Chaotic correlations in barrier billiards with arbitrary barriers

A. H. Osbaldestin* and L. N. C. Adamson†

Department of Mathematics

University of Portsmouth

Portsmouth

PO1 3HF, UK

June 19, 2013

Abstract

We study autocorrelation functions in symmetric barrier billiards for golden mean trajectories with arbitrary barriers. Renormalization analysis reveals the presence of a chaotic invariant set and thus that, for a typical barrier, there are chaotic correlations. The chaotic renormalization set is the analogue of the so-called orchid that arises in a generalized Harper equation.

PACS: 05.45.-a, 64.60.Ak

Keywords: Barrier billiards, renormalization

1 Introduction

There is a remarkable commonality between the following seemingly disparate physical problems:

- strange nonchaotic attractors in quasiperiodically forced systems [16], [3] (for a review see [5]);
- quasiperiodically forced two-level quantum system [4];
- barrier billiards [21];
- a generalized Harper equation [6].

Each in its own way may be considered as a “quasiperiodically forced” system, and our following remarks apply primarily to the case in which the “forcing frequency” is the golden mean $\omega = (\sqrt{5} - 1)/2$. (However, we shall briefly mention other frequencies later.) Further, each in its own way displays evidence of singular continuous spectra. For certain configurations, in the first three of these systems, correlations have been shown to display a self-similarity that is indicative of dynamical behavior on the border between regularity and chaos. In the latter case (generalized Harper equation), in the localized phase, the fluctuations in the wave function about the exponential decay display similar characteristics. These self-similarities have been explained by a renormalization analysis that we shall discuss later in this article. In brief, in each case, the self-similarity arises from the presence of periodic orbits of a renormalization operator. Rigorous analyses explaining this phenomenon have been carried out by the first author and coworkers in [9], [10], [1], and [14], respectively.

*email: andrew.osbaldestin@port.ac.uk

†email: luke.adamson@port.ac.uk

In the case of the generalized Harper equation, however, much more has been discovered. In [6] Ketoja and Satija reveal the presence of a universal renormalization strange set—the orchid—the existence of which implies that typically the fluctuations in the wave function are spatially chaotic and describable in terms of the dynamics of a renormalization operator on a certain functional space. In [11] a full explanation of this correspondence is given. Although the analysis for the generalized Harper equation is in the strong-coupling limit of the localized phase, and is for *fluctuations* in the wave function, Ketoja and Satija [6] conjecture that an orchid-like set might describe the wave function itself in the model’s critical phase.

For a brief review of previous works in these systems see [15]. Key to our understanding of renormalization in all of these problems is the dynamics of the functional recurrence

$$Q_n(x) = Q_{n-1}(-\omega x)Q_{n-2}(\omega^2 x + \omega), \quad (1.1)$$

or its additive version. The self-similarity is a consequence of periodic behavior in this recurrence from a given set of initial conditions (which are dictated by the system parameters). More generally, however, the dynamics of this recurrence are much more complicated.

In this article we continue the renormalization analyses by turning to a more general description of correlations in barrier billiards. The (asymptotically) exact self-similarity previously witnessed in [21] and explained in [1] is for the case of single-piece barriers whose height lies in the field $\mathbb{Q}(\omega)$ of rationals over ω . It was shown in [1] that such a configuration always leads to a periodic orbit of the renormalization operator (equivalently recurrence (1.1)) and thus (asymptotically) self-similar correlations. Here we look at more general barriers. Once again, orbits of the recurrence (1.1) are central to our work, the novelty being that the initial conditions will not give rise to periodic orbits.

Our consideration of barrier billiards follows from work of Wiersig [21] who first provided numerical evidence that for such billiards typical phase space functions exhibit a spectrum with a singular continuous component. Key to Wiersig’s analysis is a skew-product evolution equation for the sign of a phase-space variable. Previously Riley [17] had shown that these same skew-products do indeed typically display singular continuous spectra. (See [23] and references therein for results on the weak-mixing nature of such billiards using interval-exchange transformations.)

In Section 2 we briefly recap the derivation and role of Equation (1.1) for correlations in barrier billiards and show how it will be sufficient to consider a simple class of barriers consisting of a single piece. In Section 3 we report our numerical calculations for correlations of arbitrary barriers at Fibonacci times, revealing their chaotic nature (which we justify in Appendix C by calculation of the largest Lyapunov exponent of the associated time series). In addition to this we reconstruct the dynamics of the renormalization map in three dimensions revealing a “surface” on which these correlations lie. We end with some conclusions and an indication of future directions.

