

# Value semigroups, value quantales, and positivity domains

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## Abstract

In 1981 and 1997 Kopperman and Flagg, respectively, proved that every topological space is metrisable, provided the symmetry and separation axioms are removed from the requirements on the metric, and the metric is allowed to take values in, respectively, a value semigroup or a value quantale. Seeking to construct a value quantale from a value semigroup we focus on a small portion of the structure present in a value semigroup, comprising what we call a positivity domain, and we construct its enveloping value quantale, forming part of a detailed comparison between value semigroups and value quantales. We obtain a representation theorem for value quantales in terms of positivity domains, and we outline how products of positivity domains can be used in the theory of continuity spaces instead of (the non-existent) products of value quantales.

*Keywords:* value quantale, value semigroup, continuity space, ordered semigroup, ordered monoid  
*2000 MSC:* 06F03, 06B23, 06D05, 54B10, 54E35

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## 1. Introduction

Evidently, the structure present on  $[0, \infty]$  is far richer than the bare minimum required in order to serve as the codomain for a metric function. Clearly, an ordered set  $S$  with a smallest element  $0$ , and a binary operation  $+$  on  $S$  suffice in order to state the axioms of a metric function  $d: X \times X \rightarrow S$ . It is equally evident that at such a level of generality a development of a useful theory of generalised metric spaces is hopeless.

Less evident is the nature of theories of generalised metric spaces taking values in axiomatically defined structures richer than just the meager skeleton outlined above, yet allowing for many models which may resemble  $[0, \infty]$  very little. In particular, that all topological spaces may become metrisable in this more general sense, and that such generalised metric spaces are in fact models for topology, is not an a-priori observation

In [1] Kopperman introduced the concept of continuity space, i.e., a metric space taking values in a value semigroup with a set of positives, motivated by investigations into the first order nature of topology. In particular, while it is well-known that topological spaces do not form an elementary class, Kopperman’s formalism, within which every topological space is metrisable (a result later popularised in [2]), admits a first order axiomatisation, and thus Kopperman’s continuity spaces do form an elementary class.

In [3] Flagg introduced another version of continuity spaces, i.e., as metric spaces taking values in a value quantale. The motivation seems to have been not so much for the similar metrisability result obtained, as much as coming from more general considerations of domain theory (and none of the logic considerations that motivated Kopperman are at play since Flagg’s axioms of a value quantale are not first order). Flagg notices in [3, Section 5] the differences between value quantales and value semigroups, particularly the lack of closure of value quantales under certain constructions for which value semigroups are closed, offering a rudimentary comparison, and some constructions to

remedy the situation. We give a more detailed comparison, obtain a general construction of value quantales from value semigroups, study its functoriality, and recover some form of closure of value quantales under products.

The publications [2, 3] in, respectively, 1988 and 1997 spawned considerable interest. For instance, [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] cites recent work referencing at least one of the two articles. It is probably commonly assumed that any work using one of the formalisms can be adapted to yield analogous results phrased in the other formalism. Below we present a comparison between Kopperman’s value semigroups and Flagg’s value quantales, partly motivated by the need of a more formal bridge between the two formalisms, as we test the limitations of that common assumption. Preserving the topological *raison d’être* of the formalisms, we make the necessary adjustments to the definition of Kopperman’s value semigroup so that the statement that every value quantale is a value semigroup with a canonical choice of positives, becomes true. We endow value semigroups and value quantales with a notion of morphism, and present the main construction, namely the enveloping value quantale functor. The results we present provide a practical means of relating the two formalisms, allowing for greater flexibility in applications, and easing the interaction between the two theories.

The technical aim of the paper is to construct the diagram

$$\begin{array}{ccccc}
 \mathbf{K} & \longrightarrow & \mathbf{P}_\Lambda & \hookrightarrow & \mathbf{P}_\Lambda \\
 \Gamma \downarrow & & \downarrow \Theta & & \downarrow \Theta \\
 \mathbf{F} & \xleftrightarrow{-\circ} & \mathbf{F}_\Lambda & \hookrightarrow & \mathbf{F}_\Lambda
 \end{array}$$

and primarily the enveloping value quantale contravariant functor  $\Gamma$ . Section 2 introduces the concepts of Kopperman’s value semigroup and Flagg’s value quantale, emphasising the topological aspects of continuity spaces. The section goes on to define the relevant categories  $\mathbf{K}$  and  $\mathbf{F}$  and proceeds to establish some of their fundamental aspects. Section 3 is where the enveloping value quantale construction is given, including the introduction of the auxiliary concepts of positivity semigroup and positivity domain, forming the upper half of the diagram, and the concepts of deficient and semi-deficient value quantales, forming the lower half of the diagram. Section 4 is a study of the functorial properties related to the enveloping value quantale construction and comparisons between the auxiliary structures. Section 5 contains the first of two applications, namely a representation theorem for value quantales. Section 6 utilises the general machinery to define and study a notion of product of value quantales, acting as a surrogate system to the non-existent categorical products. Finally, Section 7 presents the second application, to obtain a metric definition of topological products of continuity spaces.

## 2. Kopperman’s value semigroups and Flagg’s value quantales

In this section we present the categories we wish to compare. The first subsection fixes commonly used terminology about ordered structures. Subsection 2.2 presents Kopperman’s definition of value semigroups and Subsection 2.3 is a treatment of Flagg’s value quantales. Subsection 2.4 is concerned with the topology oriented usage of these structures, primarily focusing on the result that every topological space is metrisable when the metrics take values in either a value semigroup or value quantale. Subsection 2.5 is a preliminary comparison between the structures, including a necessary slight modification to Kopperman’s definition. Finally, in Subsection 2.6 we provide definitions of morphism of value semigroups and of value quantales. Treatment of such morphisms in the literature is hard to find, if it exists at all. We motivate our choices, keeping the topological perspectives in mind.

### 2.1. Ordered algebraic structures

Let us fix some terminology. A *semigroup*  $(S, \cdot)$  is a set  $S$  together with an associative binary operation  $\cdot$  on  $S$ . If  $ab = ba$  for all  $a, b \in S$ , then  $(S, \cdot)$  is an *abelian semigroup*, in which case the operation is typically denoted by  $+$ . A *monoid*  $(M, \cdot, e)$  is a semigroup  $(M, \cdot)$  together with an identity element  $e \in M$  with  $ea = ae = a$  for all  $a \in M$ . A monoid  $M$  is commutative if the semigroup  $(M, \cdot)$  is abelian, and then the identity element is typically denoted by  $0$ . In this work we are primarily interested in the commutative case. An *ordered abelian semigroup*  $(S, \leq, +)$  is an abelian semigroup  $(S, +)$  with an ordering  $\leq$  on  $S$  such that the conditions  $a \leq a'$  and  $b \leq b'$  imply  $a + b \leq a' + b'$ , for all  $a, b, a', b' \in S$  (or, equivalently,  $a \leq b$  implies  $a + x \leq b + x$ , for all  $a, b, x \in S$ ). An *ordered commutative monoid*  $(M, \leq, +, 0)$  is a monoid  $(M, +, 0)$  such that  $(M, \leq, +)$  is an ordered abelian semigroup, and such that  $0 \leq a$  holds for all

$a \in M$ . We note that these definitions are largely standard, but whereas we require the neutral element of an ordered monoid to also be its smallest element, other texts, e.g., [18], do not impose that restriction.

We shall also require the multivalued analogue of part of the above, namely ordered multisemigroups, and we refer to [19] and [20] for further information. A *multisemigroup* is a set  $S$  together with a multivalued operation  $\cdot: S \times S \rightarrow \mathcal{P}(S)$ , where  $\mathcal{P}(S)$  is the power set of  $S$ . The operation naturally gives rise to an ordinary binary operation  $\cdot: \mathcal{P}(S) \times \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ , namely  $A \cdot B = \bigcup\{a \cdot b \mid a \in A, b \in B\}$ , and further we identify  $S$  in  $\mathcal{P}(S)$  as the collection of singletons. The multivalued operation on  $S$  is required to be associative in the sense that this associated binary operation is associative, which amounts to the usual condition  $a(bc) = (ab)c$ . The multisemigroup is *abelian* if  $a \cdot b = b \cdot a$  holds for all  $a, b \in S$ , and then the operation is often denoted by  $+$ . Generally, for subsets  $A, B$  of an ordered set, we write  $A \leq B$  to mean that for all  $\beta \in B$  there exists  $\alpha \in A$  with  $\alpha \leq \beta$ . An *ordered abelian multisemigroup* is an abelian multisemigroup  $S$  equipped with an ordering  $\leq$  on  $S$  such that  $a \leq a'$  and  $b \leq b'$  imply  $a + b \leq a' + b'$ . Explicitly, the condition is that for all  $x \in a' + b'$  there exists  $y \in a + b$  with  $y \leq x$ .

As morphisms  $f: S \rightarrow S'$  for abelian semigroups we take the additive homomorphisms, i.e., those functions satisfying  $f(a + b) = f(a) + f(b)$ , and for ordered semigroups we take the monotone additive homomorphisms, i.e., those satisfying  $f(a) \leq f(b)$  for all  $a \leq b$  in  $S$ . A morphism  $f: S \rightsquigarrow S'$  of multisemigroups is a strong multihomomorphism, i.e., a function  $f: S \rightarrow \mathcal{P}(S')$  of the underlying sets, satisfying  $f(a + b) = f(a) + f(b)$ , for all  $a, b \in S$ . A morphism  $f: S \rightsquigarrow S'$  of ordered abelian multisemigroups is a strong multihomomorphism satisfying  $f(a) \leq f(b)$  whenever  $a \leq b$ , in the same sense as above, i.e., for all  $y \in f(b)$  there exists  $x \in f(a)$  with  $x \leq y$ . We use  $\rightarrow$  and  $\rightsquigarrow$  to make a distinction between, respectively, single-valued and multivalued functions. Abelian semigroups, ordered abelian semigroups, abelian multisemigroups, and ordered abelian multisemigroups, with their respective morphisms, form the four categories

$$\begin{array}{ccc} \mathbf{AbS} & \longrightarrow & \mathbf{AbS}_{\leq} \\ \downarrow & & \downarrow \\ \mathbf{mAbS} & \longrightarrow & \mathbf{mAbS}_{\leq} \end{array}$$

where the horizontal functors, left adjoint to the evident forgetful functors, are obtained by imposing the trivial ordering on the underlying set.

## 2.2. Kopperman's value semigroups

Following [2] a *value semigroup* is an abelian semigroup  $(L, +)$  with neutral element  $0$  and absorbing element  $\infty$ , with  $0 \neq \infty$ , such that the following conditions hold.

- Defining, for  $a, b \in L$ , that  $a \leq b$  precisely when the equation  $b = a + x$  admits a solution in  $L$ , results in an ordering of  $L$  (reflexivity and transitivity are automatic) which is a meet semilattice (only binary meets need to be established, since  $\infty = a + \infty$  implies that  $a \leq \infty$  holds for all  $a \in L$ , so the empty meet exists).
- For all  $a \in L$  there exists a unique  $b \in L$  with  $a = b + b$  ( $b$  is denoted by  $\frac{a}{2}$ ).
- $(a \wedge b) + c = (a + c) \wedge (b + c)$ , for all  $a, b, c \in L$ .

A subset  $P \subseteq L$  of *positives* in a value semigroup  $L$  is any subset for which the following conditions hold.

- $P$  is a filter, namely  $\varepsilon \wedge \delta \in P$  and  $a \in P$  for all  $\varepsilon, \delta \in P$  and  $a \in L$  with  $a \geq \varepsilon$ .
- If  $\varepsilon \in P$ , then  $\frac{\varepsilon}{2} \in P$ .
- For all  $a, b \in L$ , if  $a \leq b + \varepsilon$  for all  $\varepsilon \in P$ , then  $a \leq b$ .

**Remark 1.** Note that the last condition on  $P$  and the requirement that  $|L| \geq 2$  guarantee that  $P \neq \emptyset$ .

Obviously,  $[0, \infty]$  with its usual structure and with  $P = (0, \infty]$  is a value semigroup with positives. It is straightforward that if  $\{(L_i, P_i)\}_{i \in I}$  is a family of value semigroups with positives, then the cartesian product of the sets  $\{L_i\}_{i \in I}$  with the obvious pointwise structure is a value semigroup with positives. There are in fact two natural choices for

the set of positives, namely the product of the  $P_i$ , or its subset consisting of those elements with only finitely many coordinates different than  $\infty$ . In Theorem 7 below we identify the latter as the construction giving rise to categorical products.

**Remark 2.** We note that a value semigroup actually has an underlying ordered commutative monoid and not just an ordered abelian semigroup. Below we are required to make adjustments to the definition and we then (cf. Definition 1) take the opportunity of choosing a more streamlined terminology.

Whenever we speak of a value semigroup  $L$  we mean a value semigroup together with a choice of a set  $P$  of positives.

### 2.3. Flagg's value quantales

Following [3] a *value quantale* is a complete lattice  $(L, \leq)$  with  $0 \neq \infty$ , together with a commutative and associative binary operation  $+$  for which the following properties hold.

