Contribution of Individual Judgments towards Inconsistency in Pairwise Comparisons

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Abstract

Pairwise comparison (PC) is a well-established method to assist decision makers in estimating their preferences. In PCs, the acquired judgments are used to construct a PC matrix (PCM) that is used to check whether the inconsistency in judgments is *acceptable* or requires revision. The use of Consistency Ratio (CR) - a widely used measure for inconsistency - has been widely debated and the literature survey has identified a need for a more appropriate measure. Considering this need, a new measure, termed *congruence*, is proposed in this paper. The measure is shown to be useful in finding the contribution of individual judgments towards overall inconsistency of a PCM and, therefore, can be used to detect and correct cardinally inconsistent judgments. The proposed measure is applicable to incomplete sets of PC judgments without modification, unlike CR which requires a complete set of PC judgments. To address ordinal inconsistency, another measure termed *dissonance*, is proposed as a supplement to the *congruence* measure. The two measures appear useful in detecting both *outliers* and the phenomenon of *consistency deadlock* where all judgments equally contribute towards the overall inconsistency.

Keywords: Decision support systems; Multi-criteria decision making; Pairwise comparison; Intuitionistic preference modeling; Analytic hierarchy process.

1. Introduction

The method of pairwise comparison (PC), introduced by (Thurstone, 1927), has been used in several decision making methods to assess the relative importance of *criteria* and *alternatives*, see (Saaty, 2008; Siskos et al., 2005; Figueira et al., 2004; Greco et al., 2010) for examples. The main benefit of using this

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approach is to convert both objective and subjective judgments into relative weights of importance. This paper focuses on the method of pairwise comparisons that is often used to help DM in describing his/her preferences. A new technique is proposed here to measure the level of inconsistency in a set of pairwise judgments.

The pairwise judgments can be used to construct a matrix and a prioritization procedure can be applied to derive a corresponding *priority vector*. If the judgments are consistent then all prioritization methods give the same result. However, in the case of inconsistent judgments, different prioritization methods derive different priority vectors (Choo and Wedley, 2004). It is therefore important to measure consistency of judgments prior to the process of prioritization.

Although widely used, current measures for consistency have been criticized for their shortcomings (see details in Section 2.2). We contend that the redundancy in PC judgments can be used to measure the contribution of each judgment towards the overall inconsistency. Based on this, two new measures for consistency are proposed here and are shown to be useful in measuring overall inconsistency in PC judgments. Moreover, these measures can also be used to suggest changes in the judgment values in order to improve overall consistency.

The rest of the paper is structured as follows: Section 2 formulates the problem of prioritization and reviews the literature related to consistency in PC judgments; Section 3 collects an empirical evidence that existing measures are insufficient; the new measures are then proposed in Sections 4 and 5 respectively. Section 6 discusses the use of these measures for improving the overall consistency in PC judgments and then illustrated through examples in Section 7; finally, Section 8 concludes the paper.

2. Background

Consider a prioritization of n elements $E_1, E_2, \dots, E_n$. In the PC method, DMs assess the relative importance of any two elements E_i and E_j by providing a ratio judgment a_{ij} , specifying by how much E_i is preferred to E_j . If the element E_i is preferred to E_j then $a_{ij} > 1$, if the elements are equally preferred then $a_{ij} = 1$ and if E_j is preferred to E_i then $a_{ij} < 1$. These judgments can be used to construct a PC matrix (PCM), $A = [a_{ij}]$, of the order $n \times n$. This includes all self-comparison ($a_{ii} = 1$) and reciprocal judgments ($a_{ji} = \frac{1}{a_{ij}}$). The prioritization problem is to determine a priority vector $w = (w_1, w_2, \dots, w_n)^T$ from the given set of judgments, a_{ij} . There exist many prioritization methods that can be applied to derive a priority vector from a set of PC judgments (Choo and Wedley, 2004).

The relationships among elements can also be depicted by means of a directed graph (digraph), $G =$

(E, J) , where E is the set of nodes representing n elements $E_1, E_2, \dots, E_n$, and J represents the set of all ratio judgments $\{a_{ij}\}$ as weighted edges (Ali et al., 1986; Harker and Vargas, 1987). When a complete set of judgments is provided, the digraph G becomes fully connected.

2.1. Judgment Scale

The cardinal information, a_{ij} , is provided with respect to some predetermined preference scale. In the case of *tangible* criteria, this can be derived from the directly measured information as, for example, weights (in kgs) or price (in euros). In the case of *intangibles*, Saaty (1977) proposed the use of a set of verbal judgments that correspond to the ratio-scale of 1 to 9. Pairwise judgments can be provided using an additive-scale, however, the use of ratio-scale is considered to be more appropriate for measuring the relative intensities (Harker and Vargas, 1987; Krantz, 1972; Stevens, 1961). Despite its criticism and shortcomings (Dyer, 1990; Belton and Gear, 1983), the 1 to 9 scale has been widely used in various applications (Vaidya and Agarwal, 2006; Vargas, 1990).

Harker and Vargas (1987) compared Saaty's scale with other proposed scales and considered it to be more appropriate than others. In their comparison, the proposed scales included two linear scales, 1-5 and 1-15, and two non-linear ones, x^2 and $\sqrt{x}$. Their claim was, however, supported by only one example, leaving the argument largely unresolved. Lootsma (1993) preferred the use of a geometric scale over the linear one, while Salo and Hamalainen (1997) proposed a balanced scale claiming that the weights generated by their proposed scales are more evenly dispersed as compared to Saaty's scale. Finan and Hurley (1999) proposed to re-calibrate the verbal scale, transforming it into a geometric scale and claimed to achieve better performance compared with linear scales. Saaty (1994) also proposed a finer-grained version of the 1 to 9 scale, where he suggested the use of 1.1 to 1.9 for comparing stimuli with *close resemblance*. Ishizaka et al. (2011) summarized a comprehensive list of transformations applied to Saaty's 1-9 scale with a conclusion that there exists no single scale appropriate for all situations.

A complete set of judgments in the PC method creates an opportunity to have inconsistent information, primarily due to the redundancy inherent in its structure. One may argue to try to avoid redundancies present in data, however, redundancy may prove valuable in validating information acquired from imperfect human minds.

2.2. Inconsistencies in Judgments

There are several causes of inconsistency including psychological reasons, clerical errors and an insufficient model structure (Sugden, 1985). The psychological reasons further break down into factors such as

incomplete information, uncertainty and/or lack of concentration during the judgment process.

