Bounds on the unstable eigenvalue for the asymmetric renormalization operator for period doubling

B. D. Mestel*, A. H. Osbaldestin† and A. V. Tsygvintsev‡
* Department of Computing Science and Mathematics, University of Stirling, Stirling, FK9 4LA, UK, b.d.mestel@stir.ac.uk
† Department of Mathematics, University of Portsmouth, Portsmouth, PO1 3HE, UK, andrew.osbaldestin@port.ac.uk
‡ Unite de Mathématiques Pures et Appliquées, Ecole Normale Supérieure de Lyon, 46 Allée d'Italie, F-69364 Lyon Cedex 07, France, alexei.tsygvintsev@umpa.ens-lyon.fr

December 9, 2003

Abstract

We establish rigorous bounds for the unstable eigenvalue of the period-doubling renormalization operator for asymmetric unimodal maps. Herglotz-function techniques and cone invariance ideas are used. Our result generalizes an established result for conventional period doubling.

1 Introduction

The remarkable universality of the scalings witnessed in the period-doubling route to chaos now has a well-established, mathematically rigorous, basis. Soon after discovery by Feigenbaum [8] and Coullet-Tresser [3], the first (computer-assisted) proof was given by Lanford [11], closely followed by the analytic proofs of Epstein [5] and coworkers. More recently the rigorous analysis has reached new levels of sophistication in the works of Sullivan [17] and McMullen [12].

Contemporaneously, Arneodo *et al* [2] initiated the investigation of *asymmetric* unimodal maps. In a recent series of articles [13, 14, 15], we have given a rigorous renormalization analysis of period doubling in degree-d asymmetric unimodal maps. These are unimodal maps possessing a degree-d maximum, but with differing left and right dth derivatives. The maps we have in mind take the form

$$f(x) = \begin{cases} f_L(x) = 1 - a_1|x|^d & \text{if } x \le 0\,; \\ f_R(x) = 1 - a_2|x|^d & \text{if } x \ge 0\,. \end{cases} \tag{1.1}$$

(The case of differing left- and right-hand degrees appears to be somewhat different in nature. See, e.g., [9].) In brief, for each $d > 1$, the standard Feigenbaum period-doubling renormalization operator has been shown to possess a family of period-two orbits, parametrized by an invariant asymmetry modulus, μ, measuring the ratio of the left and right dth derivatives at the maximum. The period-two orbit is then given by a quartet of functions $(f_L, f_R, \tilde{f}_L, \tilde{f}_R)$ satisfying the functional equations

$$\tilde{f}_L(x) = -\lambda^{-1} f_R f_R(-\lambda x)\,, \tag{1.2a}$$
$$\tilde{f}_R(x) = -\lambda^{-1} f_R f_L(-\lambda x)\,, \tag{1.2b}$$
$$f_L(x) = -\tilde{\lambda}^{-1} \tilde{f}_R \tilde{f}_R(-\tilde{\lambda} x)\,, \tag{1.2c}$$
$$f_R(x) = -\tilde{\lambda}^{-1} \tilde{f}_R \tilde{f}_L(-\tilde{\lambda} x)\,, \tag{1.2d}$$

with the normalizations $f_L(0) = f_R(0) = \tilde{f}_L(0) = \tilde{f}_R(0) = 1$ so that $\lambda = -f_R(1) > 0$ and $\tilde{\lambda} = -\tilde{f}_R(1) > 0$.

The solutions of (1.2) depend on two parameters, viz., the degree d of the critical point and the modulus μ, which (for the case when d is an even integer) is the ratio

$$\mu = \frac{f_L^{(d)}(0-)}{f_R^{(d)}(0+)}\,. \tag{1.3}$$

The case $\mu = 1$ is the standard Feigenbaum scenario in which case the period-two orbit is in fact a fixed point.

Let us denote by R the period-doubling renormalization operator acting on a unimodal map f with $f(0) = 1$, so that

$$R(f)(x) = -\lambda^{-1} f(f(-\lambda x)), \qquad \lambda = -f(1)\,. \tag{1.4}$$

Then R acts on both symmetric and asymmetric unimodal maps, preserving the degree d and inverting the asymmetry modulus μ. The scaling of the parameters in (1.1) undergoing a period-doubling cascade is determined by the expanding eigenvalue of the derivative of R^2 at the period-2 point f. This derivative $dR^2(f)$ is compact on a suitable Banach space of tangent functions δf and numerical results suggest that it is hyperbolic with a single expanding eigenvalue δ^2. It is this expanding eigenvalue which we investigate in this paper. More precisely, we study an associated operator T (defined below), which has a positive expanding eigenvalue δ. We give a brief description of the relationship between T and $dR^2(f)$ in Section 4.

The operator T is defined on a pair of functions $(v, \tilde{v})$ and is given by:

$$T \begin{pmatrix} v(x) \\ \tilde{v}(x) \end{pmatrix} = \begin{pmatrix} \tilde{t}^{-1}(\tilde{v}(\tilde{t}x) + \tilde{v}(\tilde{L}(\tilde{t}x))\tilde{L}'(\tilde{t}x)^{-1}) \\ t^{-1}(v(tx) + v(L(tx))L'(tx)^{-1}) \end{pmatrix}, \tag{1.5}$$

where $t = \mu\lambda^d$, $\tilde{t} = \mu^{-1}\tilde{\lambda}^d$, and $L(x) = F(x)^d$, $\tilde{L}(x) = \tilde{F}(x)^d$. In this article we analyze the positive unstable eigenvalue of T, and, in particular, we shall establish the following theorem. Our work mirrors closely the analysis of Eckmann and Epstein [4] on the expanding eigenvalue of the symmetric Feigenbaum fixed-point. We shall establish the following result,

Theorem 1. *There exists a Banach space of function pairs on which the operator T is well defined, compact and has an eigenvalue $\delta > 0$ satisfying*

$$1 < \frac{1}{(\lambda\tilde{\lambda})^{(d-1)/2}(1 + \sqrt{\lambda\tilde{\lambda}})} < \delta < \frac{1}{(\lambda\tilde{\lambda})^{d/2}}\,. \tag{1.6}$$

Several remarks are appropriate for this theorem. Firstly, the theorem establishes the existence of an expanding eigenvalue but does not prove the hyperbolicity of the operator dR^2_f. Secondly, the lower bound for δ, whilst greater than 1, is suboptimal and, indeed, is worse than the bounds $1/\lambda^d - 1/\lambda$ obtained in [4] for the symmetric period-doubling case. Unfortunately, some of the estimates in that paper do not readily generalize to the asymmetric case and our results are accordingly weaker, although they do apply to all degree d and modulus μ.

2 Notation and background material

In this section we establish our notation and give a brief summary of previous results from [13, 14, 15] that we shall use in this paper.

The Herglotz function approach [5] has been an extremely fruitful technique in the analysis of the accumulation of period-doubling. It was used in [14] to prove the existence of a solution of the equations (1.2) for all real $\mu > 0$ and $d > 1$. We recall here how equations (1.2) may be recast as an anti-Herglotz function problem.

Firstly we build the singularity into our functions by defining

$$f_R(x) = F_R(|x|^d)\,, \qquad \tilde{f}_R(x) = \tilde{F}_R(|x|^d)\,. \tag{2.1}$$

The left-hand functions are given in terms of the right-hand ones by

$$f_L(x) = F_R(\mu|x|^d)\,, \qquad \tilde{f}_L(x) = \tilde{F}_R(\mu^{-1}|x|^d)\,. \tag{2.2}$$

We then consider the inverses of these functions by defining

$$F_R(x) = U^{-1}(x)\,, \qquad \tilde{F}_R(x) = \tilde{U}^{-1}(x)\,. \tag{2.3}$$

The functions U and $\tilde{U}$ satisfy the conditions $U(1) = 0$, $U(-\lambda) = 1$, $\tilde{U}(1) = 0$, $\tilde{U}(-\tilde{\lambda}) = 1$. We may further normalize by setting $U(x) = k\psi(x)$, $\tilde{U}(x) = \tilde{k}\tilde{\psi}(x)$, where $k = U(0)$, $\tilde{k} = \tilde{U}(0)$, so that the functions ψ and $\tilde{\psi}$ satisfy $\psi(1) = 0$, $\psi(0) = 1$, $\tilde{\psi}(1) = 0$, $\tilde{\psi}(0) = 1$. We then have $U(x) = {z_1}^d\psi(x)$, $\tilde{U}(x) = \tilde{z}_1^d\tilde{\psi}(x)$, where $z_1 = \psi(-\lambda)^{-1/d}$, $\tilde{z}_1 = \tilde{\psi}(-\tilde{\lambda})^{-1/d}$.