- $a + 0 = a$ , for all  $a \in L$ .
- $a + \bigwedge \mathcal{A} = \bigwedge (a + \mathcal{A})$ , for all  $a \in L$  and  $\mathcal{A} \subseteq L$ .
- $a = \bigwedge \{b \in L \mid b > a\}$ .
- $a \wedge b > 0$  for all  $a, b \in L$  with  $a, b > 0$ .

Here  $0$  is the smallest element of  $L$ ,  $\bigwedge$  is the meet in  $L$ ,  $a + \mathcal{A} = \{a + x \mid x \in \mathcal{A}\}$ , and  $b > a$  reads as *b is well above a*, meaning that for all  $S \subseteq L$  if  $a \geq \bigwedge S$ , then there exists  $s \in S$  with  $b \geq s$ . We denote the set  $\{\varepsilon \in L \mid \varepsilon > 0\}$  by  $P = P_L$ , and call it the set of *positives* in  $L$ .

**Remark 3.** The definition of value quantale amounts precisely to a monoid object  $L$  in the category of complete meet semilattices where the unit is the smallest element and where  $\{a \in L \mid a > 0\}$  is a filter. The term *quantale*, a portemanteau on *quantum locale*, was introduced by Mulvey in [21]. In [21], as well as in other texts (e.g., and this is far from an exhaustive list, [22, 23], which are also general references for quantales, [12, 24, 25, 26, 27] for quantales in the context of topology, [28, 29, 30] for quantales in the context of categorical logic and computer science, and [31, 32] for quantales in the context of  $C^*$ -algebras) the notion of a (unital) quantale is defined as a monoid object in the category of complete join semilattices. Consequently, if  $L$  is a value quantale, then  $L^{\text{op}}$ , the opposite of  $L$ , is a quantale in the sense of, e.g., [22], which is in fact an integral quantale. More precisely, a complete lattice  $L$  with a monoid structure is a value quantale if, and only if,  $L^{\text{op}}$  is a commutative integral quantale in which  $\{a \in L^{\text{op}} \mid a \ll \infty\}$  is an ideal (the meaning of  $a \ll b$  in  $L^{\text{op}}$  is the same as  $a > b$  in  $L$ ). The direction Flagg chose, and the one we adhere to, is most suitable for the notion of metric space taking values in a value quantale dictated by the historical choice of considering distance functions rather than similarity functions (see [33]). The fourth condition in the definition of value quantale, namely that  $a \wedge b > 0$  whenever  $a, b > 0$ , conflicts with many of the standard constructions in the general theory of quantales in the sense that when these constructions are applied to value quantales the result is rarely a value quantale again. Of course, that condition is crucial for many aspects of metric analysis as can be seen in [7, 15, 34, 35].

**Remark 4.** The requirement that  $0 \neq \infty$  excludes, up to isomorphism, just one structure, namely the singleton set with its uniquely determined structure. For the reader preferring reasoned expulsion, rather than ad-hoc banishment, we note that the inherent peculiarity of the singleton as a value quantale is that it would be the only value quantale with no positive elements (cf. Remark 1 above). In particular,  $\infty > 0$  holds in every value quantale.

The following claims are easily verified (see [3, Lemma 1.2, Lemma 1.3, Theorem 1.6]).

**Proposition 1.** Let  $L$  be a value quantale,  $a, b, c \in L$ , and  $\mathcal{A} \subseteq L$ . Then

1.  $b > a$  implies  $b \geq a$ .
2. Either one of  $c > b \geq a$  or  $c \geq b > a$  implies  $c > a$ .
3.  $b > \bigwedge \mathcal{A}$  implies  $b > a$  for some  $a \in \mathcal{A}$ .

4. If  $c > a$ , then there exists  $b \in L$  with  $c > b > a$  (interpolation property).

**Example 1.** A primary example of a value quantale is  $[0, \infty]$  with its usual structure. The lattice  $\mathbb{B} = \{0 < \infty\}$  of boolean truth values with the obvious structure is a value quantale (see [3, Example 2.5]). Another simple example is the ordinal  $\omega + 1 = \{0 < 1 < 2 < \dots < n < \dots < \infty\}$  with ordinary addition. More examples are obtained, for all  $n \geq 0$ , by considering the set  $\{0, 1, \dots, n\}$  with the natural ordering and with addition given by  $i + j = \max\{i, j\}$ , a value quantale denoted by  $\mathbf{n}$ .

The next class of examples appears in [3, Example 1.11], and the construction of the enveloping value quantale presented in Section 3 is a generalisation of it.

**Example 2.** For every set  $S$ , let  $\Omega(S) = \{a \subseteq \downarrow S \mid A \in a \implies \downarrow A \subseteq a\}$  (where  $\downarrow A$ , for any  $A \subseteq S$ , is the collection of all finite subsets of  $A$ ) with the following structure. The ordering is given by reverse inclusion, and thus meets are given by union, and  $a + b = a \cap b$ .

Some care is required while computing with the well above relation. For instance, it is not generally true that  $a + \varepsilon > a$  even when  $\varepsilon > 0$ , nor does  $a + b > c + d$  necessarily follow from  $a > c$  and  $b > d$ , and nor is  $a \wedge b > c$  guaranteed by  $a, b > c$ .

For an element  $a \in L$  we write  $\uparrow(a) = \{b \in L \mid b > a\}$ . The following property is useful.

**Lemma 1.** Let  $L$  be a value quantale and  $a, b, x \in L$ . If  $x > a + b$ , then  $x > \alpha + \beta$  for some  $\alpha > a$  and  $\beta > b$ .

PROOF. Since  $\wedge(\uparrow a + \uparrow b) = a + b$  the result follows from Proposition 1.

#### 2.4. Continuity spaces

Kopperman and Flagg introduced value semigroups and value quantales, respectively, to serve the same purpose, namely defining continuity spaces. A Kopperman (respectively Flagg) *continuity space*  $(X, L, d)$  consists of a value semigroup (respectively value quantale)  $L$ , a set  $X$ , and a function  $d: X \times X \rightarrow L$  such that  $d(x, x) = 0$  and  $d(x, z) \leq d(x, y) + d(y, z)$ , for all  $x, y, z \in X$ . We shall also refer to such a continuity space as a *metric space valued in  $L$* , or simply an  *$L$ -valued metric space*. Given  $x \in X$  and a positive  $\varepsilon \in L$ , let  $B_\varepsilon(x) = \{y \in X \mid d(x, y) \leq \varepsilon\}$ . Then declare a subset  $U \subseteq X$  to be *open* if for all  $x \in U$  there exists  $\varepsilon \in P$  with  $B_\varepsilon(x) \subseteq U$ . The collection of all open sets is the topology on  $X$  associated with the continuity space.

The results [2, Theorem 7] and [3, Theorem 4.15] may be quoted simultaneously as follows.

**Theorem 1.** Let  $(X, \tau)$  be a topological space. There exists then a value semigroup (respectively a value quantale)  $L$  and a continuity space  $(X, L, d)$  whose induced topology is  $\tau$ .

Flagg defines the induced topology differently, namely by using the sets  $B_\varepsilon^F(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ , instead of  $B_\varepsilon(x)$ , using the well above relation. For the induced topology the difference is immaterial since  $B_\varepsilon^F(x) \subseteq B_\varepsilon(x)$  (seeing that  $a < b$  implies  $a \leq b$ ) and since  $B_\delta(x) \subseteq B_\varepsilon^F(x)$  for all  $0 < \delta < \varepsilon$  (seeing that  $a \leq b < c$  implies  $a < c$ ), and by the interpolation property in a value quantale (Proposition 1), at least one such  $\delta$  exists.

Kopperman also considers the set  $B_\varepsilon^K(x) = \{y \in X \mid d(x, y) < \varepsilon\}$  instead of  $B_\varepsilon(x)$ , and notices at once that  $B_\varepsilon^K(x)$  need not be open ([2, page 96]).

**Lemma 2.** Consider a Kopperman or Flagg continuity space  $(X, L, d)$ , a point  $x \in X$ , and a positive  $\varepsilon \in L$ . In the Kopperman formalism, the set  $B_\varepsilon^K(x)$  need not be open, but it is always a neighborhood of  $x$ . In the Flagg formalism, the set  $B_\varepsilon^F(x)$  is always open.

PROOF. For the Kopperman formalism this is [2, Proposition 10]. In the Flagg formalism, assume  $y \in B_\varepsilon^F(x)$ . Clearly, if  $\delta > 0$  is such that  $d(x, y) + \delta < \varepsilon$ , then  $B_\delta^F(y) \subseteq B_\varepsilon^F(x)$ . Since  $\varepsilon > d(x, y) = \bigwedge_{\delta > 0} d(x, y) + \delta$ , the existence of such a  $\delta$  is guaranteed by Proposition 1.

As a corollary one obtains the following result (which for the Flagg formalism is the content of [16]). If  $(L_i, X_i, d_i)$ ,  $i = 1, 2$ , are continuity spaces, then a function  $f: X_1 \rightarrow X_2$  is *continuous* if for all  $x \in X_1$  and for all positive  $\varepsilon$  in  $L_2$  there exists a positive  $\delta$  in  $L_1$  such that  $d_2(fx, fy) \leq \varepsilon$  for all  $y \in X$  with  $d_1(x, y) \leq \delta$ . Let  $\mathbf{Met}_K$  (respectively  $\mathbf{Met}_F$ ) be the category whose objects are all Kopperman (respectively Flagg) continuity spaces and with morphisms the continuous functions.

**Theorem 2.** *The assignments  $(X, L, d) \mapsto (X, \tau)$ , where  $\tau$  is the induced topology, and  $f \mapsto f$  on morphisms, give rise to functors  $\mathbf{Met}_K \rightarrow \mathbf{Top}$  and  $\mathbf{Met}_F \rightarrow \mathbf{Top}$ , each of which is an equivalence of categories.*

**PROOF.** The standard textbook proof of the equivalency between the  $\varepsilon - \delta$  formulation of continuity and the topological formulation shows functoriality and fullness (faithfulness is immediate). Essential surjectivity follows from Theorem 1 (in fact the functors are surjective on objects).

Let us observe a marked difference between the two formalisms. For  $(X, L, d)$  a continuity space,  $x \in X$ , and  $S \subseteq X$  let  $d(x, S) = \bigwedge \{d(x, s) \mid s \in S\}$ . For a Flagg continuity space the meet is always defined, while for a Kopperman continuity space it may be undefined.

**Theorem 3.** *Let  $(X, L, d)$  be a Flagg continuity space,  $x \in X$ , and  $S \subseteq X$ . Then  $d(x, S) = 0$  if, and only if,  $x$  belongs to the closure  $\overline{S}$  of  $S$ . The same does not hold for a Kopperman continuity space, even when the meet is defined.*

**PROOF.** Assuming a Flagg continuity space, if  $d(x, S) = 0$  and  $\varepsilon > 0$ , then  $\varepsilon > d(x, s)$  for some  $s \in S$ , and thus  $B_\varepsilon(x) \cap S \neq \emptyset$ , so that  $x \in \overline{S}$ . On the other hand, if  $d(x, S) \neq 0$ , then there exists some  $\varepsilon > 0$  with  $d(x, S) \not\leq \varepsilon$ , and thus  $x \notin \overline{S}$ , since  $d(x, s) < \varepsilon$  for some  $s \in S$  would imply  $d(x, S) \leq d(x, s) \leq \varepsilon$ .

The failure of the claim for Kopperman continuity spaces can be observed as follows. Let  $L = [0, \infty] \times [0, \infty]$  with the pointwise structure, where  $P = (0, \infty] \times (0, \infty]$ , and consider  $\mathbb{R}^2$  as a continuity space with  $d((x, y), (u, v)) = (|x - u|, |y - v|)$ . The induced topology is the Euclidean one. Taking  $x = (0, 0)$  and  $S = \{(0, 1), (1, 0)\}$  then shows there is no hope for the analogous result, not even when  $S$  is finite.

Another aspect of the sharper control provided by Flagg's formalism when it comes to the relationship between the metric and the induced topology is the following. Given a topological space  $(X, \tau)$ , Flagg's metrisation is valued in the value quantale  $\Omega(\tau)$  by means of the metric  $d(x, y) = \downarrow \{U \in \tau \mid x \in U \implies y \in U\}$  (where  $\downarrow S$  is the set of all finite subsets of  $S$ ). A careful, yet straightforward, study shows that not only is the open ball topology induced by  $d$  equal to  $\tau$ , in fact every non-empty  $U \in \tau$  satisfies  $U = B_\varepsilon^F(x)$  where  $x \in U$  is arbitrary and  $\varepsilon = \downarrow \{U\}$ . In other words,  $\{B_\varepsilon^F(x) \mid x \in X, \varepsilon > 0\}$  is not just a basis for  $\tau$ , but is, apart for the empty set, equal to  $\tau$ . The other side of the coin is that Kopperman's formalism lends itself more straightforwardly than Flagg's formalism to performing some elementary constructions. For instance, it was already pointed out that Kopperman's formalism allows for pointwise constructions of products. However, given value quantales  $L_1$  and  $L_2$ , the pointwise structure on  $L_1 \times L_2$  is never a value quantale due to the failure of one axiom. This claim becomes evident by considering  $L_1 = L_2 = [0, \infty]$  with the usual structure. Since  $(\infty, 0) \wedge (0, \infty) = (0, 0) = 0$  it follows that  $(a, b) > 0$  implies  $a \geq \infty$  or  $b \geq \infty$ . But  $(\infty, 1) > 0$  and  $(1, \infty) > 0$ , yet  $(\infty, 1) \wedge (1, \infty) = (1, 1) \not\geq 0$ . A more general argument is given in Theorem 9 below. In this sense there is some tension between the algebraic properties of value semigroups and the topologically sharper control furnished by value quantales.

