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An extended cohesive damage model for simulating arbitrary damage propagation in engineering materials

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Abstract

This paper introduces the extended cohesive damage model (ECDM) for simulating arbitrary damage propagation in engineering materials. By embedding the micromechanical cohesive damage model (CDM) into the eXtended Finite Element Method (XFEM) and eliminating the enriched degree of freedoms (DoFs), the ECDM defines the cohesive crack path at a low scale in the condensed equilibrium equations and enables the local enrichments of approximation spaces without enriched DoFs. In this developed ECDM, a new equivalent damage scalar as a function of strain field is introduced to avoid the appearance of enriched DoFs, and to substitute the conventional characterization in the approximation of displacement jump. The embedment of CDM is no longer required by the ECDM, which allows discontinuities to exist within a finite element rather than the element boundaries. This feature enables the ECDM to simulate the reality of arbitrary cracks. Initial applications of the ECDM in simulation of arbitrary cracks shows that the developed ECDM works very well when compared to experiment work and XFEM analysis.

Keywords: Extended cohesive damage model; XFEM; CDM; Fracture; Arbitrary crack propagation

1. Introduction

In structural analysis, typical examples of strong or weak discontinuities are cracks or interfacial behaviour between different domains or materials. Modelling arbitrary discontinuities and their propagation is currently one of the major concerns in the fracture related academic community. One of the earliest techniques used to model such discontinuities is Adaptive Mesh Refining (AMR) scheme which was developed between 1987 and 2000 [1, 2]. In AMR, a finite element mesh can be adaptively modified according to crack propagation, and errors during numerical simulation can be efficiently minimized. However, the treatment for weak discontinuities is still very challenging for AMR. In addition, the modification of mesh topology in AMR is tedious and brings a considerable computational burden [3]. To address all these deficiencies, Babuska and Melenk [4, 5] proposed the Partition of Unity Method (PUM), for which no mesh regeneration is needed in simulating crack growth. The finite element boundary would no longer need to be the discontinuity surface, which is a significant benefit to the work of modelling fracture.

In the past decades, based on PUM, a rapid development has been made in conducting arbitrary cracking problems within continuum solids [6-10]. Among all these PUM-based approaches, eXtended Finite Element Method (XFEM) was originally introduced by Belytschko et al. [6] and subsequently enhanced by Moës et al. [11]. XFEM is a combination of the classical Finite Element Method (FEM) and PUM. By enriching the classical piecewise polynomial approximation basis within the

FEM framework, XFEM can thoroughly capture non-smooth features independently. Such features would include jumps, kinks, singularities and inhomogeneity [6, 11]. Unlike AMR, XFEM no longer needs remeshing or adapting finite element mesh when discontinuities or crack propagations occur. The mesh adaption process can therefore be substituted by introducing enriched or additional DoFs and partitioning the domain with some triangular and quadrilateral sub-elements. The Gauss points of these sub-elements would be those that were used for sub-integration of the domain of elements [11, 12].

The appropriate treatment for the approximation space in XFEM realizes the accurate characterization for non-smooth features without dependence on the mesh topology. XFEM has been applied in describing discontinuity in solids for more than 10 years. Moës and Belytschko [13] and Mergheim et al. [14] combined the traction-separation law, known as cohesive zone model (CZM), with XFEM to model the cohesive cracks within quasi-brittle materials. Benvenuti [15] provided a unified way with cohesive processes involving either a localized or finite width zone into a zero-thickness interface. Unger et al. [16] carried out a discrete crack simulation for concrete materials employing XFEM in conjunction with an adaptive crack growth algorithm. By introducing continuum damage mechanics in the framework of XFEM, characterization of failure behaviours ranging from diffuse material degradation to discrete cracks can be achieved [17-19]. Hansbo et al. [20] presented a phantom-node

method to model arbitrary discontinuities, which is essentially another implementation of XFEM [21].

Although the numerical approaches mentioned above have similar or different advantages, they in most cases also increase the computational burden and the possibility of failed convergence. This is because they require additional external nodes or degrees of freedom (DoFs) which would be a barrier when applying them in structural level modelling. Moreover, tracking the path of crack propagation would be costly for XFEM based algorithms. Attempts to incorporate such theories in large scale structural calculations are confronted with the challenges of computational efficiency, complexity and robustness. This is especially the case for heterogeneous materials such as those used in composite structures. Oliver et al. [22] investigated an embedded finite element method (E-FEM) which is an alternative approach to XFEM. A comparative study [23] between E-FEM and XFEM demonstrated that both numerical accuracy and efficiency are evidently better in E-FEM. This is achieved through the implementation of elemental enrichments rather than the nodal enrichments required by XFEM. Even if the mesh size is relatively coarse, E-FEM is still able to predict reasonably accurate results. Yang et al. [24-26] have recently carried out an interesting augmented finite element method (A-FEM) that can account for path-arbitrary, multiple intra-elemental discontinuities. Their study demonstrated an improvement in numerical efficiency by two orders of magnitude when compared to XFEM [26]. Their work used a fully condensed elemental equilibrium equation

with standard DoFs to simulate the discontinuities. However, as A-FEM uses four internal nodes to account for the crack displacements in approximating both weak and strong intra-elemental discontinuities, specified algorithms for solving the crack displacements of internal nodes are required.

Besides XFEM, E-FEM and A-FEM, CZM has been extensively used as an interface cohesive element in the study of localisation and fracture in engineering materials and structures for more than 15 years. The application of CZM requires that FEA meshes conform to the potential cracks [28]. It must be used in such a way that corresponding interface cohesive elements are embedded along the crack path known a priori to analyse progressive failure.