The judgment scale of 1 to 9 (Saaty, 1980) is an appropriate example of an insufficient model structure. Consider that if A is 5 times better than B and B is 3 times better than C, then the consistent value of 15 for comparing A to C is unavailable when using the 1 to 9 scale. Also, replacing these values with verbal statements will not change the underlying model/scale, and therefore, using the linear scale has this limitation in its structure.

There are situations where inconsistencies are unavoidable e.g. a tournament ranking problem including results where A beats B and B beats C, but then A loses to C (*intransitivity*). In this situation, ranking is required without amending any data, as teams may be unable to play again. Sugden (1985) suggested such inconsistencies be considered as rational.

Another possible cause of inconsistency is incorrect data entry e.g. entering a reciprocal value into a PCM. This happens when a ratio for E_i to E_j is mistakenly provided at a transpose location in a PCM i.e. a_{ji} instead of a_{ij} . Lipovetsky and Conklin (2002) termed such errors *Unusual and False Observations* (UFO).

Consistency in PCs is generally of two types i.e. *cardinal consistency* (CC) and *ordinal consistency* (OC). The judgments of DMs are cardinally consistent, if $a_{ij} = \frac{1}{a_{ji}}$ and $a_{ij} = a_{ik}a_{kj}$ for all i, j and k . OC states that if E_i is preferred to E_j and E_j is preferred to E_k , then E_i should be preferred to E_k i.e. If $E_i \rightarrow E_j \rightarrow E_k$ then $E_i \rightarrow E_k$.

The preference judgments are ordinally inconsistent (or *intransitive*) if $E_k \rightarrow E_i$ when $E_i \rightarrow E_j \rightarrow E_k$. Therefore, ordinal inconsistency can be defined as $E_i \rightarrow E_j \rightarrow E_k \rightarrow E_i$, which represents a *circular triad* of preferences (Kendall and Smith, 1940).

When two comparison elements E_i and E_j are equally preferred, $E_i \sim E_j$ (*preference equivalence* or a *tie*), then OC requires one of the following three conditions to be satisfied: $E_i \rightarrow E_k$ and $E_j \rightarrow E_k$, or $E_i \leftarrow E_k$ and $E_j \leftarrow E_k$, or $E_i \sim E_k$ and $E_j \sim E_k$. For example, if $a_{ij} = 1$ and $a_{ki} > 1$, then OC demands that $a_{kj} > 1$. This implies that a judgment $a_{kj} < 1$ will introduce ordinal inconsistency. The ties present in PCMs can be shown as undirected edges between corresponding elements (Jensen and Hicks, 1993).

2.2.1. Eigenvalue-based Measures

Using the fact that for a consistent positive reciprocal matrix, the largest Eigenvalue λ_{max} is equal to n , Saaty defined a measure of consistency, termed a Consistency Index (CI):-

$$CI = \frac{\lambda_{max} - n}{n - 1} \quad (1)$$

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n	(Saaty, 1980)	(Aguaron and Moreno-Jimenez, 2003)
3	0.58	0.5245
4	0.90	0.8815
5	1.12	1.1086
6	1.24	1.2479
7	1.32	1.3417
8	1.41	1.4056
9	1.45	1.4499

Table 1: Random Consistency Indices

where perfect consistency implies that $CI = 0$ and $CI > 0$ for inconsistent matrices.

To define a unique consistency measure which does not depend on the dimension of the PCM, Saaty (1980) further introduced the Consistency Ratio (CR) criterion. This is the ratio between the Consistency Index and a Random Consistency Index (RI):-

$$CR = \frac{CI}{RI} \quad (2)$$

where RI represents the average CI of a randomly generated PCM of the same dimension, n . The values of RI are statistically calculated from thousands (or millions) of randomly generated PCMs.

Table 1 shows the two sets of CR values given in (Saaty, 1980) and (Aguaron and Moreno-Jimenez, 2003). Aguaron and Moreno-Jimenez (2003) calculated the values of CR by increasing the number of iterations for their experiments and, therefore, claiming higher precision.

Saaty (1980) claimed that if the value of CR is smaller than or equal to 0.1, the estimated priority vector w can adequately approximate the unknown preference vector r . Therefore, the PCM is considered to be *acceptable* when the threshold of $CR \leq 0.1$ is met. However, if $CR > 0.1$, the estimated priorities could be erroneous and, therefore, DMs should be asked to improve the consistency by revising their subjective judgments. This threshold is used in various applications to accept or reject a PCM (Vaidya and Agarwal, 2006).

Although widely used, CR has been much debated for its two major drawbacks i.e. sensitivity to scale (Murphy, 1993; Stein and Mizzi, 2007) and the threshold value of $CR \leq 0.1$ (Bozóki et al., 2013; Bozóki and Rapcsak, 2008; Monsuur, 1997). Consequently, a number of alternative methods have been suggested to address these drawbacks. Some of these methods are discussed below.

2.2.2. Distance-based Measure

Crawford (1987) proposed the logarithmic residual mean square as a natural measure of consistency

whilst proposing the geometric mean (GM) method. Aguaron and Moreno-Jimenez (2003) formalized the logarithmic residual mean square, termed the Geometric Consistency Index (GCI), as:-

$$GCI = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j>i}^n \left(\log(a_{ij}) - \log\left(\frac{w_i}{w_j}\right) \right)^2 \quad (3)$$

where w is a priority vector estimated using the GM method. They compared GCI to CR, and proposed threshold values for GCI equivalent to CR .

2.2.3. Measuring the Worst Error

Koczkodaj (1993) described CR as a global measure that lacks a proper justification for its use. He proposed a measure, termed *Consistency Measure* (CM), in part to eliminate the drawbacks of CR, such as the inability to identify inconsistent judgments and its arbitrary threshold value of $CR \leq 0.1$.

CM is calculated by considering a set of three judgments (*triple*) from J at a time. The value of CM is chosen from the most inconsistent *triple* amongst all possible combinations i.e.

$$CM = \max_{i \neq j \neq k \neq i} (\overline{CM}_{i,j,k}) \quad (4)$$

where $\overline{CM}_{i,j,k}$ is the inconsistency for the triple $\{a_{ij}, a_{jk}, a_{ik}\}$ and is calculated using the following formula:-

$$\overline{CM}_{i,j,k} = \min \left(\frac{|a_{ij} - a_{ik}a_{kj}|}{a_{ij}}, \frac{|a_{ij} - a_{ik}a_{kj}|}{a_{ik}a_{kj}} \right) \quad (5)$$

CM has been shown to be useful for improving consistency by revising judgments (Koczkodaj and Szarek, 2010). However, similarly to the CR index, this measure cannot capture the ordinal inconsistency of the comparison judgments.