### 2.5. Preliminary comparison

It is evident that a value quantale  $L$  already satisfies properties very close to the axioms of a value semigroup, and that in particular there is a canonical, intrinsically identified, set of positives, namely  $P_L = \{\varepsilon \in L \mid \varepsilon > 0\}$ . We now turn this observation into a precise statement.

Anticipating the details below, we slightly modify Kopperman's definition of value semigroup.

**Definition 1.** *A value monoid  $(L, \leq, +, 0, \infty, P)$  is an ordered commutative monoid  $(L, \leq, +, 0)$ ,  $\infty \in L$  with  $0 \neq \infty$ , and  $P \subseteq L$  such that the following conditions hold.*

- $0 \leq a \leq \infty$  and  $a + 0 = a$ , for all  $a \in L$ .

- $(L, \leq)$  is a meet semilattice.
- For all  $\varepsilon \in P$  there exists  $\delta \in P$  with  $\delta + \delta \leq \varepsilon$ .
- $(a \wedge b) + c = (a + c) \wedge (b + c)$ , for all  $a, b, c \in L$ .
- $P$  is a filter.
- If  $a \leq b + \varepsilon$  for all  $\varepsilon \in P$ , then  $a \leq b$ .

In the obvious sense, every value semigroup is a value monoid.

**Remark 5.** We extend the meaning of Kopperman continuity space to allow for metric spaces valued in a value monoid rather than a value semigroup.

**Theorem 4.** If  $L = (L, \leq, +, 0)$  is a value quantale, then  $(L, \leq, +, 0, \infty, P_L)$  is a value monoid, denoted by  $k(L)$ .

PROOF. The proof is elementary and the relevant results can all be found (some without proof) in [3]. We present a self-contained proof.

$0 \leq a \leq \infty$  and  $a + 0 = a$  hold by definition. A value quantale  $L$  is a complete lattice, so certainly also a meet semilattice. To prove that for a given  $\varepsilon > 0$  there exists  $\delta > 0$  with  $\delta + \delta \leq \varepsilon$ , apply Lemma 1 to the equality  $0 + 0 = 0$  to obtain  $\delta_1, \delta_2 \in L$  with  $\varepsilon > \delta_1 + \delta_2$  and  $\delta_1, \delta_2 > 0$ . Taking  $\delta_1 \wedge \delta_2$  completes the argument. In a value quantale, addition distributes over arbitrary meets, so certainly also over binary meets, and monotonicity of additivity follows. The claim that  $a \geq \varepsilon > 0$  implies  $a > 0$  follows at once from the definition of  $>$ : if  $\bigwedge S = 0$  for some  $S \subseteq L$ , then by virtue of  $\varepsilon > 0$  there exists  $s \in S$  with  $\varepsilon \geq s$ , and thus  $a \geq s$ . That  $P_L$  is a filter now follows axiomatically. Finally, if  $a \leq b + \varepsilon$  for all  $\varepsilon > 0$ , then  $a \leq \bigwedge_{\varepsilon > 0} b + \varepsilon = b + \bigwedge_{\varepsilon > 0} \varepsilon = b + 0 = b$ .

## 2.6. Morphisms

We now endow value monoids and value quantales with a notion of morphism, obtaining the categories  $\mathbf{K}$  and  $\mathbf{F}$  with objects, respectively, all value monoids as in Definition 1 and all value quantales as defined in Subsection 2.3.

A value monoid or a value quantale  $L$ , being a poset, can be viewed as a category, and then the operation  $+$  is a monoidal structure. Continuity spaces valued in  $L$  are then precisely categories enriched in  $L$  (though the enriched functors are stricter than the continuous functions, see, e.g., [36, 37] and in that context we also mention quantaloid enrichment, e.g., [38]). Functors  $\psi: L_1 \rightarrow L_2$  correspond precisely to monotone functions, and  $\psi$  is *subadditive*, if  $\psi(0) = 0$  and  $\psi(a + b) \leq \psi(a) + \psi(b)$ , for all  $a, b \in L_1$ . If equality holds for all  $a, b \in L_1$ , then  $\psi$  is *additive*.

**Definition 2.** A morphism  $\psi: L_1 \rightarrow L_2$  between value monoids is a subadditive monotone function which is continuous at 0 in the sense that for all positive  $\varepsilon \in L_2$  there exists a positive  $\varepsilon' \in L_1$  with  $\psi(\varepsilon') \leq \varepsilon$ . Similarly, a morphism  $\psi: L_1 \rightarrow L_2$  between value quantales is a subadditive monotone function which is continuous at 0.

With their respective morphisms the categories  $\mathbf{K}$  and  $\mathbf{F}$  are now defined. We record the following immediate follow up to Theorem 4.

**Theorem 5.** The assignments  $(L, \leq, +, 0) \mapsto (L, \leq, +, 0, \infty, P_L)$  for all value quantales, and  $f \mapsto f$  for all morphisms, form a functor  $k: \mathbf{F} \rightarrow \mathbf{K}$ .

**Remark 6.** If a subadditive function  $\psi: L_1 \rightarrow L_2$  between value quantales is a complete meet homomorphism, then it is a morphism in  $\mathbf{F}$ , namely it is automatically continuous at 0. Indeed, for a fixed  $\varepsilon > 0$  in  $L_2$  we have  $\varepsilon > \bigwedge \{\psi(\delta) \mid \delta > 0_{L_1}\}$ , where  $0_{L_1}$  denotes the  $0 \in L_1$ , and the result follows by Proposition 1.

There are numerous plausible definitions for morphisms between value monoids and between value quantales to be entertained, and so we must offer some justification for the choice above. Subadditivity in the definition of morphism  $\psi: L_1 \rightarrow L_2$  guarantees that if  $(X, L_1, d)$  is a continuity space, then so is  $\psi_*(X) = (X, L_2, \psi_*d)$ , where  $\psi_*d: X \times X \rightarrow L_2$  is given by  $\psi_*d(x, y) = \psi(d(x, y))$ . A requirement that a morphism of value quantales be a complete meet homomorphism would be quite strong. The importance of continuity at 0 can be appreciated as

follows. Somewhat arguably (see Mathoverflow discussion [39]) the most natural notion of morphism for classical metric spaces is that of a *short function* or *non-expansive function*, i.e.,  $f : X_1 \rightarrow X_2$  satisfying  $d_2(fx, fx') \leq d_1(x, x')$ , for all  $x, x' \in X_1$ . When  $X_i = (X_i, L_i, d_i)$ ,  $i = 1, 2$ , are continuity spaces valued in  $L_1 = L_2$ , the formally identical notion of shortness is valid, and we then say that such an  $f$  is *absolutely short*. The following definition takes care of the general case.

**Definition 3.** Let  $X_1 = (X_1, L_1, d_1)$  and  $X_2 = (X_2, L_2, d_2)$  be Kopperman (respectively Flagg) continuity spaces. A short mapping is a pair  $(f, \psi) : X_1 \rightarrow X_2$  where  $\psi : L_1 \rightarrow L_2$  is a morphism of value monoids (respectively value quantales) and  $f : X_1 \rightarrow X_2$  is a function of the underlying sets such that  $d_2(fx, fx') \leq \psi(d_1(x, x'))$ , for all  $x, x' \in X_1$ . We also say that  $f$  is short relative to  $\psi$ .

In other words,  $(f, \psi)$  is short precisely when  $f : X_1 \rightarrow \psi_*(X_2)$  is absolutely short. The concepts of absolute shortness and shortness relative to identity morphisms coincide. The demand for continuity at 0 in Definition 2 implies the following result whose proof is trivial, and thus omitted.

**Theorem 6.** If  $(f, \psi) : X_1 \rightarrow X_2$  is short, then  $f$  is uniformly continuous, and hence continuous.

### 2.7. Basic properties of $\mathbf{K}$ and $\mathbf{F}$

We briefly conduct an elementary categorical investigation of  $\mathbf{K}$  and  $\mathbf{F}$ . We first note that terminal objects in either  $\mathbf{K}$  or  $\mathbf{F}$  do not exist. Indeed, for any value quantale  $F$  there are at least two distinct morphisms  $\omega + 1 \rightarrow F$ , one with  $1 \mapsto 0$  and one with  $1 \mapsto \infty$ . Similarly, initial objects do not exist either. Of course, if singletons were allowed into the categories, then they would be zero objects; an adjustment easily seen to result in a category equivalent to  $\mathbf{F}$  with a zero object freely added, and similarly for  $\mathbf{K}$ .

In contrast, all non-empty small products in  $\mathbf{K}$  do exist, as we now show alongside a closer look at the choice of positive elements in a value monoid. Let us say that a value monoid  $L$  is *trivial* if  $0 \in L$  is positive. Let  $\mathbf{K}_0$  be the full subcategory of  $\mathbf{K}$  spanned by the trivial value monoids.

**Proposition 2.**  $\mathbf{K}_0$  is a coreflective subcategory of  $\mathbf{K}$ .

**PROOF.** A coreflector  $c^*$ , i.e., a right adjoint to the inclusion  $\mathbf{K}_0 \rightarrow \mathbf{K}$ , is easily seen to be given by enlarging the set of positives in a given value monoid  $L$  to include 0 (and thus become all of  $L$ ).

Given a non-empty collection  $\{K_i\}_{i \in I}$  of trivial value monoids, let  $K$  be the set-theoretic product of the underlying sets, where for  $a \in K$  we denote by  $a_i$  the  $i$ -th coordinate of  $a$ . The pointwise operation determined by  $(a + b)_i = a_i + b_i$  and the pointwise ordering given by  $a \leq b$  precisely when  $a_i \leq b_i$  for all  $i \in I$  is easily seen to give rise to a trivial value monoid, denoted by  $\prod K_i$ , equipped with the projections  $\pi_j : \prod K_i \rightarrow K_j$ . We denote by  $\coprod K_i$  the subset of  $\prod K_i$  consisting only of those elements with finitely many non-zero coordinates.

**Theorem 7.** The categories  $\mathbf{K}_0$  and  $\mathbf{K}$  have all non-empty small products,  $\mathbf{K}_0$  has all small non-empty coproducts, and  $\mathbf{K}$  has all small coproducts in which all but finitely many value monoids are trivial.

**PROOF.** Given trivial value monoids  $\{K_i\}_{i \in I}$  as above, the universal property establishing  $\prod K_i$  as the categorical product in  $\mathbf{K}_0$  is straightforward. Now, if the  $K_i$  are value monoids, the right adjoint  $c^*$  must preserve their product in  $\mathbf{K}$ , and so, if the product exists, it has  $\prod c^*(K_i)$  as its associated trivial value monoid. To obtain a product in  $\mathbf{K}$  a choice of positives must be made, evidently the smallest subset of  $\prod c^*(K_i) = \prod K_i$  rendering each projection  $\pi_j$  continuous at 0. That set is easily seen to be the set of all  $a \in \prod K_i$  such that  $a_i$  is positive for all  $i \in I$  and  $a_i = \infty$  for all but finitely many  $i \in I$ . Now, given morphisms  $\{\psi_i : K \rightarrow K_i\}_{i \in I}$ , for some value monoid  $K$ , the usual construction of  $\psi : K \rightarrow \prod K_i$ , i.e.,  $\psi(a)_i = \psi_i(a)$ , obviously uniquely commutes with the projections. The universal property will thus be established by showing that  $\psi$  is continuous at 0, so let  $\varepsilon \in \prod K_i$  be positive. Assume  $\varepsilon_i = \infty$  for all  $i \in I$  outside a finite subset  $J$ . For all  $j \in J$ , since  $\varepsilon_j$  is positive and  $\psi_j$  is continuous at 0, there exists a positive  $\delta_j \in K$  with  $\psi_j(\delta_j) \leq \varepsilon_j$ . It then holds that  $\delta = \bigwedge \{\delta_j \mid j \in J\}$  is positive (since  $J$  is finite), that  $\psi(\delta)_i \leq \varepsilon_i$  for all  $i \in I \setminus J$  (since  $\varepsilon_i = \infty$ ), and that  $\psi(\delta)_j \leq \psi_j(\delta_j) \leq \varepsilon_j$  for all  $j \in J$  (since  $\psi_j$  is monotone), and thus  $\psi(\delta) \leq \varepsilon$ , as required. As for coproducts, it is easily seen that  $\coprod K_i$  is a coproduct in  $\mathbf{K}_0$  and thus also in  $\mathbf{K}$ , and for finitely many value monoids  $\coprod K_i = \coprod K_i$  is a



coproduct in  $\mathbf{K}$ , and the rest of the claim follows. In some more details, the canonical injections  $f_j: K_j \rightarrow \coprod K_i$  are the evident functions where  $f(x)_j = x$  and  $f(x)_i = 0$  for  $i \neq j$ . For a general coproduct in  $\mathbf{K}$ , these constructions may fail to be a coproduct since the induced function from  $\coprod K_i$  need not be continuous at 0, unless the given objects are trivial, or if only finitely many are involved.