In the Extended Cohesive Damage Model (ECDM) presented in this paper, CDM is employed to determine the micro damage initiation. The damage propagation is calculated by a new equivalent damage scalar relating to a strain field. This developed ECDM has the following specific features: enriched DoFs vanished from the finally condensed equilibrium equations; corresponding cohesive properties taken into account through an equivalent stiffness matrix; and a new equivalent damage scalar as a function of strain field introduced to substitute for conventional characterization in the approximation of the displacement jump. The ECDM allows for discontinuities to exist within a finite element rather than just at the element boundaries. In this way the ECDM is able to simulate discontinuities much more realistically. Unlike standard XFEM, the ECDM does not include enriched DoFs in

the basic equilibrium equation. Instead a similar effect to that from enriched DoFs is achieved through an equivalent stiffness matrix in the finally condensed equilibrium equations of the ECDM. The derived formulation of the ECDM is presented with standard DoFs only. However, it is influenced by an equivalent damage scalar with strong and weak discontinuous characteristics within an element.

This paper is organized as follows: Section 1 gives relevant background information in the area of computational damage mechanics. Section 2 gives detailed derivation of the condensed equilibrium equations of the proposed ECDM. Section 3 presents examples of applications of the ECDM and the comparisons between modelling simulations and experimental work. Finally, Section 4 gives a conclusion to the ECDM with major highlights or achievements, and the future work.

2. Basic formulations of the ECDM

In section 2.1, the theoretical work starts from assuming a displacement field with enriched degree freedoms and a shifted Heaviside function to approximate a discontinuity in a solid field. In section 2.2, a discrete form of equilibrium equation with the standard and enriched degree freedoms is introduced first. Then the enriched degree freedoms are eliminated mathematically from the condensed equilibrium equations by accounting for the effects from enriched DoFs and corresponding cohesive force into the equivalent stiffness matrix. A transformation matrix is also introduced to simplify the calculation of cohesive force in section 2.2. Section 2.3 introduces an equivalent damage scalar as a function of strain field to avoid using a

displacement gap in approximating discontinuities. At the end of section 2.4, a final formulation of fully condensed equilibrium equation is presented.

2.1 Displacement filed and shifted Heaviside function

The theoretical derivation starts from the framework of XFEM without taking the crack-tip singularity into account, which means a crack must cross an element. There will be no such crack-tip singularity problems once a cohesive damage law is used anywhere in the mesh in which any element can hold the crack-tip but the crack-tip stress is restrained by the cohesion. A previous investigation given by Yang et al [26] has reported that ignoring crack-tip singularity will not influence the result for crack propagation as well as the overall response. Chen [29] has also recently investigated the basic concept of combining XFEM with CDM without using a specified enriched item to cope with the singularity problem at crack-tip. The XFEM test and trial function can thus be given as:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x}) \mathbf{u}_i + \sum_{i \in J} N_j(\mathbf{x}) H_{\Gamma_d}(\mathbf{x}) \mathbf{a}_i \quad (1)$$

where N_i is the standard FEM shape functions associated with node i . $\mathbf{u}_i$ and $\mathbf{a}_i$ are the nodal variables associated with standard degree freedoms and enriched degree freedoms at node i and node j , respectively. The Heaviside step function H_{Γ_d} shown in Eq. (2) can characterize the physical jump when the element is completely separated (strong discontinuities).

$$H_{\Gamma_d}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega^+ \\ 0, & \mathbf{x} \in \Omega^- \end{cases} \quad (2)$$

where, Ω^+ is the one side domain of discontinuity, while the Ω^- is the other side domain.

For the approximation using Heaviside function, the *Kronecker- δ* property (i.e., $N_I(x_J) = \delta_{IJ}$) cannot be satisfied, which requires the imposition of essential boundary conditions. Moreover, the total DoFs u_i of an enriched node i are the summation of the standard DoFs u_i and the enriched contribution $H_{\Gamma_d}(\mathbf{x})\mathbf{a}_i$, rather than the physical solution of the nodal displacement. Thus the interpretation of results is difficult [30]. A serious similar problem was raised by the phantom node method [31, 32].

A “shifted” enrichment is accepted in this investigation for retaining the *Kronecker- δ* property as the enriched degree of freedom or enrichment is zero at any node, which is illustrated in Fig. 1. This shifted displacement basis can also been found in other researchers’ work related to XFEM [30, 33]. The introduction of shifted Heaviside function does not alter the approximating basis while simplifies the implementation attributing to that resulting enrichment vanished in elements which are not cut by discontinuities. The displacement field with the “shifted” enrichment is shown in Eq. (3)

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x})\mathbf{u}_i + \sum_{i \in J} N_j(\mathbf{x})(H_{\Gamma_d}(\mathbf{x}) - H_{\Gamma_d}(\mathbf{x}_i))\mathbf{a}_i \quad (3)$$

Where $\mathbf{x}_i$ is the positional coordinate for the i_{th} node. The concept of modelling this “shifted” crack is firstly illustrated in one dimension as shown in Fig. 1. Supposing the left side of the interface is Ω^- , and the right side is Ω^+ , the Heaviside function at given nodes 1 and 2 are defined as : $H_{\Gamma_d}(x_1)=1$, $H_{\Gamma_d}(x_2)=0$. In Fig. 1a, supposing the

left side of the interface is Ω^- , and the right side is Ω^+ , the “shifted” crack modelling concept in one dimension can be illuminated as shown in Fig. 1b and 1c. The graphic of the standard DoF term is $N_1(x)u_1 + N_2(x)u_2$ which is presented by the dash line in Fig. 1d. The enriched term is expressed by Eq. (4).

