') \arrow[d, "\pi_{E'}"] \\ FE \arrow[r, "Fe"] & FE' \end{tikzcd}$$

for a morphism e follows by the definition of $\hat{F}$. As for the universal property, suppose $H \colon \mathcal{B} \to \mathcal{A}$ is a functor together with a natural transformation α as in the main diagram above. The uniqueness of a natural transformation β is immediate, since the property of β is expressed by the commutativity of the diagram

$$\begin{tikzcd} H(pE) \arrow[r, "\beta_{(pE)}"] \arrow[d, "\alpha_E"'] & \hat{F}(pE) \arrow[dl, "\pi_E"] \\ FE & \end{tikzcd}$$

for all objects E. In particular, fixing an object B in $\mathcal{B}$, we obtain the commutativity of

$$\begin{tikzcd} HB \arrow[r, "\beta_b"] \arrow[d, "\alpha_E"'] & \hat{F}B \arrow[dl, "\pi_E"] \\ FE & \end{tikzcd}$$

for all object E with $pE = B$. Since the π_E are the projections from $\hat{F}B$ to its defining diagram, β_b is uniquely determined by the α_E. To conclude the proof we establish the existence of β. For an object B in $\mathcal{B}$, objects E_1, E_2 with $pE_i = B$, and morphisms $e \colon E_1 \to E_2$ with $pe = \mathrm{id_B}$, consider the diagram

$$\begin{tikzcd} FE_1 \arrow[rr, "Fe"] & & FE_2 \\ & HB \arrow[ul, "\alpha_{E_1}"] \arrow[ur, "\alpha_{E_2}"'] & \end{tikzcd}$$

which may be re-written as

$$\begin{tikzcd} FE_1 \arrow[r, "Fe"] & FE_2 \\ HB \arrow[u, "\alpha_{E_1}"] \arrow[r, "H(pe)"] & HB \arrow[u, "\alpha_{E_2}"'] \end{tikzcd}$$

since $H(pe) = \mathrm{id}_{HB}$. The diagram commutes by the naturality of α. We have thus shown that HB forms a cone over the diagram F_B defining $\hat{F}B$, and thus obtain the morphisms $\beta_b \colon H_B \to \hat{F}B$ and the commutativity of

$$\begin{tikzcd} HB \arrow[r, "\beta_b"] \arrow[d, "\mathrm{id_B}"] & \hat{F}B \arrow[d, "\pi_E"] \\ H(pE) \arrow[r, "\alpha_E"] & FE \end{tikzcd}$$

which gives at once the desired decomposition of α. It thus remains to verify the naturality of β, for which we consider the diagram

$$\begin{tikzcd} HB \arrow[r, "\beta_B"] \arrow[d, "Hb"] \arrow[rr, bend left, dashed, "\alpha_E"] & \hat{F}B \arrow[r, dashed, "\pi_E"] \arrow[d, "\hat{F}b"] & FE \arrow[d, dashed, "Fe"] \\ HB' \arrow[r, "\beta_{B'}"] \arrow[rr, bend right, dashed, "\alpha_{E'}"'] & \hat{F}B' \arrow[r, "\pi_{E'}"] & FE' \end{tikzcd}$$

and we aim to establish the commutativity of the left square. To that end, it suffices to prove that the two morphisms $HB \to FE'$ along the solid part of the diagram are equal, for all E' with $pE' = B'$. But for such E' a simple chase around the diagram, using the definition of β and the naturality of α as needed, yields the desired equality, and with it the proof is complete. □

Noting that $q \colon \mathcal{D} \to \mathcal{C}$, other than give rise to the multivalued fibration p, played no role in the proof, we obtain the following corollary, which is in fact a restatement of the theorem.

Theorem 4.2. *Let $p \colon \mathcal{E} \to \mathcal{B}$ be a multivalued fibration satisfying the same conditions as in the theorem above. Then for every small complete category $\mathcal{A}$ the assignment $F \mapsto \hat{F}$ yields a functor $[\mathcal{E}, \mathcal{A}] \to [\mathcal{B}, \mathcal{A}]$ between the functor categories to $\mathcal{A}$.*

Proof. The assignment on natural transformation is given by the universal property, and functoriality follows by it as well. □

5. Topological invariants

In this section we consider three metric constructions giving rise, respectively, to connectedness, a variant of homology, and a variant of the fundamental groupoid. The constructions are valid for any scale system $\mathbf{\Sigma}$. The topological invariants are obtained when taking $\mathbf{\Sigma} = \mathbf{\Sigma_c}$, while choosing $\mathbf{\Sigma} = \mathbf{\Sigma_u}$ yields uniform invariants.