Recall again Flagg's construction, for a set  $S$ , of the value quantale  $\Omega(S) = \{a \subseteq \downarrow S \mid A \in a \implies \downarrow A \subseteq a\}$ , ordered by reverse inclusion and with  $+$  given by intersection. If  $f: S \rightarrow T$  is a function between two sets, then, denoting the direct image function by  $f_{\rightarrow}$ , let  $f_*(a) = \{f_{\rightarrow}(A) \mid A \in a\}$ , for all  $a \in \Omega(S)$ , and let  $f^*(b) = \{A \in \downarrow S \mid f_{\rightarrow}(A) \in b\}$ , for all  $b \in \Omega(T)$ . It is easily seen that  $f_*(a) \in \Omega(T)$  and that  $f^*(b) \in \Omega(S)$ . The following result is a simple reincarnation of elementary set-theoretic properties of functions.

**Proposition 3.** *For a function  $f: S \rightarrow T$  the following hold.*

- Each of  $f_*$  and  $f^*$  is a complete meet homomorphism.
- $f^*$  is additive and  $f_*$  is superadditive (i.e.,  $f_*(a + a') \geq f_*(a) + f_*(a')$ ).
- $f_*$  is subadditive (and thus additive) if, and only if,  $f$  is injective.
- $f^*(0) = 0$ .
- $f_*(0) = 0$  if, and only if,  $f$  is surjective.

PROOF. Elementary.

**Remark 7.** *Note that  $f_*$  is thus a morphism of value quantales if, and only if,  $f$  is a bijection.*

**Theorem 8.** *The assignments  $S \mapsto \Omega(S)$  and  $f \mapsto f^*$  give rise to a functor  $\Omega: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{F}$ .*

PROOF. Given a function  $f: S \rightarrow T$ , the verification that  $f^*$  is a morphism in  $\mathbf{F}$  is included in the preceding proposition. Given another function  $g: T \rightarrow U$ , the verification that  $(g \circ f)^* = f^* \circ g^*$  is again elementary.

For an element  $a \in \Omega(S)$ , the *support* of  $a$  is  $\sigma_S(a) = \sigma(a) = \{s \in S \mid \{s\} \in a\}$ . Recall the contravariant power set functor  $\mathcal{P}^{\leftarrow}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ .

**Proposition 4.** *The functions  $\sigma_S$  are the components of a natural transformation  $\sigma: U \circ \Omega \rightarrow \mathcal{P}^{\leftarrow}$ , where  $U: \mathbf{F} \rightarrow \mathbf{Set}$  is the forgetful functor.*

PROOF. Given a function  $f: S \rightarrow T$ , the claim is that  $\sigma(f^*(b)) = f^{\leftarrow}(\sigma(b))$ , for all  $b \in \Omega(T)$ ; a simple verification:  $s \in \sigma(f^*(b)) \iff \{s\} \in f^*(b) \iff \{f(s)\} \in b \iff f(s) \in \sigma(b) \iff s \in f^{\leftarrow}(\sigma(b))$ , for all  $s \in S$ .

In contrast to  $\mathbf{K}$ , the category  $\mathbf{F}$  has only very few products and coproducts, with the following observation being used in the proof. Any value quantale  $F$  contains an element  $a$  with  $a \neq \infty$  and  $a > 0$ . Indeed, if  $\infty$  were the only positive element, then  $0 = \bigwedge \{a \in F \mid a > 0\} = \infty$ , which is axiomatically prohibited.

**Theorem 9.** *Let  $\{F_i\}_{i \in I}$  be a collection of value quantales indexed by a set  $I$ . Then the product of the collection exists in  $\mathbf{F}$  if, and only if,  $|I| = 1$ . A similar statement holds for coproducts.*

PROOF. The stated condition trivially implies the existence of the product, so we assume the condition is not met and proceed to show the product does not exist. The case  $|I| = 0$  is the claim of the non-existence of a terminal object, which was already discussed. Assuming thus that  $|I| > 1$ , fix  $k, t \in I$  with  $k \neq t$ .

Note that the value quantale  $\mathbf{1}$  classifies the underlying set  $U(F)$  of any value quantale  $F$  in the sense that there is a canonical isomorphism between  $U(F)$  and  $\mathbf{F}(\mathbf{1}, F)$ , the set of morphisms  $\mathbf{1} \rightarrow F$  in  $\mathbf{F}$ . Similarly,  $\mathbf{2}$  classifies the ordering in  $F$  in the sense that  $a \leq b$  in  $F$  if, and only if, there exists a morphism  $\mathbf{2} \rightarrow F$  with  $1 \mapsto a$  and  $2 \mapsto b$ . Consequently, if the product  $F$  of the given collection  $\{F_i\}_{i \in I}$  existed, with projections  $\pi_i: F \rightarrow F_i$ , then  $U(F) \cong \mathbf{F}(\mathbf{1}, F) \cong \prod_{i \in I} \mathbf{F}(\mathbf{1}, F_i) \cong \prod_{i \in I} U(F_i)$ , canonically. Thus we may proceed under the assumption that  $U(F)$  is

the cartesian product of  $\{U(F_i)\}_{i \in I}$ , which we may also assume to be pair wise disjoint, with the usual projections. It then similarly further follows that the ordering in  $F$  is determined coordinate wise.

Now, for  $a \in F_i$  let  $\hat{a}$  be the element in  $F$  with  $\hat{a}_i = a$  and  $\hat{a}_j = \infty$  for all  $j \in I \setminus \{i\}$ . Since  $\bigwedge \{\hat{a} \mid a \in F_i, i \in I, a > 0\} = 0$  it follows that  $x > 0$  in  $F$  implies  $x \geq \hat{a}$  for some  $i \in I$  and  $a \in F_i$ . In short, if  $x > 0$  in  $F$ , then there exists  $i \in I$  with  $x_j = \infty$  for all  $j \neq i$  and  $x_i > 0$ . The converse follows immediately from the definition of  $>$ .

Let  $a \in F_k$  satisfy  $a \neq \infty$  and  $a > 0$ , and let  $b \in F_l$  be chosen analogously. Then  $\hat{a}$  and  $\hat{b}$  are positive, but  $\hat{a} \wedge \hat{b}$  has two non-infinity coordinates, and is thus not positive. This is in contradiction with the requirement that the positive elements in a value quantale form a filter. The claim about coproducts is more elaborate due to the lack of particularly nice co-representing objects, but similar techniques still suffice, and we omit the details.

We note that  $\Omega: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{F}$  is not a right adjoint nor a left adjoint. Indeed, the severe lack of products and coproducts in  $\mathbf{F}$  shows this functor fails to preserve products in  $\mathbf{Set}$ , or to convert coproducts into products.

### 3. The enveloping value quantale of a positivity domain

The aim of this section is to construct a value quantale from a fragment of the structure present in a value monoid. The construction is of general interest as a method for constructing value quantales. Recall from Subsection 2.1 that in an ordered set  $P$ , for subsets  $S, T \subseteq P$ , we write  $S \leq T$  to mean that for all  $t \in T$  there exists  $s \in S$  with  $s \leq t$ . We extend this notation to elements by considering an element  $x \in P$  as the singleton  $\{x\}$ .

**Definition 4.** A positivity multisemigroup is an ordered abelian multisemigroup  $(P, +, \leq)$  satisfying  $a \leq a + b$ , for all  $a, b \in P$ . A positivity multisemigroup  $P$  is a positivity domain if for all  $b \in P$  there exists  $a \in P$  with  $a + a \leq b$ .

As usual, if the operation is not a multi-operation, we may speak of a positivity semigroup or of a single-valued positivity domain.

**Example 3.** Clearly, the set  $P$  in a value monoid together with the induced structure is a positivity domain, as is the set  $P_L$  in a value quantale  $L$  when endowed with the structure from  $L$ . If  $(M, +, 0, \leq)$  is an ordered abelian monoid, then each of  $(M, +, \leq)$  and  $(M \setminus \{0\}, +, \leq)$  is a positivity semigroup. Each of  $(0, \infty)$ ,  $[0, \infty)$ ,  $(0, \infty) \cap \mathbb{Q}$ , and  $[0, \infty) \cap \mathbb{Q}$  with ordinary addition and the usual ordering is a positivity domain. Similarly, each of  $(1, \infty)$ ,  $[1, \infty)$ ,  $(1, \infty) \cap \mathbb{Q}$ , and  $[1, \infty) \cap \mathbb{Q}$  with multiplication and with the usual ordering is a positivity domain.

These examples exhibit an underlying semigroup rather than a multisemigroup. The following examples rely on the operation being multivalued.

**Proposition 5.** Let  $S$  be a set. Endowing  $S$  with the trivial ordering and defining  $a + a = \{a\}$  for all  $a \in S$ , and  $a + b = \emptyset$  when  $a \neq b$  results in a positivity domain.

The proof is immediate. We denote the resulting positivity domain by  $\widehat{S}$ . The next class of examples utilises multi-valuedness in a more substantial way, and is crucial for the representation theorem for value quantales (Theorem 15). Recall that for a value quantale  $L$  and  $a \in L$  we write  $\uparrow a = \{b \in L \mid b > a\}$ .

**Proposition 6.** Let  $L$  be a value quantale and  $P_L$  its set of positives, endowed with the induced ordering from  $L$ . Defining  $a \oplus b = \uparrow(a + b)$  turns  $P_L$  into a positivity domain.

**PROOF.** The operation  $\oplus$  is clearly commutative, so to verify that  $(P_L, \oplus)$  is a commutative multisemigroup, introduce the ternary operation  $a \oplus b \oplus c = \uparrow(a + b + c)$  and we proceed to show that  $(a \oplus b) \oplus c = a \oplus b \oplus c = a \oplus (b \oplus c)$ , for all  $a, b, c \in P_L$ . If  $x \in (a \oplus b) \oplus c$ , then  $x > y + c$  for some  $y > a + b$ . But then  $x > y + c \geq a + b + c$  and so  $x \in a \oplus b \oplus c$ . In the other direction, assuming  $x > a + b + c$  it follows by Lemma 1 that  $x > y + \gamma$  for some  $y > a + b$  and  $\gamma > c$ . Thus  $y \in a \oplus b$  and since  $\gamma > c$  it follows that  $x > y + c$ , so that  $x \in (a \oplus b) \oplus c$ . This argument establishes the first equality, and the second one follows similarly, and thus  $(P_L, \oplus)$  is a commutative multisemigroup.

To show that  $P_L$  is an ordered multisemigroup assume that  $a \leq a', b \leq b'$ , and that  $x > a' + b'$ , and we must show the existence of  $y > a + b$  with  $y \leq x$ . Since  $x > a' + b' \geq a + b$ , it follows that  $x > a + b$ , and so the required  $y$  exists by the interpolation property. It is easily seen that  $P_L$  is a positivity multisemigroup, i.e., that  $a \leq a + b$  holds. Finally, the condition for  $P_L$  to be a positivity domain is easily established using the property that in a value quantale, if  $\varepsilon > 0$  then there exists  $\delta > 0$  with  $\delta + \delta < \varepsilon$ , together with interpolation again.

We proceed to construct a value quantale from a positivity domain. The construction is given in two steps, utilising the following auxiliary concepts.

**Definition 5.** A deficient value quantale is a complete lattice  $L$  with a binary operation  $+$  satisfying the same conditions as in the definition of value quantale except, possibly, for the condition that  $0 + a = a$  for all  $a \in L$ . A semi-deficient value quantale is a deficient value quantale in which  $0 + 0 = 0$ .

**Example 4.** Let  $L$  be a value quantale and consider the set  $L_s = \{a_u, a_d \mid a \in L, a \neq 0\} \cup \{0\}$  where  $a_u, a_d$  are formal copies of  $a$ . We think of  $L_s$  as  $L$  where each element other than 0 has been split up into two parts, one directly above the other. In more detail, the order from  $L$  extends to  $L_s$  by defining 0 to remain the smallest element, and  $a_j \leq b_k$  as soon as  $a < b$ , and  $a_d < a_u$  for  $a \neq 0$ . Addition is extended by demanding that  $a_u = a_d + 0$  holds for all  $a \in L \setminus \{0\}$ . This results in a semi-deficient value quantale.

First we describe a construction  $P \mapsto \Theta(P)$  which turns a positivity multisemigroup into a deficient value quantale, and then a construction that sifts a semi-deficient value quantale  $L$  to yield a value quantale  $L_\circ$ . Noting that if  $P$  is a positivity domain, then  $\Theta(P)$  is a semi-deficient value quantale, these constructions are combined to give rise to the enveloping value quantale  $\Gamma(P) = \Theta(P)_\circ$  of a positivity domain  $P$ .