Fig. 1. The representation of the discontinuity by the shifted enrichment

$$\Delta u = a_1 N_1(x) (H_{\Gamma_d}(x) - H_{\Gamma_d}(x_1)) + a_2 N_2(x) (H_{\Gamma_d}(x) - H_{\Gamma_d}(x_2)) \quad (4)$$

Actually, the combination with two enriched items in Eq. (4) results a real displacement with a jump Δu as shown in the last illustration given in Fig. 1d. This shifted enrichment ensures that the value of displacement basis at any nodes is single-valued for any crack geometries. It should be noticed that the enriched contribution vanishes at enriched nodes, but not at integration points.

The proposed approach is based on the displacement approximation of XFEM for a 2D cracked homogeneous domain $\Omega = \bigcup_{i=1}^n \Omega^i$, and its unit normal $\mathbf{n}$, as shown in Fig. 2. The Dirichlet boundary conditions $\bar{\mathbf{u}}$ were applied at Γ_u . Considering the

total potential energy governing the problem, the weak form of equilibrium equation can be obtained as given below:

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \delta \boldsymbol{\epsilon} d\Omega = \int_{\Omega} \mathbf{f}^b \cdot \delta \mathbf{u} d\Omega + \int_{\Gamma_d} \mathbf{f}^t \cdot \delta \mathbf{u} d\Gamma \quad (5)$$

Fig. 2. A 2D domain with an arbitrary discontinuity Γ_d

Where, $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$, $\mathbf{u}$, $\mathbf{f}^b$ and $\mathbf{f}^t$ are the stress tensor, strain tensor, displacement vector, body forces and traction forces, respectively. Γ_d is the traction boundary with a normal $\mathbf{m}$, which splits the domain Ω into Ω^+ and Ω^- .

2.2 Discrete form of the equilibrium equations

Using the weak form of equilibrium equation from Bubnov-Galerkin method, the discrete form of equilibrium equation for static analysis can be written as shown in Eq. (6).

$$\begin{bmatrix} \mathbf{K}^{uu} & \mathbf{K}^{ua} \\ \mathbf{K}^{au} & \mathbf{K}^{aa} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{ext}^u \\ \mathbf{f}_{ext}^a \end{bmatrix} \quad (6)$$

Where, $\mathbf{K}^{uu}$, $\mathbf{K}^{aa}$ and $\mathbf{K}^{ua}$ are the stiffness matrix associated with the standard FE approximation, the enriched approximation and the coupling between the standard FE approximation and the enriched approximation, respectively. $\mathbf{f}_{ext}^u$ and $\mathbf{f}_{ext}^a$ are the equivalent nodal force vectors, $\mathbf{f}_{ext}^u$ is for standard FEM freedoms while $\mathbf{f}_{ext}^a$ is for enriched freedoms. $\mathbf{u}$ denotes the vector including ordinary degrees of freedom while

the enriched degrees of freedom is signified by $\mathbf{a}$. Each detailed sub-matrix of stiffness in Eq. (7) is given below.

$$\begin{aligned}\mathbf{K}^{uu} &= \int_{\Omega^h} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega \\ \mathbf{K}^{ua} &= \int_{\Omega^h} \mathbf{B}^T \mathbf{D} \mathbf{B}^a d\Omega \\ \mathbf{K}^{au} &= \int_{\Omega^h} (\mathbf{B}^a)^T \mathbf{D} \mathbf{B} d\Omega \\ \mathbf{K}^{aa} &= \int_{\Omega^h} (\mathbf{B}^a)^T \mathbf{D} \mathbf{B}^a d\Omega\end{aligned}\quad (7)$$

where $\mathbf{D}$ is the constitutive matrix, $\mathbf{B}$ is the deformation matrix for strain calculation.

In the 2D domain,

$$\mathbf{B} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \dots \quad \mathbf{B}_i \quad \dots \quad \mathbf{B}_m] \quad (8)$$

$$\text{Where, } \mathbf{B}_i = \begin{bmatrix} N_i^{x'} & 0 \\ 0 & N_i^{y'} \\ N_i^{y'} & N_i^{x'} \end{bmatrix},$$

$$\mathbf{B}^a = (\mathbf{H}_{\Gamma_d}(x, y) - \mathbf{H}_{\Gamma_d}(x_i, y_i)) \mathbf{B} \quad (9)$$

For solving a 2D problem, if the crack shown in Fig. 2 is a cohesive crack, the discontinuous boundary is a cohesive crack boundary, i.e. $\Gamma_d = \Gamma_{coh}^{crack}$. In Eq. (5), the equivalent nodal force vectors without body force can be expressed as:

$$\begin{aligned}\mathbf{f}_{ext}^u &= \int_{\Gamma^h} \mathbf{N}^T \bar{\mathbf{t}} d\Gamma \\ \mathbf{f}_{ext}^a &= \int_{\Gamma^h} \mathbf{N}^T (\mathbf{H}_{\Gamma_{coh}^{crack}}(x, y) - \mathbf{H}_{\Gamma_{coh}^{crack}}(x_i, y_i)) \bar{\mathbf{t}} d\Gamma + \mathbf{f}_{coh}\end{aligned}\quad (10)$$