2 Golden mean renormalization for symmetric barrier billiards

2.1 Brief derivation of key equations

The barrier billiard problem is described as follows. A point unit mass undergoes elastic collisions according to the law that angle of incidence equals angle of reflection in a rectangular chamber $[-1, 1] \times [0, 1]$ in which a vertical barrier (perhaps consisting of many pieces) is located centrally as in Figure 1. Note that the evolution of $(|x|, y)$ is merely the simple integrable case of rectangular billiards in $[0, 1] \times [0, 1]$ which may be written in terms angle-coordinates (θ_x, θ_y) . (We ignore trajectories that hit corners we also ignore those that hit barrier ends.) Writing

$$x(t) = s(t)|x(t)|, \quad (2.1)$$

so that s is the sign of x (which we leave undefined whenever $x = 0$), the evolution is fully understood if we understand the evolution of s .

Figure 1: A symmetric barrier billiard and its corresponding barrier function $B(y)$.

It suffices to consider the discrete times at which the particle crosses the x -axis (equivalently a section at $x = 0$), and we define s_n to be the value of s just before the n th such time. The sign s_n will evolve according to the barrier: if there is a gap it will flip to the opposite sign so that $s_{n+1} = -s_n$; if there is a barrier it remains the same. We thus write

$$s_{n+1} = s_n B(y_n), \quad (2.2)$$

where y_n is the value of y at step n and B is the barrier function shown in Figure 1 describing the barrier.

Setting θ_n to be the value of the angle coordinate θ_y at the n th time step, the system may be considered to be “driven” by θ_n evolving as $\theta_{n+1} = \theta_n + \omega \pmod{1}$, where ω is the ratio of the signed (constant) one-dimensional velocities (momenta). The rotation number ω may equivalently be thought of as the angle of the trajectory. For full details of this derivation see [1].

Returning to the sign s_n , since we are now in angle coordinates. We define

$$\Phi(\theta) = B(f(\theta)) = B(y), \quad (2.3)$$

where $f(\theta) = 2\theta$, $\theta \in [0, 1/2]$, and $f(\theta) = 2 - 2\theta$, $\theta \in [1/2, 1]$. (Note that Φ possesses the symmetry $\Phi(1 - x) = \Phi(x)$ reflecting the fact that the particle sees the barrier both “on the way up” and then “on the way down”.) The barrier billiard dynamics may thus be understood in terms of the skew-product system

$$\theta_{n+1} = \theta_n + \omega \pmod{1}, \quad (2.4)$$

$$s_{n+1} = s_n \Phi(\theta_n). \quad (2.5)$$

The system (2.4–2.5) has solution

$$\theta_n = \theta_0 + n\omega \pmod{1}, \quad (2.6)$$

$$s_n = s_0 \prod_{k=0}^{n-1} \Phi(\theta_0 + k\omega). \quad (2.7)$$

However, in the case of irrational rotation number this does not illuminate the behavior of the sign s_n . We therefore consider its autocorrelation function

$$C(t) = \langle s_n s_{n+t} \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N s_n s_{n+t}, \quad (2.8)$$

(Note that $\langle s_n^2 \rangle = 1$ since $s_n = \pm 1$, and $\langle s_n \rangle = 0$ since θ_n is uniformly distributed for irrational ω . Also our average may also be considered to be taken over all initial s_0 and θ_0 .)

As in [3] (and [4], [9], [10]), we have (using the fact that θ_n is uniformly distributed)

$$C(t) = \int_0^1 S_t(\theta) d\theta,$$

where

$$S_t(\theta) = \prod_{k=0}^{t-1} \Phi(\theta + k\omega), \quad t \geq 1, \quad S_0(\theta) = 1. \quad (2.9)$$

We concentrate now on the case of golden mean rotation number, so that $\omega = (\sqrt{5}-1)/2$. We refer to the trajectory of the particle as the “golden mean trajectory.” The rational approximants to ω are now the ratios of successive Fibonacci numbers (F_n) where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The period-1 functions S_t then satisfy a recurrence relation at Fibonacci indices. Indeed we have

$$S_{F_n}(\theta) = S_{F_{n-1}}(\theta) S_{F_{n-2}}(\theta + F_{n-1}\omega) \quad (2.10)$$