For a set  $S$  recall that  $\downarrow S$  denotes the set of all finite subsets of  $S$ . The notation  $A \subseteq_f S$  will be used to indicate that  $A$  is a finite subset of  $S$ .

**Definition 6.** For an ordered set  $P$  consider the functions  $\mathcal{P}(P) \xrightarrow{\uparrow} \mathcal{P}(P) \xrightarrow{\downarrow} \mathcal{P}(\downarrow P)$ , where  $\uparrow S = \{b \in P \mid \exists s \in S : s \leq b\}$ , and their composition  $\Downarrow = \downarrow \circ \uparrow$ .

For a singleton set  $\{x\}$  we abbreviate  $\Downarrow\{x\}$  to  $\Downarrow x$ , and similarly for  $\uparrow$  and  $\downarrow$ .

**Definition 7.** For an ordered set  $P$  let  $\Theta(P) = \{a \subseteq \downarrow P \mid A \in a \implies \Downarrow A \subseteq a\}$ , ordered by reverse inclusion.

Clearly,  $\Theta(P)$  is a complete lattice with  $\bigwedge = \bigcup$ . Indeed, all that is required to verify is that  $\Theta(P)$  is closed under arbitrary unions, a triviality. Consequently, by well-known results of Raney ([40, 41, 42]), it follows that  $\Theta(P)$  is a completely distributive complete lattice, and so, equivalently, that  $a = \bigwedge_{b > a} b$ , for all  $a \in \Theta(P)$ . Note that in  $\Theta(P)$ ,  $0 = \downarrow P$  while  $\infty = \emptyset$ . Generally,  $\Downarrow S \in \Theta(P)$  for all  $S \subseteq P$ .

**Remark 8.** Note that for all sets  $S$ , the sets  $\Omega(S)$  and  $\Theta(\widehat{S})$  are equal.

**Proposition 7.** If  $P$  is an ordered set, then  $a = \bigwedge \{\Downarrow A \mid A \in a\}$  for all  $a \in \Theta(P)$ .

**PROOF.** If  $X \in \bigwedge \{\Downarrow A \mid A \in a\}$ , then  $X \in \Downarrow A$  for some  $A \in a$ , and since  $\Downarrow A \subseteq a$ , it follows that  $X \in a$ . The argument in the other direction is immediate.

The meaning of  $b > a$  in a poset is sensitive to the ordering and the ambient poset. We say that  $b > a$  in  $\Theta(P)$  to mean that for all  $S \subseteq \Theta(P)$  with  $a > \bigwedge S$ , there exists  $s \in S$  with  $b \geq s$ .

**Lemma 3.** Let  $P$  be an ordered set. An element  $b \in \Theta(P)$  satisfies  $b > 0$  if, and only if,  $b \geq \Downarrow F$  for some  $F \subseteq_f P$ . More generally, for all  $a, b \in \Theta(P)$ ,  $b > a$  if, and only if,  $b \geq \Downarrow A$  for some  $A \in a$ .

**PROOF.** We establish the more general claim. Since  $\bigwedge_{A \in a} \Downarrow A = a$  it follows that if  $b > a$ , then  $b \geq \Downarrow A$  for some  $A \in a$ . Conversely, suppose that  $b \geq \Downarrow A$ ,  $A \in a$ , and that  $S \subseteq \Theta(P)$  satisfies  $a \geq \bigwedge S$ . Then since  $A \in a$  it follows that  $A \in s$  for some  $s \in S$ . But then  $\Downarrow A \subseteq s$ , and thus  $s \leq \Downarrow A \leq b$ , as required.

**Corollary 1.** Let  $P$  be an ordered set. If  $a, b > 0$  in  $\Theta(P)$ , then  $a \wedge b > 0$ .

**PROOF.** If  $a \geq \Downarrow F$  and  $b \geq \Downarrow G$ , then  $a \wedge b \geq \Downarrow F \wedge \Downarrow G = \Downarrow F \cup \Downarrow G \geq \Downarrow(F \cup G)$ .

Clearly  $S \subseteq T \subseteq P$  implies  $\Downarrow T \leq \Downarrow S$  in  $\Theta(P)$ , and, more precisely,  $S \subseteq \uparrow T$  if, and only if,  $\Downarrow T \leq \Downarrow S$ .

**Definition 8.** Let  $P$  be a positivity multisemigroup. For all  $a, b \in \Theta(P)$  let  $a + b = \bigcup_{A \in a, B \in b} \Downarrow(A + B)$ , where  $A + B = \bigcup\{\alpha + \beta \mid \alpha \in A, \beta \in B\}$ . In other words,  $a + b = \bigwedge_{A \in a, B \in b} \Downarrow(A + B)$ , so in particular  $+$  is a binary operation on  $\Theta(P)$  (obviously a commutative one).

**Remark 9.** Further to Remark 8, note that addition in  $\Theta(\widehat{S})$  corresponds to intersection. In other words, the operation on  $\Theta(\widehat{S})$  is the addition of  $\Omega(S)$ .

**Proposition 8.** If  $P$  is a positivity multisemigroup, then  $\Uparrow A + \Uparrow B = \Uparrow(A + B)$  for all  $A, B \subseteq P$ . Moreover,  $(\Theta(P), +)$  is a commutative ordered semigroup.

**PROOF.** The proof of the stated equality in case the sets are finite is trivial, and in the general case only slightly less trivial. As for the associativity of  $+$ , it is easily seen that  $a + (b + c) = \bigwedge\{\Downarrow(A + B + C) \mid A \in a, B \in b, C \in c\}$ , and we merely remark that some attention needs to be paid to the fact that due to multivaluedness of the operation in  $P$ , the set  $A + B + C$  may be infinite even when  $A, B$ , and  $C$  are finite.

We show that  $+$  is monotone, namely that  $b \leq b'$  implies  $a + b \leq a + b'$ . Indeed, to show the latter it suffices to show that  $a + b \leq \Downarrow(A + B')$ , for fixed  $A \in a$  and  $B' \in b'$ , namely that  $\Downarrow(A + B') \subseteq a + b$ , which would follow from  $A + B' \in a + b$ . Now, for all  $B \in b$  we have  $\Downarrow(A + B) \subseteq a + b$ , so if  $b \leq b'$ , then  $B' \in b$  and thus  $A + B' \in \Downarrow(A + B') \subseteq a + b$ , as needed. It remains to verify that  $a \leq a + b$ , for which it suffices to show that  $a \leq \Downarrow(A + B)$  for fixed  $A \in a$  and  $B \in b$ . Let  $F \subseteq_f \uparrow(A + B)$  be given, and we must show that  $F \in a$ . Note that for all  $x \in F$  there exists  $\alpha_x \in A, \beta_x \in B$ , and  $y_x \in \alpha_x + \beta_x$  with  $x \geq y_x$ . Since  $P$  is a positivity multisemigroup it follows that  $\alpha_x \leq y_x$ , and thus  $x \geq \alpha_x$ . We now have that  $F \in \Uparrow A \subseteq a$ , as required.

**Theorem 10.** If  $P$  is a positivity multisemigroup, then  $(\Theta(P), +)$  is a deficient value quantale.

**PROOF.** So far we have established that  $\Theta(P)$  is a complete lattice, that  $+$  is commutative and associative, and that the elements  $a \in \Theta(P)$  satisfying  $a > 0$  form a filter. That  $\Theta(P)$  is completely distributive follows by the computation, for all  $a \in \Theta(P)$ ,  $\bigwedge\{b \in \Theta(P) \mid b > a\} \leq \bigwedge\{\Downarrow A \mid A \in a\} = a$ , noting that  $\Downarrow A > a$  holds for all  $A \in a$ .

It remains to show that  $+$  distributes over arbitrary meets. Let  $a \in \Theta(P)$  and  $\mathcal{S} \subseteq \Theta(P)$  be given. That  $a + \bigwedge \mathcal{S} \leq \bigwedge a + \mathcal{S}$  follows at once from monotonicity. To show that  $\bigwedge a + \mathcal{S} \leq a + \bigwedge \mathcal{S} = \bigwedge\{\Downarrow(A + B) \mid A \in a, B \in \bigwedge \mathcal{S}\}$ , we show, for fixed  $A \in a, b \in \mathcal{S}$ , and  $B \in b$ , that  $\bigwedge a + \mathcal{S} \leq \Downarrow(A + B)$ . But that is trivial since  $a + \bigwedge \mathcal{S} \leq a + b \leq \Downarrow(A + B)$ , again due to monotonicity of addition.

**Remark 10.** It may be the case that  $0 = \Downarrow P$  is neutral with respect to  $+$ , for instance if  $P$  is an ordered monoid, but, of course, that is not the generic case

We now attend to the second construction, turning a semi-deficient value quantale into a value quantale. Generally, for a deficient value quantale  $L$  and  $a \in L$  let  $a_\circ = a + 0$ , giving rise to the function  $-_\circ : L \rightarrow L$ , obviously a complete meet homomorphism since  $(\bigwedge_{b \in \mathcal{B}} b)_\circ = \bigwedge_{b \in \mathcal{B}} b + 0 = \bigwedge_{b \in \mathcal{B}} b_\circ$ . Let  $L_\circ = \{a \in L \mid a_\circ = a\}$ ; obviously  $L_\circ \subseteq \text{Im}(-_\circ)$ . In the presence of lattice inclusions  $L_1 \subseteq L_2$ , we shall write, for elements  $a, b \in L_1$ ,  $b >_{L_1} a$  to indicate that  $b$  is well above  $a$  in  $L_1$ .

**Lemma 4.** If  $L$  is a semi-deficient value quantale, then  $L_\circ = \text{Im}(-_\circ)$ . In particular,  $L_\circ$  is a complete lattice where meets are computed as in  $L$ . Moreover,  $L_\circ$  is completely distributive.

**PROOF.**  $(a_\circ)_\circ = a + 0 + 0 = a_\circ$  shows that  $a_\circ \in L_\circ$  for all  $a \in L$ .  $L_\circ$  is thus the homomorphic image under the complete meet semilattice homomorphism  $-_\circ : L \rightarrow L$ .  $L_\circ$  consists precisely of the fixed-points of  $-_\circ$ , and it is thus immediate that  $L_\circ$  is a complete meet sublattice of  $L$ . Finally, we show that  $L_\circ$  is completely distributive, for which it suffices, for a given  $a \in L_\circ$ , to show that  $a = \bigwedge\{b \in L_\circ \mid b >_{L_\circ} a\}$ . The proof reduces to showing that for all  $a \in L_\circ$  and  $b \in L$ , if  $b >_L a$ , then  $b_\circ >_{L_\circ} a$ . Indeed, with that property assumed, we obtain  $\bigwedge\{b \in L_\circ \mid b >_{L_\circ} a\} \leq \bigwedge\{b_\circ \in L_\circ \mid b \in L, b >_L a\} = (\bigwedge\{b \in L \mid b >_L a\})_\circ = a_\circ = a$ , which suffices to show that  $a = \bigwedge\{b \in L_\circ \mid b >_{L_\circ} a\}$ . To now verify the assumed property for some  $a \in L_\circ$  and  $b \in L$  with  $b >_L a$ , suppose that  $a \geq \bigwedge \mathcal{S}$  for some  $\mathcal{S} \subseteq L_\circ$ . Then  $b \geq s$  for some  $s \in \mathcal{S}$  and thus  $b_\circ \geq s_\circ = s$ , as required.

**Proposition 9.** Let  $L$  be a semi-deficient value quantale. Then the addition operation  $+$  on  $L$  restricts to  $L_\circ$ , and  $0 \in L_\circ$  is neutral with respect to  $+$ .

PROOF. Obvious.

**Lemma 5.** *Let  $L$  be a semi-deficient value quantale, and  $a, b \in L_\circ$ . Then  $b \succ_L a$  if, and only if,  $b \succ_{L_\circ} a$ .*

PROOF. That  $b \succ_L a$  implies  $b \succ_{L_\circ} a$  is simply due to the fact that  $L_\circ$  is a complete meet sublattice of  $L$ . Suppose now that  $b \succ_{L_\circ} a$ , and that  $a \geq \bigwedge S$  for some  $S \subseteq L$ . Applying  $-_\circ$  element wise yields  $a_\circ \geq \bigwedge S_\circ$ . Consequently,  $b \geq s_\circ$  for some  $s \in S$ , and since  $s_\circ \geq s$  it follows that  $b \geq s$ , showing that  $b \succ_L a$ .

**Corollary 2.** *Let  $L$  be a semi-deficient value quantale. If  $a, b \succ_{L_\circ} 0$ , then  $a \wedge b \succ_{L_\circ} 0$ , for all  $a, b \in L_\circ$ .*

We summarise the above results in the following theorem.

**Theorem 11.** *If  $L$  is a semi-deficient value quantale, then  $L_\circ = \{a \in L \mid a + 0 = a\}$ , with the structure inherited from  $L$ , is a value quantale.*

The auxiliary constructions are thus concluded and the next step is to relate them.