In this new setting our equations become

$$\psi(x) = \tilde{\tau}^{-1}\tilde{\psi}(\tilde{\phi}(x))\,, \qquad \tilde{\psi}(x) = \tau^{-1}\psi(\phi(x))\,, \tag{2.4}$$

where $\phi(x) = z_1\psi(-\lambda x)^{1/d} = U(-\lambda x)^{1/d}$, $\tilde\phi(x) = \tilde z_1\tilde\psi(-\tilde\lambda x)^{1/d} = \tilde U(-\tilde\lambda x)^{1/d}$, and $\tau = \psi(z_1)$, $\tilde\tau = \tilde\psi(\tilde z_1)$ satisfy

$$\lambda^d = \frac{\tau z_1^d}{\mu \tilde z_1^d}\,, \qquad \tilde\lambda^d = \frac{\mu\tilde\tau\tilde z_1^d}{z_1^d}\,. \tag{2.5}$$

Note that $\tau\tilde\tau = (\lambda\tilde\lambda)^d$. In terms of the functions U, $\tilde U$, equations (2.4) become

$$U(x) = \mu\tilde\lambda^{-d}\tilde U(\tilde U(-\tilde\lambda x)^{1/d})\,, \qquad \tilde U(x) = \mu^{-1}\lambda^{-d}U(U(-\lambda x)^{1/d})\,. \tag{2.6}$$

The method of the existence proof is now to show that (2.4) has a solution in a space of anti-Herglotz functions.

Let $\mathbb{C}_+$, $\mathbb{C}_-$ denote the upper and lower half planes in $\mathbb{C}$. Recall that a complex analytic function on $\mathbb{C}_+ \cup \mathbb{C}_-$ is said to be Herglotz (resp. anti-Herglotz) if $f(\mathbb{C}_+) \subset \bar{\mathbb{C}}_+$ and $f(\mathbb{C}_-) \subset \bar{\mathbb{C}}_-$ (resp. $f(\mathbb{C}_+) \subset \bar{\mathbb{C}}_-$ and $f(\mathbb{C}_-) \subset \bar{\mathbb{C}}_+$).

For $A < B \in \mathbb{R}$, we let $\Omega(A,B)$ denote $\mathbb{C}_+ \cup \mathbb{C}_- \cup (A,B)$. We denote by $\mathrm{H}(A,B)$ and $\mathrm{AH}(A,B)$ (respectively) the space of Herglotz and anti-Herglotz functions (respectively) analytic on the interval (A,B). Furthermore, if $[0,1] \subset (A,B)$, let $\mathrm{E}(A,B)$ denote the space of anti-Herglotz functions $\psi \in \mathrm{AH}(A,B)$ which satisfy the normalizations $\psi(0) = 1$, $\psi(1) = 0$. As is normal, we equip $\mathrm{H}(A,B)$, $\mathrm{AH}(A,B)$ and $\mathrm{E}(A,B)$ with the topology of uniform convergence on compact subsets of $\Omega(A,B)$.

In [14] we prove the following existence theorem:

Theorem. *For each $\mu > 0$ and for each $d > 1$, there exists a solution pair $(\psi, \tilde\psi)$ for (2.4) with $\psi \in E(-\tilde\lambda^{-1}, (\lambda\tilde\lambda)^{-1})$ and $\tilde\psi \in E(-\lambda^{-1}, (\lambda\tilde\lambda)^{-1})$.*

From this it is straightforward to reverse the transformation above to show that (1.2) has a solution. See [14].

One crucial feature of the Herglotz and anti-Herglotz functions is that they satisfy the so-called *a priori* bounds. (See [5, 6, 14].) For the solution pair $(\psi, \tilde\psi)$ these bounds are, for $x < 0$ and $x > 1$:

$$\frac{1-x}{1-\lambda\tilde\lambda x} \le \psi(x) \le \frac{1-x}{1+\tilde\lambda x}\,, \qquad \frac{1-x}{1-\lambda\tilde\lambda x} \le \tilde\psi(x) \le \frac{1-x}{1+\lambda x}\,; \tag{2.7}$$

and for $0 < x < 1$:

$$\frac{1-x}{1+\tilde\lambda x} \le \psi(x) \le \frac{1-x}{1-\lambda\tilde\lambda x}\,, \qquad \frac{1-x}{1+\lambda x} \le \tilde\psi(x) \le \frac{1-x}{1-\lambda\tilde\lambda x}\,. \tag{2.8}$$

In addition, as in [5], it is straightforward to derive *a priori* bounds on the first and second derivatives:

$$\frac{-2\tilde\lambda}{(1+\tilde\lambda x)} \le \frac{\psi''(x)}{\psi'(x)} = \frac{U''(x)}{U'(x)} \le \frac{2\lambda\tilde\lambda}{(1-\lambda\tilde\lambda x)}\,, \qquad x \in (-\tilde\lambda^{-1}, (\lambda\tilde\lambda)^{-1})\,, \tag{2.9a}$$

$$\frac{-2\lambda}{(1+\lambda x)} \le \frac{\tilde\psi''(x)}{\tilde\psi'(x)} = \frac{\tilde U''(x)}{\tilde U'(x)} \le \frac{2\lambda\tilde\lambda}{(1-\lambda\tilde\lambda x)}\,, \qquad x \in (-\lambda^{-1}, (\lambda\tilde\lambda)^{-1})\,. \tag{2.9b}$$

Let us define $t = \mu\lambda^d$, $\tilde t = \mu^{-1}\tilde\lambda^d$. Then we have the following properties which are a consequence of the definitions and the results of [14].

1. t, $\tilde t$, z_1, $\tilde z_1$, τ, $\tilde\tau \in (0,1)$;
2. $t < z_1^d$, and $\tilde t < \tilde z_1^d$.

Following [5], we define the functions $V(x) = \tau^{-1}\psi(z_1 x^{1/d})$, $\tilde V(x) = \tilde\tau^{-1}\tilde\psi(\tilde z_1 x^{1/d})$. Let us further define $\alpha = \psi(-\lambda)$, $\tilde\alpha = \tilde\psi(-\tilde\lambda)$. Then $V \in \mathrm{AH}(0, \alpha(\lambda\tilde\lambda)^{-d})$ and $\tilde V \in \mathrm{AH}(0, \tilde\alpha(\lambda\tilde\lambda)^{-d})$ and, in view of equations (2.4), we have

$$\psi(x) = \tilde V(\tilde\psi(-\tilde\lambda x))\,, \qquad \tilde\psi(x) = V(\psi(-\lambda x)\,. \tag{2.10}$$

Note that $V(1) = \tilde V(1) = 1$ and $V(\alpha) = \tilde V(\tilde\alpha) = 0$. Differentiating (2.10), and evaluating at 0, gives

$$V'(1) = \frac{-\tilde\psi'(0)}{\lambda\psi'(0)}\,, \qquad \tilde V'(1) = \frac{-\psi'(0)}{\tilde\lambda\tilde\psi'(0)}\,, \qquad V'(1)\tilde V'(1) = \frac{1}{\lambda\tilde\lambda}\,. \tag{2.11}$$

Lemma 1. *The functions $U(x)$ and $\tilde U(x)$ are injective respectively in domains $\Omega = \Omega(-\tilde\lambda^{-1}, (\lambda\tilde\lambda)^{-1})$ and $\tilde\Omega = \tilde\Omega(-\lambda^{-1}, (\lambda\tilde\lambda)^{-1})$.*

Proof. Our proof is based on the ideas of Epstein given in [6] and [7]. It is sufficient to show the injectivity of U, for the injectivity of $\tilde{U}$ then easily follows from the equations (2.6). From the same equations we have

$$U(x) = (\lambda\tilde{\lambda})^{-d} U(\Phi_3(x)), \quad x \in \Omega\,, \tag{2.12}$$

where $\Phi_3(x) = u(-\mu^{-1/d} u(u(\lambda\tilde{\lambda}x)))$ with $u(x) = U(x)^{1/d}$. To elucidate the proof we further define $\Phi_2(x) = -\mu^{-1/d} u(u(\lambda\tilde{\lambda}x))$, $\Phi_1(x) = u(\lambda\tilde{\lambda}x)$, $\Phi_0(x) = \lambda\tilde{\lambda}x$.

The functions Φ_0, Φ_3 are Herglotz in Ω and Φ_1, Φ_2 are anti-Herglotz in the same domain. Indeed, it is clear that they are Herglotz and anti-Herglotz respectively on $\mathbb{C}_+ \cup \mathbb{C}_-$ and any interval on which they are well-defined. Indeed they are Herglotz and anti-Herglotz respectively on $J_0 = (-\tilde{\lambda}^{-1}, (\lambda\tilde{\lambda})^{-1})$ as follows from the inclusions proved below. Let $s \in (1, (\lambda\tilde{\lambda})^{-1})$ be arbitrarily chosen and let $J = (-\tilde{\lambda}^{-1}, s)$. Our first aim is to show that $\Phi_i(\Omega(J)) \Subset \Omega(J)$ for $0 \le i \le 3$, where we adopt the notation that $A \Subset B$ means that A is *strictly* inside of B, i.e., $\overline{A} \subset B$. Taking into account the Herglotz (resp. anti-Herglotz) character of the functions Φ_i it is sufficient to prove the inclusions $\Phi_i(J) \Subset J$. We consider each function in turn.

(a) $\Phi_0(J) \Subset J$ is seen from the inequality $0 < \lambda\tilde{\lambda} < 1$.