Renormalizing S_{F_n} , defining $Q_n(x) = S_{F_n}((-\omega)^n x)$, the recurrence (2.10) becomes

$$Q_n(x) = Q_{n-1}(-\omega x) Q_{n-2}(\omega^2 x + \omega), \quad (2.11)$$

the recurrence (1.1) mentioned in the introduction. The initial conditions dictate the class of functions on which it is defined. In this case they are

$$Q_0(x) = 1, \quad Q_1(x) = \Phi(-\omega x), \quad (2.12)$$

piecewise-constant functions taking the values ± 1 . An important observation is that Q_1 inherits a symmetry from Φ , so that Q_1 cannot be chosen with arbitrary discontinuity locations.

In terms of the renormalized functions, the autocorrelation function $C(t)$ at Fibonacci times is

$$C(F_n) = \int_0^1 S_{F_n}(\theta) d\theta = \frac{1}{(-\omega)^{-n}} \int_0^{(-\omega)^{-n}} Q_n(x) dx, \quad (2.13)$$

and it is these integrals that are our primary concern in this article.

2.2 The barrier function Φ for an arbitrary barrier

For a barrier B consisting of several pieces, such as that shown in Figure 1, we may decompose its barrier function Φ_B in terms of the barrier functions of more elementary barriers. We make the following definitions.

Definition. A *barrier* B is a finite set of intervals $[l_i, u_i]$, $i = 1, 2, \dots, n$ satisfying $0 \leq l_1 < u_1 < l_2 < u_2 < \dots < l_n < u_n \leq 1$.

Definition. For $\alpha \in [0, 1]$, the α -*barrier* is the barrier consisting of a single interval $[0, \alpha]$.

Figure 2: The α -barrier and its associated barrier function $\Phi_\alpha(\theta)$.

The *barrier function* $\Phi_\alpha(\theta)$ of the α -barrier is defined to be

$$\Phi_\alpha(\theta) = \begin{cases} +1, & \theta \in [0, \alpha/2], [1 - \alpha/2, 1] \\ -1, & \text{otherwise.} \end{cases} \quad (2.14)$$

See Figure 2.

Note that the barrier function of the complementary barrier to the α -barrier—namely a single piece barrier of length $1 - \alpha$ attached to the top—is $-\Phi_\alpha(x)$. This observation enables us to deduce that the barrier function Φ_B for an arbitrary barrier B is given by the product

$$\Phi_B(x) = - \prod_{i=1}^n \Phi_{l_i}(x) \Phi_{u_i}(x). \quad (2.15)$$

A proof of this is given in Appendix A. The multiplicative nature of this expression and of the recurrence (1.1) means that we may study separately the behavior of simple α -barriers, combining such results for a more complicated barrier.

3 Chaotic correlations

To begin this section in Figure 3 we display a numerical evaluation of the autocorrelation function $C(t)$ for two different choices of α : $\alpha = \sqrt{2} - 1$ (the case which will be used for the remainder of this section) and $\alpha = \pi - 3$. These choices are somewhat arbitrary, but we believe them to be representative of the autocorrelation function for $\alpha \notin \mathbb{Q}(\omega)$.

Figure 3: Numerical evaluation of the autocorrelation function for a) $\alpha = \sqrt{2} - 1$ and b) $\alpha = \pi - 3$.

For a number of these α -barriers we have numerically approximated the integrals (2.13) for the autocorrelation function at Fibonacci times, $C(F_n)$. Further details on these calculations are given in Appendix B. Figures 4a and 4b show a plot of $C(F_{n+1})$ and $C(F_{n+2})$ respectively versus $C(F_n)$ for an α -barrier with $\alpha = \sqrt{2} - 1 \notin \mathbb{Q}(\omega)$ as above. An identical pair of images is seen for other such choices of α . As shown in [1], for a choice of α in $\mathbb{Q}(\omega)$ the resulting plot would display a finite set of points since the correlation function $C(F_n)$ is then (asymptotically) periodic in n . The chaotic nature here is apparent, further evidenced by our calculation that the largest Lyapunov exponent for this time series is approximately 0.2, see Appendix C. We thus claim that the dynamics show *chaotic correlations*.