**Lemma 6.** *If  $P$  is a positivity domain, then  $\Theta(P)$  is a semi-deficient value quantale.*

PROOF. We need to show that  $0_\circ = 0 + 0 = 0$ , namely that  $\downarrow P \subseteq \downarrow P + \downarrow P$ . Suppose  $F \subseteq_f P$ , and for each  $x \in F$  let  $y_x \in P$  and  $z_x \in y_x + y_x$  satisfy  $z_x \leq x$ . Let  $G = \{y_x \mid x \in F\} \subseteq_f P$ , so clearly  $y_x + y_x \subseteq G + G$ , and so  $\{z_x \mid x \in F\} \subseteq G + G$ . Thus  $F \subseteq \uparrow(G + G)$ , and so  $F \in \uparrow(G + G) = \uparrow G + \uparrow G \subseteq \downarrow P + \downarrow P$ .

Combining Theorem 10, Lemma 6, and Theorem 11, if  $P$  is a positivity domain, then  $\Gamma(P) = \Theta(P)_\circ$  is a value quantale, the *enveloping value quantale* of the positivity domain  $P$ .

**Remark 11.** *It is interesting to note that it is only in the proof of Lemma 6 that the ability to divide by 2, namely that for  $b \in P$  there is  $a \in P$  and  $x \in a + a$  with  $x \leq b$ , plays a role. In a value monoid (or value semigroup) division by 2 is an axiomatic requirement while in a value quantale it is a consequence of Lemma 1 applied to the property  $0 + 0 = 0$ . Lemma 6 is in a sense a converse result, obtaining a well-behaved 0 from the presence of division by 2.*

The notion of support for elements in  $\Omega(S)$  is meaningful for elements in  $\Theta(P)$  as well.

**Definition 9.** *Let  $P$  be a positivity multisemigroup. The support of an element  $a \in \Theta(P)$  is  $\sigma_P(a) = \sigma(a) = \{x \in P \mid \{x\} \in a\}$ .*

**Proposition 10.** *Let  $P$  be a positivity multisemigroup. Then  $\uparrow\sigma(a) = \downarrow\sigma(a) \leq a$  for all  $a \in \Theta(P)$ , and the following are equivalent.*

1.  $P$  is totally ordered.
2.  $\downarrow\sigma(a) = a$  for all  $a \in \Theta(P)$ .
3.  $\sigma: P \rightarrow \mathcal{P}(P)$  is injective.

PROOF. Clearly,  $\uparrow\sigma(a) = \sigma(a)$ , and thus  $\uparrow\sigma(a) = \downarrow\sigma(a)$ . If  $A \in a$ , then clearly  $A \subseteq \sigma(a)$ , namely  $A \in \downarrow\sigma(a)$ , as required for the inequality.

To show that condition 1 implies condition 2 suppose that  $P$  is totally ordered, and let  $F \in \downarrow\sigma(a)$ . Thus  $F$  is a finite subset of  $\sigma(a)$ , necessarily a chain, so let  $x \in F$  be the smallest element. But then  $F \subseteq \uparrow\{x\}$ , and so  $F \in \uparrow\{x\}$ . Since  $x \in \sigma(a)$  it follows that  $\{x\} \in a$ , and thus that  $\uparrow\{x\} \subseteq a$ , and we may conclude that  $F \in a$ .

Condition 2 easily implies condition 3, which leaves the verification of condition 1 once  $\sigma$  is injective. Let  $x, y \in P$  be arbitrary and consider the elements  $a = \uparrow\{x\}$ ,  $b = \uparrow\{y\}$ , and  $c = \uparrow\{x, y\}$ . Recall that  $a \wedge b = a \cup b$  and thus  $\sigma(a \wedge b) = \sigma(c)$ . By assumption it follows that  $a \wedge b = c$ , and thus that  $\{x, y\} \in a \cup b$ . Noting that if  $\{x, y\} \in a = \uparrow\{x\}$ , then  $\{x, y\} \subseteq \uparrow\{x\}$ , and thus that  $x \leq y$ , with a similar argument for  $b$ , completes the proof.

As an illustrative example of the constructions above let us consider the positivity domain  $\mathbb{Q}_+ = \mathbb{Q} \cap (0, \infty)$  with the usual ordering and addition. It is then easy to see that for all  $a \in \Theta(\mathbb{Q}_+)$ , if  $a < \infty$  then  $\sigma(a)$  is a Dedekind cut. In more detail, a Dedekind cut in  $\mathbb{Q}_+$  is a non-empty  $S \subseteq \mathbb{Q}_+$  with  $\uparrow S = S$ . We say a Dedekind cut  $S$  is proper if  $S$  does not have a minimum. Classically, the collection of proper Dedekind cuts in  $\mathbb{Q}_+$  is one of numerous constructions of the set  $\mathbb{R}_+$  of non-negative real numbers. The easily established claim is thus that  $\sigma(a)$  is a Dedekind cut for all  $a \in \Theta(\mathbb{Q}_+)$  except for  $a = \infty$ . In fact,  $\sigma$  establishes a canonical bijective correspondence between  $\Theta(\mathbb{Q}_+) \setminus \{\infty\}$  and the set of all, proper as well as non-proper, Dedekind cuts in  $\mathbb{Q}_+$ .  $\Theta(\mathbb{Q}_+)$  is a semi-deficient value quantale which fails to be a quantale. The sifting process yields  $\Theta(\mathbb{Q}_+)_\circ$ , which consists precisely of those  $a \in \Theta(\mathbb{Q}_+)$  with  $\sigma(a)$  a proper Dedekind cut, as well as  $a = \infty$ . In other words,  $\Gamma(\mathbb{Q}_+)$ , the enveloping value quantale of  $\mathbb{Q}_+$ , is canonically isomorphic to  $[0, \infty]$ , the value quantale of the extended non-zero real numbers with their usual structure. Note that if instead one starts with the positivity domain  $\mathbb{Q}_+ \cup \{0\}$ , then the sifting stage in the enveloping value quantale construction is redundant, and the enveloping value quantale  $\Gamma(\mathbb{Q}_+ \cup \{0\}) = \Theta(\mathbb{Q}_+ \cup \{0\})$  is significantly different than  $[0, \infty]$ .

#### 4. Functoriality

We now turn to study the functorial properties of the constructions presented in Section 3, and we thus firstly define the relevant categories.

A morphism  $\psi: F \rightarrow F'$  between deficient value quantales is defined to be a monotone subadditive function which is continuous at 0, so that  $\mathbf{F}_\Lambda$ , the category of deficient value quantales and their morphisms, contains  $\mathbf{F}$  as a full subcategory. Further, let  $\mathbf{F}_\Lambda$  be the full subcategory of  $\mathbf{F}_\Lambda$  spanned by the semi-deficient value quantales. A morphism  $\psi: P \rightarrow Q$  between positivity multisemigroups is defined to be a monotone subadditive multifunction valued in up-closed sets. In more detail,  $\psi$  is to satisfy  $\psi(x) \leq \psi(y)$  whenever  $x \leq y$ , in the sense that for all  $t \in \psi(y)$  there exists  $v \in \psi(x)$  with  $v \leq t$ ,  $\psi(x+y) \leq \psi(x) + \psi(y)$  with a similar interpretation, and  $\uparrow\psi(x) = \psi(x)$ , for all  $x \in P$ . Let  $\mathbf{P}_\Lambda$  be the category of all positivity multisemigroups and their morphisms, and let  $\mathbf{P}_\Lambda$  be its full subcategory spanned by the positivity domains. Identity morphisms  $\text{id}_P$  are given by  $x \mapsto \uparrow x$ , and composition is given by  $\varphi \circ \psi(x) = \bigcup \{\varphi(y) \mid y \in \psi(x)\}$ .

The construction  $P \mapsto \Theta(P)$

Here we concentrate on the first of the two auxiliary constructions. Let  $\psi: P \rightsquigarrow Q$  be a morphism of positivity semigroups and  $b \in \Theta(Q)$ . Define  $\psi^*(b) = \bigwedge \{\uparrow\psi^{\leftarrow}(B) \mid B \in b\}$ , where  $\psi^{\leftarrow}(B) = \{x \in P \mid \psi(x) \subseteq \uparrow B\}$ .  $\psi_*(a)$  can also be defined and similar properties to those given in Proposition 3 can be formulated, however (recalling Remark 7) we are primarily interested in  $\psi^*$ , for the following result.

**Theorem 12.** *The assignments  $P \mapsto \Theta(P)$  and  $\psi \mapsto \psi^*$  give rise to the two functors in the diagram*

$$\begin{array}{ccc} \mathbf{P}_\Lambda^{\text{op}} & \hookrightarrow & \mathbf{P}_\Lambda^{\text{op}} \\ \downarrow \Theta & & \downarrow \Theta \\ \mathbf{F}_\Lambda & \hookrightarrow & \mathbf{F}_\Lambda \end{array}$$

which is a commutative diagram.

**PROOF.** It is easily seen that  $\psi^*(0) = \downarrow P = 0$ . The fact that  $\psi^*$  preserves arbitrary meets, i.e., arbitrary unions, is immediate and implies continuity at 0. Subadditivity, namely  $\psi^*(b + b') \leq \psi^*(b) + \psi^*(b') = \bigwedge \{\uparrow\psi^{\leftarrow}(B) + \psi^{\leftarrow}(B') \mid B \in b, B' \in b'\}$ , follows from  $\psi^*(b + b') \leq \uparrow\psi^{\leftarrow}(B + B') \leq \uparrow(\psi^{\leftarrow}(B) + \psi^{\leftarrow}(B'))$ , resulting from the monotonicity of  $\psi$ . We have established that  $\psi^*$  is a morphism of (deficient) value quantales. Functoriality is immediate.

Typically,  $\Theta(P)$  will not land in  $\mathbf{F}$ , even if  $P$  is a positivity domain. There is though a class of positivity domains for which this is the case, in fact recovering Flagg's functor  $\Omega: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{F}$  from Theorem 8, as follows. Recall the construction  $S \mapsto \widehat{S}$  from Example 3, which obviously extends to a functor  $\widehat{\cdot}: \mathbf{Set} \rightarrow \mathbf{P}_\Lambda$  acting trivially on morphisms. The following result is the combination of Remark 8 and Remark 9.

**Proposition 11.** *For all sets  $S$  the deficient value quantale  $\Theta(\widehat{S})$  is a value quantale, and thus  $\Gamma(\widehat{S}) = \Theta(\widehat{S})$ . Further,  $\Gamma \circ \widehat{\quad} = \Omega$  as functors  $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{F}$ .*

The notion of support for elements in  $\Theta(P)$  (Definition 9) recovers the notion for elements in  $\Omega(S)$  (Subsection 2.7) in the following precise sense. Recall that in Proposition 4 it is shown that the support functions constitute a natural transformation from  $U \circ \Omega$  to the contravariant power set functor.

**Proposition 12.** *The functions  $\sigma_P: \Theta(P) \rightarrow \mathcal{P}(P)$  are the components of a natural transformation  $\sigma: U \circ \Theta \rightarrow \mathcal{P}^{\leftarrow} \circ U$ , where  $U$  stands for the respective forgetful functors  $\mathbf{P}_\Lambda \rightarrow \mathbf{Set}$  and  $\mathbf{F}_\Lambda \rightarrow \mathbf{Set}$ .*

PROOF. Formally identical to the proof of Proposition 4.

*The sifting process  $F \mapsto F_\circ$*

We now turn our attention to the sifting process  $F \mapsto F_\circ$  for semi-deficient value quantales  $F$ , and note first that it extends to morphisms  $\psi: F \rightarrow F'$  by defining  $\psi_\circ: F_\circ \rightarrow F'_\circ$  to be  $\psi_\circ(a) = \psi(a)_\circ$ . Functoriality is clear (e.g., for an identity morphism  $\text{id}_F: F \rightarrow F$ , one has  $(\text{id}_F)_\circ(a) = a_\circ = a$  for all  $a \in F_\circ$ , and hence  $(\text{id}_F)_\circ = \text{id}_{F_\circ}$ ) and we obtain the functor  $-\circ: \mathbf{F}_\Lambda \rightarrow \mathbf{F}$ . To exhibit  $(-\circ)$  as a right adjoint, without delving into details, recall the construction  $L \mapsto L_s$  from Example 4.

**Theorem 13.** *The construction  $L \mapsto L_s$  is functorial in the evident way, giving rise to the functor  $-_s: \mathbf{F} \rightarrow \mathbf{F}_\Lambda$  which is left adjoint to the functor  $-\circ$ .*

PROOF. We omit the straightforward verification.

*Comparing  $P$  with  $\Theta(P)$  and with  $\Gamma(P)$*

For a positivity multisemigroup  $P$  consider the functions  $\mu, \nu: P \rightarrow \Theta(P)$  given by  $\mu(a) = \uparrow(a + P)$  and  $\nu(a) = \uparrow a$ .

**Proposition 13.** *Let  $P$  be a positivity multisemigroup. The following hold.*

- $\nu$  is injective.
- $\nu(a) \leq \mu(a)$ , for all  $a \in P$ .
- $\mu$  and  $\nu$  are monotone and sub-additive.