(b) $\Phi_1(J) \Subset J$.
We have $u(\lambda\tilde{\lambda}x) \in (z_1\psi((\lambda\tilde{\lambda})(\lambda\tilde{\lambda})^{-1})^{1/d}, z_1\psi((\lambda\tilde{\lambda})(-\tilde{\lambda})^{-1})^{1/d}) = (0, z_1\psi(-\lambda)^{1/d}) = (0,1)$ for all $x \in J$. The result follows from this and the fact that $s > 1$.

(c) $\Phi_2(J) \Subset J$.
We have $-\mu^{-1/d} < \Phi_2(x) < 0$ for all $x \in J \subset J_0$ since $0 < u(u(\lambda\tilde{\lambda}x)) < 1$ for arbitrary $x \in J_0$. The inequality $\mu^{-1/d}\tilde{\lambda} < 1$ implies $-\mu^{-1/d} > -\tilde{\lambda}^{-1}$ and we obtain $\Phi_2(J) = (\Phi_2(s), \Phi_2(-\tilde{\lambda}^{-1})) \Subset (-\tilde{\lambda}^{-1}, 0)$ as required.

(d) $\Phi_3(J) \Subset J$.
We have $\Phi_3(J) = (\Phi_3(-\tilde{\lambda}^{-1}), \Phi_3(s))$, $0 < \Phi_3(-\tilde{\lambda}^{-1}) < \Phi_3(s)$. But $\Phi_3(s) < s$. Indeed, by the Schwarz Lemma, $x = 1$ is the unique attracting fixed point of Φ_3 in the interval J_0. Hence, $\Phi_3(1) = 1$, $\Phi_3'(1) < 1$ and the graph of $y = \Phi_3(x)$ lies below the line $y = x$ for all $x \in (1, (\lambda\tilde{\lambda})^{-1})$.

Following [6] we define

$$D(a,b,\theta) = \left\{ z \in \mathbb{C} : 0 < \arg \frac{z-a}{b-z} < \theta \right\}\,, \tag{2.13}$$

which is the domain between the real segment (a,b), $a < b$ and the circular arc in $\mathbb{C}_+$ with ends a and b whose tangent at a has argument $\theta \in (0,\pi)$. $\widehat{D}(a,b,\theta)$ will denote the domain between the same arc and its reflection in the real axis. We also use the notation $\delta(c,\theta,r)$ (resp. $\hat{\delta}(c,\theta,r)$) for $D(a,b,\theta)$ (resp. $\widehat{D}(a,b,\theta)$), where c denotes the mid-point of (a,b) and $r = (b-a)/2$ denotes its radius. Let C and R be the corresponding mid-point and radius of the interval J. Then, according to the conditions $\Phi_i(J) \Subset J$, $0 \le i \le 3$ and Lemma 4 from [6], there exists $k \in (0,1)$ such that each Φ_i maps $\hat{\delta}(C,\theta,R)$ into $\hat{\delta}(C,\theta,kR)$ for an arbitrary $\theta \in (0,\pi)$. In addition, as follows from Lemma 5 of [6], each Φ_i maps $\hat{\delta}(C,\theta,R)$ into $\hat{\delta}(C,\theta',R)$, with

$$\tan\frac{\theta'}{2} = k \tan\frac{\theta}{2}\,. \tag{2.14}$$

Suppose that z_0 and w_0 are distinct points in $\Omega(J)$ such that $U(z_0) = U(w_0)$. There exists a $\theta_0 \in (0,\pi)$ such that z_0 and w_0 are both in $\hat{\delta}(C,\theta_0,R)$. For $n = 1,2,...$ we define inductively a pair (z_n, w_n) of distinct points, both lying in $\hat{\delta}(C,\theta_0,kR)$ such that $U(z_n) = U(w_n)$. This we do as follows. Assume (z_n, w_n), $n \ge 0$ constructed. Then we set

$$(z_{n+1}, w_{n+1}) = \begin{cases} (\Phi_3(z_n), \Phi_3(w_n)) & \text{if} \quad \Phi_3(z_n) \ne \Phi_3(w_n) \\[2ex] (\Phi_2(z_n), \Phi_2(w_n)) & \text{if} \quad \begin{cases} \Phi_3(z_n) = \Phi_3(w_n) \\ \Phi_2(z_n) \ne \Phi_2(w_n) \end{cases} \\[2ex] (\Phi_1(z_n), \Phi_1(w_n)) & \text{if} \quad \begin{cases} \Phi_1(z_n) \ne \Phi_1(w_n) \\ \Phi_2(z_n) = \Phi_2(w_n) \\ \Phi_3(z_n) = \Phi_3(w_n) \end{cases} \\[2ex] (\Phi_0(z_n), \Phi_0(w_n)) & \text{if} \quad \begin{cases} \Phi_1(z_n) = \Phi_1(w_n) \\ \Phi_2(z_n) = \Phi_2(w_n) \\ \Phi_3(z_n) = \Phi_3(w_n) \end{cases} \end{cases}$$

According to (2.14) applied to Φ_i, $0 \le i \le 3$, z_n and w_n are both in $\hat{\delta}(C, \theta_n, R)$, with

$$\tan\frac{\theta_n}{2} = k^n \tan\frac{\theta_0}{2}\,. \tag{2.15}$$

Moreover, for $n > 1$ one has $(z_n, w_n) \in \hat{\delta}(C, \theta_0, kR)$. We note that the restriction of U to the real interval $I = \overline{\hat{\delta}(C, \theta_0, kR)} \cap \mathbb{R}$ is injective. This implies that there exists a small complex neighborhood $\mathcal{U}_\mathcal{I} \subset \mathbb{C}$ of I where U will be also injective. As $n \to \infty$ we have $\theta_n \to 0$ and the both points z_n and w_n will enter eventually $\mathcal{U}_\mathcal{I}$ for n sufficiently large. This contradiction proves that U is injective in Ω. $\square$

3 Lower bound for d

In this section we shall prove the inequality $d > (1+\lambda\tilde{\lambda})/(1-\lambda\tilde{\lambda})$ which is important in the proof of convexity of the functions L, $\tilde{L}$, V and $\tilde{V}$. The inequality is analogous to Epstein's result for the symmetric Feigenbaum function, viz., $d > (1+\lambda^2)/(1-\lambda^2)$. However, our proof differs somewhat from that in [5].

Lemma 2.

$$d > (1+\lambda\tilde{\lambda})/(1-\lambda\tilde{\lambda})\,. \tag{3.1}$$

Proof. We start from the estimates contained in the paper [15]

$$\psi(x) \le \frac{1-x}{(\tilde{\tau}+\tilde{\lambda}\tilde{\tau}-\tilde{\lambda})(1-x)+(1-\tilde{\tau})(1+\tilde{\lambda})}\,, \quad x \in (-\tilde{\lambda}^{-1}, 0)\,, \tag{3.2a}$$

$$\tilde{\psi}(x) \le \frac{1-x}{(\tau+\lambda\tau-\lambda)(1-x)+(1-\tau)(1+\lambda)}\,, \quad x \in (-\lambda^{-1}, 0)\,. \tag{3.2b}$$

The first inequality gives

$$\psi(-\lambda) \le \frac{1+\lambda}{(\lambda+\lambda\tilde{\lambda})\tilde{\tau}+1-\lambda\tilde{\lambda}}\,, \tag{3.3}$$

so that

$$z_1 = \psi(-\lambda)^{-1/d} \ge \left(\frac{(\lambda+\lambda\tilde{\lambda})\tilde{\tau}+1-\lambda\tilde{\lambda}}{1+\lambda}\right)^{1/d} = (\rho\tilde{\tau}+(1-\rho))^{1/d} \ge \rho\tilde{\tau}^{1/d}+1-\rho = 1-\rho(1-\tilde{\tau}^{1/d})\,, \tag{3.4}$$

where we have set $\rho = (\lambda+\lambda\tilde{\lambda})/(1+\lambda) \in (0,1)$ and have used the property $(\rho x + (1-\rho)y)^{1/d} \ge \rho x^{1/d} + (1-\rho)y^{1/d}$. Similarly, we have $\tilde{z}_1 \ge 1-\tilde{\rho}(1-\tau^{1/d})$, where $\tilde{\rho} = (\tilde{\lambda}+\lambda\tilde{\lambda})/(1+\tilde{\lambda}) \in (0,1)$. Note $\rho\tilde{\rho} = \lambda\tilde{\lambda}$. Using the *a priori* bounds (2.8) and writing $c = \lambda\tilde{\lambda}$, we have

$$c^d = \tau\tilde{\tau} = \psi(z_1)\tilde{\psi}(\tilde{z}_1) \le \left(\frac{1-z_1}{1-cz_1}\right)\left(\frac{1-\tilde{z}_1}{1-c\tilde{z}_1}\right) \tag{3.5}$$

$$\le \frac{c(1-\tilde{\tau}^{1/d})(1-\tau^{1/d})}{(1-c+c\rho(1-\tilde{\tau}^{1/d}))(1-c+c\tilde{\rho}(1-\tau^{1/d}))}\,, \tag{3.6}$$

using the fact that $x \mapsto (1-x)/(1-cx)$ is monotonic decreasing on $(0,1)$. Now, noting that $\rho\tilde{\rho} = c$, we see that the denominator is minimized when $\rho = (c(1-\tau^{1/d})/(1-\tilde{\tau}^{1/d}))^{1/2}$, so that

$$c^d \le \frac{c(1-\tilde{\tau}^{1/d})(1-\tau^{1/d})}{(1-c+c^{3/2}\sqrt{(1-\tilde{\tau}^{1/d})(1-\tau^{1/d})})^2}\,. \tag{3.7}$$