We observe that the boundary of the region displayed in Figure 4a appears to consist of four symmetric hyperbolic curves, crossing the axes at $1 - 1/\sqrt{5} \approx 0.55$, the value calculated in [1] as the height of the peaks in the correlation function of the half-barrier. As yet, we have no explanation for this fact. Indeed we conjecture that the left and

Figure 4: Chaotic correlations.

right sides of the boundary are the hyperbola

$$C(F_{n+1})^2 - (1 - a^2)C(F_n)^2 = a^2, \quad (3.1)$$

where $a = 1 - 1/\sqrt{5}$. The remaining two curves follow by reflection along the diagonal. For Figure 4b we observe that the boundary is piecewise linear and intersects the axes at ω . These boundary curves are superposed on the figure (red on web version). Points lying outside these curves are attributed to the approximation of the integrals. We further observe that the correlations take values close to both zero and one. In particular, we conjecture that typical data is dense in the set displayed so that the correlations repeatedly return arbitrarily close to one.

Due to the symmetry of this image, and the multiplicative decomposition of an arbitrary barrier into α -barriers of Subsection 2.2, the attractor obtained for an arbitrary barrier will be identical.

Given the detailed explanation of the renormalization strange set (the orchid) arising in the generalized Harper equation [11], we anticipate an interpretation of the renormalization action here in terms of a skew-conjugacy to a subshift of finite type. A major simplifying feature of the present problem is that the function space in which we need to work is merely made up of (pairs of) piecewise constant functions with values ± 1 . This leads one to speculate that the description of this attractor might be much simpler than that of the orchid. The seemingly regular geometry in Figure 4 supports this expectation.

3.1 Reconstruction in three dimensions

If we look at the plots from Figure 4 we see that there appear to be dense clusters of points within each invariant set perhaps suggesting that we are looking through a projection of a higher dimensional object. In addition to this, in Appendix C we find the minimum embedding dimension to be 3 or 4 and so it seems prudent to plot our reconstructed “attractor” by plotting the triple $(C(F_n), C(F_{n+1}), C(F_{n+2}))$ as suggested in [20] in addition to a color bar representing the values of the fourth dimension $(C(F_{n+3}))$. This is shown in Figure 5.

We note that, as expected, looking in from any side of the cube at this object gives a view of one of the two boundary types from Figure 4, in particular there are four sides giving a view of the form shown in Figure 4a and two giving the form in Figure 4b. There is also a “void” around the origin in which no points lie; indeed there are no points within distance $\simeq 0.267$ of the center of the object. The closed dense set of points visible in Figure 4b is

a 2D projection of part of the boundary of this vacuum of points. The object has eight vertices each at the corners of the surrounding unit cube (for clarity we have depicted them as large colored dots) which are the endpoints of eight distorted “cones”, and by panning the object we can see that twelve pairs of these vertices appear to be “connected” i.e. there is a *smooth* path between them.

Figure 5: Plot of invariant set in three dimensions ($C(F_{n+3})$ is represented by the color bar.)

We see that the coloring representing $C(F_{n+3})$ takes high magnitude values along the edges of the cones leading to these vertices and more neutral values toward the center. Inspection of the aforementioned cones reveals that they are hollow. It is evident that the object in the center of the plot (the suspected boundary of the void around the origin) consists of the intersection of these eight cones, thus suggesting that the object is a surface.

4 Conclusion

We have extended our previous renormalization analysis of correlations in barrier billiards [1] to a consideration of arbitrary (finite) barriers. Numerically, we have seen that, in general, the renormalization transformation appears to behave chaotically, and thus the correlations in typical barrier billiards are chaotic. In many ways this phenomenon

is identical to the nature of the fluctuations in a generalized Harper equation, for which case the orchid of Ketoja and Satija [6] is a beautiful projection of the universal chaotic renormalization attractor. The analysis of this object in [11] should find application to the problem at hand. Moreover, the same works should find application to the structure of strange nonchaotic attractors themselves [7], and not just their correlations ([3], [9]).

It will be straightforward to generalize the results of this article to quadratic irrationals, as has been done for correlations in a quantum two-level system in [12], and for the generalized Harper equation in [13]. (See also [2] where the existence of a renormalization fixed point associated with the Harper equation at quadratic irrationals is established.) An appropriate generalization of recurrence (1.1) governs the behavior.

The generalization to arbitrary irrational numbers should also be possible. In this case a renormalization scheme in which the recurrence changes from step-to-step is appropriate. We intend to treat this problem in future works. Such a study should throw light on the challenging question of understanding the precise nature of the “pseudo-integrable” dynamics of barrier billiards: one would like to understand the problem of a fixed barrier, but consider (almost) all trajectories within it.