2.2.4. Kendall's ζ

In order to measure ordinal inconsistency, Kendall (1955) introduced an ordinal Coefficient of Consistence, ζ , for comparisons between elements with no preference equivalencies. Kendall's ζ is calculated using the following equation:-

$$\zeta = 1 - \frac{L}{L_{max}} \quad (6)$$

where L is the number of *circular triads* present in given PCM, and L_{max} is the maximum number of circular triads possible. The value of L_{max} is $(n^3-n)/24$ for odd values of n , and $(n^3-4n)/24$ when n is even, formulated

in (Kendall, 1955).

Gass (1998) formulated the prioritization problem for PCs as a tournament ranking problem and calculated the total number of cyclic judgments as:-

$$L = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2} \sum_i s_i^2 \quad (7)$$

where s_i is the total number of wins claimed by E_i (also termed the *out-degree*). It should be noted that this definition of L does not support *preference equivalence*.

In order to include *preference equivalence*, Jensen and Hicks (1993) extended Kendall's work and proposed the generalized coefficient of consistence. They suggested it be used as a supplement to CR, in the presence of intransitive judgments. The number of circular triads considering *preference equivalence* is always greater than L for obvious reasons.

2.3. Critique

From their definitions, it can be seen that the CR and CM measures of consistency do not take into account the ordinal inconsistency (or intransitivity) of the PCM. This is also true for GCI, which has a linear relationship with CR (Aguaron and Moreno-Jimenez, 2003). Bozóki and Rapcsák (2008) compared CR and CM with a conclusion that no single measure is adequate in all situations. They also questioned the CR work for using randomly generated PCMs as the reference point and, hence, considered this reference to be inappropriate for analyzing human-based decisions.

Generally, if the comparison matrices are ordinally consistent, most prioritization methods derive priorities with the same ranking, only with different intensities. If, however, the matrices are ordinally inconsistent (*intransitive*), there exists no priority vector satisfying all contradictory preferences. Therefore, different prioritization methods provide different ordinal rankings that partially correspond to the ordinal comparison judgments.

3. Judgment Scales and Inconsistency

Experiments have been performed to investigate how well inconsistency in a PCM is measured when different scales of measurement are used. The authors have used Monte-Carlo simulation for this purpose. Monte-Carlo experiments are based on repeated random sampling to compute results that are usually not achieved through a deterministic algorithm. Such methods are useful to model phenomena with significant

Linear	1	2	3	4	5	6	7	8	9
Balanced	1	1.111	1.125	1.143	1.167	1.20	1.25	1.333	1.5
Geometric	1	2	4	8	16	32	64	128	256
Inverse	1	1.125	1.286	1.5	1.8	2.25	3.0	4.5	9.0
Logarithmic	1	1.585	2	2.322	2.585	2.807	3	3.17	3.322
Power	1	4	9	16	25	36	49	64	81
Root	1	1.414	1.732	2	2.236	2.449	2.646	2.828	3

* *Reciprocals for these values are not shown here*

Table 2: The sets of values used for each judgment scale

uncertainty in inputs. Monte-Carlo methods have often been used previously for analyzing PCMs (Seaman et al., 1991; Herman and Koczkodaj, 1996; Koczkodaj, 1998; Ishizaka and Lusti, 2006).

PCMs were randomly generated using the sets of values given in Table 2. Each scale consists of equivalence (1), dominance (>1) and their reciprocal values. For each n ranging from 3 to 6, 50000 PCMs were generated by randomly picking the values from the selected scale. The CR of each PCM was measured using CR while the OC was measured with the number of circular triads, L .

The results from these experiments are discussed below.

3.1. Experimental Results

The percentage of the *acceptable* PCMs found for each scale is shown in Table 3. Considering the *linear* scale of 1 to 9, the percentage of *acceptable* PCMs decreases with the increase in n i.e. 22.21% for $n = 3$, 3.36% for $n = 4$ and only 0.26% for $n = 5$.

The *geometric* and *power* scales are least likely to produce *acceptable* PCMs. In the performed experiments, no PCM was found *acceptable* for $n = 5$ when using the *power* scale.

For the *inverse* and *root* scales, the percentage decreases monotonically as the value of n is increased. The possibility of having an acceptable PCM remains around 60% when using the *logarithmic* scale and is invariant to n . The chances of obtaining an acceptable PCM with the *balanced* scale are higher than with all the other tested scales and goes even higher for greater values of n (see Table 3). When PCMs were randomly generated for $n = 5$ using the *balanced* scale, all had CR below 0.1 (100%).

It is inferred from these results that the possibility of having an *acceptable* PCM is highly sensitive to the selected scale. In other words, the value of CR is sensitive to the measurement scale chosen for generating PCMs.

In order to investigate the use of CR for measuring OC, 15000 *acceptable* PCMs were generated for each scale. These matrices were then tested as to whether *transitive* ($L = 0$) or *intransitive* ($L \neq 0$). The

n	Linear	Balanced	Geometric	Inverse	Log	Power	Root
3	22.21%	94.59%	12.97%	41.78%	67.61%	10.10%	51.72%
4	3.36%	98.88%	0.47%	18.37%	58.71%	0.29%	29.91%
5	0.26%	99.93%	0.01%	7.34%	58.15%	0.00%	14.64%

Table 3: Percentage of *acceptable* PCMs in randomly generated matrices

Figure 1: Percentage of IAPs found in *acceptable* PCMs

intransitive acceptable PCMs (IAPs) were then counted for each scale. The results for $n = 5$ are shown in Fig. 1. Similar results were found for other values of n .

The percentage of IAPs was found to be very low for the *linear* and *geometric* scales (below 1%). No IAP was generated for the *power* scale. The *balanced* scale produced the highest percentage of IAPs, followed by the *logarithmic* scale. When the *inverse* scale was used, more than half (53.3%) of the generated PCMs were found to be *intransitive*. Similarly, 23.4% of *acceptable* PCMs were found to be *intransitive* when using the *root* scale.