Since $x \mapsto cx/(1-c+c^{3/2}\sqrt{x})^2$ is increasing, a further upper bound is obtained by maximizing $(1-\tilde{\tau}^{1/d})(1-\tau^{1/d})$. Since $\tilde{\tau}^{1/d}\tau^{1/d} = c$, $(1-\tilde{\tau}^{1/d})(1-\tau^{1/d})$ is maximized when $\tau^{1/d} = \sqrt{c}$, and we obtain

$$c^d \le \frac{c(1-\sqrt{c})^2}{(1-c+c^{3/2}(1-\sqrt{c}))^2}\,. \tag{3.8}$$

Writing $c = g^2$ and taking square roots, we have, dividing g out from both sides,

$$g^{d-1} \le \frac{(1-g)}{(1-g^2+g^3-g^4)} = \frac{1}{1+g+g^3}\,. \tag{3.9}$$

Now suppose $d \le (1+\lambda\tilde{\lambda})/(1-\lambda\tilde{\lambda}) = (1+g^2)/(1-g^2)$. Then, since $g \in (0,1)$,

$$g^{\frac{2g^2}{1-g^2}} \le g^{d-1} \le \frac{1}{1+g+g^3}\,. \tag{3.10}$$

so that $\log(1+g+g^3) + 2g^2/(1-g^2)\log g \le 0$. However, we have a contradiction with the following result which we prove in Appendix A:

Lemma 3. *For all* $x \in (0,1)$, $(1-x^2)\log(1+x+x^3) + 2x^2\log(x) > 0$.

Thus the lemma is proved. $\square$

4 The operator T

In this section we first of all discuss informally the relationship between $dR^2(f)$ and the operator T given in Section 1.

When analyzing asymmetric maps it is often convenient to work (as in [14]) with a map of pairs, rather than the doubling operator R. Let R_P denote the map $R_P(f,\tilde{f}) = (R(\tilde{f}), R(f))$. Then a fixed point of R_P, with $f \neq \tilde{f} = R(f)$, corresponds to a period-2 point of R and vice versa. The spectra of the derivatives $dR^2(f)$ and $dR_P(f,\tilde{f})$ are related: an eigenvalue ρ^2 of $dR^2(f)$ corresponds to a pair of eigenvalues $\pm\rho$ of $dR_P(f,\tilde{f})$. Indeed, if $\rho^2 \in \mathbb{C}$ is an eigenvalue of $dR^2(f)$ with eigenvector δf, then the pair $(\delta f, \pm\rho^{-1}\,\delta\tilde{f})$, where $\delta\tilde{f} = dR_f\delta f$, is eigenvector of $dR_P(f,\tilde{f})$ with eigenvalue $\pm\rho$, and vice versa. We may therefore study the spectrum of $dR_P(f,\tilde{f})$ in lieu of $dR^2(f)$.

As in [4], a further simplification can be made by studying the operator $\bar{R}_P$ given by R_P with the parameters λ and $\tilde{\lambda}$ held constant at their values at the fixed-point pair $(f,\tilde{f})$. This introduces eigenvalues ± 1 into the spectrum of $d\bar{R}_P(f,\tilde{f})$ but otherwise leaves the spectrum undisturbed. Acting on pairs of tangent functions $(\delta f(x), \delta\tilde{f}(x))$, the operator $d\bar{R}_P(f,\tilde{f})$ is given by:

$$d\bar{R}_P(f,\tilde{f})\begin{pmatrix}\delta f(x)\\ \delta\tilde{f}(x)\end{pmatrix} = \begin{pmatrix}-\tilde{\lambda}^{-1}\delta\tilde{f}(\tilde{f}(-\tilde{\lambda}x)) - \tilde{\lambda}^{-1}\tilde{f}'(\tilde{f}(-\tilde{\lambda}x))\delta\tilde{f}(-\tilde{\lambda}x)\\ -\lambda^{-1}\delta f(f(-\lambda x)) - \lambda^{-1}f'(f(-\lambda x))\delta f(-\lambda x)\end{pmatrix}. \tag{4.1}$$

Furthermore, it is convenient to build in the degree of criticality d by writing $f(x) = F(|x|^d)$, $\tilde{f}(x) = \tilde{F}(|x|^d)$ leading to an induced map $\bar{R}_P$ on pairs $(F,\tilde{F})$ and derivative $d\bar{R}_P(F,\tilde{F})$. Following [4], as a final simplification, we consider tangent vector pairs $(v,\tilde{v}) = (\delta F/F', \delta\tilde{F}/\tilde{F}')$. Following [4] we define a map from $\mathbb{R}$ to $\mathbb{R}$ given by

$$q(x) = \mathrm{sign}(x)|x|^d\,. \tag{4.2}$$

We then define L, $\tilde{L}$ by

$$L(x) = q(F(x)), \quad \tilde{L}(x) = q(\tilde{F}(x)), \quad x \in [0,1]\,, \tag{4.3}$$

and use also the notation

$$L(x) = \begin{cases} L_+(x) = F(x)^d\,, & x \in [0, z_1^d]\\ -L_-(x) = -|F(x)|^d\,, & x \in [z_1^d, 1]\,,\end{cases} \tag{4.4a}$$

$$\tilde{L}(x) = \begin{cases} \tilde{L}_+(x) = \tilde{F}(x)^d\,, & x \in [0, \tilde{z}_1^d]\\ -\tilde{L}_-(x) = -|\tilde{F}(x)|^d\,, & x \in [\tilde{z}_1^d, 1]\,.\end{cases} \tag{4.4b}$$

These functions satisfy the identities

$$L(x) = -\frac{1}{\tilde{\lambda}^d}\tilde{L}(\tilde{L}(\tilde{t}x)), \quad \tilde{L}(x) = -\frac{1}{\lambda^d}L(L(tx))\,, \tag{4.5}$$

or, equivalently,

$$L_+(x) = \frac{1}{\tilde{\lambda}^d}\tilde{L}_-(\tilde{L}_+(\tilde{t}x)), \quad \forall\, x \in [0, z_1^d]\,, \quad \tilde{L}_+(x) = \frac{1}{\lambda^d}L_-(L_+(tx)), \quad \forall\, x \in [0, \tilde{z}_1^d] \tag{4.6}$$

$$L_-(x) = \frac{1}{\tilde{\lambda}^d}\tilde{L}_+(\tilde{L}_+(\tilde{t}x)), \quad \forall\, x \in [z_1^d, 1]\,, \quad \tilde{L}_-(x) = \frac{1}{\lambda^d}L_+(L_+(tx)), \quad \forall\, x \in [\tilde{z}_1^d, 1]\,. \tag{4.7}$$

The linear operator induced on $(v,\tilde{v})$ by $d\bar{R}_P(F,\tilde{F})$ is the operator T described in the introduction:

$$T\begin{pmatrix}v(x)\\ \tilde{v}(x)\end{pmatrix} = \begin{pmatrix}v_1(x)\\ \tilde{v}_1(x)\end{pmatrix} = \begin{pmatrix}\tilde{t}^{-1}(\tilde{v}(\tilde{t}x) + \tilde{v}(\tilde{L}(\tilde{t}x))\tilde{L}'(\tilde{t}x)^{-1})\\ t^{-1}(v(tx) + v(L(tx))L'(tx)^{-1})\end{pmatrix}. \tag{4.8}$$

In view of Lemma 1, the functions $F(x)$, $v(x)$ are analytic in the domain $\Delta = U(\Omega)$ and $\tilde{F}(x)$, $\tilde{v}(x)$ are analytic in $\tilde{\Delta} = \tilde{U}(\tilde{\Omega})$.

We recall that $U(x)$, $\tilde{U}(x)$ satisfy the following functional equations

$$\tilde{t}U(x) = \tilde{U}(\tilde{u}(-\tilde{\lambda}\,x)), \quad \tilde{u}(x) = \tilde{U}(x)^{1/d}, \quad x \in \Omega\,, \tag{4.9a}$$

$$t\tilde{U}(x) = U(u(-\lambda\,x)), \quad u(x) = U(x)^{1/d}, \quad x \in \tilde{\Omega}\,. \tag{4.9b}$$

The following equations are a direct consequence of (4.9):

$$L(t\tilde{U}(x)) = U(-\lambda x), \quad x \in \tilde{\Omega}, \qquad \tilde{L}(\tilde{t}U(x)) = \tilde{U}(-\tilde{\lambda}x), \quad x \in \Omega, \tag{4.10}$$

which provide (by the injectivity of U and $\tilde{U}$) a holomorphic extension of the restriction $L|(0, z_1^d)$ (resp. $\tilde{L}|(0, \tilde{z}_1^d)$) to the complex domain $t\tilde{\Delta}$ (resp. $\tilde{t}\Delta$).