A Derivation of formula for an arbitrary barrier

In section 2.2 it was claimed that an arbitrary barrier B can be decomposed into a product of so-called α barriers (2.14). In this appendix we present a proof of this fact.

Theorem 1. *The barrier function for an arbitrary barrier B is given by;*

$$\Phi_B(x) = - \prod_{i=1}^n \Phi_{l_i}(x) \Phi_{u_i}(x). \quad (\text{A.1})$$

Where, as shown in Fig. 2, for $0 \leq \alpha \leq 1$,

$$\Phi_\alpha = \begin{cases} +1, & x \in [0, \frac{\alpha}{2}], [1 - \frac{\alpha}{2}, 1] \\ -1, & \text{otherwise.} \end{cases} \quad (\text{A.2})$$

Proof. Using the notation of section 2.2, we begin by looking at an arbitrary barrier attached to the base so in all cases $l_1 = 0$ and $-\prod_{i=1}^n \Phi_{l_i} \Phi_{u_i} = \prod_{i=2}^n \Phi_{l_i} \prod_{i=1}^n \Phi_{l_i}$. Firstly we observe that since $\Phi(1-x) = \Phi(x)$ we only need to consider the interval between 0 and $\frac{1}{2}$. The interval on which Φ_α is negative is therefore $(\frac{\alpha}{2}, \frac{1}{2})$.

If we take an arbitrary barrier B the corresponding $\Phi_B(x)$ will have $+1$ intervals between $[\frac{l_i}{2}, \frac{u_i}{2}]$, for $i = 1, 2, \dots, n$, the number of pieces, and -1 everywhere else on $[0, \frac{1}{2}]$. We are attached to the base so $l_1 = 0$ and hence $\Phi_{l_1} = -1$ so this multiplied with the minus sign outside the product gives us a $+1$ in the interval $[0, \frac{1}{2}]$. Next we multiply by Φ_{u_1} and so the resulting function is $+1$ on the interval $[\frac{l_1}{2} = 0, \frac{u_1}{2}]$ and -1 on the interval $[\frac{u_1}{2}, \frac{1}{2}]$. Then we multiply by Φ_{l_2} which is $+1$ from $[0, \frac{l_2}{2}]$ and -1 elsewhere so nothing changes up until $\frac{l_2}{2}$ which is greater than $\frac{u_1}{2}$ so that we get that the resulting function is $+1$ on $[\frac{l_1}{2}, \frac{u_1}{2}]$, -1 on $[\frac{u_1}{2}, \frac{l_2}{2}]$ and $+1$ everywhere else. Carrying on inductively we see that the barrier given by the formula (A.1) is $+1$ on $[\frac{l_i}{2}, \frac{u_i}{2}]$ and -1 elsewhere for $i \leq n$ as required.

In order to understand a general barrier B we need only look at barriers which are attached at the base since if we take one which is not attached at the base there is a corresponding attached one where we replace barriers by gaps which is equivalent to multiplying by minus one.

We can now see why the minus sign is on the outside of the product symbol. That is if we are not attached at the base $l_1 \neq 0$ so $\Phi_{l_1} \neq -1$ so we multiply the whole product by -1 . $\square$

Figure 6: A barrier attached at the base and its unattached partner.

B Summary of algorithm for the approximation of $C(F_n)$ for large n

In this Appendix we give a summary of the algorithm we have implemented for the approximation of the integrals $C(F_n)$:

1. Select a number X such that the average of Q_n over the interval $[-X, X]$ remains similar to the average value over $[-x, x]$ for $x > X$. Indeed, by selecting increasing values of X we have deduced that these integrals converge for large enough X . The values we have tried range between 20 and 1500, and we have found $X = 200$ to be a value which gives a good approximation whilst being small enough to ensure a respectable computing time.
2. Locate the discontinuities $\Phi(-\omega x) = Q_1(x)$ in the fundamental interval $[-\omega, 1]$ and thus locate the discontinuities of $Q_2, \dots, Q_{n_{\max}}$ inside $[-X, X]$ where $n_{\max}$ is the desired number of $C(F_n)$ values to calculate. Also note the values of the functions at $x = 0$.
3. Calculate N to be the first value of n such that Q_N has discontinuities which “fill up” the interval $[-X, X]$, thus ensuring any transient behavior has been eradicated.
4. We now calculate $C(F_N), \dots, C(F_{n_{\max}})$. If n is even then discard discontinuities in $[-X, 0]$, if n is odd discard discontinuities in $[0, X]$. We now calculate the area A from “left to right” starting at the first discontinuity of Q_n in the remaining interval. Then