From the results given above, it is concluded that *intransitive* PCMs are not always rejected by Saaty's criterion of $CR \leq 0.1$.

4. Congruence

Considering the CC test between E_i and E_j i.e. $a_{ij} = a_{ik}a_{kj}$ (for all i, j, k), if the DM's judgments are cardinally inconsistent, then we obtain at least one indirect judgment $a_{ik}a_{kj} \neq a_{ij}$ for some i, j and k where $i \neq k \neq j$. Fig. 2 illustrates this concept by locating direct and indirect judgments together on a judgment

Figure 2: The indirect judgments $(a_{ik}a_{kj})$ on a judgment scale

scale. Two of the indirect judgments in Fig. 2 suggest preference ratios higher than the direct judgment, while the third one (on the left) suggests a preference reversal (to be discussed in Section 5). When a DM provides a complete set of judgments in $J = \{a_{ij}\}$, a total of $\frac{n(n-1)(n-2)}{2}$ indirect judgments can be inferred.

Considering the dispersion of indirect judgments across the measurement scale, the deviation between a_{ij} and the indirect judgment $a_{ik}a_{kj}$ can be formulated in a variety of ways, such as:-

$$\delta(a_{ij}, a_{ik}a_{kj}) = |a_{ij} - a_{ik}a_{kj}| \quad (8)$$

$$\delta(a_{ij}, a_{ik}a_{kj}) = (a_{ij} - a_{ik}a_{kj})^2 \quad (9)$$

$$\delta(a_{ij}, a_{ik}a_{kj}) = \left| \frac{a_{ij} - a_{ik}a_{kj}}{a_{ij}} \right| \quad (10)$$

$$\delta(a_{ij}, a_{ik}a_{kj}) = \left(\frac{a_{ij} - a_{ik}a_{kj}}{a_{ij}} \right)^2 \quad (11)$$

$$\delta(a_{ij}, a_{ik}a_{kj}) = |\log a_{ij} - \log(a_{ik}a_{kj})| \quad (12)$$

$$\delta(a_{ij}, a_{ik}a_{kj}) = (\log a_{ij} - \log(a_{ik}a_{kj}))^2 \quad (13)$$

As multiple indirect judgments are possible for $n \geq 4$, the deviations need to be aggregated in order to define a measure of *congruence*, θ_{ij} for each judgment:-

$$\theta_{ij} = \frac{1}{K_{ij}} \sum_k \delta(a_{ij}, a_{ik}a_{kj}) \forall \{a_{ij}, a_{ik}, a_{kj}\} \subseteq J \quad (14)$$

where $k \neq i \neq j \neq k$ and K_{ij} is the total number of indirect paths available between E_i and E_j . When a complete set of judgments is available, the value of K_{ij} is equal to $(n-2)$ for all judgments. In the case of an incomplete set of judgments, the values of K_{ij} may have different values for different judgments.

The aggregation can also be formulated in other ways, for example, θ_{ij} can be taken as the worst deviation

among all the indirect judgments i.e.

$$\theta_{ij} = \max_k (\delta(a_{ij}, a_{ik}a_{kj}) \forall \{a_{ij}, a_{ik}, a_{kj}\} \subseteq J) \quad (15)$$

or the geometric mean of all the indirect judgments i.e.

$$\theta_{ij} = \left(\prod_k \delta(a_{ij}, a_{ik}a_{kj}) \right)^{\frac{1}{K_{ij}}} \forall \{a_{ij}, a_{ik}, a_{kj}\} \subseteq J$$

It is clear that a single global value measuring inconsistency, whether cardinal or ordinal, is insufficient to locate all the inconsistent judgments present in J . Although Koczkodaj's CM has the property of detecting the most inconsistent judgment, it fails to describe the contribution of other judgments to the overall inconsistency. It is, therefore, proposed to measure inconsistency with the help of these *individual congruence* values for a given set of judgments.

For example, consider the following set of judgments:-

$$\begin{aligned} J_1 : \quad & a_{12} = \frac{7}{4}, a_{13} = \frac{3}{4}, a_{14} = \frac{5}{2}, a_{15} = \frac{7}{4} \\ & a_{23} = \frac{3}{4}, a_{24} = \frac{9}{4}, a_{25} = \frac{9}{4} \\ & a_{34} = \frac{3}{4}, a_{45} = \frac{3}{4} \\ & a_{45} = \frac{5}{8} \end{aligned}$$

The set of judgments J_1 can be used to construct the following PCM:-

$$A_1 = \begin{bmatrix} 1 & \frac{7}{4} & \frac{3}{4} & \frac{5}{2} & \frac{7}{4} \\ \frac{4}{7} & 1 & \frac{3}{4} & \frac{9}{4} & \frac{9}{4} \\ \frac{4}{3} & \frac{4}{3} & 1 & \frac{3}{4} & \frac{3}{4} \\ \frac{2}{5} & \frac{4}{9} & \frac{4}{3} & 1 & \frac{5}{8} \\ \frac{4}{7} & \frac{4}{9} & \frac{4}{3} & \frac{8}{5} & 1 \end{bmatrix}$$

The value of CR for A_1 is equal to 0.083 and, therefore, A_1 is considered to be acceptable in AHP terms.

A *congruence* matrix for A_1 can be calculated using (14) and (12) as follows:-

$$\theta = \begin{bmatrix} - & 0.61 & 1.06 & 0.69 & 0.69 \\ & - & 1.11 & 0.77 & 0.89 \\ & & - & 1.12 & 1.00 \\ & & & - & 0.35 \\ & & & & - \end{bmatrix}$$

where the values for self-comparison and reciprocal judgments are not shown, being redundant. The use of logarithmic deviations here removes the possibility of having different values in the upper and lower triangles of the matrix. The values in θ can be sorted in descending order as follows:-

θ_{34}	θ_{23}	θ_{13}	θ_{35}	θ_{25}	θ_{24}	θ_{14}	θ_{15}	θ_{12}	θ_{45}
1.12	1.11	1.06	1.00	0.89	0.77	0.69	0.69	0.61	0.35

The judgment, a_{34} is found to be most inconsistent ($\theta_{34} = 1.12$), while a_{45} is considered the most consistent one. The two judgments a_{14} and a_{15} are found to be equally inconsistent (i.e. $\theta_{14} = \theta_{15} = 0.69$). Similarly, this can be calculated for other PCMs generated using different scales.