We now consider rigorously the properties of the operator T. Our first task is to show that T is well defined on function pairs $(v, \tilde{v})$ on suitable domains. We first of all show that the domains Δ and $\tilde{\Delta}$ map nicely.

Lemma 4. *The domains* Ω, $\tilde{\Omega}$, Δ, $\tilde{\Delta}$ *satisfy:*

1. $-\tilde{\lambda}\Omega \subset \tilde{\Omega}, \quad -\lambda\tilde{\Omega} \subset \Omega$.
2. $\tilde{t}\,\Delta \subset \tilde{\Delta}, \quad t\,\tilde{\Delta} \subset \Delta$.
3. $L(t\tilde{\Delta}) \subset \Delta, \quad \tilde{L}(\tilde{t}\Delta) \subset \tilde{\Delta}$.

Proof. Statement 1 follows directly from the definition of Ω, $\tilde{\Omega}$. We shall prove the first inclusion of statement 2; the proof of the second one is similar. Let $x \in \tilde{t}\Delta$, i.e., suppose there exists $y \in \Omega$ such that $x = \tilde{t}U(y)$. Then, according to the first of equations (4.9), we have $x = \tilde{t}\,U(y) = \tilde{U}(\tilde{u}(-\tilde{\lambda}\,y))$. However, $\tilde{u}(-\tilde{\lambda}y) \in \tilde{\Omega}$ and so $x \in \tilde{\Delta}$ by the definition of $\tilde{\Delta}$.

We now establish statement 3. We shall prove the first inclusion; the proof of the second one is analogous. Let $x \in t\tilde{\Delta}$ i.e. suppose there exists $y \in \tilde{\Omega}$ such that $x = t\,\tilde{U}(y)$. Then, the first of equations (4.10) gives us $L(x) = U(-\lambda\,y)$, and hence, by statement 1, $-\lambda\,y \in \Omega$ and $L(x) \in \Delta$. □

The domains Δ, $\tilde{\Delta}$ and $t\tilde{\Delta}$, $\tilde{t}\Delta$ are natural domains on which to define F, $\tilde{F}$ and L, $\tilde{L}$. However, to ensure that T is well defined and compact, we must obtain smaller domains on which T is bounded and analyticity improving. This we do in the next section.

5 Analyticity-improving domains

Let a, $b \in \mathbb{R}$, $a < b$ and $D(a, b)$ be an open disc in $\mathbb{C}$ with diameter (a, b). We introduce the domains $\Delta_1 = U(D(\alpha_1, \beta_1))$, $\Delta_0 = U(D(\alpha_0, \beta_0))$, $\tilde{\Delta}_1 = \tilde{U}(D(\tilde{\alpha}_1, \tilde{\beta}_1))$, $\tilde{\Delta}_0 = \tilde{U}(D(\tilde{\alpha}_0, \tilde{\beta}_0))$, where

$$\alpha_1 = -\tilde{\lambda}^{-1}, \quad \tilde{\alpha}_1 = -\lambda^{-1}, \quad \beta_0 = u(-\tilde{\lambda}^{-1}), \quad \tilde{\beta}_0 = \tilde{u}(-\lambda^{-1}), \tag{5.1}$$

$$\alpha_0 = -3\tilde{\lambda}^{-1}/4 - \lambda\tilde{u}(-\lambda^{-1})/4, \quad \tilde{\alpha}_0 = -3\lambda^{-1}/4 - \tilde{\lambda}u(-\tilde{\lambda}^{-1})/4, \tag{5.2}$$

$$\beta_1 = (\lambda\tilde{\lambda})^{-1}/2 + u(-\tilde{\lambda}^{-1})/2, \quad \tilde{\beta}_1 = (\lambda\tilde{\lambda})^{-1}/2 + \tilde{u}(-\lambda^{-1})/2. \tag{5.3}$$

The following inequalities will be important in what follows:

$$1 < u(-\tilde{\lambda}^{-1}) < (\lambda\tilde{\lambda})^{-1}, \quad 1 < \tilde{u}(-\lambda^{-1}) < (\lambda\tilde{\lambda})^{-1}. \tag{5.4}$$

We shall now prove (5.4). To show that $u(-\tilde{\lambda}^{-1}) > 1$ we use the fact that $u(x)$ is an anti-Herglotz function which is decreasing in $(-\tilde{\lambda}^{-1}, 1)$ and satisfies the condition $u(-\lambda) = 1$, $\lambda \leq \tilde{\lambda}^{-1}$. This gives $u(-\tilde{\lambda}^{-1}) = \lim_{x \to -\tilde{\lambda}^{-1}} u(x) > 1$. Next, $u(-\tilde{\lambda}^{-1}) = \tilde{z}_1/\tilde{t}^{1/d} = \tilde{z}_1\mu^{1/d}\tilde{\lambda}^{-1}$, so that $u(-\tilde{\lambda}^{-1}) < (\lambda\tilde{\lambda})^{-1}$ if and only if $\tilde{z}_1\mu^{1/d}\lambda < 1$ which follows from the inequalities $\tilde{z}_1 < 1$, $\mu^{1/d}\lambda < 1$. The other inequalities follow similarly.

From the inequalities (5.4), it straightforward to check the following:

$$-\tilde{\lambda}^{-1} = \alpha_1 < \alpha_0 < -\lambda < 0 < 1 < \beta_0 < \beta_1 < (\lambda\tilde{\lambda})^{-1} \tag{5.5a}$$

$$-\lambda^{-1} = \tilde{\alpha}_1 < \tilde{\alpha}_0 < -\tilde{\lambda} < 0 < 1 < \tilde{\beta}_0 < \tilde{\beta}_1 < (\lambda\tilde{\lambda})^{-1}, \tag{5.5b}$$

and from these it is easy to check that $\Delta_0 \Subset \Delta_1 \subset \Delta$ and $\tilde{\Delta}_0 \Subset \tilde{\Delta}_1 \subset \tilde{\Delta}$.

We have the following lemma concerning the domains Δ_0, Δ_1, $\tilde{\Delta}_1$, $\tilde{\Delta}_0$.

Lemma 5. *The domains* Δ_0, Δ_1, $\tilde{\Delta}_0$, $\tilde{\Delta}_1$ *satisfy:*

1. $\tilde{L}(\tilde{t}\,\Delta_1) \subset \tilde{\Delta}_0, \quad L(t\,\tilde{\Delta}_1) \subset \Delta_0$.
2. $\tilde{t}\,\Delta_1 \subset \tilde{\Delta}_0, \quad t\,\tilde{\Delta}_1 \subset \Delta_0$.
3. $[0, 1] \subset \Delta_0, \quad [0, 1] \subset \tilde{\Delta}_0$.

Proof. We shall prove the first inclusion of statement 1, the proof of the second is similar. Let $x = U(\zeta) \in \Delta_1$, $\zeta \in D(\alpha_1, \beta_1)$. By (4.10), we need to show that $\tilde{U}(-\tilde{\lambda}\,\zeta) \subset \tilde{\Delta}_0$. This is equivalent to $-\tilde{\lambda}D(\alpha_1, \beta_1) \subset D(\tilde{\alpha}_0, \tilde{\beta}_0)$, where we have used the property that if an anti-Herglotz function is holomorphic on a real segment (A, B) and maps it into the real segment (A', B'), then it maps $D(A, B)$ into $D(A', B')$ (see [6]). This in turn gives the inequalities $-\tilde{\lambda}\,\beta_1 \geq \tilde{\alpha}_0$ and $-\tilde{\lambda}\alpha_1 \leq \tilde{\beta}_0$ which are easy to check with help of the inequalities (5.4).

We shall now outline the proof of the first inclusion of statement 2. The proof of the second one is similar. Consider $x = U(\zeta) \in \Delta_1$ for some $\zeta \in D(\alpha_1, \beta_1)$. From the first of equations of (4.9) we have $\tilde{t}\,x = \tilde{t}\,U(\zeta) = \tilde{U}(\tilde{u}(-\tilde{\lambda}\,\zeta))$. We note that $\tilde{u}(x)$ is an anti-Herglotz function, analytic in the domain $\mathbb{C}_+ \cup \mathbb{C}_- \cup (-\lambda^{-1}, 1)$. It is sufficient to show that $\tilde{u}(-\tilde{\lambda}\zeta) \in D(\tilde{\alpha}_0, \tilde{\beta}_0)$. We have $-\tilde{\lambda}\zeta \in D(a, b)$, where $a = -\tilde{\lambda}\beta_1$, $b = -\tilde{\lambda}\alpha_1$ and $\tilde{u}(-\tilde{\lambda}\zeta) \in D(\tilde{u}(b), \tilde{u}(a))$. We thus verify that $D(\tilde{u}(b), \tilde{u}(a)) \subset D(\tilde{\alpha}_0, \tilde{\beta}_0)$. Finally, we prove statement 3. We have $\Delta_0 = U(D(\alpha_0, \beta_0))$. Thus, $\Delta_0 \subset D(U(\beta_0), U(\alpha_0))$. Using the property that $U(x)$ is decreasing in $(-\tilde{\lambda}^{-1}, (\lambda\tilde{\lambda})^{-1})$ and that $U(-\lambda) = 1$, $U(1) = 0$ we conclude that the condition $[0, 1] \subset \Delta_0$ is equivalent to the inequalities $\alpha_0 < -\lambda$ and $\beta_0 > 1$ which are given by (5.5). The proof of $[0, 1] \subset \tilde{\Delta}_0$ is similar. □

From Lemma 5 we have that if $(v, \tilde{v})$ are analytic on $\Delta_0 \times \tilde{\Delta}_0$ then $\tilde{v}(\tilde{t}x)$, $\tilde{v}(\tilde{L}(\tilde{t}x))$ are analytic on Δ_1 and $v(tx)$, $v(L(tx))$ are analytic on $\tilde{\Delta}_1$. Furthermore, differentiating the first equation of (4.10) gives $L'(t\tilde{U}(x)) \neq 0$ for all $x \in \tilde{\Omega}$, since U is univalent on $-\lambda\tilde{\Omega} \subset \Omega$. We deduce that $L'(tx) \neq 0$ for all $x \in \tilde{\Delta} = \tilde{U}(\tilde{\Omega})$. Similarly $\tilde{L}'(tx) \neq 0$ for all $x \in \Delta$.