$\bar{\mathbf{t}}$ is the external nodal force vector as shown in Fig. 2. Because of the existence of cohesive segment, the internal nodal force vector due to cohesive traction $\mathbf{t}$ shown in Fig. 2 on the crack surface Γ_{coh}^{crack} can be expressed as:

$$\begin{aligned}\mathbf{f}_{coh} &= \int_{\Gamma_{coh}^{crack+}} (H_{\Gamma_d}(x, y) - H_{\Gamma_d}(x_i, y_i)) \mathbf{N}^T \mathbf{t} d\Gamma + \int_{\Gamma_{coh}^{crack-}} (H_{\Gamma_d}(x, y) - H_{\Gamma_d}(x_i, y_i)) \mathbf{N}^T (-\mathbf{t}) d\Gamma \\ &= \int_{\Gamma_{coh}^{crack}} \mathbf{N}^T \mathbf{t} d\Gamma\end{aligned}\quad (11)$$

where $\mathbf{N}$ is standard shape function applied at the enriched degree freedoms. CDM is then introduced here to characterize the nonlinear cohesive segment evolution. In CDM, the normal traction component t_n and shearing traction component t_s are used to account for the damage state of cohesive segment. The cohesive tractions as a function of the damage scalar d in both normal and tangential directions at crack surface decrease monotonically from their initial values $\mathbf{t}_0$ to zero, which is mathematically expressed by Eq. (12).

$$\mathbf{t} = \begin{bmatrix} t_{coh}^n \\ t_{coh}^s \end{bmatrix} = \begin{bmatrix} (1-d)t_0^n \\ (1-d)t_0^t \end{bmatrix} \quad (12)$$

where $\mathbf{t}_0$ is the cohesive traction on the cracked surface when crack propagates, which is composed of the calculated normal and tangential stresses. It is known the t_{coh} is given in the directions n and s at crack surface, so the transformation of coordinates from local coordinate n and s to x and y (global coordinate) is necessarily required and expressed by Eq. (13).

$$\mathbf{t}_{coh}^{global} = \begin{bmatrix} t_{coh}^x \\ t_{coh}^y \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} t_{coh}^n \\ t_{coh}^s \end{bmatrix} \quad (13)$$

The angle θ is the angle between the coordinate n - s and the coordinate x - y . There is not a physical relative displacement $\Delta\delta$ before a crack formed within element. When the damage increases, the cohesive traction decreases, following a linear softening damage law as shown in Fig. 3.

Fig. 3. A micromechanical linear softening damage law

Substituting the expression of equivalent nodal force vector in Eq. (10) into Eq. (6) results Eq. (14) as below.

$$\begin{bmatrix} \mathbf{K}^{uu} & \mathbf{K}^{ua} \\ \mathbf{K}^{au} & \mathbf{K}^{aa} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u^{ext} \\ \int_{\Gamma_{coh}^{crack}} \mathbf{N}^T (H_{\Gamma_{coh}^{crack}}(x, y) - H_{\Gamma_{coh}^{crack}}(x_i, y_i)) \bar{\mathbf{t}} d\Gamma \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{f}_{coh} \end{bmatrix} \quad (14)$$

With the purpose of reaching a fully condensed equilibrium system, the additional enrichment term $\mathbf{a}$, is eliminated, thus the equilibrium equation with the standard FEM unknown quantities can be consequently obtained as shown in Eq. (15).

$$\left(\mathbf{K}^{uu} - \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{K}^{au} \right) \mathbf{u} = \mathbf{f}_{ext}^u - \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{f}_{ext}^a - \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{f}_{coh} \quad (15)$$

In the equation above, the nodal force on additional DoFs attributing to the contribution of the external load is denoted as $\mathbf{f}_{ext}^a$. Suppose the external nodal force arrays in Eq. (15) can be expressed by Eq. (16),

$$\mathbf{f}_{ext}^u = \begin{bmatrix} f_1^u & \cdot & \cdot & f_i^u & f_j^u & \cdot & f_k^u & \cdot & f_m^u \end{bmatrix}_{1 \times m}^T \quad (16)$$

which is a m by 1 vector, and

$$\mathbf{f}_{ext}^a = \begin{bmatrix} f_i^a & f_j^a & \cdot & \cdot & f_k^a \end{bmatrix}_{1 \times n}^T \quad (1 \leq i < j < k \leq n) \quad (17)$$

which contains n items. m and n are the numbers for standard DoFs and additional DoFs, respectively. Then a transformation matrix is required to be introduced in order to replace $\mathbf{f}_{ext}^a$ by $\mathbf{f}_{ext}^u$ in Eq. (15). This transformation matrix is constructed as:

$$\mathbf{M} = \begin{bmatrix} m_{1,1} & \cdot & m_{1,i} & m_{1,j} & \cdot & m_{1,k} & \cdot & m_{1,m} \\ m_{2,1} & \cdot & m_{2,i} & m_{2,j} & \cdot & m_{2,k} & \cdot & m_{2,m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{n,1} & \cdot & m_{n,i} & m_{n,j} & \cdot & m_{n,k} & \cdot & m_{n,m} \end{bmatrix}_{n \times m} \quad (18)$$

It should be noticed that there is only one non-zero item at each row of $\mathbf{M}$. According to the array of $\mathbf{f}_{ext}^a$ expressed in Eq. (17), the non-zero items are defined as $m_{1,i} = f_i^a / f_i^u$, $m_{2,j} = f_j^a / f_j^u$ and $m_{n,k} = f_k^a / f_k^u$. With this matrix, the required transformation can be expressed by Eq. (19).