It is emphasized here that the values of θ_{ij} in this example are calculated using logarithmic deviation between direct and indirect judgments, as given in (12). The use of other definitions may produce different results and the appropriateness of each definition may be further investigated.

4.1. Overall Congruence

If need be, the set of values $\{\theta_{ij}\}$ can further be used to derive a single global measure - the overall *congruence* - through an aggregation procedure, for example:-

$$\Theta = \frac{1}{m} \sum \theta_{ij} \forall a_{ij} \in J \quad (16)$$

where m is the total number of judgments acquired in J . Like θ_{ij} , the overall congruence, Θ can also be formulated using other approaches, such as choosing the worst deviation amongst all the measured ones i.e.

$$\Theta = \max(\theta_{ij} \forall a_{ij} \in J) \quad (17)$$

It should be highlighted that Θ can have many definitions, depending on the way individual congruence values are aggregated and the formulation used for δ . For example, the overall congruence value for J_1 (in

the above example) will be equal to $\Theta = 0.828$ when using (16), or $\Theta = 1.12$ when using (17).

Recall: the value of CR is sensitive to the measurement scale and is only suitable for a linear scale of 1 to 9. In contrast, the *congruence* measure is applicable to any scale in a straight-forward manner. However, the threshold of acceptance for each scale may differ and requires investigation.

4.1.1. CM and GCI: Examples of Congruence

As discussed earlier, Aguaron and Moreno-Jimenez (2003) formalized GCI as the aggregated logarithmic residuals, defined in (3). It can be shown that the GCI measure can be formulated as a congruence measure by using logarithmic squared deviations (given in (13) for δ), (14) for θ and (16) for Θ .

Similarly, CM can also be formulated as a congruence measure. As discussed earlier, the value of CM is chosen from the most inconsistent *triple* amongst all possible combinations of indirect judgments (Koczkodaj and Szarek, 2010). On carefully observing the multiple definitions of δ , θ and Θ , it can be concluded that CM can be obtained with δ in (10), θ in (15), and Θ defined in (17).

In other words, it can be concluded that CM and GCI can be seen as the two possible aggregates for inconsistencies present in individual judgments.

4.2. Incomplete Sets of Judgments

Harker (1987) investigated an incomplete set of judgments, where DMs are allowed to respond with “don’t know” or “not sure” to some judgments. It has been highlighted that the probability of acquiring incomplete PCs increases as n increases (Fedrizzi and Giove, 2007; Fedrizzi, 2012). The proposed *congruence* measure is applicable to incomplete sets of PC judgments without modification, unlike CR which requires a complete set of PC judgments.

Consider the following incomplete form of A_1 :-

$$\check{A}_1 = \begin{bmatrix} 1 & \frac{7}{4} & - & \frac{5}{2} & \frac{7}{4} \\ \frac{4}{7} & 1 & \frac{3}{4} & \frac{9}{4} & \frac{9}{4} \\ - & \frac{4}{3} & 1 & \frac{3}{4} & \frac{3}{4} \\ \frac{2}{5} & \frac{4}{9} & \frac{4}{3} & 1 & \frac{5}{8} \\ \frac{4}{7} & \frac{4}{9} & \frac{4}{3} & \frac{8}{5} & 1 \end{bmatrix}$$

Here, the DM has not provided the judgment a_{13} and, therefore, the value of CR cannot be calculated for the incomplete PCM. In contrast, *congruence* values for the provided judgments are shown below:-

$$\check{\theta} = \begin{bmatrix} - & 0.42 & - & 0.19 & 0.31 \\ & - & 0.92 & 0.77 & 0.89 \\ & & - & 0.62 & 0.62 \\ & & & - & 0.35 \\ & & & & - \end{bmatrix}$$

The most inconsistent judgment is found to be a_{23} with the congruence value of $\check{\theta}_{23} = 0.92$. The judgment a_{25} is found to be the next most inconsistent while a_{45} is still considered to be the most consistent one. The overall congruence for $\check{A}_1$ is $\check{\Theta} = 0.51$.

4.3. Outlier Detection

The set of *congruence* values, θ_{ij} , can be used to help spot the outlier with the highest deviation value. In order to identify an outlier with the help of *congruence*, consider the following example:-

$$A_2 = \begin{bmatrix} 1 & 2 & 4 & 8 & \frac{1}{3} \\ \frac{1}{2} & 1 & 2 & 4 & 8 \\ \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 2 \\ 3 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

The *congruence matrix* for A_2 can be calculated using (12) and (14) as follows:-

$$\theta = \begin{bmatrix} - & 1.29 & 1.29 & 1.29 & 3.87 \\ & - & 0 & 0 & 1.29 \\ & & - & 0 & 1.29 \\ & & & - & 1.29 \\ & & & & - \end{bmatrix}$$

The value of $\theta_{15} = 3.87$ shows that a_{15} is the least congruent with other judgments. The three judgments of a_{23} , a_{34} and a_{24} are found to be perfectly consistent. All the other judgments make an equal contribution to the overall inconsistency. As there exists a single judgment that shows the highest deviation, this suggests the presence of an outlier i.e. a_{15} .

4.4. Consistency Deadlock

Consider another example of inconsistent PCM as follows:-

$$A_3 = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ \frac{1}{2} & 1 & 2 & 2 & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 & 2 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

The *congruence* matrix for A_3 is as follows:-

$$\theta = \begin{bmatrix} - & .69 & .69 & .69 & .69 \\ & - & .69 & .69 & .69 \\ & & - & .69 & .69 \\ & & & - & .69 \\ & & & & - \end{bmatrix}$$

This matrix tells a completely different story: all given judgments are found to be equally inconsistent. This type of situation can be termed “*consistency deadlock*” as no improvement can be suggested. A single global measure of consistency (e.g. CR) cannot describe the presence of consistency deadlock. In such cases, any of the provided judgments can be selected for revision.