Since $\Delta_1 \subset \Delta$ and $\tilde{\Delta}_1 \subset \tilde{\Delta}$ we conclude that if $(v, \tilde{v})$ are analytic on $\Delta_0 \times \tilde{\Delta}_0$ then then $T(v, \tilde{v})$ is defined and analytic on $\Delta_1 \times \tilde{\Delta}_1$.

We note that the derivative $L'(tx)$ (resp. $\tilde{L}'(\tilde{t}x)$) vanishes at $x = z_1^d/t \in \partial\tilde{\Delta}_1$ (resp. $x = \tilde{z}_1^d/\tilde{t} \in \partial\Delta_1$). But its reciprocal $1/L'(tx)$ (resp. $1/\tilde{L}'(\tilde{t}x)$) is bounded in any $\tilde{\Delta}' \Subset \tilde{\Delta}_1$ (resp. $\Delta' \Subset \Delta_1$). Hence $T(v, \tilde{v})$ is well defined and bounded on any domain $\Delta_1' \times \tilde{\Delta}_1'$ with $\Delta_1' \Subset \Delta_1$, $\tilde{\Delta}_1' \Subset \tilde{\Delta}_1$.

From this we immediately have the following lemma, which shows that T is analyticity-improving.

Lemma 6. *If $(v(x), \tilde{v}(x))$ is a pair of real functions on $[0, 1]$ which extend to a holomorphic functions on $\Delta_0 \times \tilde{\Delta}_0 \subset \mathbb{C}^2$ then the pair $(v_1(x), \tilde{v}_1(x))$, defined in (4.8), extend to holomorphic functions on $\Delta_1 \times \tilde{\Delta}_1$.*

We now define the Banach space in which we shall work.

Definition. Let B denote the Banach space of pairs of functions $(v(x), \tilde{v}(x))$ holomorphic and bounded on $\Delta_0 \times \tilde{\Delta}_0 \subset \mathbb{C}^2$ which are real on $[0, 1]$. We equip B with the norm

$$||(v, \tilde{v})|| = \max\left(\sup_{x\in\Delta_0} |v(x)|, \sup_{x\in\tilde{\Delta}_0} |\tilde{v}(x)|\right). \tag{5.6}$$

The results of this section enable us to conclude that T is compact. Indeed, we have the following, which is a direct consequence of Lemma 4 and Lemmas 6.

Corollary 1. *T is a compact operator on B and $TB \subset B$. Moreover, for every $\Delta' \Subset \Delta_1$, $\tilde{\Delta}' \Subset \tilde{\Delta}_1$ we have*

$$\sup_{x\in\Delta'} |v_1(x)| \leq \tilde{t}^{-1}\left(1 + \sup_{x\in\Delta'}\left|\frac{1}{\tilde{L}'(\tilde{t}x)}\right|\right) \sup_{y\in\tilde{\Delta}_0} |\tilde{v}(y)|, \tag{5.7a}$$

$$\sup_{x\in\tilde{\Delta}'} |\tilde{v}_1(x)| \leq t^{-1}\left(1 + \sup_{x\in\tilde{\Delta}'}\left|\frac{1}{L'(tx)}\right|\right) \sup_{y\in\Delta_0} |v(y)|. \tag{5.7b}$$

6 Properties of the functions $L(x)$ and $\tilde{L}(x)$

In this section, we prove several properties of the functions L and $\tilde{L}$; in particular we show that they are convex.

Lemma 7. *The function L_+ is convex on $[0, z_1^d]$, and the function L_- is convex on $[z_1^d, 1]$. The function $\tilde{L}_+$ is convex on $[0, \tilde{z}_1^d]$, and the function $\tilde{L}_-$ is convex on $[\tilde{z}_1^d, 1]$.*

Proof. In order to prove the lemma, we prove that $S_\pm = L_\pm^{-1}$ are convex. The proof is analogous to the proof of Lemma 3.2 from [4].

Let $S_\pm(\zeta) = U(\pm\zeta^{1/d})$, $\tilde{S}_\pm(\zeta) = \tilde{U}(\pm\zeta^{1/d})$. Then

$$-\frac{S_\pm''(\zeta)}{S_\pm'(\zeta)} = \frac{1}{d\zeta}\left(d - 1 - x\frac{U''(x)}{U'(x)}\right), \quad x = \pm\zeta^{1/d}. \tag{6.1}$$

Using Lemma 2 and the *a priori* bounds (2.9a) we have for $x = \zeta^{1/d} > 0$

$$-\frac{S''_+(\zeta)}{S'_+(\zeta)} > \frac{1}{d\zeta}\left(\frac{1+\lambda\tilde{\lambda}}{1-\lambda\tilde{\lambda}} - \frac{1+\lambda\tilde{\lambda}x}{1-\lambda\tilde{\lambda}x}\right). \tag{6.2}$$

This is positive for $x < 1$. For $x = -\zeta^{1/d} \leq 0$, we get:

$$-\frac{S''_-(\zeta)}{S'_-(\zeta)} > \frac{1}{d\zeta}\left(\frac{1+\lambda\tilde{\lambda}}{1-\lambda\tilde{\lambda}} - \frac{1-\tilde{\lambda}x}{1+\tilde{\lambda}x}\right). \tag{6.3}$$

This is positive for $-\lambda \leq x \leq 0$. This gives us the inequalities

$$-\frac{S''_+(\zeta)}{S'_+(\zeta)} > 0, \quad \zeta \in [0,1], \quad -\frac{S''_-(\zeta)}{S'_-(\zeta)} > 0, \quad \zeta \in [0,\lambda^d]. \tag{6.4}$$

The analogous inequalities hold for the functions $\tilde{S}_\pm(x)$. This completes the proof of Lemma 7. □

The next two lemmas give important estimates on L, $\tilde{L}$ and their derivatives.

Lemma 8. *For all $x \in [0,1]$, we have*

$$L_+(tx) > tx, \quad \tilde{L}_+(\tilde{t}x) > \tilde{t}x\,. \tag{6.5}$$

Proof. The proof is similar to that of Corollary 3.3 of [4]. By the monotonicity and convexity of L, $\tilde{L}$, it suffices to prove the lemma for $x = 1$. Using the equations (4.5) at $x = 0$ we find $L(1) = -\lambda^d$, $\tilde{L}(1) = -\tilde{\lambda}^d$. Reapplying them at $x = 1$ we obtain

$$L(L(t)) = (\lambda\tilde{\lambda})^d\,, \quad \tilde{L}(\tilde{L}(\tilde{t})) = (\lambda\tilde{\lambda})^d\,. \tag{6.6}$$

The inequalities $\lambda < z_1\mu^{-1/d}$, $\tilde{\lambda} < \tilde{z}_1\mu^{1/d}$, and z_1, $\tilde{z}_1 < 1$ imply

$$z_1^d > \frac{(\lambda\tilde{\lambda})^d}{\tilde{t}}\,, \quad \tilde{z}_1^d > \frac{(\lambda\tilde{\lambda})^d}{t}\,, \tag{6.7}$$

so that, in particular,

$$z_1^d > (\lambda\tilde{\lambda})^d\,, \quad \tilde{z}_1^d > (\lambda\tilde{\lambda})^d\,. \tag{6.8}$$

Let $\zeta = L(t)/t$. Then the inequality $L(t) > t$ is equivalent to $\zeta > 1$. The function $\tilde{L}(x)$ is decreasing for $x \in [0,1]$ and takes values in the interval $[-\tilde{\lambda}^d, 1]$. Suppose that $\zeta \leq 1$. Then from (6.6) and (4.5) we have

$$\tilde{L}(\zeta) = -\frac{1}{\lambda^d}L((\lambda\tilde{\lambda})^d) < -\frac{1}{\lambda^d}L(t\tilde{z}_1^d) = -\frac{z_1^d}{\lambda^d}\,, \tag{6.9}$$

where we have used (6.7). Thus, we must have $-z_1^d/\lambda^d \in [-\tilde{\lambda}^d, 1]$ i.e. $z_1^d < (\lambda\tilde{\lambda})^d$. This contradicts (6.8), so $\zeta > 1$ and $L(t) > t$. The inequality $\tilde{L}(\tilde{t}) > \tilde{t}$ can be shown in a similar manner. □