$$\mathbf{f}_{ext}^a = \mathbf{M} \mathbf{f}_{ext}^u \quad (19)$$

Substituting Eq. (19) into Eq. (15) results

$$\mathbf{f}_{ext}^a = \mathbf{M} \mathbf{f}_{ext}^u \quad (20)$$

Substituting Eq. (20) into Eq. (15) results

$$\left(\mathbf{K}^{uu} - \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{K}^{au} \right) \mathbf{u} = \mathbf{f}_{ext}^u - \mathbf{K}^{uu} (\mathbf{K}^{aa})^{-1} \mathbf{M} \mathbf{f}_{ext}^u - \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{f}_{coh} \quad (21)$$

Then, the equivalent equilibrium equations with standard FEM degree freedoms can be expressed as shown in Eq. (22).

$$\begin{aligned} & \left(\mathbf{I} - \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{M} \right)^{-1} \left(\mathbf{K}^{uu} - \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{K}^{au} \right) \mathbf{u} \\ & = \mathbf{f}_{ext}^u - \left(\mathbf{I} - \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{M} \right)^{-1} \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{f}_{coh} \end{aligned} \quad (22)$$

In Eq. (22), the calculation of the equivalent nodal force due to the existence of internal cohesive segment for a cracked element is given by Eq. (23).

$$\mathbf{f}_{\text{int}}^{\text{coh}} = \left(\mathbf{I} - \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{M} \right)^{-1} \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \mathbf{f}_{\text{coh}} \quad (23)$$

2.3 Equivalent damage scalar

Normally, the released fracture energy as shown in Fig. 3 should be used to describe the crack propagation. Therefore, the damage scalar d in Eq. (12) for the cohesive behavior along the crack length can be expressed as:

$$d = \frac{\frac{1}{2} \int_{\Gamma_{\text{crack}}} \mathbf{t}_0 \Delta \delta d\Gamma}{G_c l_{\text{crack}}} \quad (24)$$

In Eq. (24), G_c is fracture energy; l_{crack} is defined as a crack length in a failed element, within which, the damage status (the status of cohesion) is assumed to be the same along the crack length. In the implementation, the crack distance is calculated from the start point to the end point of a crack in the failed element. This means the damage status throughout one element is consistent, and crack propagates along a straight line within the failed element. This can be seen from failed elements 1-4 in Fig. 4. The failed elements 1-4 have crack lengths defined by the lengths of the lines AB, BC, CD and DE, respectively. As shown in Eq. (24), the defined crack length l_{crack} is one of variables to calculate the damage scalar d ; the displacement jump $\Delta \delta$ relating to additional DoFs is also a variable in calculating the damage scalar d . The displacement jump $\Delta \delta$ is required to obtain the state of the cohesive segment by calculating the damage scalar d , and determine if $d=1$ then the cohesive traction vanishes, and the cohesive segment behaves as a strong discontinuity (element separates completely).

Fig. 4. The illustration of the crack lengths calculated by the ECDM

In this proposed ECDM, a new equivalent damage scalar is used to avoid the appearance of the additional DoFs related displacement gap $\Delta\delta$. In the ECDM based FEM modelling, the degradation of cohesive zone will present a micro behavior of strain softening, so the strain energy released due to the fracture should be equal to the released work done by cohesive traction. Therefore, in the utilization of the ECDM, the damage scalar can be expressed equivalently by a released strain energy which is shown in Eq. (25).

$$d = \frac{\int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon d\Omega - \int_{\Omega} \frac{1}{2} \sigma \varepsilon^0 d\Omega}{G_c l_{crack}} \quad (25)$$

where σ^0 and ε^0 are the material strength and the initial damage strain, respectively, when the damage onsets. Herein, a softening constitutive law is used for reducing the cohesive traction, i.e. $\sigma = (1-d)\sigma^0$. Bringing this calculation into Eq. (25), an explicit expression of the equivalent damage scalar can be achieved as shown below.

$$d = \frac{\int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon d\Omega - \int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon^0 d\Omega}{G_c l_{crack} - \int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon^0 d\Omega} \quad (26)$$

It should be noticed that the damage scalar d in Eq. (26) is a function of the strain ε only, which has no need to calculate the additional DoFs related displacement gap $\Delta\delta$.

In order to simplify the problem, it is supposed that there is no distributed external load applied on the cracked element, thus transformation matrix $\mathbf{M}$ would be a zero matrix. Then using the Eqs. (11), (12), (22), (23) and (26), the nodal force given in Eq. (23) with the existence of cohesive traction at the crack can be expressed as:

$$\begin{aligned} \mathbf{f}_{int}^{coh} &= \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \int_{\Gamma_{coh}^{crack}} \mathbf{N}_{enr}^T \mathbf{t}_0 (1-d) d\Gamma \\ &= \mathbf{K}^{ua} (\mathbf{K}^{aa})^{-1} \int_{\Gamma_{coh}^{crack}} \mathbf{N}_{enr}^T \mathbf{t}_0 \left(\frac{G_c l_{crack} - \int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon d\Omega}{G_c l_{crack} - \int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon^0 d\Omega} \right) d\Gamma \end{aligned} \quad (27)$$