5. Dissonance

The concept of indirect judgments can also be used to calculate OC, based on the ordinal violation between a given judgment and the corresponding indirect judgments. The preference relation $C_i \rightarrow C_j$ (implying $a_{ij} > 1$) should enforce $a_{ik}a_{kj} > 1$ for all $k = 1, 2, \dots, n$. Therefore, whenever an indirect judgment contradicts this preference relation, a *latent violation* occurs. The concept of latent violation can be visualized in Fig. 2, which shows a violation of a_{ij} by an indirect judgment through k_1 . This is mathematically formulated as:-

$$\psi_{ij} = \frac{1}{K_{ij}} \sum_k \text{step}(-\log a_{ij} \log a_{ik}a_{kj}) \forall \{a_{ij}, a_{ik}, a_{kj}\} \subseteq J \quad (18)$$

where $i \neq k \neq j \neq i$ and the *step* function is defined as:-

$$step(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

The set of values ψ_{ij} construct a *dissonance matrix* that can supplement the *congruence matrix* in order to spot the most inconsistent judgments. The term proposed here relates to the notion of *cognitive dissonance* i.e. the holding of two contradictory beliefs simultaneously.

It is proposed to calculate OC as the average of all the elements in $\{\psi_{ij}\}$ as *overall dissonance* (Ψ):-

$$\Psi = \frac{1}{m} \sum \psi_{ij} \forall a_{ij} \in J \quad (19)$$

where m is the total number of judgments acquired in J . The average is proposed instead of the aggregation to remove its dependence on n .

Although the values of CR , CM and Θ measure the level of inconsistency in judgments, Ψ uniquely measures ordinal inconsistency (as with Kendall's ζ), suggesting a number of possible preference reversals. This *dissonance* can be used to detect priority violations, as illustrated through examples in the next section.

The calculation of *congruence matrix*, $[\theta_{ij}]$, has been useful to analyze *cardinal* inconsistency. The use of *dissonance matrix*, $[\psi_{ij}]$, can further supplement the visualization of *ordinal* inconsistency. This is explained here with the help of examples already discussed above.

Considering A_1 again, the dissonance matrix is obtained using (18) as follows:-

$$\psi = \begin{bmatrix} - & 0.33 & 1.00 & 0.33 & 0.33 \\ & - & 0.67 & 0.33 & 0.33 \\ & & - & 1.00 & 0.67 \\ & & & - & 0.35 \\ & & & & - \end{bmatrix}$$

where transpose elements are not shown, as they are redundant. The value of $\psi_{34} = 1$ suggests a_{34} to be completely dissonant with all the related judgments in A_1 . The judgment of a_{34} , which was declared to be the most inconsistent, has also been found to be the most dissonant.

Consider another interesting example where the most inconsistent and the most dissonant judgments are different, as follows:-

	a_{12}	a_{13}	a_{15}	a_{23}	a_{35}	a_{14}	a_{24}	a_{45}	a_{34}	a_{25}
$\theta \Rightarrow$	2.10	1.87	1.66	1.60	1.53	1.43	1.33	1.16	1.13	0.95
$\psi \Rightarrow$	0.33	0.67	0.33	0.67	0	1	0.67	0.33	0.67	0

Table 4: Individual *congruence* & *dissonance* values for the judgments in A_4

$$A_4 = \begin{bmatrix} 1 & 9 & \frac{3}{4} & \frac{3}{4} & 9 \\ \frac{1}{9} & 1 & \frac{5}{4} & \frac{3}{4} & 4 \\ \frac{4}{3} & \frac{4}{5} & 1 & 2 & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{1}{2} & 1 & 3 \\ \frac{1}{9} & \frac{1}{4} & \frac{3}{4} & \frac{1}{3} & 1 \end{bmatrix}$$

The two consistency measures, calculated for each judgment in A_4 , are given in Table 4 where the judgments are arranged with the values of θ_{ij} in descending order. The value $\psi_{14} = 1$ for a_{14} implies that no judgment in A_4 supports the order of preference suggested by a_{14} . Whilst the most dissonant judgment is a_{14} (i.e. $\psi_{14} = 1$), the most inconsistent is found to be a_{12} according to the CC measure. This paradox has generated the question of whether CC should be given precedence over OC. An answer to this question remains an area of further investigation.

6. Approaches for Improving Consistency

When both the *ordinal* and the *cardinal* inconsistency in PC judgments are found, the next obvious step is to improve the consistency prior to the process of prioritization. Different techniques have been introduced to detect and rectify inconsistency, either by using a local operator or a matrix-based global operation (Zeshui and Cuiping, 1999; Cao et al., 2008; Ergu et al., 2011; Siraj et al., 2012). The improvements can be done manually by asking the DM to revise the judgments, or alternatively, improvements can be done using an automated approach.

6.1. Manual Revision of Judgments

In situations where the provided judgments are allowed to be revised, it is suggested that the DM is offered both θ_{ij} and ψ_{ij} . For example, consider the PCM, A_4 in the form of a table, shown in Fig. 3a. The DM is supposed to manually revise by changing the values of a_{ij} .

The two consistency measures, θ and ψ , are shown along with the original judgments in Fig. 3a. The values of θ_{ij} are shown above a_{ij} while ψ_{ij} is shown below a_{ij} in each cell. In practical situations, the DM

Figure 3: Offering θ and ψ for A_4

may find it difficult to make reasonably correct assessments as several numeric values on the table may cause information overload.

6.1.1. θ and ψ as a Graphical Aid

A better approach is to visualize the consistency of each judgment graphically as a bar graph. Fig. 3b shows this approach for A_4 , where *congruence* is shown as a grey bar with each judgment and *dissonance* is shown as a white bar. A DM can easily spot the most inconsistent judgment ($a_{12} = 9$) to manually improve its value. When OC has precedence, the value of a_{14} should be rectified first as it is ordinaly the most inconsistent.

This graphical approach is useful to spot both *outliers* and the phenomenon of *consistency deadlock*. Consider the two matrices A_2 and A_3 , already discussed in Section 4. Fig. 4a highlights the presence of an outlying judgment in A_2 where a_{15} is shown to be the most inconsistent, both cardinally and ordinaly. Similarly, the phenomenon of *consistency deadlock* for A_3 is visually represented in Fig. 4b. All the judgments in A_3 are shown here to be equally responsible for inconsistency. There is no ordinal inconsistency present in A_3 .

6.2. Reducing Inconsistency using an Automated Approach

There exist situations in practice where acquired data cannot be either changed or not allowed to be changed and, hence, manual revision of judgments is not possible. One way to address this is to automatically

Figure 4: Visualizing the phenomena of outlier detection and consistency deadlock

remove inconsistent judgments with minimal deviation from the initial judgments. This can be achieved using the *congruence matrix*; choosing the judgment with the highest value of θ . When a set of judgments is ordinaly inconsistent, the use of the *dissonance matrix* proves supplemental, suggesting a two-step process i.e. firstly, choose the set of judgments with the highest value of ψ and then, from these candidates, select the judgment with the highest value of θ .