Lemma 9. *For all $x \in [0,1]$, we have*

$$L'(tx) < -1, \quad \tilde{L}'(\tilde{t}x) < -1\,. \tag{6.10}$$

Proof. Using the convexity of $L(x)$ and $\tilde{L}'(x)$ it is sufficient to prove the above inequalities for $x = 1$ only. We will show that $L'(t) < -1$. Let $y = L(t)$, then $L(y) = (\lambda\tilde{\lambda})^d$ as follows from (6.6). According to the previous lemma $t < y$. By the mean value theorem there exists $t_0 \in [t, y]$ such that $L'(t_0) = (L(t) - L(y))/(t - y) = (y - (\lambda\tilde{\lambda})^d)/(t - y)$. The inequality $|L'(t_0)| > 1$ can be easily established then with help of $t > (\lambda\tilde{\lambda})^d$ which in its turn follows from $\mu^{-1}\tilde{\lambda} < 1$. Since $t_0 > t$, then, from the convexity of $L(x)$, we see that $L'(t) < -1$. The same proof holds for $\tilde{L}'(x)$. □

7 Invariant cone for T

Generalizing the result obtained in [4], now we can derive the existence of an invariant cone for the operator T in the space of functions $(v(x), \tilde{v}(x))$. We shall then be able to apply the Krein-Rutman theorem.

Definition. Define Γ_1 to be the set of pairs $(v(x), \tilde{v}(x))$ of real smooth functions on $[0,1]$ which, for $x \in [0,1]$, satisfy (i) $v(x) \geq 0$, $\tilde{v}(x) \geq 0$, and (ii) $v'(x) \leq 0$, $\tilde{v}'(x) \leq 0$.

The following lemma is a generalization of Lemma 3.4 of [4].

Lemma 10. *Let* $\Gamma = B \cap \Gamma_1$. *Then* T *maps* Γ_1 *into itself and* T^2 *maps any non-zero vector in* Γ *into the interior of* Γ.

Proof. We have $T(v,\tilde{v}) = (v_1,\tilde{v}_1)$. From Lemma 8 we have

$$\tilde{t}v_1(x) \geq \tilde{v}(\tilde{t}x)(1+1/\tilde{L}'(\tilde{t}x)), \quad t\tilde{v}_1(x) \geq v(tx)(1+1/L'(tx))\,. \tag{7.1}$$

Both of these expressions are non-negative since $L'_+(tx) < -1$, $\tilde{L}'_+(\tilde{t}x) < -1$ according to Lemma 9.
Differentiation of (4.8) gives

$$v_1'(x) = \tilde{v}'(\tilde{t}x) + \tilde{v}'(\tilde{L}(\tilde{t}x)) - \tilde{v}(\tilde{L}(\tilde{t}x))\tilde{L}''(\tilde{t}x)\tilde{L}'(\tilde{t}x)^{-2}\,, \tag{7.2a}$$

$$\tilde{v}_1'(x) = v'(tx) + v'(L(tx)) - v(L(tx))L''(tx)L'(tx)^{-2}\,. \tag{7.2b}$$

These two expressions are non-positive since each one is a sum of three non-positive terms. Thus we have proved that $(v_1,\tilde{v}_1) \in \Gamma_1$. Repeating the arguments from the proof of Lemma 3.4 from the paper [4], we notice that the interior of Γ is composed of $(v(x),\tilde{v}(x))$ for which the inequalities defining Γ are all strict. Suppose that $(v(x),\tilde{v}(x)) \not\equiv (0,0)$. If $v(x)$ (resp. $\tilde{v}(x)$) vanished for some $x \in [0,1)$, it would have to vanish on $[x,1]$, hence everywhere by analyticity, i.e. 1 is the only place in $[0,1]$ where $v(x)$ (resp. $\tilde{v}(x)$) can vanish. But according to (7.1) $v_1(x)$ and $\tilde{v}_1(x)$ cannot vanish even at 1. Considering now (7.2), we observe that their last terms cannot vanish in $(0,1]$, and can vanish at 0 only if $v(1) = 0$ or $\tilde{v}(1) = 0$. This proves that $T^2(v,\tilde{v})$ is in the interior of Γ. □

From the theorem of Krein and Rutman [10] we thus have the following result.

Theorem 2. *The operator* T*, acting on* B*, has an eigenvalue of largest modulus* $\delta > 0$. *The spectral subspace corresponding to* δ *is one-dimensional and is generated by an element from the interior of* Γ *which is the only eigenvector of* T *in* Γ.

In the next section we give some bounds on this eigenvalue δ.

8 Bounds on the expanding eigenvalue

Let $(v,\tilde{v})$ be an eigenvector with eigenvalue δ in the cone $v \geq 0$, $\tilde{v} \geq 0$, $v' \leq 0$, $\tilde{v}' \leq 0$. We have further that $v(0)$, $\tilde{v}(0) > 0$, since $(v,\tilde{v})$ is in the interior of the cone.
The eigenvector equations are

$$\delta\tilde{v}(x) = t^{-1}\left(v(tx) + \frac{v(L(tx))}{L'(tx)}\right)\,, \qquad \delta v(x) = \tilde{t}^{-1}\left(\tilde{v}(\tilde{t}x) + \frac{\tilde{v}(\tilde{L}(\tilde{t}x))}{\tilde{L}'(\tilde{t}x)}\right)\,. \tag{8.1}$$

Evaluating these at 0 we obtain

$$\delta\tilde{v}(0) = t^{-1}\left(v(0) + \frac{v(1)}{L'(0)}\right)\,, \qquad \delta v(x) = \tilde{t}^{-1}\left(\tilde{v}(0) + \frac{\tilde{v}(1)}{\tilde{L}'(0)}\right)\,. \tag{8.2}$$

Now we have $L'(0)$, $\tilde{L}'(0) < -1$ and $v(1)$, $\tilde{v}(1) > 0$ so that, neglecting the second term on the right hand sides of these equations, and multiplying, we immediately obtain the bound $\delta^2 v(0)\tilde{v}(0) < (t\tilde{t})^{-1}v(0)\tilde{v}(0)$ so that $\delta^2 < (t\tilde{t})^{-1} = (\lambda\tilde{\lambda})^{-d}$, which is the upper bound in Theorem 1.
To obtain the lower bound, we use the convexity of L and $\tilde{L}$. Since v', $\tilde{v}' \leq 0$, we have that $v(1) \leq v(0)$ and $\tilde{v}(1) \leq \tilde{v}(0)$ so that, multiplying the eigenvector equations (8.2), we have

$$\delta^2 v(0)\tilde{v}(0) \geq (t\tilde{t})^{-1}v(0)\tilde{v}(0)\left(1+\frac{1}{L'(0)}\right)\left(1+\frac{1}{\tilde{L}'(0)}\right)\,. \tag{8.3}$$

From the convexity of L and $\tilde{L}$ we have $L'(0) < -1/z_1^d < -1$, and $\tilde{L}'(0) < -1/\tilde{z}_1^d < -1$ so that $1 - z_1^d < 1 + 1/L'(0)$ and $1 - \tilde{z}_1^d < 1 + 1/\tilde{L}'(0)$, and, hence,

$$\delta^2 > \frac{1}{t\tilde{t}}(1-z_1^d)(1-\tilde{z}_1^d)\,. \tag{8.4}$$

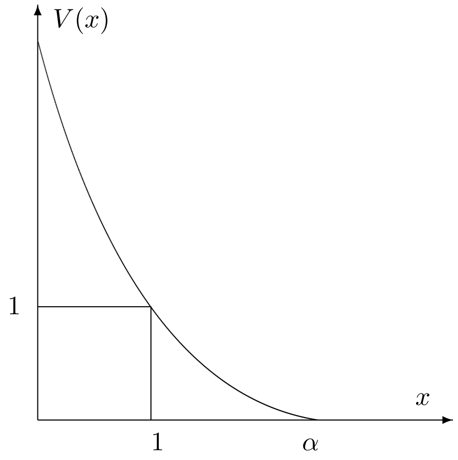

Figure 1: The convexity of V gives the lower bound $-1/V'(1)$ for $(\alpha - 1)$.

Recall that we have $V'(1)\tilde{V}'(1) = (\lambda\tilde{\lambda})^{-1}$. Then both V and $\tilde{V}$ are convex since they are scaled versions of S_+ and $\tilde{S}_+$ respectively. We also have $V(1) = 1$, $V(\alpha) = 0$, where $\alpha = z_1^{-d} > 1$. The graph of V is sketched in Figure 1.