2.4 Final formulation of the fully condensed equilibrium equations

Herein, an operator is introduced for Eq. (27) to obtain the implicit expression for achieving the ultimate equilibrium equations. Considering the symmetric property of stiffness matrix, an operator is chosen as below:

$$\mathfrak{R} = \frac{1}{\left(\mathbf{K}_r^{ua} (\mathbf{K}_r^{aa})^{-1} \int_{\Gamma_{coh}^{crack}} \mathbf{N}_{enr}^T \mathbf{t}_0 d\Gamma \right)^T} \left(\mathbf{K}_r^{ua} (\mathbf{K}_r^{aa})^{-1} \int_{\Gamma_{coh}^{crack}} \mathbf{N}_{enr}^T \mathbf{t}_0 d\Gamma \right)^T \quad (28)$$

which satisfies $\mathfrak{R}\mathbf{u} = 1$. Using this operator, Eq. (27) can be rewritten as Eq. (29).

$$\begin{aligned}
 \mathbf{f}_{\text{int}}^{\text{coh}} &= \mathbf{K}^{ua} \left(\mathbf{K}^{aa} \right)^{-1} \int_{\Gamma_{\text{crack}}^{\text{coh}}} \mathbf{N}_{\text{enr}}^T \mathbf{t}_0 \left(\frac{G_c l_{\text{crack}} - \int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon d\Omega}{G_c l_{\text{crack}} - \int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon^0 d\Omega} \right) d\Gamma \Re \mathbf{u} \\
 &= \mathbf{K}^{ua} \left(\mathbf{K}^{aa} \right)^{-1} \int_{\Gamma_{\text{crack}}^{\text{coh}}} \mathbf{N}_{\text{enr}}^T \mathbf{t}_0 \frac{G_c l_{\text{crack}}}{G_c l_{\text{crack}} - \int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon^0 d\Omega} d\Gamma \Re \mathbf{u} \\
 &\quad - \mathbf{K}^{ua} \left(\mathbf{K}^{aa} \right)^{-1} \int_{\Gamma_{\text{crack}}^{\text{coh}}} \mathbf{N}_{\text{enr}}^T \mathbf{t}_0 \frac{\int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon d\Omega}{G_c l_{\text{crack}} - \int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon^0 d\Omega} d\Gamma \Re \mathbf{u}
 \end{aligned} \tag{29}$$

Considering a convenient expression, the following two symbols are defined.

$$\begin{aligned}
 \mathbf{L}_1 &= \int_{\Gamma_{\text{crack}}^{\text{coh}}} \mathbf{N}_{\text{enr}}^T \mathbf{t}_0 \frac{G_c l_{\text{crack}}}{G_c l_{\text{crack}} - \int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon^0 d\Omega} d\Gamma \Re \\
 \mathbf{L}_2 &= \int_{\Gamma_{\text{crack}}^{\text{coh}}} \mathbf{N}_{\text{enr}}^T \mathbf{t}_0 \frac{\int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon d\Omega}{G_c l_{\text{crack}} - \int_{\Omega} \frac{1}{2} \sigma^0 \varepsilon^0 d\Omega} d\Gamma \Re
 \end{aligned} \tag{30}$$

Using Eq. (30), the nodal force vector can be simplified as shown in Eq. (31).

$$\mathbf{f}_{\text{int}}^{\text{coh}} = \mathbf{K}^{ua} \left(\mathbf{K}^{aa} \right)^{-1} (\mathbf{L}_1 - \mathbf{L}_2) \mathbf{u} \tag{31}$$

Substituting Eq. (31) into Eq. (21) for calculating the cohesive nodal force, the finally condensed equilibrium equations can be expressed by Eq. (32).

$$\left(\mathbf{K}^{uu} - \mathbf{K}^{ua} \left(\mathbf{K}^{aa} \right)^{-1} \mathbf{K}^{au} + \mathbf{K}^{ua} \left(\mathbf{K}^{aa} \right)^{-1} (\mathbf{L}_1 - \mathbf{L}_2) \right) \mathbf{u} = \mathbf{f}_{\text{ext}}^u \tag{32}$$

Now the basic equilibrium expression of the ECDM without the enriched or additional DoFs has been obtained, in which the nonlinearity must be accounted because of CDM embedded. The equivalent stiffness matrix and the right hand side (i.e. the nodal force vector) are constructed at elemental level and transferred into ABAQUS to assemble the system equilibrium. The nonlinear incremental iteration is performed using a displacement control procedure. The Newton Raphson algorithm in

conjunction with line search scheme [34] provided by ABAQUS is employed as the technique to carry out nonlinear iteration. As far as authors know, this combined technique is widely used in iteration procedure for solving strong nonlinearity. Numerical iteration will end when the residual nodal force reduces to the prescribed tolerance. It should be noticed that the sub-matrix $\mathbf{K}^{aa}$ in Eq. (32) is symmetric but not necessarily invertible, hence the generalized inversed matrix (Moore–Penrose pseudoinverse) [35] is applied to complete the calculation of the inversed matrix. The ECDM formulation is a lower order equilibrium system compared to the standard XFEM, which permits nodal displacement calculation of the cracked element using the standard FEM DoFs only. This proposed rigorous mathematical procedure can fully cover the damage evolution from a weak discontinuity to a strong discontinuity. This developed novel ECDM has been implemented using the user subroutine UEL in the commercial finite element package ABAQUS.

3. Numerical examples

In this section, four numerical examples are modelled by the ECDM to validate the performance of the developed ECDM. In these examples, homogeneous quasi-brittle materials are used and a cohesive strength based criteria is used to characterize the damage initiation. Based on this criteria, damage starts when the maximum principal stress at Gauss points of any element is beyond the cohesive strength of any individual fracture mode. The perpendicular direction to the maximum principal stress is adopted to be the crack direction within elements. A strain field

related damage scalar is used to account for damage accumulation until it satisfies the crack criteria determined by the fracture energy. Overlapping elements with zero stiffness are placed in the FEM mesh to visualize the discretization of the ABAQUS user elements with the ECDM. Quasi-static crack propagation is mainly considered in this investigation. The accuracy, mesh-independence, robustness and capability have been presented and discussed through numerical examples in simulating arbitrary crack propagation using the proposed ECDM.