In the presence of an *outlier*, a revised value for the blamed judgment can be calculated as:-

$$\tilde{a}_{ij} = \left(\prod_k a_{ik} a_{kj} \right)^{\frac{1}{n-2}} \quad (20)$$

where $i \neq k \neq j$. Another way to calculate a consistent value is to use the arithmetic mean i.e. $a_{ij} = \frac{1}{n-2} \sum_k a_{ik} a_{kj}$ where $i \neq k \neq j$. The use of the arithmetic or geometric mean is also an area for further investigation.

In the case of ordinal inconsistency, the preference order of the most inconsistent judgment needs to be reversed. The suggested averaging process may not work in the presence of ordinal inconsistency. Considering the minimum deviation from the original value, a revised value can be calculated as:-

$$\tilde{a}_{ij} = \begin{cases} 0.99 & \text{if } a_{ij} > 1 \\ 1.01 & \text{if } a_{ij} < 1 \end{cases} \quad (21)$$

Consider A_2 for example; the value for a_{15} can be changed from $\frac{1}{3}$ to 16 using (20). If minimal change is to be expected, then the value of a_{15} should be changed to 1.01 using (21). Considering the phenomenon of *consistency deadlock*, as all the judgments in A_3 are equally inconsistent; the only possible improvement is to randomly select a judgment and replace its value using (20) which is equal to 4.

In the case of A_1 , a_{12} is cardinally the most inconsistent and the suggested correction is $(a_{13}a_{32}a_{14}a_{42}a_{15}a_{52})^{\frac{1}{3}} = 1.1$. If ordinal inconsistency has priority, then a_{14} should instead be corrected and the suggested value for a_{14} is 1.01 using (21).

Holsztynski and Koczkodaj (1996) proved that removing local inconsistencies incrementally will improve the overall consistency of PCMs. The same proof justifies convergence of the improvement process using θ_{ij} and ψ_{ij} .

7. Illustrative Examples

To illustrate the concept of indirect judgments and the use of *congruence* and *dissonance*, four different types of PCMs are discussed below.

7.1. Consistent PCM

Consider an example of a consistent PCM given below:-

$$A_{ex1} = \begin{bmatrix} 1 & 2 & 4 & 12 \\ \frac{1}{2} & 1 & 2 & 6 \\ \frac{1}{4} & \frac{1}{2} & 1 & 3 \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{3} & 1 \end{bmatrix}$$

The set of all indirect judgments calculated for A_{ex1} are identical to the direct judgments, therefore $\Theta = 0$ and $\Psi = 0$. The other consistency measures, CR and CM, also produce the same value, as expected. Hence, A_{ex1} contains a set of *congruent* judgments.

This can be visualized in Fig. 5a where each judgment lying in the upper triangular part of A_{ex1} is shown on a logarithmic scale. A circle in the figure depicts a direct judgment while indirect judgments are shown as an upwards arrow. All the indirect judgments in this case appear aligned with their direct judgment, following the consistency rule of $a_{ij} = a_{ik}a_{kj}$. Hence, all the prioritization methods generate the same weights i.e. $w = [0.545 \ 0.273 \ 0.136 \ 0.045]$.

(a) A_{ex1} with consistent judgments

(b) A_{ex2} having inconsistent judgments

(c) A_{ex3} having transitive but dissonant judgments

(d) A_{ex4} having *intransitive* set of judgments

Figure 5: Graphical views of the direct and indirect judgments for the illustrated examples

Figure 6: Reducing Inconsistency in A_{ex2} using the Automated Approach

7.2. Transitive PCM

Consider the following inconsistent PCM with $CR < 0.1$ and $L = 0$:-

$$A_{ex2} = \begin{bmatrix} 1 & 2\frac{1}{2} & 4 & 9\frac{1}{2} \\ \frac{2}{5} & 1 & 3 & 6\frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & 1 & 5 \\ \frac{2}{19} & \frac{2}{13} & \frac{1}{5} & 1 \end{bmatrix}$$

Unlike A_{ex1} , the indirect judgments in A_{ex2} are not equal to the direct ones, yet the order of *preference dominance* remains. As can be seen in Fig. 5b, no indirect judgment has caused latent violations implying that the dissonance value is 0. Therefore, a preference vector with $NV = 0$ is possible for A_{ex2} . The values of $CR=0.045$, $CM=0.567$ and $\Theta=0.686$ all suggest cardinal inconsistency among judgments, while Ψ confirms that A_{ex2} is ordinally consistent.

Inconsistency in A_{ex2} can be improved using the automated approach as shown in Fig. 6. The contribution of each judgment in A_{ex2} towards overall inconsistency is shown on the left of the figure, where a_{34} is highlighted to be the most inconsistent. In the first iteration, the value of a_{34} is updated using (20). This reduces the overall inconsistency from $\Theta=0.686$ to $\Theta=0.314$. During next iteration, the judgment a_{12} is found to be the second most inconsistent and eventually been changed from $a_{12} = 2.5$ to $a_{12} = 1.4$ using (20). This step has further reduced the overall inconsistency from $\Theta=0.314$ to $\Theta=0.046$ (as shown on right side of Fig. 6).

Figure 7: Reducing Inconsistency in A_{ex3} using the Automated Approach

7.3. Transitive PCM with Latent Violations

Another PCM with $CR < 0.1$ and $L = 0$ is considered below:-

$$A_{ex3} = \begin{bmatrix} 1 & 3 & 2 & 6 \\ \frac{1}{3} & 1 & 1\frac{1}{5} & 2 \\ \frac{1}{2} & \frac{5}{6} & 1 & 3 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix}$$

A_{ex3} is included here to investigate second-order intransitivity. Hartvigsen (2005) pointed out that this comparison is transitive yet no method produces the *ideal ranking*. Considering the indirect judgments shown in Fig. 5c, we can easily identify latent violations between E_2 and E_3 . The direct judgment for a_{23} is 1.2, however, the two indirect judgments suggest 0.66, causing a priority violation whilst estimating weights. Although the values $CR = 0.031$, $CM = 0.555$ and $\Theta = 0.405$ highlight judgment inconsistency, the only measure suggesting a possibility of a priority violation is $\Psi = 0.167$.