From the convexity of V we have $V'(1) \leq -1/(\alpha - 1)$ so that $\alpha - 1 \geq -1/V'(1)$. Similarly, we have $\tilde{\alpha} - 1 \geq -1/\tilde{V}'(1)$, and thus $(\alpha - 1)(\tilde{\alpha} - 1) \geq \lambda\tilde{\lambda}$. Now if $x, y > 1$ and we have $(x - 1)(y - 1) \geq C > 0$, then a straightforward application of Lagrange multipliers shows that $(1 - x^{-1})(1 - y^{-1}) \geq C/(1 + \sqrt{C})^2$. We conclude that

$$\delta^2 \geq \frac{1}{t\tilde{t}}(1 - z_1^d)(1 - \tilde{z}_1^d) = \frac{1}{t\tilde{t}}(1 - \alpha^{-1})(1 - \tilde{\alpha}^{-1}) \geq \frac{1}{(\lambda\tilde{\lambda})^{(d-1)}(1 + \sqrt{\lambda\tilde{\lambda}})^2}\,. \tag{8.5}$$

Now, recall that for $g = \sqrt{\lambda\tilde{\lambda}}$ we have from (3.9) that $g^{(d-1)} \leq (1 + g + g^3)^{-1} < (1 + g)^{-1}$. It follows that $(\lambda\tilde{\lambda})^{(d-1)}(1 + \sqrt{\lambda\tilde{\lambda}})^2 = g^{2(d-1)}(1 + g)^2 < 1$ and thus

$$1 < \frac{1}{(\lambda\tilde{\lambda})^{(d-1)}(1 + \sqrt{\lambda\tilde{\lambda}})^2} < \delta^2 < \frac{1}{(\lambda\tilde{\lambda})^d}\,, \tag{8.6}$$

so, in particular, $\delta > 1$. This completes the proof of Theorem 1.

Acknowledgement

This research is supported by The Leverhulme Trust (grant number F/00144/W).

References

[1] Acton, F.S.: *Numerical Methods that Work. 2nd printing.* Math. Assoc. Amer., Washington, DC, 1990

[2] Arneodo, A., Coullet, P., Tresser, C.: *A renormalization group with periodic behaviour.* Phys. Lett. A **70**, 74–76 (1979)

[3] Coullet, P., Tresser, C.: *Itération d'endomorphismes et groupe de renormalisation.* J. de Physique C **5**, 25–28 (1978)

[4] Eckmann, J.-P., Epstein, H.: *Bounds on the unstable eigenvalue for period doubling.* Commun. Math. Phys. **128**, 427–435 (1990)

[5] Epstein, H.: *New proofs of the existence of the Feigenbaum functions.* Commun. Math. Phys. **106**, 395–426 (1986)

[6] Epstein, H.: *Fixed points of composition operators.* In: *Nonlinear Evolution and Chaotic Phenomena.* Gallavotti, G., and Zweifel P. (eds.) Plenum Press, New York 1988

[7] Epstein, H.: *Existence and properties of p-tupling fixed points.* Commun. Math. Phys. **215**, 443–476 (2000)

[8] Feigenbaum, M. J.: *Quantitative universality for a class of nonlinear transformations.* J. Stat. Phys. **19**, 25–52 (1978)

[9] Jensen, R. V., Ma, L. K. H.: *Nonuniversal behavior of asymmetric unimodal maps.* Phys. Rev. A **31**, 3993–3995 (1985)

[10] Krein, M. G., Rutman, M. A.: *Linear operators leaving invariant a cone in a Banach space.* Usp. Mat. Nauk **3**, 1, 3–95 (1948); English Translation: Functional analysis and measure theory. Am. Math. Soc., Providence 1962

[11] Lanford, O. E.: *A computer-assisted proof of the Feigenbaum conjectures.* Bull. Am. Math. Soc. **6**, 427–434 (1982)

[12] McMullen, C. T.: *Renormalization and 3-Manifolds which Fiber over the Circle*, Annals of Mathematical Studies, vol. 142, Princeton University Press, Princeton 1996

[13] Mestel, B. D., Osbaldestin, A. H.: *Feigenbaum theory for unimodal maps with asymmetric critical point.* J. Phys. A **31**, 3287–3296 (1998)

[14] Mestel, B. D., Osbaldestin, A. H.: *Feigenbaum theory for unimodal maps with asymmetric critical point: rigorous results.* Commun. Math. Phys. **197**, 211–228 (1998)

[15] Mestel, B. D., Osbaldestin, A. H.: *Asymptotics of scaling parameters for period-doubling in unimodal maps with asymmetric critical points.* J. Math. Phys. **41**, 4732–4746 (2000)

[16] Mestel, B. D., Osbaldestin, A. H., Tsygvintsev, A.V.: *Continued fractions and solutions of the Feigenbaum-Cvitanovic equation.* C.R. Acad. Sci. Paris, **334**, Série I, 683–688 (2002)

[17] Sullivan, D.: *Bounds, quadratic differentials and renormalization conjectures.* In: Mathematics into the Twenty-first Century. Browder, F. E. (ed.) Amer. Math. Soc. 1992

[18] Wall, H. S.: *Analytic Theory of Continued Fractions.* Van Nostrand, New York 1948

A The proof of Lemma 3

In this Appendix we give the proof of the inequality

$$(1-x^2)\log(1+x+x^3)+2x^2\log(x)>0\,,\quad x\in(0,1)\,. \tag{A.1}$$

Dividing both sides of (A.1) by $2x^2$ we obtain the equivalent inequality

$$F(x)=\frac{1}{2}\left(\frac{1}{x^2}-1\right)\log(1+x+x^3)+\log(x)>0,\quad x\in(0,1)\,. \tag{A.2}$$

Differentiation gives

$$F'(x)=-\frac{1}{x^3}\log(1+x+x^3)+\frac{(1+2x+4x^2-x^4)}{2x^2(1+x+x^3)}\,. \tag{A.3}$$

Since $F(1)=0$ in order to prove (A.2) it is sufficient to show that $F'(x)<0$ for $x\in(0,1)$. Firstly we shall obtain a lower rational bound on the function $\log(1+x)$.

We have the following integral formula

$$g(x)=\frac{\log(1+x)}{x}=\int_0^1\frac{dt}{1+xt}, \tag{A.4}$$

which implies (see [18], p. 279), that $g(x)$ can be expressed as a *g-fraction*

$$g(x)=\cfrac{1}{1+\cfrac{g_1x}{1+\cfrac{(1-g_1)g_2x}{1+\cfrac{(1-g_2)g_3x}{1+\cdots}}}}\,, \tag{A.5}$$

with coefficients $g_i\in[0,1]$ defined uniquely by derivatives of $g(x)$ calculated at $x=0$. These may be calculated with help of the so called Stieltjes formulas. (See [18] p. 203.) We have $g_1=1/2$, $g_2=1/3$,

$g_3 = 1/2$, $g_4 = 2/5$. According to the *a priori* bounds for $g(x)$ obtained in [16] (Theorem 2.2, case $k = 2$) we have

$$g(x) = \frac{\log(1+x)}{x} \geq \frac{(6+5x)}{2(x+3)(1+x)}\,, \quad x \geq 0\,, \tag{A.6}$$

so that, multiplying both sides by x and replacing x by $x + x^3$, we obtain

$$\log(1+x+x^3) \geq \frac{x(1+x^2)(6+5x+5x^3)}{2(x+x^3+3)\,(1+x+x^3)}\,, \quad x \geq 0\,. \tag{A.7}$$

Substituting the last inequality in (A.3) we obtain

$$F'(x) < R(x) = \frac{-(3-2x-8x^2+5x^3+x^4+2x^5+x^7)}{2x^2(x+x^3+3)(1+x+x^3)}\,. \tag{A.8}$$

Obviously, in order to prove $F'(x) < 0$ it is sufficient to prove that $R(x) < 0$ for $x \in (0,1)$. But this is equivalent to the polynomial inequality

$$P(x) = 3 - 2\,x - 8\,x^2 + 5\,x^3 + x^4 + 2\,x^5 + x^7 > 0, \quad x \in (0,1)\,. \tag{A.9}$$

Since $P(0) > 0$ we can establish (A.9) by showing that $P(x)$ does not have any roots in the interval $(0,1)$. Below we calculate its Sturm sequence [1]

$$s_1(x) = -\frac{21}{4} + 3x + 10x^2 - 5x^3 - \frac{3}{4}x^4 - x^5\,, \tag{A.10}$$

$$s_2(x) = -\frac{409}{337} + \frac{1096}{337}x + \frac{264}{337}x^2 - \frac{1604}{337}x^3 + x^4\,, \tag{A.11}$$

$$s_3(x) = \frac{12104}{30867} - \frac{19981}{30867}x - \frac{35629}{61734}x^2 + x^3\,, \tag{A.12}$$

$$s_4(x) = -\frac{4822700}{11118563} - \frac{1726274}{11118563}x + x^2\,, \tag{A.13}$$

$$s_5(x) = -\frac{418811267}{558846882} + x\,, \tag{A.14}$$

$$s_6(x) = -1\,. \tag{A.15}$$

Counting the number of sign changes between the Sturm functions $s_i(x)$ evaluated at two points $x = 0$ and $x = 1$ we conclude that $P(x)$ has no roots in the interval $(0,1)$.