PROOF. The proofs are straightforward:

- If  $\uparrow a = \uparrow b$ , then  $\{a\} \subseteq \downarrow b$ , and thus  $a \geq b$ , and similarly  $b \geq a$ .
- Note that  $a + P \subseteq \uparrow a$ .
- Monotonicity is obvious. For sub-additivity of  $\nu$  we must show that  $\uparrow(a + b) \leq \uparrow a + \uparrow b = \bigwedge \{\uparrow(S + T) \mid S \in \uparrow a, T \in \uparrow b\}$ , namely that  $\uparrow(S + T) \subseteq \uparrow(a + b)$  for some fixed  $S$  and  $T$ , but that follows at once from  $S + T \subseteq \uparrow(a + b)$ . As for  $\mu$ , one similarly needs to show that  $S + T \subseteq \uparrow(a + b + P)$  for fixed  $S \in \uparrow(a + P)$  and  $T \in \uparrow(b + P)$ . The argument for a positivity semigroup is the following. Given  $s \in S$  and  $t \in T$ , write  $s \geq a + \delta$  and  $t \geq b + \varepsilon$ , for  $\delta, \varepsilon \in P$ , and then  $s + t \geq a + b + \delta + \varepsilon$ . Minor adjustments yield the multisemigroup case.

We record the information about  $\nu$  separately.

**Lemma 7.** *For every positivity multisemigroup  $P$  there is a deficient value quantale  $L$  in which  $P$  embeds sub-additively, though not necessarily as a meet sublattice.*

When  $P$  is a positivity domain the behaviour of  $\mu$  improves, as follows.

**Proposition 14.** *If  $P$  is a positivity domain, then  $\mu$  is additive and  $\text{Im}(\mu) \subseteq \Gamma(P)$ .*

PROOF. To verify additivity we only need to show that  $\mu(a) + \mu(b) \leq \mu(a + b)$ , namely that  $\Downarrow(a + b + P) \subseteq \bigcup\{\Downarrow(S + T) \mid S \in \Downarrow(a + P), T \in \Downarrow(b + P)\}$ , so let  $F \in \Downarrow(a + b + P)$ . Thus for each  $x \in F$  there is  $\varepsilon_x \in P$  with  $x \geq a + b + \varepsilon_x$ , and let  $\delta_x \in P$  satisfy  $\delta_x + \delta_x \leq \varepsilon_x$ . Let  $G = \{\delta_x \mid x \in F\}$ , and set  $S = a + G$  and  $T = b + G$ . Then  $x \geq a + b + \varepsilon_x \geq a + \delta_x + b + \delta_x \in S + T$  holds for all  $x \in F$ , and thus  $F \subseteq \uparrow(S + T)$ , as required. To show that  $\mu$  lands in  $\Gamma(P)$  it suffices to show that  $\Downarrow(a + P) \geq \Downarrow(a + P)_\circ = \Downarrow(a + P) + \Downarrow P$ , namely that  $\Downarrow(a + P) \subseteq \bigcup\{\Downarrow(S + T) \mid S \in \Downarrow(a + P), T \in \Downarrow P\}$ , so let  $F \in \Downarrow(a + P)$ . For each  $x \in F$  one has  $x \geq a + \varepsilon_x$ ,  $\varepsilon_x \in P$ , and let  $\delta_x \in P$  satisfy  $\delta_x + \delta_x \leq \varepsilon_x$ . Let  $G = \{\delta_x \mid x \in F\}$ , and set  $S = \Downarrow(a + G)$  and  $T = \Downarrow G$ . It then easily follows that  $F \subseteq \uparrow(S + T)$ , completing the proof.

Since  $\mu$  may fail to be injective, the analogue of Lemma 7 is less universal.

**Lemma 8.** *For every positivity domain  $P$  in which the condition  $\uparrow(a + P) = \uparrow(b + P)$  implies  $a = b$  holds, there is a value quantale  $L$  in which  $P$  embeds additively, but not necessarily as a meet sublattice.*

PROOF. The condition immediately implies the injectivity of  $\mu$ , and  $\Gamma(P)$  is a value quantale.

## 5. Representations of value quantales

We present two applications of positivity domains, one, the subject of this section, in the form of a representation theorem for value quantales, and the other, given in Section 7, to the theory of continuity spaces.

If  $L$  is a value quantale, then the requirement that  $a = \bigwedge\{b \in L \mid b > a\}$  implies that the positive elements  $P_L$  should suffice to determine the entire value quantale  $L$ . The fact that  $a = \bigwedge\{a + \varepsilon \mid \varepsilon > 0\}$  suggests two possibilities for an embedding and the results above allow us to turn this into precisely formulated representations.

By an embedding of a value quantale  $L_1$  in a value quantale  $L_2$  we mean an injective morphism  $\psi: L_1 \rightarrow L_2$  in  $\mathbf{F}$ . If, moreover,  $\psi$  is additive, we speak of an additive embedding.

**Theorem 14 (Subadditive Representation Theorem for Value Quantales).** *Let  $L$  be a value quantale and consider  $P = (P_L, +)$  as a single-valued positivity domain with the structure inherited from  $L$ . Then  $\mu: L \rightarrow \Gamma(P_L)$  given by  $\mu(a) = \Downarrow(a + P)$  is a subadditive embedding.*

PROOF. To show that  $\mu$  lands in  $\Gamma(P)$  we must show that  $\Downarrow(a + P) + \Downarrow P \leq \Downarrow(a + P)$ , so choose  $F \in \Downarrow(a + P)$ . For all  $x \in F$  we have  $x \geq a + \varepsilon_x \geq a + \delta_x + \delta_x$ , with  $\varepsilon_x, \delta_x > 0$ . Taking  $A = \{a + \delta_x \mid x \in F\}$  and  $B = \{\delta_x \mid x \in F\}$  yields sets with  $A \in \Downarrow(a + P)$  and  $B \in \Downarrow P$ . Thus  $\Downarrow(a + P) + \Downarrow P \leq \Downarrow(A + B) \leq \Downarrow F$ , as required.

Clearly,  $\mu$  is monotone and  $\mu(0) = \Downarrow(0 + P) = \Downarrow P = 0$ . To verify that  $\mu$  is subadditive, namely that  $\Downarrow(a + b + P) \leq \Downarrow(a + P) + \Downarrow(b + P) = \bigwedge\{\Downarrow(A + B) \mid A \in \Downarrow(a + P), B \in \Downarrow(b + P)\}$ , fix some such  $A, B$ . Since  $x \geq a + \varepsilon_x$  and  $y \geq b + \varepsilon_y$ , for all  $x \in A, y \in B$ , we have that  $A + B \subseteq \uparrow(a + b + P)$ , since  $\varepsilon_x + \delta_x \in P$ . Thus  $\Downarrow(a + b + P) \leq \Downarrow(A + B)$ , as needed.

To verify continuity at 0 let  $\varepsilon > 0$  in  $\Gamma(P)$  be given. Then, by Lemma 3,  $\varepsilon \geq \Downarrow F$  for some finite  $F \subseteq P$ . It thus follows that  $\bigwedge F > 0$ , and let  $0 < \gamma < \bigwedge F$ . Now, it follows that there exists  $\eta > 0$  in  $L$  with  $\bigwedge F \geq \gamma + \eta$ , and therefore  $F \subseteq \uparrow(\gamma + P)$ , leading to  $\mu(\gamma) = \Downarrow(\gamma + P) \leq \Downarrow F \leq \varepsilon$ .

$\mu$  is thus a morphism in  $\mathbf{F}$ . Lastly, for injectivity, assume  $\mu(a) = \mu(b)$ , and fix  $\varepsilon > 0$  in  $L$ . The existence of  $0 < \delta < \varepsilon$  implies that  $a + \varepsilon \in \mu(a)$ , and so  $a + \varepsilon \geq b$ , showing that  $a + P \subseteq b + P$ , with equality of the sets following immediately, and thus  $a = b$ .

The reason for considering multivalued semigroups instead of just single-valued ones is to allow for the following stronger embedding theorem.

**Theorem 15 (Additive Representation Theorem for Value Quantales).** *Let  $L$  be a value quantale, and consider  $P = (P_L, \oplus)$  with the structure of a positivity domain as in Proposition 6. Then  $\rho: L \rightarrow \Gamma(P_L)$  given by  $\rho(a) = \Downarrow a = \bigwedge\{x \in L \mid x > a\}$  is an additive embedding.*

PROOF. We shall denote by  $\oplus$  all operations directly derived from the addition  $\oplus$  on  $P_L$ , to distinguish it from the addition  $+$  on  $L$ . To show that  $\rho$  lands in  $\Gamma(P)$  we show that  $\Downarrow a \oplus \Downarrow P \leq \Downarrow a$ , so choose  $F \in \Downarrow a$ . For all  $x \in F$  we have  $x > a$  and thus  $x > a + \varepsilon_x$  for some  $\varepsilon_x > 0$ , and, by Lemma 1,  $x > y_x + \delta_x$  with  $y_x > a$  and  $\delta_x > \varepsilon_x$ , namely  $x \in y_x \oplus \delta_x$ . Taking  $A = \{y_x \mid x \in F\}$  and  $B = \{\delta_x \mid x \in F\}$  we thus have that  $F \subseteq A \oplus B$ , and since  $A \in \Downarrow a$  and  $B \in \Downarrow P$ , we obtain  $\Downarrow a \oplus \Downarrow P \leq \Downarrow(A + B) \leq \Downarrow F$ , as required.



Clearly,  $\rho$  is monotone and  $\rho(0) = \Downarrow 0 = \Downarrow P = 0$ . To verify that  $\rho$  is additive we first show that  $\Downarrow(a+b) \leq \Downarrow a \oplus \Downarrow b = \bigwedge \{\Downarrow(A \oplus B) \mid A \in \Downarrow a, B \in \Downarrow b\}$ , and so fix some such  $A, B$ . Given  $F \in \Downarrow(A+B)$ , each  $x \in F$  satisfies  $x > \alpha_x + \beta_x$  where  $\alpha_x > a$  and  $\beta_x > b$ , and in particular  $\alpha_x \geq a$  and  $\beta_x \geq b$ . It follows that  $x > a+b$ , and so  $F \subseteq \Downarrow(a+b)$ , as required. To establish that  $\rho$  is additive, we now show that  $\Downarrow a \oplus \Downarrow b \leq \Downarrow(a+b)$ , so fix  $F \in \Downarrow(a+b)$ . Each  $x \in F$  satisfies  $x > a+b$  and thus, by Lemma 1 again,  $x > \alpha_x + \beta_x$  where  $\alpha_x > a$  and  $\beta_x > b$ . Taking  $A = \{\alpha_x \mid x \in F\}$  and  $B = \{\beta_x \mid x \in F\}$  we see that  $A \in \Downarrow a$ ,  $B \in \Downarrow b$ , and  $F \subseteq A \oplus B$ . It follows that  $\Downarrow a \oplus \Downarrow b \leq \Downarrow(A+B) \leq \Downarrow F$ , as needed.

To verify continuity at 0 let  $\varepsilon > 0$  in  $\Gamma(P)$  be given, namely  $\varepsilon \geq \Downarrow F$  for some finite  $F \subseteq P$ . Thus  $\bigwedge F > 0$  and by interpolation there exists  $0 < y < \bigwedge F$ . It follows that  $F \subseteq \Downarrow y$  and thus  $\rho(y) = \Downarrow y \leq \Downarrow F \leq \varepsilon$ .  $\rho$  is thus a morphism in  $\mathbf{F}$ . Injectivity of  $\rho$  is immediate since clearly  $\Downarrow a = \Downarrow b \implies \Downarrow a = \Downarrow b \implies a = b$ .

The behaviour of  $\rho$  with respect to the complete meet lattice structure is given in the following result.

**Theorem 16.** *The embedding  $\rho: L \rightarrow \Gamma(P_L, \oplus)$  preserves all meets in  $L$  of sets closed under finite meets. If  $L$  satisfies the condition that  $a, b > t$  implies  $a \wedge b > t$ , then  $\rho$  is a complete meet homomorphism.*

**PROOF.** Let  $S \subseteq L$ . Since  $\rho$  is monotone we only need to show that  $\bigwedge \{\rho(s) \mid s \in S\} \leq \rho(\bigwedge S)$ , so let  $F \in \Downarrow \bigwedge S$ . Assume either one of the conditions of the theorem. Note that each  $x \in F$  satisfies  $x > s_x$  with  $s_x \in S$ . Let  $S_0 = \{s_x \mid x \in F\}$ . If  $S$  is closed under finite meets, then  $x > s_x \geq \bigwedge S_0 = s_0 \in S$  implies that  $F \subseteq \Downarrow s_0$ , and thus  $\bigwedge \{\rho(s) \mid s \in S\} \leq \Downarrow s_0 \leq \Downarrow F$ . If, instead, the stated condition on the quantale holds, then since  $x > \bigwedge S$  holds for all  $x \in F$ , it follows that  $\bigwedge F > \bigwedge S$ , and thus  $\bigwedge F > s_0$  for some  $s_0 \in S$ . Consequently, we again conclude that  $F \subseteq \Downarrow s_0$ , from which the claim follows.