In more detail, let $\mathcal{A}$ be an arbitrary category which is small complete. Given any scale system $\mathbf{\Sigma}$ consider the forgetful functor $q \colon \mathbf{\Sigma} \to \mathbf{Met_{all}}$. Let $p \colon \mathbf{\Sigma_{cont}} \to$

$\mathbf{Met_\Sigma}$ be the multivalued fibration associated to q (cf. Lemma 3.4). It is straightforward to verify that the conditions of Theorem 4.1 are satisfied (we note that the requirement on the scale system to be closed under point-wise meets is crucial), and thus any functor $F\colon \mathbf{\Sigma} \to \mathcal{A}$ gives rise to a functor $\hat{F}\colon \mathbf{Met_\Sigma} \to \mathcal{A}$. $\hat{F}$ computes objects in $\mathcal{A}$ which are invariant with respect to isomorphisms in $\mathbf{Met_\Sigma}$. In particular, given a single functor $\mathbf{\Sigma_c} \to \mathcal{A}$, one obtains, for every scale system $\mathbf{\Sigma}$ the functor $\mathbf{\Sigma} \to \mathbf{\Sigma_c} \to \mathcal{A}$, and thus corresponding invariants. For $\mathbf{\Sigma} = \mathbf{\Sigma_c}$, these are topological invariants since the isomorphisms in $\mathbf{Met_{\Sigma_c}}$ are continuous functions with continuous inverses, namely homeomorphisms. For $\mathbf{\Sigma_u}$ the isomorphisms in $\mathbf{Met_{\Sigma_u}}$ are uniformly continuous functions with uniformly continuous inverses, and thus the invariants are uniform invariants. In the three examples below we simply present one functor $\mathbf{\Sigma_c} \to \mathcal{A}$, and then speak freely of the resulting topological and uniform invariants. We keep the discussion somewhat informal.

All of the constructions pass through the category **Tol** of *tolerance spaces*, namely pairs (X, T) where T is a symmetric and reflexive relation on X. The morphisms $f\colon (X, T) \to (Y, S)$ in **Tol** are the *tolerant* functions, namely functions $f\colon X \to Y$ satisfying $(fx)S(fy)$ for all $x, y \in X$ with xTy. The concept goes back to Poincaré (see [16]) but more formally introduced by Zeeman in [20] (see [15] for a modern perspective, including historical remarks). It is straightforward that, given a scaled space (X, L, d, R), defining xTy precisely when $d(x, y) \prec Rx$ is a reflexive relation. Taking its symmetric closure yields a tolerance space, giving rise to a functor $\mathbf{ScMet} \to \mathbf{Tol}$. The reference of importance for the invariants we introduce below is [17] where the homotopy and homology of tolerance spaces are studied (see also [7] for deeper results on the homology of relations).

5.1. **Connectedness.** Firstly, we recast the metric characterisation of connectedness given in [18] in the language of the machinery above. Consider the category $\mathcal{A} = \{\text{False} \to \text{True}\}$ with only two objects and three morphisms. For a tolerance space (X, T) let $\bar{T}$ be the transitive closure of T. Consider the diagram

$$\begin{tikzcd}
\mathbf{ScMet_c} \arrow[r, hook] \arrow[d, two heads] & \mathbf{ScMet} \arrow[r] \arrow[d] \arrow[ddr, "c_?"] & \mathbf{Tol} \arrow[dd] \\
\mathbf{Met_c} \arrow[r, hook] \arrow[drr, "\widehat{c_?}"'] & \mathbf{Met_{all}} & \\
 & & \mathcal{A}
\end{tikzcd}$$

where the functor $\mathbf{Tol} \to \mathcal{A}$ maps (X, T) to True if, and only if, $\hat{T}$ is the trivial relation $X \times X$. Then, by the results of [18], for $X = (X, L, d)$, $\widehat{c_?}(X) = \text{True}$ precisely when X is connected in the classical topological sense. The uniform invariant corresponding to $\widehat{c_?}$ is precisely the notion of uniform connectedness (also known as Cantor connectedness).