Inconsistency in A_{ex3} can be improved using the automated approach as shown in Fig. 7. The congruence value of $\theta_{23} = 0.59$ and the dissonance value of $\psi_{23} = 1.0$ both suggest changing the value of a_{23} . The value of a_{23} can be changed from 1.2 to $\frac{2}{3}$ using (20). This makes the matrix A_{ex3} (fully) consistent.

7.4. Intransitive PCM

An *intransitive* PCM ($L \neq 0$) always contains dissonant judgments. This is illustrated below with the following *intransitive* PCM:-

Figure 8: Reducing Inconsistency in A_{ex4} using Automated Approach

$$A_{ex4} = \begin{bmatrix} 1 & 2 & \frac{3}{4} & 1\frac{1}{2} \\ \frac{1}{2} & 1 & 1\frac{1}{2} & 3 \\ \frac{4}{3} & \frac{2}{3} & 1 & 2 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}$$

There is a circular triad present in A_{ex4} i.e. $E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_1$. We discuss this example in detail with the help of Fig. 5d. The two indirect judgments for a_{34} are perfectly consistent, while the judgment a_{14} has one consistent and one inconsistent judgment. Each of the three judgments, a_{13} , a_{23} and a_{24} , show one latent violation. Finally looking at a_{12} , we see that both indirect judgments are in conflict with the direct judgment. Thus the dissonance value ϕ_{12} is maximum (equals to 1), suggesting the judgment a_{12} to be the outlier in this PCM.

Inconsistency in A_{ex4} can be improved using the congruence and dissonance matrices as shown in Fig. 8. The congruence value of $\theta_{12} = 1.39$ and the dissonance value of $\psi_{12} = 1.0$ both suggest changing the value of a_{12} . The judgment a_{12} can be updated from 2 to $\frac{1}{2}$ using (20). This makes the matrix A_{ex4} (fully) consistent as all the indirect judgments become aligned with the direct judgments after this revision.

The given examples have suggested measuring consistency as a pair of values i.e. (Θ, Ψ) . Further investigation is required to explore the properties of these two consistency measures.

7.5. Discussion

The *congruence* measure requires a threshold value as with, for example, $CR \leq 0.1$ that declares a PCM to be *acceptable*. Such a threshold would also enable individual judgments to be declared to be *acceptable* or *unacceptable*.

For purpose of demonstration, assume that the threshold value of 0.6 for *congruence* has been selected for the following two PCMs generated with the linear scale of 1 to 9:-

$$A_1 = \begin{bmatrix} 1 & 2 & 4 & 8 & \frac{1}{3} \\ \frac{1}{2} & 1 & 2 & 4 & 8 \\ \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 2 \\ 3 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 2 & 4 & 8 & \mathbf{9} \\ \frac{1}{2} & 1 & 2 & 4 & 8 \\ \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 2 \\ 3 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

The value of CR for A_1 is 0.55 suggesting that the PCM is *unacceptable* and the value of CR for A_2 is 0.009 that qualifies A_2 as an *acceptable* PCM. However, the use of CR does not reveal the sources of inconsistency in these PCMs.

The *congruence* matrices for A_1 and A_2 are shown below:-

$$A_1 : \theta = \begin{bmatrix} - & 1.29 & 1.29 & 1.29 & \mathbf{3.87} \\ & - & 0 & 0 & 1.29 \\ & & - & 0 & 1.29 \\ & & & - & 1.29 \\ & & & & - \end{bmatrix}$$

$$A_2 : \theta = \begin{bmatrix} - & 0.19 & 0.19 & 0.19 & \mathbf{0.58} \\ & - & 0 & 0 & 0.19 \\ & & - & 0 & 0.19 \\ & & & - & 0.19 \\ & & & & - \end{bmatrix}$$

The *congruence* measure detects a_{15} to be an outlier in both PCMs. The value of $\theta_{15} = 3.87$ for A_1 suggests that judgment a_{15} is highly inconsistent. The value of $\theta_{15} = 0.58$ for A_2 suggests that a_{15} is less

inconsistent in A_2 than in A_1 .

The overall *congruence*, Θ , for A_1 is equal to 1.16 using (16) (i.e. the average of the individual *congruence* values). The threshold of 0.6 suggests that the PCM is *unacceptable* as $\Theta > 0.6$. Based on this threshold, only three judgments in A_1 are found to be *acceptable* i.e. a_{23} , a_{34} and a_{24} .

A_2 has an overall *congruence* value of $\Theta = 0.17$. The threshold of 0.6 declares A_2 to be an *acceptable* PCM. All the judgments in A_2 are also considered to be *acceptable* when using this threshold value.

As demonstrated above, a properly justified threshold value for *congruence* will offer increased utility in comparison to the threshold available for CR. We consider this to be an important area for future work: finding an appropriate threshold value for the *congruence* measure to determine *acceptable* and *unacceptable* judgments in a PCM.

8. Conclusions

This paper has investigated a number of widely used consistency measures with the help of Monte-Carlo simulations. This has shown that a single global value for measuring inconsistency, whether cardinal or ordinal, is insufficient to locate all the inconsistent judgments present in a PCM. Hence, it is proposed to measure inconsistency at a fine-grained level to quantify the contribution of each judgment towards the overall inconsistency. A graph-theoretic approach is suggested for measuring level of inconsistency in the individual judgments. First, a generalized model is proposed to measure individual inconsistencies; and then, two new measures have been introduced: *congruence* for CC and *dissonance* for OC. The proposed measures are also applicable to incomplete sets of PC judgments without modification, unlike CR which requires a complete set of PC judgments. In addition, the *congruence* measure is applicable to any scale in a straight-forward manner, unlike CR which is only suitable for the linear and geometric scales.

The proposed measures are illustrated through examples, and it is concluded that the two matrices showing individual *congruence* and *dissonance* can be used to locate and (possibly) correct inconsistent (and/or outlying) judgments. The given examples have suggested measuring consistency as a pair of values i.e. (Θ, Ψ) . It is suggested that the use of these matrices should form a useful addition to PC-based decision support tools. Further investigation is required to explore the properties of these two consistency measures. For example, the threshold of acceptance/rejection for (Θ, Ψ) should be investigated for different scales. A numerical comparison with other measures should be performed in order to compare these thresholds.

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