$$C(F_n) \simeq \frac{Q_n(0) \times A}{X}.$$

C Calculation of the Largest Lyapunov Exponent

Some methods for generating the Largest Lyapunov Exponent (LLE) from an experimental time series have been suggested in [22] and [18] but here we adopt the method developed in [19] (“Rosenstein’s method”). We choose this method because it is generally robust for different choices of the embedding dimension and for noisy data with signal-to-noise ratio between 10 and 100. It also gives a qualitative assurance that the exponent is positive. The idea is to take our time series and use the method of delays [20] to reconstruct the attractor in m dimensional phase space. Given a time series $x_1, \dots, x_n$ we define

$$X_i = (x_i, x_{i+1}, \dots, x_{m+(i-1)})^T \quad i = 1, \dots, k, \quad (\text{C.1})$$

where $k = n - m + 1$. Before we can use Rosenstein’s method we first need to calculate the minimum embedding dimension, d , for our time series ($C(F_n)$). We do this using the method proposed in [8] based on Symplectic Geometry, which we choose for similar reasons as those stated above i.e. that it has been shown to be robust for short noisy time series. We define the $m \times m$ matrix $A = XX^T$ where $X = (X_1, \dots, X_k)$. This matrix is then repeatedly transformed using a Householder matrix H . We list the *squares* of the eigenvalues of the transformed

matrix as $\sigma_1 > \sigma_2 > \dots > \sigma_d > \sigma_{d+1} \geq \dots \geq \sigma_m$, and d is selected as the embedding dimension (we are looking for the values to “flatten out”). The algorithm does this for different values of m (here $m = 3, 4, \dots, 20$).

The output of this is displayed in Figure 7a. This shows *Symplectic Geometry Spectra* [8] and the observation we make is that all of these curves have “flattened out” after the dimension has reached 4, and so we select minimum dimension $d = 3$. An argument could be made for selecting $d = 4$ but because Rosenstein’s method is robust for choices of dimension $m > d$ it is best to choose $d = 3$ and check that the results gained from this selection concur when we select a higher dimensions.

Figure 7: Calculating embedding dimension and Lyapunov exponent.

We now summarize Rosenstein’s method [19]. The determination of d allows us to locate the nearest neighbor $X_{\hat{i}}$ to each X_i . Using the same notation as in [19] we write

$$d_i(0) = \min_{x_{\hat{i}}} \|X_i - X_{\hat{i}}\|. \quad (\text{C.2})$$

The Lyapunov exponent is now estimated by examining the rate of separation of these neighbors. In particular we define $d_i(j)$ to be the distance between the between the i^{th} pair of neighbors after j time steps. The largest Lyapunov measures the exponential growth of these separations and so $d_i(j) \simeq \zeta_i e^{\lambda_1 j}$ where ζ_i is the initial separation and λ_1 is the LLE. Taking logs we have

$$\ln d_i(j) = \ln \zeta_i + \lambda_1 j. \quad (\text{C.3})$$

A least squares fit is used to calculate the slope of the “average” line $y(j) = \langle \ln d_i(j) \rangle$ where $\langle \dots \rangle$ denotes the average over all i . Thus a plot of j versus $y(j)$ should contain a roughly linear region from which we extract our exponent. The obvious advantage of this method is the inclusion of *all* available data (a frequent criticism of prior methods was that they looked at only one trajectory) allowing it to be used for small amounts of experimental data.

A plot of time step j versus $y(j)$ is given in Figure 7b ($n = 30000$), and the key point to note is that the *slope is positive* giving us qualitative confirmation of a positive exponent and hence chaos. Note that the curve *saturates* as j increases because the average divergence is unable to exceed the length of the “attractor”, a feature which should be present if we have selected our embedding dimension correctly. Using $d = 4, 5$ and 6 as opposed to $d = 3$ gives a qualitatively similar plot which supports our selection of $d = 3$. The value of the LLE is estimated to be $\lambda_1 \simeq 0.4$ (to one decimal place).

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