3.1 A concrete beam under three-point bending

One of the most popular specimens in concrete material testing, three-point bending, is modelled as a preliminary verification of the proposed ECDM. The geometry and boundary conditions of beam modelled can be seen from Fig. 5. This simply supported beam is made of Ultra High Performance Fibre Reinforced Concrete (UHPERC) and subjected to displacement-controlled loading [36]. The following material properties for UHPERC are used in modelling simulation [36]: Young's modulus $E = 47000.0\text{MPa}$, Poisson's ratio $\nu = 0.2$, tensile strength $f_t = 8.0\text{MPa}$ and fracture energy release rate $G_f = 30000.0\text{N/m}$. In order to reduce computational work, only the potential fracture zone marked by dash lines in Fig. 5 is discretized with the proposed ECDM. For the purpose of investigating the mesh-independence of the proposed ECDM, four sets of different mesh-sized models for modelling the potential fracture zone with the area $18.5 \times 100\text{mm}^2$, named as M1, M2, M3 and M4, are investigated. The average element side lengths for modelling the potential fracture

zone are 1.65, 1.82, 2.07 and 2.45mm, and the total numbers of elements in this zone are 682, 558, 434 and 310 for M1, M2, M3 and M4, respectively. Other regions are modelled by elements whose average size is 8mm. All these models are constructed by the four-node plane strain user elements with the ECDM. It should be noted that the control parameters of line search scheme are set to be 4.0, 4.0, 0.25, 0.25 and 0.15, respectively for nonlinear fracture analysis in all models in this section.

Fig. 5. A concrete beam with a notch under three-point bending

Fig. 6. Load-displacement curves given by the ECDM and experimental work

It can be seen from Fig. 6 that the ECDM predicted initial stiffness of the notched beam agrees with test work well. The predicted failure load is slightly higher than the tested one. This is because the data for fracture property and material strength

was collected from a limited number of test specimens. In the crack propagation or post failure stage, the ECMD prediction almost agrees with experimental work. Because of the relatively high fracture toughness of the UHPERC, after the failure load is reached, the response remains at a higher level in the post-failure response rather than dropping down immediately. The predicted load-displacement curves are plotted in terms of four models with different mesh densities shown in Fig. 6. It can be seen from Fig. 6 that the load-displacement curves obtained from the meshes M1, M2, M3 and M4 are almost same, only the result from mesh M3, containing 434 elements, is slightly higher than others.

Fig. 7 shows the deformation of the beam model with the finest mesh together with a cracked region in the middle section presented by a maximum principal strain field under applied displacement of 2.25mm. The deformation due to the macro crack opening can be observed in the area with large deformation. The crack propagation basically follows the grey band in the maximum principal strain contour shown in the contours. It can be concluded that the predicted failure response is independent from the mesh densities, and coincident with the experimental measurement. Therefore, the ECMD based modelling provides a reliable prediction of fracture behaviour of the UHPERC sample.

Fig. 7. Maximum principal strain contour field in a concrete beam under three-point bending (Mesh M4) simulated by the ECDM at the final state of analysis.

3.2 A notched concrete beam under asymmetric bending

A comparative study is carried out on a benchmark test using both the ECDM and the standard XFEM in FEM commercial package ABAQUS (shorten as XFEM below) to assess the performance of the ECDM. In the test work [37], a single notched beam as shown in Fig. 8, made of plain concrete, was subjected to four-point bending. A mixed failure mode with a curved cracking path was observed experimentally. The dimensions of the beam is $440 \times 100 \times 100\text{mm}^3$; a notch with the size $5 \times 20 \times 100\text{mm}^3$ is located at the top centre. The loading and boundary conditions can be seen from Fig. 8. A rigid bar between two loading points is set up for the purpose of imposing a proportionally distributed load P at the bottom of beam. Through the rigid bar, the proportional loads applied on the left and right loading point are $P/11$ and $10P/11$, respectively. Following the previous work given by references [38] and [39], the material properties used in modelling are: Young's modulus $E = 35000.0\text{MPa}$, Poisson's ratio $\nu = 0.2$, tensile strength $f_t = 3.0\text{MPa}$ and fracture energy $G_f = 0.1\text{ N/mm}$.

Fig. 8. Configuration of the Single-notched concrete beam under asymmetric bending load.

Four meshes, named as M1, M2, M3 and M4 with different element sizes are used in both the ECDM and XFEM for mesh sensitivity investigation. It should be noted that only the potential fracture zone of $125 \times 100\text{mm}^2$ marked by dash line in Fig. 8 is discretised with different element sizes. The average element side lengths used in the potential fracture zone are 8, 4, 2 and 1mm, respectively. The numerical simulations are carried out without setting the discontinuity evolution path beforehand, so cracks could propagate arbitrarily. Fig. 9 shows the deformation after failure predicted by the ECDM using four different meshes. The cracked elements are presented in terms of maximum principle strain (buckled zone are shown in grey), from which a curved feature of the crack can be observed. The predicted crack profiles from four meshes are similar. At the end of analysis, the exact crack paths predicted by the ECDM together with the experimental envelop are shown in Fig. 10. It can be seen from Fig. 10 that the predicated crack paths from four different meshes are almost identical and are certainly within the experimental envelop. This investigation proves the mesh-independence of the ECDM in simulation of crack propagation. It should be noted that each crack path given in Fig. 10 is calculated according to the location of the crack tip in each failed element. In both the ECDM and XFEM modelling, the crack propagations are determined using a principal stress failure criterion.