More generally, consider the diagram

$$\begin{tikzcd}
\mathbf{ScMet_c} \arrow[r, hook] \arrow[d, two heads] & \mathbf{ScMet} \arrow[r] \arrow[d] \arrow[rdd, "cc"] & \mathbf{Tol} \arrow[d] \\
\mathbf{Met_c} \arrow[r, hook] \arrow[rrd, "\widehat{cc}"] & \mathbf{Met_{all}} & \mathbf{Eq} \arrow[d] \\
 & & \mathbf{Set}
\end{tikzcd}$$

where $cc(X,T)$ is the set of equivalence classes of $\hat{T}$. Then $\widehat{cc}(X)$ is the set of quasi-components of X in the classical topological sense (i.e., the quasi-component of $x \in X$ is the intersection of all clopen subsets containing x). The uniform variant of $\widehat{cc}$ is given by the uniform quasi-components.

5.2. **homology.** To address homology we consider the category **sSet** of simplicial sets (see, e.g., [10]) and the category **cSet** of cubical sets (see, e.g., [11] or [12]), recalling that an object in each of these categories yields a chain complex, and thus homology group functors $\mathbf{sSet}, \mathbf{cSet} \to \mathbf{Ab}$. Let Δ^n (resp. $\square^n$) be the set of vertices of the standard n-simplex (resp. n-cube) endowed with the tolerance relation given by adjacency. For a tolerance space (X,T) one may follow ([17]) and define a simplicial set whose simplicies are the tolerance functions $\Delta^n \to X$ or one may define a cubical set whose cubes are the tolerant functions $\square^n \to X$ (which is essentially what [1] does, though not explicitly factored through tolerance spaces). Consider the diagram

$$\begin{tikzcd}
\mathbf{ScMet_c} \arrow[r, hook] \arrow[d, two heads] & \mathbf{ScMet} \arrow[rr] \arrow[d] \arrow[rrdd, bend left, "cH_n"] \arrow[rrdd, bend right, "sH_n"'] & & \mathbf{Tol} \arrow[dl] \arrow[dr] & \\
\mathbf{Met_c} \arrow[r, hook] \arrow[rrrd, bend right, "\widehat{cHn}"] \arrow[rrrd, bend right=40, "\widehat{sH_n}"'] & \mathbf{Met_{all}} & \mathbf{sSet} \arrow[dr] & & \mathbf{cSet} \arrow[dl] \\
 & & & \mathbf{Ab} &
\end{tikzcd}$$

where cH_n is the n-th homology computed via **cSet** and sH_n is the n-th homology computed via **sSet**. The authors of [1] refer to cH_n as discrete homology of (classical) metric spaces, and their remarks in [1, p. 904, Section 7, (3)] is the statement that the uniform invariant $\widehat{cH_0}$ of a punctured disk is trivial. It is not hard to see that the topological invariant $\widehat{cH_0}$ of a punctured disc is isomorphic to $\mathbb{Z}$. More generally, both topological invariants $\widehat{sH_0}(X)$ and $\widehat{cH_0}(X)$ are isomorphic to the free abelian group on the quasi-components of X. At this point, while an equivalence is certainly expected, we do not know the precise relationship between $\widehat{cH_n}$ and $\widehat{sH_n}$ for n.

5.3. **homotopy.** In [17] the notion of homotopy for tolerance spaces is given, briefly, as follows. Two tolerant functions $f, h \colon X \to Y$ between tolerance spaces are *homotopic* if there exists a finite sequence $g_1, \ldots, g_n \colon X \to Y$ of tolerant functions such that $g_1 = f$, $g_n = h$, and, for all $1 \le k < n$, xTx' implies $(g_k(x))T(g_{k+1}(x))$ for all $x, x' \in X$. Such a sequence is called a *homotopy*. Relative homotopy is defined in the obvious way. One may then consider tolerant functions $[n] \to X$, where $[n]$ is the set $\{0, 1, \ldots, n\}$ with tolerance given by kTm precisely when $|k - m| \le 1$