**Remark 12.** *It is possible to obtain an embedding which is always a complete meet homomorphism, but at the cost of losing subadditivity (but retaining superadditivity) and losing the property that  $0 \mapsto 0$ . Since such a construction is of lesser importance to the intended applications of value quantales, we merely note the construction, avoiding any further details. Given a value quantale  $L$  consider the positivity domain  $(P_L, \oplus)$ . The complete meet embedding  $L \rightarrow \Gamma(P_L)$  is then given by  $a \mapsto \bigwedge \{\Downarrow x \mid x > a\}$ .*

**Remark 13.** *For representations and embedding of quantales see [43, 44].*

## 6. Products of value quantales

With the representation theory in place we now consider two notions of product of value quantales.

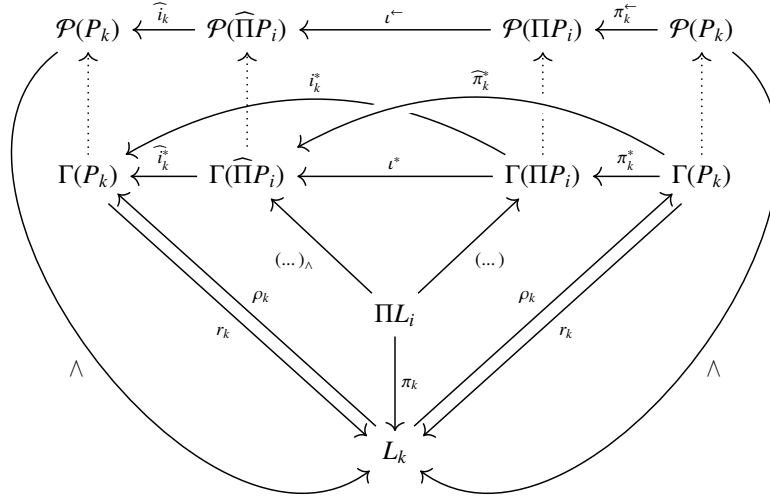
**Theorem 17.** *The categories  $\mathbf{P}_\Delta$  and  $\mathbf{P}_\Lambda$  admit all small products.*

**PROOF.** The obvious pointwise structure on the cartesian product of the underlying sets is easily seen to form a categorical product in each of the categories.

We denote the product in these categories by  $\prod P_i$  and the canonical projections by  $\pi_k: \prod P_i \rightarrow P_k$ . We define the *Tychonoff* product  $\widehat{\prod} P_i$  to be the subset of the product consisting only of the elements with finitely many coordinates different than  $\infty$ , with the induced structure. In the case of infinitely many factors the Tychonoff product is, obviously, not a categorical product. However, we still write  $\pi_k: \widehat{\prod} P_i \rightarrow P_k$  for the restriction of the projections. We now assume that a collection  $\{L_i\}_{i \in I}$  of value quantales is given, and  $P_i$  is then the set of positives in  $L_i$ , viewed as a positivity domain as in Proposition 6. We then consider the morphisms  $\widehat{i}_k: P_k \rightarrow \widehat{\prod} P_i$  given by  $\widehat{i}_k(a) = \hat{a}$ , where  $\hat{a}$  has  $a$  at the  $k$ -th coordinate and all other coordinates equal to  $\infty$ , and its extension to  $\prod P_i$  along the inclusion. We obtain the diagram

$$\begin{array}{ccccc}
 & & \widehat{i}_k & & \widehat{\pi}_k \\
 & & \curvearrowright & & \curvearrowleft \\
 P_k & \xrightarrow{\widehat{i}_k} & \widehat{\prod} P_i & \xrightarrow{\iota} & \prod P_i & \xrightarrow{\pi_k} & P_k
 \end{array}$$

in  $\mathbf{P}_\Lambda$ . Applying the contravariant power set functor and  $\Gamma$  yields the top two layers of



where  $\rho_k$  is the embedding established above,  $r_k: \Gamma(P_k) \rightarrow L_k$  is given by  $r_k(a) = \bigwedge \sigma(a)$ , and all of the dotted lines are the components of the natural transformation  $\sigma$ . We denote  $\boxtimes L_i = \Gamma(\Pi P_i)$  and call it the *box product* of the value quantales, while  $\widehat{\boxtimes} L_i = \Gamma(\widehat{\Pi} P_i)$  is called the *Tychonoff product* of the value quantales. In the diagram, the product  $\Pi L_i$  is of the underlying sets of the value quantales, and the functions  $(\dots)_\Lambda$  and  $(\dots)$  are the *coordinates mappings* given as follows. For  $a \in \Pi L_i$  let  $\boxtimes(a) = \{x \in \Pi P_i \mid x_k > a_k, \forall k \in I\} = \Pi_i \uparrow a_i$ , giving rise to the *box injection*  $\Pi L_i \rightarrow \mathcal{P}(\Pi P_i)$ . For any set  $S \in \Pi P_i$ , let  $T(S)$  be the subset of  $S$  consisting only of those elements with finitely many coordinates different than  $\infty$ . Denoting  $\widehat{\boxtimes}(a) = T(\boxtimes(a))$ , gives rise to the *Tychonoff injection*  $\Pi L_i \rightarrow \mathcal{P}(\widehat{\Pi} P_i)$ . The *box coordinates* are then defined to be  $(a) = \uparrow \boxtimes(a)$ , and the *Tychonoff coordinates* are given by  $(a)_\Lambda = \uparrow \widehat{\boxtimes}(a)$ . Similarly to the case in previous proofs, it is easy to verify that the coordinates functions indeed land in  $\Gamma$  and not just in  $\Theta$ . Note that both in the box and in the Tychonoff sense, coordinates determine a unique element in the respective product of value quantales, but not every element in the product has coordinates. When possible we shall allow context to determine whether the box or Tychonoff coordinates are meant by the notation  $(a)$ , avoiding the subscript. For an element  $a$  in either product which has coordinates, we write  $a_i$  for the  $i$ -th coordinate.

Various commutativity relations in the diagram above can be established, but a systematic listing of all of them is not informative. Instead, we leave it to the reader to appreciate the capabilities present in the diagram by establishing three properties which we then use in the next section.

**Proposition 15.** *For each of the box and Tychonoff coordinates,  $0$  has coordinates all  $0$ , if  $a \leq b$  coordinate-wise, then  $(a) \leq (b)$ , and finally coordinates are additive, i.e.,  $(a + b) = (a) + (b)$ .*

**PROOF.**  $(0) = 0$ : in box coordinates  $(0) = \uparrow(\Pi_i \uparrow 0) = \Pi P_i = 0$ ; in Tychonoff coordinates  $(0) = \uparrow T(\Pi P_i) = 0$ . Assume that  $a \leq b$  coordinate-wise. It is then clear that both  $\boxtimes(b) \subseteq \boxtimes(a)$  and  $\widehat{\boxtimes}(b) \subseteq \widehat{\boxtimes}(a)$ , from which the inequalities in the respective products of the value quantales follow. The proof of the additivity of coordinates is now routine, and we merely mention that it, again, depends on Lemma 1 and the construction of Proposition 6.

We conclude this section by noting that neither of the two products on  $\mathbf{F}$  leads to a monoidal structure. In fact, no unit exists, nor are there possible candidates for associativity isomorphisms. However, it is not hard, but a bit tedious, to verify that each product endows  $\mathbf{F}$  with the structure of a weak unbiased monoidal category (see [45]).

## 7. Positivity domains in the service of continuity spaces

In this section we exploit the box and Tychonoff products of value quantales introduced in Section 6 in order to obtain a purely metric interpretation of topological products. In more detail, recall Theorem 2, stating that the usual

formalism of topology (e.g., in terms of abstract open sets) is equivalent to the formalism of Flagg continuity spaces (as well as to the formalism of Kopperman continuity spaces). One of the main difficulties with classical metric spaces is the non-existence of arbitrary products (under any reasonable choice of morphisms), in sharp contrast to the situation in **Top**. In light of the equivalency between **Top** and the category of all value quantale valued continuity spaces with all continuous mappings, it follows that the latter has products. Of course, one way to obtain the product of  $(X_1, L_1, d_1)$  and  $(X_2, L_2, d_2)$  is as the space  $(X_1 \times X_2, L, d)$  where  $L = \Omega(\tau)$ , with  $\tau$  the product topology of  $\mathcal{O}(d_1)$  and  $\mathcal{O}(d_2)$ , and  $d$  the metrisation mentioned in Theorem 1. However, this is deeply unsatisfactory as the metric flavour is completely removed from the picture. We show how to obtain the product of continuity spaces directly, without first passing to **Top**. We follow the notation of Section 6.

**Definition 10.** Let  $\{(X_i, L_i, d_i)\}_{i \in I}$  be a collection of Flagg continuity spaces. Their box product is the continuity space  $(X, L, d)$  where  $X = \prod X_i$ ,  $L = \boxtimes L_i$  is the box product of the value quantales, and  $d = \boxtimes d_i: X \times X \rightarrow L$  is given by  $d(x, y)_i = d_i(x_i, y_i)$ , using box coordinates. Similarly, the Tychonoff product of the Flagg continuity spaces is  $(X, L, d)$  where this time  $L = \widehat{\boxtimes} L_i$  is the Tychonoff product of the value quantales, and  $d = \widehat{\boxtimes} d_i$  is given Tychonoff coordinates.

It is immediate from Proposition 15 that these metric products each gives rise to a continuity space. The objective of this section is the following result. Considering the monoidal category  $(\mathbf{Top}, \boxtimes)$ , where  $\boxtimes$  refers to the box product of topological spaces, the equivalence  $\mathcal{O}: \mathbf{Met}_{\mathbf{F}} \rightarrow \mathbf{Top}$  is a monoidal functor when  $\mathbf{Met}_{\mathbf{F}}$  is given the box product of Flagg continuity spaces. Further, the same functor is monoidal when  $\mathbf{Top}$  is given the Tychonoff product of topological spaces, and  $\mathbf{Met}_{\mathbf{F}}$  is given the Tychonoff product of Flagg continuity spaces. Generally, for topological space  $\{(X_i, \tau_i)\}_{i \in I}$ , we write  $\boxtimes \tau_i$  for the box topology on  $\prod X_i$ , and  $\prod \tau_i$  for the Tychonoff product of the topologies.

**Theorem 18.** Let  $\{X_i = (X_i, L_i, d_i)\}_{i \in I}$  be a collection of Flagg continuity spaces. Then  $\mathcal{O}(\prod X_i, \boxtimes L_i, \boxtimes d_i) = \boxtimes \mathcal{O}(X_i, L_i, d_i)$  and  $\mathcal{O}(\prod X_i, \widehat{\boxtimes} L_i, \widehat{\boxtimes} d_i) = \prod \mathcal{O}(X_i, L_i, d_i)$ .

**PROOF.** The standard textbook argument for classical metric spaces can now be repeated nearly verbatim, with the aid of Lemma 3. We omit the details.

**Remark 14.** Since any topological space is metrisable for a suitable value quantale, the above result implies that coordinate techniques can be applied to the box and Tychonoff products of topological spaces, and moreover that the naive notion of product of metrics, without involving a Euclidean type norm (or any other norm), simply by recording the distance per coordinate becomes a definition equivalent to the topological notions. This result is in line with the thread of results presented in [46, 7, 47, 15, 34].

Recall the discussion revolving Theorem 3, namely that the point-to-point topological capability of the value monoid formalism is adequate, but it is too crude to correctly capture point-to-set topological notions using products of value monoids. Notice that the example given in the proof of Theorem 3 involves a finite product of Kopperman continuity spaces, thus showing that the trend continues: while product topologies are captured metrically in the Kopperman formalism, point-to-set information is too crude. This problem is resolved in the Flagg formalism with the use of the computationally more refined box and Tychonoff products of value quantales. In the example from Theorem 3 the point  $x$  has 0 distance from  $S$ , yet it is not in the closure of  $S$ . By the result of this section, using the product of the metrics does capture the topology correctly, and thus the distance  $d(x, S)$  cannot be 0. We conclude this work by observing these details.

We take  $[0, \infty]$  as a value quantale and treat  $\mathbb{R}$  with the Euclidean metric. For all  $a \in [0, \infty]$  we have  $\uparrow a = \{x \in [0, \infty] \mid x > a\}$ . Let  $d: \mathbb{R}^2 \rightarrow [0, \infty] \boxtimes [0, \infty]$  be the product of the metrics. Then  $t_1 = d((0, 0), (1, 0)) = \downarrow(\uparrow 1 \times \uparrow 0)$  while  $t_2 = d((0, 0), (0, 1)) = \downarrow(\uparrow 0 \times \uparrow 1)$ . It is now clear that  $d(x, S) = t_1 \wedge t_2 = t_1 \cup t_2 \neq 0 = \downarrow(\uparrow 0 \times \uparrow 0)$  since  $(\varepsilon, \varepsilon) \notin \sigma(t_1 \wedge t_2)$ , for all  $0 < \varepsilon \leq 1$ . Referring to the main diagram of Section 6, we see that the topological information is captured when working in the lower part of the diagram. However, in its upper part, namely by taking supports, the computations become too crude, i.e., even though  $t_1 \wedge t_2 \neq 0$  we do have that  $\bigwedge \sigma(t_1 \wedge t_2) = 0$ , when the meet is computed coordinate-wise.

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