Fig. 9. Maximum principal strain contour field in the single-notched concrete beam under asymmetric bending load simulated with the ECDM at the final state of analysis.

Fig. 10. Predicted crack paths by the ECDM at the final state of analysis and experimental crack envelop [37].

Fig. 11 displays the sequences of predicted failure responses provided by the ECDM and XFEM with progressively refined meshes. In these responses, the relative difference of vertical displacements between two sides of the notch, i.e., crack mouth sliding displacement (CMSD), is computed as the abscissa, while the reaction force on the loading point is ordinate. The load - CMSD curves given by the ECDM and XFEM are compared to the experimental measurements. It can be seen from Fig. 10 that the ECDM modelling with four different meshes give almost identical responses; the predicted peak load agrees with experimental envelope very well. In the post-failure softening stage, the predicted residual stiffness is slightly lower than the experimental observation. This is because a residual load capacity was still recorded in experimental work due to the compressive stresses around the rigid cap [37], however, at the later post-failure softening stage, the ECDM prediction has same tendency with experimental measurement. This benchmark modelling indicates that the proposed ECDM is capable of predicting arbitrary crack propagation with accuracy and robustness along with the change of mesh size. In contrast, the results of peak load are obviously overestimated by XFEM compared to experimental measurement, and is varied to some extent in different meshes. The post-failure softening behaviour predicted by XFEM is also changed some extent in different meshes compared to experimental observation. This implies that the outcome from XFEM simulation relatively relates to the mesh size in this investigation. Moreover,

XFEM simulation tends to overestimate the structural strength at the failure point and the most softening stage in all four predicted load - CMSD curves.

Fig. 11. Load - CMSD curves obtained from the ECDM, XFEM and experimental work [37].

3.3 A steel plate specimen with two holes and two notches under tension

This example is investigated to evaluate the capability of the ECDM in simulating multiple crack propagations. The geometry, boundary conditions and material properties of a rectangular steel plate with two holes and two initial cracks are taken from previous work [33] as shown in Fig.12a. The thickness of the plate is 1 mm. The Young's modulus and Poisson's ratio are, respectively, $E = 210000\text{MPa}$ and $\nu=0.3$. Material tensile strength $f_t = 235.0\text{MPa}$. Plane stress material model is used in the investigation. Displacement is prescribed on the upper side of the specimen shown

in Fig. 12a. It should be noted that this analysis focuses on multiple crack propagations, only the crack paths are presented in this paper, and the cohesive effect is ignored in simulation for quick tracking of the crack paths.

Fig.12. (a) A steel plate with two holes and two initial cracks under tension; (b) The ECDM solution presented by principal strain contour at the final state of analysis.

In this modelling work, two cracks are tracked with the assistance of a common data block COMXYC in coding, by which the crack tip information can be shared by different elements and updated at each loading increment. Fig. 12b shows the simulated two-crack propagations using a mesh with 17414 elements. It is expected that the two simulated crack paths are symmetric and look similar. The simulated crack paths by the ECDM reflect the reality of the actual cracking pattern in this steel plate. This example basically confirms the capability of the ECDM in tracking the multiple crack propagations, although there is no experimental work reported in literatures for comparison.

4. Conclusions and future work

The finite element framework and detailed rigorous derivation in forming the condensed equilibrium equations of the ECDM have been presented in this paper. This is a new model for simulating arbitrary crack propagation within a continuum solid. Based on the fundamental concept of XFEM, the ECDM introduces a displacement field with a shifted Heaviside function and enriched DoFs to approximate discontinuities at the beginning. In the final condensed equilibrium equation of the ECDM, the additionally enriched DoFs are eliminated at the element level. This enables ECDM to improve the overall convergence in numerical calculation, and to be capable of carrying out nonlinear fracture analysis with large computing work at structural level. Considering the length of this paper, detailed investigation of the computing efficiency of the ECDM will be discussed in different papers.

The ECDM embeds the micro-mechanical cohesive damage model CDM into the macro-mechanical finite element formula. Unlike the classic CDM which requires the displacement gap to qualify the state of cohesive segment, the ECDM employs a new equivalent damage scalar as a function of strain field. This feature allows it to avoid the requirement of explicit displacement gap and to make it possible to embed the CDM into the ECDM. With this new model, the crack nucleation as well as crack propagation can be characterized without any re-meshing effort. In the numerical implementation via user subroutine UEL in ABAQUS, the maximum principal stress based criteria is employed for the judgement of damage initiation and the

determination of the crack direction. The introduced equivalent damage scalar and fracture energy criteria are used for the judgement of crack propagation. More details for determining the crack direction as well as the integration scheme of stiffness matrix of the ECDM based user element will be given in different papers.

Through all investigated examples in this paper, it has been validated that the developed ECDM has the capability of capturing multiple crack propagations in solids with sufficient accuracy, efficiency and stability. The mesh-independence of the ECDM has been verified by a number of models with different meshes which produce identical solutions. This developed ECDM supplies a promising numerical approach for simulating arbitrary damage propagation in engineering materials. In the future, the developed ECDM will be applied in simulating multiple crack propagations in heterogeneous materials such as fibre composites. This will include multiple layered delamination, multiple matrix crack and fibre breakage to fully validate the capability of the ECDM in simulating the reality of arbitrary damage propagation.

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