as discrete analogues of paths. The authors of [6] perform essentially the same construction directly on uniformly scaled (classical) metric spaces under the name of discrete homotopy. It is straightforward to define the fundamental groupoid $\pi_1^g(X,T)$ of a tolerance space: the objects are the points of X, and the morphisms are equivalence classes of paths $[n] \to X$ modulo homotopy relative to end-points (paths of different lengths can always be augmented without affecting the homotopy class to obtain two representatives with the same domain $[n]$). This gives rise to the functor $\pi_1^g\colon \mathbf{Tol} \to \mathbf{Grpd}$. Consider now the diagram

$$\begin{tikzcd}
\mathbf{ScMet_c} \arrow[r, hook] \arrow[d, two heads] & \mathbf{ScMet} \arrow[r] \arrow[d] & \mathbf{Tol} \arrow[dd, "\pi_1^g"] \\
\mathbf{Met_c} \arrow[r, hook] \arrow[drr, "\widehat{\pi_1^g}"'] & \mathbf{Met_{all}} & \\
& & \mathbf{Grpd}
\end{tikzcd}$$

It is not hard to see that the uniform invariant $\widehat{\pi_1^g}$ of the punctured disk is contractible, while the topological invariant is isomorphic to the fundamental groupoid of S^1 in the classical sense. Going back to [6], where the authors concentrate on the uniform picture, we note that the general discussion of the following subsection are related to the concept of critical points presented in [6].

5.4. **Persistence and stability.** We conclude this work with a short discussion of the naturally arising concepts of persistence and stability for invariants $\hat{F}$ produced via a multivalued fibration $p\colon \mathcal{E} \to \mathcal{B}$ according to the main construction

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook] \arrow[d, two heads, "p"] & \mathcal{D} \arrow[d, "q"] \arrow[ddr, "F"] & \\
\mathcal{B} \arrow[r, hook] \arrow[drr, "\hat{F}"'] & \mathcal{C} & \\
& & \mathcal{A}
\end{tikzcd}$$

Let us recall that $\hat{F}B$ is the limit of the diagram $F\colon \mathcal{E}_B \to \mathcal{A}$, with $\mathcal{E}_B$ the fiber over B, namely all objects projecting to B and all morphisms projecting to id_B. For the rest of this section, fix the above, including an object B whose invariant $\hat{F}B$ is of interest.

Definition 5.1. Let $\mathcal{I}$ be a non-empty full subcategory of $\mathcal{E}_B$. Say that $\mathcal{I}$ is an *interval* if for all $E_1 \to E_2 \to E_3$ is in $\mathcal{E}_B$ if $E_1, E_2 \in \mathcal{I}$, then $E_3 \in \mathcal{I}$. If F is constant on an interval $\mathcal{I}$, then we say that FE, the common value on the objects of the interval, is a *persistent approximation* of $\hat{F}B$ over $\mathcal{I}$. Say that an interval $\mathcal{I}$ is a *ray* if for all E' in $\mathcal{E}_B$ there exists E in $\mathcal{I}$ with at least one morphism $E \to E'$ in $\mathcal{E}_B$. Then $\hat{F}B$ is *stable* if there exists a persistent approximation FE on a ray $\mathcal{I}$ such that $\pi\colon \hat{F}B \xrightarrow{\pi_E} FE$ is the identity, for all E in the interval. More generally, define $\hat{F}B$ to be stable over any non-empty subcategory J of $\mathcal{E}_B$, if $\pi_E\colon \hat{F}B \to FE$ is the identity for all E in $\mathcal{J}$.

In the context of the topological and uniform invariants above, it is obvious that persistence and stability are highly sensitive to the metric, and are far from being topological or uniform invariants. A detailed study of persistence and stability will be carried out in future work, including the elucidation of the relationship with computational topology ([8]). At this point let us just point out that some aspects of stability are topological. For instance, connectedness is highly stable. In fact, if $X = (X, L, d)$ is connected, then the topological invariant $\widehat{c_?}(X)$ is stable over the entire diagram $\mathcal{E}_B$, independently of the metric, simply because $\widehat{c_?}(X) = \text{True}$, for all scales $R \in \mathbf{\Sigma}$ on X. On the other extreme, for a totally disconnected and nowhere discrete space $X = (X, L, d)$, $\widehat{cc}(X)$ is, regardless of the metric, highly unstable. In fact, it is not hard to see that $\widehat{cc}(X)$ is not stable over any ray.

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